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Jordan Decomposition in Lattices and Quasi Unipotence

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To Professor G.A Margulis with admiration

We prove that some power of the semi-simple and unipotent parts of the Jordan decomposition of an element of a lattice of a semi-simple linear real Lie group belong to the lattice.

As a corollary, we deduce that if the real rank of the Lie group is one, and the above unipotent part is non-trivial, then the semi-simple part is of finite order (i.e. the element of the lattice is **quasi-unipotent**).

In the special case when the symmetric space associated to the Lie group is of Hermitian type, this will imply that any holomorphic map of the punctured unit disc into the locally symmetric variety corresponding to the lattice (provided the lattice is a neat subgroup) takes the generator of the fundamental group of the punctured disc into a unipotent element of the lattice. This extends a result of A.Borel to the case when the lattice is not necessarily arithmetic.

1. Introduction

The Big Picard Theorem says that any holomorphic function of the punctured unit disc $\Delta^* = \{z \in \mathbb{C} :\mid z \mid< 1, z \neq 0\}$ which misses two values can not have an essential singularity at the puncture. This is, of course, equivalent to saying that any holomorphic map from the punctured unit disc into $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ extends to a holomorphic map from the full unit disc $\Delta = \{z \in \mathbb{C} :\mid z \mid< 1\}$ into $\mathbb{P}^1(\mathbb{C})$. Note that the Riemann surface $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ is the quotient of the

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upper half plane by the action of the discrete subgroup $SL(2,2\mathbb{Z})$ (the principal congruence subgroup of level two in $SL(2,\mathbb{Z})$), and hence has finite hyperbolic volume. Picard's proof shows also that the image of the loop around the puncture (in the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\})$, viewed as an element of $SL(2,2\mathbb{Z})/\{\pm 1\}$, is **unipotent** (a power of the loop around one of the three points in $\{0,1,\infty\}$).

If the upper half plane is replaced by any Hermitian symmetric domain D, and $SL(2,2\mathbb{Z})$ is replaced by any (torsion free) lattice Γ in the group of holomorphic automorphisms of the domain D, we may ask if any holomorphic map from Δ^* into the locally Hermitian symmetric complex manifold $\Gamma\backslash D$ (in fact, a quasi-projective variety, by [Bai-Bor]), takes the loop around the puncture into a (possibly trivial) unipotent element of Γ. It is a well known result of A.Borel (see [Del], [Gri], [Sch]) that if Γ is an **arithmetic subgroup**, then this is indeed the case. Borel's proof makes crucial use of the fact that Γ is arithmetic. The Arithmeticity Theorems of [Mar], [Cor], [Gro-Sch] show that Γ is almost always arithmetic.

In the present note, we will extend the result of Borel to the remaining case of quotients of the unit ball in $\mathbb{C}^n$ by lattices (which are not necessarily arithmetic groups).

To this end, we first analyse the Jordan decomposition of an element $\gamma \in G$ as a product $\gamma_s\gamma_u$, of two commuting elements such that γ_s is semi-simple and γ_u is unipotent. We prove that the semi-simple and unipotent parts of an element of Γ virtually belong to the lattice Γ. Precisely, we prove

Theorem 1. *Let Γ be a lattice in G, the group of real points of a semi-simple algebraic group over $\mathbb{R}$. Let $\gamma = \gamma_s\gamma_u$ be the Jordan decomposition of an element γ of Γ. Then there exists an integer $m \geq 1$ such that $\gamma_s^m, \gamma_u^m \in \Gamma$.*

If the real rank of G is one, and $\gamma \in \Gamma$ is such that $\gamma_u \neq 1$, then there exists an integer $m \geq 1$ such that $\gamma^m = \gamma_u^m$ i.e. γ is quasi-unipotent.

Assume now that K is a maximal compact subgroup of G such that the quotient $D = G/K$ is a Hermitian symmetric domain (of non-compact type). Assume that Γ is a **neat** subgroup of G, which is also a lattice in G ("Neat" [cf. [Ra], Chapter 4] means that in every algebraic linear complex representation of G, if an element of Γ has a root of unity as an eigenvalue, then that root of unity is one; in particular, Γ is torsion free). Since a lattice is a finitely generated subgroup of the linear group ([Ra], Remark (13.21)), it contains a neat subgroup of finite index ([Ra], Theorem (6.11)). We may replace Γ by a subgroup of finite index, without loss of generality.

