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Diophantine Approximation and Dynamics of Unipotent Flows on Homogeneous Spaces

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Dedicated to G.A. Margulis on the occasion of his sixtieth birthday

It was around 1980 that Margulis got interested in the area of Diophantine approximation. The main focus at that time was on the Oppenheim conjecture, and the Raghunathan conjecture formulated in that connection for flows on homogeneous spaces induced by unipotent one-parameter subgroups. Raghunathan's conjecture was formulated around 1975, and some of my work during the interim was driven by it. In print it was introduced in my paper on invariant measures of horospherical flows [6], where its connection with the Oppenheim conjecture, pointed out to me by Raghunathan, was also noted. By mid-eighties it was in the air that Margulis had proved the Oppenheim conjecture. I got a preprint, with complete proof, in 1987. An announcement with a sketch appeared in Comptes Rendus in 1987 [34], and expositions of proof followed [35], [36].

From then on Margulis has played a leading role in the study of problems in Diophantine approximation via study of dynamics of flows on homogeneous spaces, with an impressive array of results on a variety of problems in the area. The aim of this article is to review some of this work, tracing along the way the development of ideas on the themes concerned. To be sure there are various survey articles, including by Margulis as also the present author, giving accounts of the area. The present article, apart from including some recent results in the area, is also different from the existing accounts in various respects: exposition from a historical point of view, the measure of details with regard to various concepts and results, focus on the contributions of Margulis etc. and the author hopes that it would help the reader in getting a quick introduction to the topic.

The article is organised in terms of sections on the major themes involved.

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§1. Oppenheim conjecture

The first result to be proved was the following:

1.1. Theorem (Oppenheim conjecture) *Let $Q(x_1, \dots, x_n) = \Sigma_{i,j=1}^n a_{ij} x_i x_j$ be a nondegenerate indefinite quadratic form in $n \geq 3$ real variables, and suppose that it is not a scalar multiple of a rational form, i.e. a_{ij}/a_{kl} is irrational for some distinct pairs (i,j) and (k,l), with $a_{kl} \neq 0$. Then*

$$\min\{Q(x) \mid x \in \mathbb{Z}^n, x \neq 0\} = 0.$$

The conjecture goes back to a 1929 paper of Oppenheim (for $n \geq 5$). Extensive work was done on the conjecture by methods of analytic number theory in the 1930s (Chowla, Oppenheim), 40s (Davenport-Heilbronn), 50s (Oppenheim, Cassels & Swinnerton-Dyer, Davenport, Birch-Davenport, Davenport-Ridout) .... Papers of Birch-Davenport, Davenport-Ridout and one of Ridout in 1968 together confirmed the validity of the conjecture for $n \geq 21$, together with partial results for lower values of n, involving conditions on signature, diagonalisability etc. (see [31] and [38] for details). Partial results continued to be obtained in the 70s and 80s (Iwaniec, Baker-Schlickewei) by number-theoretic methods, but there was a gradual realisation that the methods of analytic number theory may not be adequate to prove the conjecture for small number of variables.

Margulis obtained the result by proving the following:

1.2. Theorem *Let $G = SL(3, \mathbb{R})$, $\Gamma = SL(3, \mathbb{Z})$. Let H be the subgroup consisting of all elements of G preserving the quadratic form $Q_0(x_1, x_2, x_3) = x_1 x_3 - x_2^2$ (viz. the special orthogonal group of Q_0). Let $z \in G/\Gamma$ and H_z be the stabilizer $\{g \in G \mid gz = z\}$ of z in G. Suppose that the orbit Hz is relatively compact in G/Γ. Then H/H_z is compact (equivalently, the orbit Hz is compact).*

By the well-known Mahler criterion (see [47] for instance) Theorem 1.2 implies Theorem 1.1, a priori for $n = 3$ and then by a simple restriction argument for all $n \geq 3$; for $n = 3$ the two statements involved are in fact equivalent. The possibility of proving the Oppenheim conjecture via this route was observed by Raghunathan, which inspired Margulis in his work. Margulis discovered later, as reported in his survey article in Fields Medallists' Lectures (1997) [41], that in implicit form the equivalence as above appears already in an old paper of Cassels and Swinnerton-Dyer [5].

In response to Margulis's preprint [35] on the above mentioned theorems A. Borel pointed out that Oppenheim was in fact interested, in his papers in the fifties, in concluding the set of values $Q(\mathbb{Z}^n)$ to be dense in $\mathbb{R}$, under the hypothesis as in Theorem 1.1. By a modification in the original argument this was also upheld by Margulis in a later preprint [36].

In 1988 Margulis was visiting the Max Planck Institute, Bonn, and he arranged, with the help of G. Harder, for me to visit there and we could work together. We strengthened Theorem 1.2 to the following [14]:

1.3. Theorem *Let the notation be as in Theorem 1.2. Then for every $z \in G/\Gamma$ the H-orbit Hz is either closed or dense in G/Γ.*

This implies Theorem 1.1, and it also implies the stronger assertion of density of $Q(\mathbb{Z}^3)$ more directly, without recourse to the Mahler criterion used earlier: the quadratic form Q may be assumed to be given by $v \mapsto Q_0(gv)$ for all $v \in \mathbb{R}^3$, where $g \in G$; then $Q(\mathbb{Z}^3) = Q_0(g\mathbb{Z}^3) = Q_0(Hg\Gamma\mathbb{Z}^3)$, and hence if $Hg\Gamma$ is dense in G then $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$; it turns out that if $Hg\Gamma$ is closed then Q is a multiple of a rational form. The theorem also implies the following strengthening of the Oppenheim conjecture.

1.4. Corollary *If Q is as in Theorem 1.1 and $\mathcal{P}$ denotes the set of all primitive integral n-tuples, then $Q(\mathcal{P})$ is dense in $\mathbb{R}$.*

To prove the corollary, and in particular the Oppenheim conjecture, one does not need the full strength of Theorem 1.3. Let ν be the matrix of the nilpotent linear transformation given by $e_1 \mapsto 0$, $e_2 \mapsto 0$ and $e_3 \mapsto e_1$ ($\{e_i\}$ denotes the standard basis). Then it suffices to prove that for $z \in G/\Gamma$ such that Hz is not closed, the closure $\overline{Hz}$ in G/Γ contains a point y such that either $\{(\exp t\nu)y \mid t \geq 0\}$ or $\{(\exp t\nu)y \mid t \leq 0\}$ is contained in $\overline{Hz}$. Based on this observation we gave an elementary proof of Corollary 1.4, involving only basic knowledge of topological groups and linear algebra [16]; see also [13].

One of the main ideas in the proof of Theorem 1.2 consists of the following: Let U be a connected unipotent Lie subgroup of G and X be a compact U-invariant subset of G/Γ which is not a U-orbit. The goal then is to show that X contains an orbit of a larger connected Lie subgroup of G. To this end one studies the minimal U-invariant subsets of X and the topological limits of orbits of points from a sequence in X tending to one of the minimal sets. When we get a larger subgroup as above, if it is unipotent we can continue further along the same lines; if it is not unipotent, the strategy cannot be readily continued, but in the cases considered the argument is complemented by structural considerations. For example in the proof Theorem 1.2 given the compact H-invariant subset $X = \overline{Hz}$, this strategy is applied with respect to a unipotent one-parameter subgroup U contained in H, and we get a larger subgroup W and a W-orbit contained in X. The argument is then completed by showing that if W is not contained in H the W-orbit cannot have compact closure, while W being contained in H implies that X is a H-orbit; this is achieved by a closer look at the subgroups of $SL(3,\mathbb{R})$. The argument thus shows that the compact invariant subset $\overline{Hz}$ has to be a compact orbit of H.

In proving Theorem 1.3, as $X = \overline{Hz}$ may not be compact, in applying the strategy one first needs to ensure that it contains a compact minimal U-invariant subset. This depends on "non-divergence" properties of orbits of these actions, which I shall discuss later, in §4.

§2. Raghunathan conjecture

I will next discuss the developments around the Raghunathan conjecture, which had been formulated as means for proving the Oppenheim conjecture. In this respect it will be convenient to consider a general connected Lie group G and a lattice Γ in G, viz. a discrete subgroup such that the quotient space G/Γ admits a finite measure invariant under the G-action, even though our primary interest will be in the case of $G = SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$ which is a lattice in $SL(n,\mathbb{R})$. In the general case we shall say that an element $g \in G$ is unipotent, if the adjoint transformation $\mathrm{Ad}\, g$ of the Lie algebra of G is unipotent.

The following is a more general form of the Raghunathan conjecture, formulated by Margulis in his ICM address at Kyoto, 1990 [40].

Conjecture *Let G be a connected Lie group and Γ be a lattice in G. Let H be a Lie subgroup of G with the property that it is generated by the unipotent elements contained in it. Then for any $z \in G/\Gamma$ there exists a closed subgroup F of G such that $\overline{Hz} = Fz$.*

(The Raghunathan conjecture is the special case of this, with H a unipotent one-parameter subgroup of G).

In the case when $G = SL(2,\mathbb{R})$, and H a unipotent one-parameter subgroup this is a classical result due to Hedlund (1936). In this case the orbits are either dense or closed, so $F = G$ or H, and only the first case occurs if the quotient $SL(2,\mathbb{R})/\Gamma$ is compact. It is instructive to see a proof of this, especially in the case when Γ is a cocompact lattice, using the overall strategy introduced by Margulis, described at the end of the last section; see [1], Chapter IV, for details. A distinguishing feature of this case is that H is a "horospherical subgroup", namely there exists a $g \in G$ such that $H = \{h \in G \mid g^j u g^{-j} \to e, \text{ as } j \to \infty\}$, e being the identity element of G. The case of the conjecture with G any reductive Lie group and H a horospherical subgroup was proved in [9], generalising Hedlund's result to this case. For solvable (connected) Lie groups G the conjecture was proved by A.N. Starkov, in 1984 (see [52] for details).