Then, the quotient $\Gamma\backslash D$ is locally Hermitian symmetric. Let $f : \Delta^* \to \Gamma\backslash D$ be a holomorphic map of the punctured disc; denote by f_* the associated maps of fundamental groups (based at some point of Δ^* and its image under f). The fundamental group of the punctured disc is isomorphic to $\mathbb{Z}$; fix a generator θ of this $\mathbb{Z}$. The fundamental group of $\Gamma\backslash D$ is isomorphic to Γ.

Theorem 2. *Under the foregoing assumptions, the image $\gamma = f_*(\theta)$ in Γ is a unipotent element in Γ.*

The proof imitates Borel's proof of quasi-unipotence of the Picard-Lefschetz transformation (see [Gri], [Del], section 6, [Sch], Lemma (4.5)) associated to a variation of Hodge structure on Δ^*. It uses the fact that if Δ^* is equipped with the hyperbolic metric and $\Gamma\backslash D$ is equipped with a suitable multiple of the G-invariant metric on D arising from the Killing form, then the **holomorphic** map f decreases distances (see [Kob], [Kwa]). In section 4, we give an elementary proof of this distance decreasing property.

It then follows from Borel's argument that closure of the conjugacy class of the element γ (of Theorem 2) in G intersects the compact group K. We then use Theorem 1 to conclude that the semi-simple part γ_s is of finite order. The fact that Γ is a neat subgroup implies that γ is actually unipotent.

Remark 1. If Γ is not assumed to be a **lattice**, but only a discrete subgroup, then Theorem 2 is false in general. For example, suppose $G = SL_2(\mathbb{R}) \times SU(1,1)$, and $k \in SU(1,1)$ is a diagonal element $k = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}$, where θ is **irrational**, and $0 < \theta < 1/2$. Now k acts on Δ, the unit disc in $\mathbb{C}$, as multiplication by the scalar $e^{4\pi i\theta}$. Let $u \in SL_2(\mathbb{R})$, where u is the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. On the upper half plane its action is $\tau \mapsto \tau + 1$.

Set $\gamma = (u, k) \in G$. The group generated by γ is discrete in G. The map $F : \tau \mapsto (\tau, e^{4\pi i\theta\tau})$ maps the upper half plane H into the product $H \times \Delta$ (the symmetric space of G). Moreover, $F(\tau + 1) = \gamma F(\tau)$. This shows that F yields a holomorphic map from Δ^* into the quotient manifold $\gamma^{\mathbb{Z}}\backslash(H \times \Delta)$ and the monodromy being γ, is not quasi-unipotent.

Remark 2. However, suppose $G = SL_2(\mathbb{R})$, and Γ **any** (not necessarily of finite covolume) discrete torsion-free subgroup. If $f : \Delta^* \to \Gamma\backslash H$ is a holomorphic, then the monodromy $f_*(\theta)$ is indeed a unipotent element of Γ. For example, this follows from the fact that there are no holomorphic maps from Δ^* into a bounded annulus in $\mathbb{C}$ inducing isomorphisms of the fundamental groups (the singularity of any such map at the point 0 is removable).

2. Proof of Theorem 1

2.1. **Preliminary Remarks.** In the statement of Theorem 1, one may replace Γ by a subgroup of finite index, and the conclusion remains the same. If $G = G_1 \times G_2$ is a product and $\Gamma = \Gamma_1 \times \Gamma_2$ with each Γ_i a lattice in G_i, it is easy to see that if Theorem 1 holds for each $\Gamma_i (i = 1, 2)$, then Theorem 1 holds for Γ as well. Hence we can (and we do) assume that Γ is an irreducible lattice.