Theorem 1.2 confirmed the conjecture for $G = SL(3,\mathbb{R})$ and H the special orthogonal group of a nondegenerate indefinite quadratic form.

Pursuing further the methods involved in the results of the last section Margulis and I proved in [15] the following special case of the Raghunathan conjecture.

2.1. Theorem *Let $G = SL(3, \mathbb{R})$ and Γ be a lattice in G. Let U be a unipotent one-parameter subgroup of G such that $u - I$ has rank 2 (as a matrix) for all $u \in U\backslash\{I\}$. Then for every $z \in G/\Gamma$ there exists a closed subgroup F such that $\overline{Uz} = Fz$.*

Unlike in the case of $SL(2, \mathbb{R})$ mentioned above, F can have more possibilities in this case; when $\Gamma = SL(3, \mathbb{Z})$, the following subgroups of G have closed orbits on G/Γ: the subgroup consisting of all elements of $SL(3, \mathbb{R})$ fixing a nonzero vector under the natural action on $\mathbb{R}^3$, the subgroup of elements leaving invariant a linear functional on $\mathbb{R}^3$ under the contragradient action, and the special orthogonal group consisting of elements leaving invariant the form Q_0 as in Theorem 1.2 - these subgroups can occur in the place of the subgroup F in the above discussion.

Theorem 2.1 has the following consequence in the study of values of quadratic forms (see [15] and also [12]).

2.2 Corollary *Let Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^3$. Let L be a linear form on $\mathbb{R}^3$. Let*

$$C = \{v \in \mathbb{R}^3 \mid Q(v) = 0\} \quad \textit{and} \quad P = \{v \in \mathbb{R}^3 \mid L(v) = 0\},$$

and suppose that the plane P is tangential to the cone C. Suppose also that no linear combination $\alpha Q + \beta L^2$, with $(\alpha, \beta) \neq (0, 0)$ is a rational quadratic form. Then $\{(Q(x), L(x)) \mid x \in \mathcal{P}\}$ (with $\mathcal{P}$ as before) is dense in $\mathbb{R}^2$, viz. given any $a, b \in \mathbb{R}$ and $\epsilon > 0$ there exists $x \in \mathcal{P}$ such that

$$|Q(x) - a| < \epsilon \quad \textit{and} \quad |L(x) - b| < \epsilon.$$

It may be mentioned here that for $n \geq 4$ the answer to the analogous problem is not completely understood yet; see Theorem 2.5 and the remark following it.

Margulis nurtured the hope that the overall method of "building up" orbits of larger subgroups inside a given closed set invariant under the action should lead to a proof of Raghunathan's conjecture. This however has not yet materialised.

The Raghunathan conjecture was in the meantime proved by Marina Ratner, in 1990-91 ([48], [49]), where she also proved the above-mentioned general conjecture under the additional condition that every connected component of H contains a unipotent element. The general statement of the conjecture was proved by Nimish Shah, building up on her work [51].

Ratner's work involved classifying the invariant measures of actions of unipotent subgroups [48], proving a conjecture that I had formulated in connection with the Raghunathan conjecture. The following classification theorem was proved by Ratner [48] under the additional condition as mentioned above, and was completed by N.A. Shah [51].

2.3 Theorem (Ratner) *Let G be a connected Lie group and Γ be a discrete subgroup of G (not necessarily a lattice). Let H be a closed subgroup of G which is generated by the unipotent elements contained in it. Let μ be a finite H-invariant and H-ergodic measure on G/Γ. Then there exists a closed subgroup F of G and a F-orbit Φ such that μ is F-invariant and supported on Φ (the two conditions determine the measure up to a scalar multiple).*

Though a proof of the Raghunathan conjecture eluded Margulis, later he contributed a more transparent proof of Theorem 2.3 (jointly with Tomanov) [45], in the crucial case of G a real algebraic group. Though the proof is influenced by Ratner's arguments it also involves new approach and methods.

The Raghunathan conjecture was deduced by Ratner from Theorem 2.3, in [49], by proving the following result on uniform distribution.

2.4 Theorem (Ratner) *Let G be a connected Lie group, Γ be a lattice in G and $U = \{u_t\}$ be a unipotent one-parameter subgroup of G. Let $z \in G/\Gamma$. Suppose that there is no proper closed connected subgroup G_1 of G, with $U \subset G_1$, such that $G_1 z$ is closed and admits a finite G_1-invariant measure. Then the U-orbit of z is uniformly distributed in G/Γ, viz. for every bounded continuous function f on G/Γ*

$$\frac{1}{T}\int_0^T f(u_t z)dt \to \int_{G/\Gamma} f(g\Gamma)dm(g\Gamma),$$

where m is the normalised G-invariant measure on G/Γ.