Suppose that Γ is arithmetic, and $\gamma = \gamma_s \gamma_u$. Then there exists a semi-simple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, such that $\mathbf{G}(\mathbb{R}) = G$ up to compact factors, and $\mathbf{G}(\mathbb{Z}) = \Gamma$ up to commensurability. We may assume (see the remarks of the foregoing paragraph) that $G = \mathbf{G}(\mathbb{R})$, and $\Gamma = \mathbf{G}(\mathbb{Z})$. The Jordan components of γ lie in $\mathbf{G}(\mathbb{Q})$, and since γ_u is unipotent, some power γ_u^m of it lies in the integral group $\mathbf{G}(\mathbb{Z})$. Hence γ_s^m lies in $\mathbf{G}(\mathbb{Z})$, and this proves Theorem 1.

If $\mathbb{R} - rank(G) \geq 2$, then by [Mar], Γ is arithmetic and Theorem 1 holds. (If G is locally isomorphic to $Sp(n, 1)$ or the real rank one form of F_4, then by [Cor], [Gro-Sch], Γ is arithmetic and Theorem 1 again holds).

We may thus assume that G has real rank one (and even that G is locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$, but we do not use this assumption since the arguments do not become any simpler under this assumption). Note that (e.g.) by [GR-PS] in the case of $SO(n, 1)$ and [Del-Mos] in the case of $SU(n, 1)$ (for small n), there do exist non-arithmetic lattices.

2.2. **Notation.** We refer to [Bor-Tit] for the following facts on algebraic groups. In what follows, for ease of notation, we identify a real algebraic group H with the group $H(\mathbb{R})$ of real points (this causes no confusion, since the group of real points is Zariski dense and completely determines the group).

Let G be a simple algebraic group over $\mathbb{R}$ of $\mathbb{R}$-rank one. Fix a minimal parabolic subgroup P of G, and let U be its unipotent radical. There exists a maximal split torus S of rank one in P such that if $Z(S)$ denotes the centraliser of S in P, then P is the product $P = Z(S)U$. Moreover, if M is a maximal compact subgroup of $Z(S)$ (then it is a maximal compact subgroup of P as well), then $Z(S)$ is the product MS. The split real torus S is isomorphic to R^*.

The group U is a maximal unipotent subgroup of G; any two maximal unipotent subgroups of G are conjugate under G. Any unipotent element of G lies in a maximal unipotent subgroup of G. There exists an element w of G which normalises S but does not centralise S, and it generates the Weyl group $N(S)/Z(S)$ (which is of order two). Here $N(S)$ (resp. $Z(S)$) is the normaliser (resp. centraliser) of S in the group G. We have the **Bruhat decomposition** $G = P\bigcup UwP$. The Bruhat decomposition shows immediately that the intersection of any two distinct maximal unipotent subgroups is the singleton $\{1\}$. Denote by U^- the conjugate wUw^{-1}.

If $\mathfrak{u}$ denotes the (real) Lie algebra of U, then the adjoint action of the split torus S on $\mathfrak{u}$ is semi-simple; in fact there exists a homomorphism (positive root) $\alpha : S \to \mathbb{G}_m$ such that we have the root space decomposition $\mathfrak{u} = \mathfrak{u}_\alpha \oplus \mathfrak{u}_{2\alpha}$. We may identify α as a map $x \mapsto x^r$ for some integer $r \geq 1$ by a suitable isomorphism $S = \mathbb{G}_m$. Moreover, the exponential map $\mathfrak{u} \to U$ is an isomorphism of varieties and is equivariant with respect to the conjugation action of S on both sides. It follows that if $s \in S = \mathbb{R}^*$ is an element with $s > 1$ (and therefore $\alpha(s) = s^r > 1$), then for all $u \in U$,

$$s^{-m}us^m \to 1$$

as $m \to +\infty$.

Lemma 3. *If $g \in G$ is an element with a Jordan decomposition $g = g_s g_u$ with $g_u \neq 1$, then the semi-simple part g_s lies in a compact subgroup.*

Proof. After a conjugation, we may assume that $g_u \in U$. Since g_s commutes with g_u, g_u also lies in the conjugate $g_s U g_s^{-1}$. Since $\mathbb{R}$-rank of G is one, this means that g_s normalises U. Hence g_s lies in P.

Let F be the connected component of the Zariski closure of the group generated by powers of the semi-simple element g_s. Therefore, Z is commutative and hence is a torus. Moreover, F centralises the element g_u since g_s does. If $\mathbb{R}$-rank of F is non-zero, then F contains an $\mathbb{R}$-split torus; but all $\mathbb{R}$-split tori in P are conjugate to S in P. Hence, replacing g_s by a conjugate, we may assume that F contains the torus S.

But the torus S operates on the Lie algebra $\mathfrak{u}$ by the characters α and α^2 neither of which is one; hence under conjugation on U, the torus S has no fixed point and therefore g_u can not be centralised by S. This shows that $\mathbb{R}$-rank of F is zero: i.e. F is compact. Hence g_s lies in a compact subgroup of G. $\square$

Lemma 4. *If G is a simple algebraic group of real rank one and H is a connected nilpotent subgroup of G generated by unipotent elements, then H is contained in a conjugate of the maximal unipotent subgroup U of G.*

Proof. Suppose that the Zariski closure H_0 of H in G has no unipotent radical. Then H_0 is reductive. The group H_0 is nilpotent since H is. Therefore, the semi-simple part of H_0 is trivial, i.e. H_0 is a torus. But then H_0 does not contain any unipotent elements.

This shows that the unipotent radical of H_0 is non-trivial. By a result of [Bor-Tit], it follows that H_0 (and hence H) is contained in a proper parabolic subgroup of G. But since the real rank of G is one, all the proper parabolic subgroups of G are conjugate to P.

However, the quotient $P/U = SM$ has no unipotent elements since M is compact and S is a torus. Since H is generated by unipotent elements, H lies in U. $\square$

Lemma 5. *Suppose G is a simple Lie group of $\mathbb{R}$-rank one. Let u and v be two non-trivial elements of the unipotent group U and let $g \in G$. If u and $g^{-1}vg$ generate a nilpotent Zariski closed subgroup of G, then g lies in the parabolic subgroup P and $g^{-1}vg \in U$.*

Proof. Since u, v are nilpotent, the Zariski closed subgroup generated by them is connected. Denote this group by $< u, g^{-1}vg >$. If $g \notin P$, then we may write its Bruhat decomposition as $g = u_1wzv_1$ with $u_1, v_1 \in U$ and $z \in Z(S)$.

If F is a subgroup of G and $f \in F, h \in G$, denote by f^h, F^h (resp. ${}^h(f), {}^h(F)$) the conjugate $h^{-1}fh, h^{-1}Fh$ (resp. hfh^{-1}, hFh^{-1}).

We write $g^{-1}vg = ((zv_1)^{-1})w^{-1}u_1^{-1})v(u_1wzv_1)$. Put $w^{-1}u_1^{-1}vu_1w = v_2$ and put $u_2 = (zv_1)u(zv_1)^{-1}$. It is then clear that the Zariski closed group $< u_2, v_2 >$ generated by u_2, v_2 is the conjugate

$$< u_2, v_2 > = <^{zv_1}(u), ^{w^{-1}u_1^{-1}}(v) > = ^{zv_1}< u, v^g > = ^{zv_1}(H),$$

where H is the group $< u, v^g >$. Since H is nilpotent by assumption, Lemma 4 shows that H is contained in a conjugate of U. But H contains $u \in U$ and since G has real rank one, every unipotent element is contained in a unique maximal unipotent subgroup of G. Hence H is contained in U and since $zv_1 \in P$ normalises U, it follows that $< u_2, v_2 >$ lies in U. But v_2 being a w-conjugate of a unipotent element of U, clearly lies in the opposite unipotent group U^- and hence can not

lie in U.
This contradiction shows that $g \in P$, and the Lemma follows. □

2.3. **Non-cocompact Lattices in rank one groups.** Assume that Γ is a lattice in the real rank one group G. We keep the notation of (2.2).

Lemma 6. *There exist finitely many maximal unipotent subgroups $U(\Gamma)$ in Γ such that*

(1) the group $U(\Gamma)$ is a cocompact lattice in its Zariski closure; the latter is a maximal unipotent subgroup of G.