Note that if there exists a proper closed connected subgroup G_1 containing U and such that $G_1 z$ is closed and admits a finite G_1-invariant measure then (by downward induction) there is a minimal one with that property and the U orbit is uniformly distributed in the orbit under that subgroup; furthermore the subgroup is unique.

Ratner also deduced from Theorem 2.4, with further work, the generalised conjecture stated in the beginning of the section, under the additional condition that every connected component of H contains a unipotent element. Her result also yields that the orbit closure $\overline{Hz} = Fz$ as in the conclusion admits a finite F-invariant measure; viz. the closure is a homogeneous space with finite invariant measure. In [51] Nimish Shah showed, following up on the work of Ratner, that the conjecture holds for any subgroup H which is contained in Zariski closure of the subgroup generated by unipotent elements contained in it. However, in the general case it was only concluded (in the place of the above assertion of Fz admitting a finite F- invariant measure) that the the connected component $F^0 z$ of Fz admits a finite F^0-invariant measure. It was proposed as a conjecture that Fz has only finitely many connected components, and hence in fact admits a finite F-invariant measure. The issue was reduced to the question whether the orbit closure is finite whenever it is discrete. This question was settled in the affirmative by Eskin and Margulis [20], thus completing the picture. I may mention that the proof in [20] involves a random walk version of recurrence properties that I will discuss in § 5.

Ratner's results have been used in various contexts, including in problems of Diophantine approximation. The reader is referred to [4], [11], [30], [46], [52] for details. I will however mention here the following recent result of A. Gorodnik [25], which complements Corollary 2.2, and is proved using Ratner's results.

2.5 Theorem (Gorodnik) *Let Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^n$, $n \geq 4$, and let L be a linear form on $\mathbb{R}^n$. Suppose that (i) the restriction of Q to the subspace $\{v \in \mathbb{R}^n \mid L(v) = 0\}$ is an indefinite quadratic form, and (ii) no linear combination $\alpha Q + \beta L^2$, with $\alpha, \beta \in \mathbb{R}$ and $(\alpha, \beta) \neq (0,0)$ is a rational quadratic form. Then $\{(Q(x), L(x)) \mid x \in \mathcal{P}\}$ (with $\mathcal{P}$ as before) is dense in $\mathbb{R}^2$.*

The analogue of the above corollary is not true for $n = 3$; this was noted in [12], and depends on Theorem 4.4 below, due to Kleinbock and Margulis. On the other hand Theorem 2.5 does not complete the picture for $n \geq 4$ since condition (i) can not be expected to be a necessary condition for the conclusion to hold. It is conjectured in [25] that the conclusion holds if (i) is replaced by a condition equivalent to the following, which is indeed a necessary condition: $\{(Q(v), L(v)) \mid v \in \mathbb{R}^n\} = \mathbb{R}^2$. The case to be settled happens to be that of a pair (Q, L) for which there exists $g \in SL(n, \mathbb{R})$ such that $v \mapsto Q(gv)$ and $v \mapsto L(gv)$ are the forms $x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}x_n$ and x_n (quadratic and linear respectively). It is stated in [25] that the proof of the result in the other case there can be adapted to prove this statement for $n = 4$. For $n \geq 5$ however, it is open.

Before concluding this section it may be remarked that the theme of addressing problems in Diophantine approximation via study of flows on homogeneous spaces, got strengthened by Margulis's work, and in turn inspired similar work on the question of values of cubic forms at integer points and a conjecture of Littlewood in the topic. It however involves actions of subgroups which are quite the contrary to being generated by unipotent elements. In this context Margulis has proposed a conjecture about the behaviour of orbits under actions of subgroups which are not generated by unipotent elements (see [42]). There has been considerable work towards the conjecture and its applications to the Littlewood conjecture; in the general form the conjecture is still open however. We will not go into the details of these topics here. The reader is referred to [19] and [32], and the references there for an exposition of the area.

§3. Uniform versions of uniform distribution.

I had the opportunity of collaborating with Margulis once again, following Ratner's proof of Theorem 2.3. We proved certain uniform versions of uniform distribution of orbits of unipotent flows, and applied them to study asymptotics of the set of solutions of quadratic inequalities as in the Oppenheim conjecture [17].

Let G be a connected Lie group and Γ be a lattice in G. For any $x \in G/\Gamma$, any unipotent one-parameter subgroup $U = \{u_t\}$ of G and $T > 0$ let $\lambda(x, U, T)$ denote the probability measure on the arc $\{u_t x \mid 0 \leq t \leq T\}$ in G/Γ, uniform along the parameter t; viz. $\lambda(x, U, T)$ is the measure such that for every continuous function φ with compact support, on G/Γ,

$$\int_{G/\Gamma} \varphi \, d\lambda(x, U, T) = \frac{1}{T} \int_0^T \varphi(u_t x) \, dt.$$

For any closed subgroup F and $x \in G/\Gamma$ such that Fx is closed and admits a finite F-invariant measure let $\mu(x, F)$ denote the F-invariant probability measure on G/Γ.