(2) If $\{h_n\}_{n\geq 1}$ is a sequence of elements in G tending to infinity in the quotient $\Gamma\backslash G$, then there exists a sequence $\{\theta_n\}_{n\geq 1}$ of elements in Γ, some $U(\Gamma)$ among these finitely many, and and an element $u = u(\Gamma)$ in $U(\Gamma)$, such that after replacing h_n by a subsequence if necessary,

$$h_n^{-1}\theta_n^{-1}u\theta_n h_n \to 1.$$

Proof. Since G has real rank one, every unipotent element is contained in a unique maximal unipotent subgroup of G. All maximal unipotent subgroups of G are conjugate. Let U be one such, and S,P be as in (2.2). By Theorem (0.6) of [Gar-Ra], there exists finite set Ξ of conjugates of U such that for each $V \in \Xi$, $V \cap \Gamma$ is a maximal unipotent subgroup of Γ and is a cocompact lattice in V and hence has V as its Zariski closure. We take the collection $U(\Gamma)$ in the Lemma to be these groups $V \cap \Gamma$ with $V \in \Xi$.

Moreover, the description of the fundamental domain in Theorem (0.6) of [Gar-Ra] is such that if a sequence $h_n \in \Gamma\backslash G$ tends to infinity, then there exist (1) a sequence θ_n of elements of Γ and (2) some $V \in \Xi$ such that

$$h_n^{-1}\theta_n^{-1}vh_n\theta_n \to 1$$

for some $v \in V \cap \Gamma, v \neq 1$. □

We record the well known Zassenhaus Lemma (see Theorem (8.16) of [Ra]).

Lemma 7. *There exists a compact neighbourhood W of identity in G such that if Δ is any discrete subgroup of G, then the intersection $W \cap \Delta$ generates a connected nilpotent subgroup of G.*

2.4. **Proof of Theorem 1.** Let $\gamma = \gamma_s\gamma_u$ be the Jordan decomposition of an element γ of Γ and assume that the unipotent part γ_u is not identity. Let U be the maximal unipotent subgroup of G containing γ_u. Let P be the normaliser of U. Let S and M be as in (2.2). Since M is a maximal compact subgroup of P, after a conjugation by an element of P, we may assume that $\gamma_s \in M$. By (2.2), it follows that there exists an element $s \in S$ such that $s^{-m}\gamma_u s^m \to 1$ as $m \to +\infty$.

Since $s \in S$ s centralises M and hence centralises the semi-simple part γ_s. By the remark at the end of (2.2), we have

(1) $s^{-m_l}\gamma s^{m_l} = \gamma_s s^{-m_l}\gamma_u s^{m_l} \to \gamma_s$,

and g_l tends to g.

We analyse two cases.

(i) There exists a sequence $\{m_l\}_{l\geq 1}$ of positive integers such that the sequence s^{m_l} converges in the quotient $\Gamma\backslash G$. That is, there is a sequence $\{\theta_l\}_{l\geq 1}$ of elements of the lattice Γ such that the sequence $g_l \overset{defn}{=} \theta_l s^{m_l}$ converges to an element g in G. We have $g_l s^{-m_l}\gamma s^{m_l} g_l^{-1} = \theta_l\gamma\theta_l^{-1}$.

Therefore, by equation(1), the sequence $\theta_l\gamma\theta_l^{-1}$ of elements of Γ converges to the semi-simple element $g\gamma_s g^{-1}$; hence this sequence is (after a certain stage) constant, and hence γ is conjugate to γ_s. Therefore, γ is semi-simple, contradicting the assumption that $\gamma_u \neq 1$.

(ii) The sequence $\{s^m\}_{m\geq 1}$ has no convergent subsequence in $\Gamma\backslash G$. By Lemma 6, there exist a subsequence $\{s^{m_l}\}_{l\geq 1}$, a sequence θ_l of elements of Γ and a fixed element $u \in U(\Gamma) \subset \Gamma$ such that, as l tends to infinity,

(2) $s^{-m_l}\theta_l^{-1}u\theta_l s^{m_l} \to 1$.