Ratner's uniform distribution theorem (viz. Theorem 2.4 above) means that for any $x \in G/\Gamma$ and any unipotent one-parameter subgroup U there exists a closed subgroup F such that Fx is closed and admits a F-invariant measure, and the family of probability measures $\{\lambda(x, U, T)\}$ converges to $\mu(x, F)$ as $T \to \infty$ (in the weak topology with respect to bounded continuous functions). In this respect one may also consider the dependence of the convergence on the point x and the one-parameter subgroup U.

We say that a point $x \in G/\Gamma$ is *generic* for the U-action of a unipotent one-parameter subgroup U if Ux is dense in G/Γ, or equivalently uniformly distributed in G/Γ, and we say that x is *singular*, for the U-action, if it is not generic.

Given a sequence $U_i = \{u_t^{(i)}\}$ of unipotent one-parameter subgroups of G and a unipotent one-parameter subgroup $U = \{u_t\}$ of G, we say that U_i converges to U, and write $U_i \to U$, if $u_t^{(i)} \to u_t$ for all $t \in \mathbb{R}$. We proved the following (see Theorem 2 in [17]; the statement there is formulated for individual integrals):

3.1 Theorem *Let $\{x_i\}$ be a sequence in G/Γ converging to $x \in G/\Gamma$, and $\{U_i\}$ be a sequence of unipotent one-parameter subgroups of G converging to a unipotent one-parameter subgroup U of G. Suppose that x is a generic point for U. Then for any sequence $\{T_i\}$ in $\mathbb{R}^+$ such that $T_i \to \infty$ the sequence of probability measures $\{\lambda(x_i, U_i, T_i)\}$ converges to the G-invariant probability measure on G/Γ.*

The main points in the proof in [17] are the following. Consider a limit point, say σ, of the sequence $\{\lambda(x_i, U_i, T_i)\}$ of probability measures on G/Γ, viewed as measures on the one-point compactification of G/Γ. Using results on non-divergence of orbits of unipotent flows one concludes that σ is a probability measure on G/Γ, viz. point at ∞ carries zero measure. Clearly σ is a U-invariant measure. By Ratner's classification of invariant measures (see Theorem 2.3) together with ergodic decomposition it follows that if the set of singular points for the U-action has zero σ-measure then σ is the G-invariant probability measure.

For any closed subgroup H such that $H\cap\Gamma$ is a lattice in H let $X(H,U)=\{g\in G \mid Ug\subset gH\}$. If $g\in X(H,U)$ then $g\Gamma$ is a singular point for the U-action, since $Ug\Gamma/\Gamma\subset gH\Gamma/\Gamma$ which is a proper closed subset. Conversely every singular point is of the form $g\Gamma$ for $g\in X(H,U)$ for some H as above. Furthermore, considering the minimal ones from the subgroups involved, one can see that H can be chosen from a countable collection. It therefore suffices to prove that $\sigma(X(H,U)\Gamma/\Gamma)=0$ for all proper closed subgroups H such that $H\cap\Gamma$ is a lattice in H. For this purpose we associate to each H a linear action of G on a finite-dimensional vector space V_H in such a way that behaviour of trajectories of U of points near the set $X(H,U)\Gamma/\Gamma$ can be compared with that of trajectories of certain associated points in V_H near a U-invariant algebraic subvariety A_H associated with X_H. For the latter one shows, using Lagrange interpolation formula together with the fact that trajectories of unipotent one-parameter subgroups are polynomial maps, that for any compact subset C of A_H and $\epsilon>0$ there exists a compact subset D of A_H such that the proportion of time spent by near C to that spent near D is at most ϵ. This means that asymptotically the trajectories spend arbitrarily little time near any fixed compact subset of the variety, which then yields that $\sigma(X(H,U)\Gamma/\Gamma)=0$, for all H as above.

Theorem 3.1 can also be proved by an argument along the lines of Ratner's proof of uniform distribution theorem, namely Theorem 2.4 which it generalises; Ratner has stated that this was pointed out to her by Marc Burger in 1990, prior to our proving the result. The above approach, and especially the idea of comparing the behaviour of trajectories on G/Γ with that of certain trajectories in finite-dimensional vector spaces, referred to as linearisation, has on the other hand been useful in proving Theorem 3.2 below, involved in quantitative versions of the Oppenheim conjecture, and also in some subsequent work of other authors.

Clearly the condition in the hypothesis that x is a generic point for U is necessary for the stated conclusion to hold. It turns out however that if we fix a bounded continuous function φ and $\epsilon>0$, and want to know if $\int\varphi\,d\lambda(x_i,U_i,T_i)$ differs from $\int\varphi\,d\mu$ by at most ϵ then a weaker hypothesis suffices. This turns out to be important in applications. The following is a slightly more general version of Theorem 3 of [17], allowing the unipotent one-parameter subgroups to vary, which can be proved along the lines of the original proof.

3.2 Theorem *Let $\{U_i\}$ be a sequence of unipotent one-parameter subgroups of G converging to a unipotent one-parameter subgroup U of G. Let K be a compact* *subset of* G/Γ, φ *be a bounded continuous function on* G/Γ *and* $\epsilon > 0$ *be given. Then there exist finitely many proper closed subgroups* $H_1, \dots, H_k$ *such that* $H_i \cap \Gamma$ *is a lattice in* H_i *for each* $i = 1, \dots, k$, *and a compact subset* K_0 *of* K *contained in* $\cup_{i=1}^k X(H_i, U)\Gamma/\Gamma$ *such that the following holds: for any compact subset* F *of* $K \backslash K_0$ *there exist* $i_0 \geq 1$ *and* $T_0 > 0$ *such that for all* $x \in F$, $i \geq i_0$ *and* $T \geq T_0$

$$\Big| \int \varphi \, d\lambda(x, U_i, T) - \int \varphi \, d\mu \Big| < \epsilon.$$

The proof consists of showing that if the assertion does not hold then there exists a sequence $\{x_i\}$ in K converging to a generic point of U for which the conclusion of Theorem 3.1 does not hold.

§4. Quantitative versions of Oppenheim conjecture

Theorem 3.2 was applied in [17] to obtain asymptotic lower estimates for the number of integral solutions in large balls, for the quadratic inequalities as in the Oppenheim conjecture.

Let ω be a positive continuous function on the unit sphere $\{v \in \mathbb{R}^n \mid \|v\| = 1\}$, and $\Omega = \{v \in \mathbb{R}^n \mid \|v\| < \omega(v)\}$. For $T > 0$ let $T\Omega$ denote the dilate $\{Tv \mid v \in \Omega\}$ of Ω by T.

While considering asymptotics of the solutions of the inequalities involving the quadratic form we shall also consider the dependence on the forms. In this respect the spaces of quadratic forms will be considered equipped with the usual topology, arising from the associated symmetric bilinear forms, or equivalently the topology of convergence as functions. We proved the following.

4.1 Theorem *Let* $\mathcal{O}(p,q)$ *denote the space of quadratic forms on* $\mathbb{R}^n$ *with discriminant* ± 1 *and signature* (p,q), *with* $p \geq 2$, $q \geq 1$, $p \geq q$ *and* $p + q = n$. *Let* $\mathcal{K}$ *be a compact subset of* $\mathcal{O}(p,q)$. *Let* $a, b \in \mathbb{R}$, *with* $a < b$ *be given. Then for any* $\theta > 0$ *there exists a finite subset* $\mathcal{F}$ *of* $\mathcal{K}$ *such that*

i) each Q *in* $\mathcal{F}$ *is a scalar multiple of a rational quadratic form, and*

ii) for any compact subset $\mathcal{C}$ *of* $\mathcal{K} \backslash \mathcal{F}$ *there exists a* $T_0 > 0$ *such that for all* $Q \in \mathcal{C}$ *and* $T \geq T_0$,

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid a < Q(x) < b\} \geq (1 - \theta) \mathrm{vol}\,\{v \in T\Omega \mid a < Q(v) < b\}.$$

The basic idea involved in proving the estimate may be explained as follows. For any function f on $\mathbb{R}^n$ vanishing outside a compact subset let $\tilde{f}$ be the function on G/Γ defined by $\tilde{f}(g\Gamma) = \sum_{v \in g\mathbb{Z}^n} f(v)$; since f vanishes outside a compact set the right hand side expression is in effect a finite sum, and yields a well-defined function on G/Γ. It can be seen that if f is a measurable function on $\mathbb{R}^n$ then $\tilde{f}$ is measurable on G/Γ. Furthermore, by a theorem of Siegel, if f is integrable on

$\mathbb{R}^n$ then $\tilde{f}$ is integrable on G/Γ and $\int_{\mathbb{R}^n} f d\lambda = \int_{G/\Gamma} \tilde{f} d\mu$, where λ is the Lebesgue measure on $\mathbb{R}^n$ and μ is the G-invariant probability measure on G/Γ.

For simplicity we consider only the case with $n = 3$, and $\mathcal{K}$ consisting of a single quadratic form which is not a multiple of a rational form. Let Q_0 be the quadratic form as before, namely $Q_0(x_1, x_2, x_3) = x_1x_3 - x_2^2$. Let $g \in G = SL(3, \mathbb{R})$ and let Q be the quadratic form $v \mapsto Q_0(gv)$ for all $v \in \mathbb{R}^3$ (as seen before, it suffices to consider only these quadratic forms). Let $U = \{u_t\}$ be a unipotent one-parameter subgroup contained in $SO(Q_0)$. Let $a, b \in \mathbb{R}$, with $a < b$ be given. We are interested in solutions of $a < Q(x) < b$, or equivalently $a < Q_0(gx) < b$ with x an integral point in a region of the form $T\Omega$ as above. Let B be a subset of $\mathbb{R}^3$ which is a "box" of the form $\{u_t s \mid |t| < \tau,\ s \in S\}$ where τ is a small positive number and S is a small open set in a plane transversal to the U-action, contained in $\{v \in \mathbb{R}^3 \mid a < Q_0(v) < b\}$. Let χ denote the characteristic function of B. Let $0 < T_1 < T_2$ and $S(T_1, T_2) = \{u_t s \mid T_1 < t < T_2,\ s \in S\}$. Then $\int_{T_1}^{T_2} \chi(u_t v)dt \le 2\tau$ for any $v \in \mathbb{R}^3$, and hence $\int_{T_1}^{T_2} \tilde{\chi}(u_t g\Gamma)dt = \int_{T_1}^{T_2} \sum_{v\in g\mathbb{Z}^3} \chi(u_t v)dt$ is bounded by $2\tau\#(S(T_1, T_2) \cap g\mathbb{Z}^3)$. Therefore,