The equations (1) and (2) show that the elements $\alpha_l \overset{defn}{=} \theta_l^{-1}u\theta_l$ and $\beta_l \overset{defn}{=} \gamma^{-1}\alpha_l\gamma$ are conjugated into an arbitrarily small nieghbourhood of identity in G, provided l is large enough. Now Lemma 7 implies that if l is large enough, α_l and β_l are (unipotent and) contained in a connected nilpotent subgroup H of G. Now, Lemma 5 implies (since $\alpha_l \in U^{\theta_l}$) that $\gamma \in P^{\theta_l}$ for some (large) integer l. We fix this integer l. We may replace γ by the conjugate γ^{θ_l} and assume thus that $\theta_l = 1$.

We therefore have: $U \cap \Gamma$ is a co-compact lattice in U. Moreover, $\gamma \in P$. By Lemma 3, the semi-simple part γ_s lies in a compact subgroup of P, hence, after a further conjugation by an element of P, we may assume that $\gamma_s \in M$ (M being the maximal compact subgroup of P. Now, the group MU is an extension of the unipotent group U by a compact group, and $U \cap \Gamma$ is a lattice in U, hence is a lattice in $MU \cap \Gamma$. It follows that $U \cap \Gamma$ is of finite index in $MU \cap \Gamma$. Since $\gamma \in M \subset MU$, it follows that some power of γ lies in $\Gamma \cap U$. That is, γ is quasi-unipotent. This proves Theorem 1.

3. Quasi-unipotence of Non-arithmetic Monodromy

3.1. **Notation.** We will now assume that G is a semi-simple real algebraic group, such that if K is a maximal compact subgroup of G, then the symmetric space G/K is a **Hermitian symmetric domain** of non-compact type. Let Γ be a lattice in G.If Γ is torsion free, then it operates without fixed points on G/K and the quotient $\Gamma\backslash G/K$ is a complex manifold with universal covering space G/K. In fact, if Γ is arithmetic, it is even a quasi projective variety ([Bai-Bor]). We consider holomorphic maps f from Δ^* into $\Gamma\backslash G/K$, where Δ^* is the punctured unit disc, i.e. unit disc in $\mathbb{C}$ with the origin removed.

As in Theorem 2, denote by f_* the associated maps of fundamental groups. The left side has fundamental group $\mathbb{Z}$ and the right side has fundamental group Γ. Fix a generator δ of $\mathbb{Z}$ and denote by γ its image in Γ. Denote by f^* a lift of f- a map of the universal coverings Δ into G/K; here Δ is the unit disc in $\mathbb{C}$. As Δ is bihomorphic to the upper half plane H, we have an associated lifted map $F : H \to G/K$. The covering $H \to \Delta^*$ is given by $\tau \mapsto e^{2\pi i\tau}$; the deck transformation group is (isomorphic to $\mathbb{Z}$ and is generated by $\tau \mapsto \tau + 1$. Therefore, the lift F has the transformation property

(3) $F(\tau + 1) = \gamma F(\tau)$.

Equip Δ by the hyperbolic metric and G/K the G-invariant metric coming from the Killing from on G. Denote the Riemannian manifolds by (Δ, d_{hyp}) and $(G/K, d_g)$.

Lemma 8. *Given any holomorphic map $h : \Delta \to G/K$, there exists a constant $C > 0$ such that for all $p, q \in \Delta$, we have*

(4) $d_g(h(p), h(q)) \leq C d_{hyp}(p, q)$.

In other words, after rescaling the metrics, any holomorphic map from the disc into the symmetric space G/K is distance decreasing.

For a proof of this result see [Kob], Theorem (4.1) of Chapter III. In section 4, we will give a simple proof of this Lemma.

3.2. **Proof of Theorem 2.**

Proof. Let F be the lift of f in Theorem 2, as a map from the upper half plane H into the Hermitian symmetric domain G/K. For $n \in \mathbb{Z}, n \geq 1$, let $\tau_n = in \in H$. Then, as n tends to $+\infty$,

(5) $d_{hyp}(\tau_n, \tau_n + 1) \to 0$.

By (3),(4) and (5) it follows that as $n \to +\infty$,

(6) $d_g(F(\tau_n), \gamma \circ F(\tau_n)) \to 0$.