$$\#(S(T_1, T_2) \cap g\mathbb{Z}^3) \ge \frac{1}{2\tau}\int_{T_1}^{T_2} \tilde{\chi}(u_t g\Gamma)dt.$$

When Q is not a multiple of a rational quadratic form, g can be chosen to be such that $g\Gamma$ is generic for the U-action. Then given $\theta > 0$ as in the hypothesis, when T_1, T_2 and $T_2 - T_1$ are sufficiently large the integral on the right hand side exceeds $(1 - \theta)(T_2 - T_1)\int \tilde{\chi} d\mu$ which, by the theorem of Siegel recalled above, equals $(1 - \theta)(T_2 - T_1)\lambda(B)$. Then the cardinality of $S(T_1, T_2) \cap g\mathbb{Z}^3$ is at least $(1 - \theta)(T_2 - T_1)\lambda(B)/2\tau$, which may be seen to be the same as $(1-\theta)\mathrm{vol}S(T_1, T_2)$. The proof of the lower estimates for Q as above essentially consists of "filling up" more than $(1 - \theta/2)$ proportion of the regions $g(T\Omega) \cap \{v \in \mathbb{R}^3 \mid a < Q_0(v) < b\}$ as above, for large enough T, efficiently (in a way that the overlaps would not matter) by subsets of the form $mS(T_1, T_2)$ with m in a certain compact subgroup M of $SO(Q_0)$; in the calculations it is convenient to use equality of the integrals

$$\int_{T_1}^{T_2}\int_E \sum_{v\in g\mathbb{Z}^3} \chi(u_t m v)d\sigma(m)dt = \int_{T_1}^{T_2}\int_E \tilde{\chi}(u_t g m\Gamma)d\sigma(m)dt,$$

for various subsets E of M, where σ is the normalised Haar measure on M. Using the comparison as above and using Theorem 3.2 we conclude that

$$\#\{y \in g(\mathbb{Z}^3 \cap T\Omega) \mid a < Q_0(y) < b\} \ge (1 - \theta)\mathrm{vol}(T\Omega \cap \{v \in \mathbb{R}^3 \mid a < Q(v) < b\}),$$

which is equivalent to the desired inequality in the case at hand. The proof in the general case is analogous, using Theorem 3.2 as above.

It is also proved in [17] that when $n \geq 5$ for any compact subset $\mathcal{K}$ of $\mathcal{O}(p,q)$ and $\epsilon > 0$ there exist $c > 0$ and $T_0 > 0$ such that for all $Q \in \mathcal{K}$ and $T \geq T_0$ the number of $x \in T\Omega \cap \mathbb{Z}^n$ for which $|Q(x)| < \epsilon$ is at least $c \,\mathrm{vol}\,\{v \in T\Omega \mid |Q(v)| < \epsilon\}$. This in particular gives a quantitative version of the classical theorem of Meyer that for $n \geq 5$ every nondegenerate indefinite rational form represents zero.

Remarks For an indefinite binary quadratic form the set of values at integral points need not be dense in $\mathbb{R}$, even when the form is not a scalar multiple of a rational form; it can be seen that for $Q(x,y) = (x+ay)(x+by)$, $\{Q(x,y) \mid x, y \in \mathbb{Z}\}$ has zero as a limit point if and only if one of a and b is an irrational number which is not badly approximable. Conditions for density of the values on the set of integral points, and also on the set of pairs with positive integer coordinates are considered in [18]. In the context of the latter it may be mentioned here that by an argument as in the first part of the sketch of the proof of Theorem 4.1 it can be shown that for $n \geq 3$ for a nondegenerate indefinite quadratic form Q on $\mathbb{R}^n$ which is not a multiple of a rational form, if the cone $\{v \in \mathbb{R}^n \mid Q(v) = 0\}$ contains vectors with all coordinates positive, then the set $\{Q(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

The lower estimates obtained in [17] were complemented in the work of Margulis with Eskin and Mozes [21] with upper estimates.