Now, $F(\tau_n)$ is a point on G/K. Hence there exists a sequence $\{g_n\}$ of elements of G such that $g_n K = F(\tau_n)$ for each n. The equation (6) (and the fact that left translation by g_n is an isometry on $(G/K, d_g)$) shows that as $n \to +\infty$,

(7) $d_g(K, g_n^{-1}\gamma g_n K) \to 0$.

Therefore, since K is compact, we may replace g_n by a subsequence and conclude that the sequence $g_n^{-1}\gamma g_n$ converges to an element of K.

We may easily reduce the proof of Theorem 2 to the case when Γ is an irreducible lattice in G; we assume this henceforth.

Case 1. The lattice Γ is arithmetic. We have a semi-simple algebraic group $\mathbf{G}$ over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{R}) = G$ up to compact factors and $\Gamma = \mathbf{G}(\mathbb{Z})$ up to commensurability. We may replace G by $G(\mathbb{R})$ and Γ by a torsion free neat subgroup of $G(\mathbb{Z})$ of finite index. Then the conclusion $g_n^{-1}\gamma g_n \to k \in K$ implies that the characteristic polynomial of γ^r (in some fixed $\mathbb{Q}$-linear embedding of G in GL_n over $\mathbb{Q}$) is that of the element k^r which lies in the compact group K for **each** $r \geq 1$. But the characteristic polynomial of γ^r has integral coefficients, which means that the characteristic polynomials of γ^r for varying r run through a finite set; therefore, all the eigenvalues of γ are roots of unity. Since Γ is a neat subgroup, this implies that γ is unipotent. This yields Theorem 2.

Case 2. The lattice is non-arithmetic. Therefore, by [Mar], $\mathbb{R} - rank\ (G) = 1$. By Theorem 1, if the Jordan decomposition γ has non-trivial unipotent component, then γ is quasi-unipotent; since Γ is neat, γ is actually unipotent.

The other possibility is that γ is semi-simple. In that case, the conjugacy class of γ is closed ([Bor-Tit]). Hence k is conjugate to γ which shows that $\gamma \in \Gamma$ lies in a conjugate of K- a compact group. Therefore, γ is of finite order; but the lattice Γ is neat, hence γ is trivial. This proves Theorem 2 in all cases. □

Remark 3. The foregoing proof follows closely, A.Borel's proof of quasi-unipotence of monodromy as explained in [Del], section 6. The proof in the non-arithmetic case needs some modification, and that is provided by Theorem 1.

4. Distance Decreasing Property of Holomorphic Mappings.

4.1. **Notation.** We first prove a result which may be of independent interest. Let D be a complex manifold, realisable as a bounded submanifold of the complex n-space $\mathbb{C}^n$ with the standard metric on $\mathbb{C}^n$. Suppose $D \subset B(0, R)$ where $B(0, R)$

is the ball of radius R centred at the origin. Suppose that there is a Hermitian metric g on D such that the group G of holomorphic automorphisms of D which are isometries of (D, g) act transitively on D. Let $h : \Delta \to D$ be a holomorphic mapping from the open unit disc into D. Equip Δ with the hyperbolic metric. We then have

Lemma 9. *If $0 \in D$ and $h(0) = 0$, then*

$$| h'(0) | \leq R.$$

Proof. This is a part of Schwarz' Lemma: fix an r with $0 < r < 1$. Since $D \subset B(0, R)$, we have $| h(z) | < R$ for all $z \in \Delta$. Hence for $z \in \Delta$ with $| z | = r$, we have $| h(z)/z | < R/r$. The function $h(z)/z$ is holomorphic on Δ and hence satisfies the maximum principle. Hence for $z \in \Delta$ with $| z | \leq r$ we have $| h(z)/z | \leq R/r$. Letting r tend to 1, we see that $| h(z)/z | \leq R$ for all $z \in \Delta$. Letting z tend to 0, we get the lemma. □

Remark 4. Note that by the Harish-Chandra Embedding Theorem, every Hermitian symmetric space of non-compact type is a bounded domain in $\mathbb{C}^n$ for some n, and satisfies the hypotheses for the manifold D.

Let $z \in \Delta$, v a non-zero vector in the tangent space $T_{f(z)}(D)$ at the point $f(z)$ of D. Then g defines a Hermitian metric on $T_{f(z)}(D)$; denote by $|| v ||_g^2$ the inner product of v with itself with respect to this metric.
We may similarly define $|| w ||_{hyp}^2$ for $w \in T_z(\Delta)$, with respect to the hyperbolic metric on Δ. Denote by df the holomorphic differential of f. If $w \neq 0$, consider the ratio

$$\phi_f(z) = \frac{|| df(w) ||_g^2}{|| w ||_{hyp}^2}.$$

Note that since $T_z(\Delta)$ is one dimensional, this ratio is independent of the non-zero vector $w \in T_x(\Delta)$ chosen.

Lemma 10. *With the foregoing notation, we have*

[1] If θ is a holomorphic automorphism of Δ (hence an isometry) of (Δ, d_{hyp})), then for all $z \in \Delta$,

$$\phi_{f\circ\theta}(z) = \phi_f(\theta(z)).$$

[2] If $\gamma \in G$, the group of holomorphic automorphisms of the space D which are isometries of (D, d_g), then for all $z \in \Delta$,

$$\phi_{\gamma\circ f}(z) = \phi_f(z).$$

[3] There exists a constant C_0 depending only on (D, d_g) such that if $0 \in D$ and $h : \Delta \to D$ is holomorphic,

$$| \phi_h(0) | \leq C_0 \, || h'(0) ||^2 .$$

Proof. The parts [1] and [2] follow immediately since θ is an isometry of (Δ, d_{hyp}) and γ is an isometry of (D, d_g). To prove [3]: let $\partial/\partial z$ is the standard generator of the tangent space at z in Δ, let $z_1, z_2, \cdots, z_n$ be the co-ordinate functions on $\mathbb{C}^n$ and ∂/∂_i the corresponding basis elements of the tangent space at $h(0) = 0$ in $\mathbb{C}^n$. Then, $h(z) = (h_1(z), h_2(z), \cdots, h_n(z))$, and $h'(0) = \sum h'_i(0)\partial/\partial_i$. The inner product g at $T_0(D)$ may be extended to a Hermitian inner product $<,>_g$ on the bigger tangent space $T_0(\mathbb{C}^n) = \mathbb{C}^n$ in some arbitrary, but fixed fashion. Then,

$$\phi_h(0) = \sum_{1 \leq i,j \leq n} h'_i(0)\overline{h'_j(0)} < \partial/\partial_i, \partial/\partial_j >_g .$$

In this equality, each term $h'_i(0)\overline{h'_j(0)}$ is bounded by $|| \ h'(0) \ ||^2$. Take $C_0 = n^2 C_1$ where C_1 is the supremum of the absolute values of the numbers $< \partial/\partial_i, \partial/\partial_j >_g$, for varying i and j . This gives [3] of the Lemma. □

Lemma 11. *If $f : \Delta \to D$ is a holomorphic map on the unit disc, and $R > 0$ is chosen as before such that $(0 \in)D \subset B(0, R) \subset \mathbb{C}^n$, then for all $z \in \Delta$,*

$$\phi_f(z) \leq C_0 R^2,$$

with C_0 chosen as in [3] of Lemma 10.

Proof. By Lemma 10, we may replace f by a composite $h = \gamma \circ f \circ \theta$ with γ and θ holomorphic isometries on the left and right without changing the conclusion of the present Lemma. Then $\phi_f(z) = \phi_h(\theta z)$. Fix $z \in \Delta$. Choose θ such that $\theta(0) = z$; choose γ such that $\gamma(f(z)) = 0$. Thus $\phi_f(z) = \phi h(0)$ with $h(0) = 0$. Then, Lemma 9 and [3] of Lemma 10 complete the proof. □

Theorem 12. *The holomorphic map h is distance decreasing. Precisely, there exists a constant $C > 0$ such that for all $p, q \in \Delta$, we have*

$$d_g(h(p), h(q)) \leq C d_{hyp}(p, q).$$

Proof. We choose R as before and assume that D contains 0 (without loss of generality). Then Lemma 11 is the infinitesimal version of our Theorem, with $C = C_0 R^2$. The theorem immediately follows by integrating along differentiable paths in Δ joining p and q. □

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