4.2 Theorem (Eskin-Margulis-Mozes) *Let the notation be as in Theorem 4.1. If $p \geq 3$ then the subset $\mathcal{F}$ as in the conclusion can also be chosen so that for any compact subset $\mathcal{C}$ of $\mathcal{K} \backslash \mathcal{F}$ there exists a $T_0 > 0$ such that for all $Q \in \mathcal{C}$ and $T \geq T_0$,*

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid a < Q(x) < b\} \leq (1+\theta)\mathrm{vol}\,\{v \in T\Omega \mid a < Q(v) < b\}.$$

The function $\tilde{f}$ as in the remarks following Theorem 4.1 is unbounded for any nonnegative nonzero function f. Therefore the relation used in obtaining the lower estimates is not amenable to computations for upper estimates. The difficulty is overcome in [21] via analysis of integrals of the form $\int_K \tilde{f}(a_t k\Gamma)^{1+\delta} dk$, where $\delta > 0$, K is a maximal compact subgroup of $SO(p,q)$ and $\{a_t\}$ is a diagonalisable one-parameter subgroup of $SO(p,q)$.

We note that the right hand side of the inequalities in Theorems 4.1 and 4.2 are asymptotic to cT^{n-2} for a constant $c > 0$ (depending on a, b and Ω and the quadratic form), and thus so is the number of solutions as on the left hand side, provided $p \geq 3$. It is also proved in [21] when $p \geq 3$, given $\mathcal{K}$, a, b and Ω as in Theorems 4.1 and 4.2 there exists an effectively computable constant $C > 0$ such that $\#\{x \in \mathbb{Z}^n \cap T\Omega \mid a < Q(x) < b\}$ is bounded by CT^{n-2}; we note here that for the results recounted earlier there are no effective proofs - the reader is referred to [42] for a discussion on this issue. The corresponding statement does not hold for $p = 2$, but $CT^{n-2}\log T$, serves as a bound, with an effective constant C.

Indeed, when $p = 2$, given $q = 1$ or 2, for every $\epsilon > 0$ and interval (a, b) in $\mathbb{R}$ there exists a quadratic form Q of signature $(2, q)$, a constant $\delta > 0$ and a sequence $T_i \to \infty$ such that

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid a < Q(x) < b\} \geq \delta T_i^q (\log T_i)^{1-\epsilon}$$

for all i (see [21]). The examples, first noticed by P. Sarnak, arise as irrational forms which are very well approximable by split rational forms. Sarnak also noted that a hypothesis suggested by Berry and Tabor, on the statistic of the eigenvalues of the quantisation of a completely integrable Hamiltonian is related to the asymptotics in the problem as above, in the case of signature $(2, 2)$. In this context, it is of interest to identify classes of quadratic forms with $p = 2$ for which the number of solutions is asymptotic to cT^{n-2}, with c the constant as above. Sarnak showed that this holds for almost all quadratic forms from the two-parameter family $(x_1^2 + \alpha x_1 x_2 + \beta x_2^2) - (x_3^2 + \alpha x_3 x_4 + \beta x_4^2)$.

Apart from the issue of very well approximability there is also another aspect of forms of signature $(2, 2)$ which precludes the asymptotics as desired. It may be seen that the c as above depends linearly on $(b - a)$. On the other hand, whenever a quadratic form of signature $(2, 2)$ has a rational isotropic subspace, say L, then for any $\epsilon > 0$,

$$\#\{x \in \mathbb{Z}^n \cap T\Omega \mid |Q(x)| < \epsilon\} \geq \#(\mathbb{Z}^n \cap T\Omega \cap L) \geq \sigma T^2,$$

where $\sigma > 0$ is a constant independent of ϵ. In this respect the following result is proved in the recent paper of Eskin, Margulis and Mozes [22].

4.2 Theorem (Eskin, Margulis, Mozes) *Let the notation be as in Theorem 3.1. Let $Q \in \mathcal{O}(2, 2)$, and suppose that it is not extremely well approximable, in the sense that there exists $N > 0$ such that for all split integral forms Q' and $k \geq 2$, $\|Q - \frac{1}{k}Q'\| > k^{-N}$. Let X be the set of points in $\mathbb{Z}^4$ which are not contained in any isotropic subspace of Q. Then, as $T \to \infty$,*

$$\#\{x \in X \cap T\Omega \mid a < Q(x) < b\} \sim \mathrm{vol}\,\{v \in T\Omega \mid a < Q(v) < b\}.$$

§5. View of orbits from infinity

When the underlying space of a flow is noncompact, questions arise about whether some of the orbits are bounded (relatively compact), or diverge to infinity etc.; these aspects I refer as view of the orbits from infinity. In the light of the Mahler criterion, in the case of flows on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ these have close connections with questions in diophantine approximation. One of the earliest results of this kind was proved by Margulis, in [33]; the statement had been conjectured earlier by Piatetski-Shapiro, and was used by Margulis in his work on the arithmeticity theorem for nonuniform lattices.

5.1 Theorem (Margulis) *Let $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$. Let $\{u_t\}$ be a unipotent one-parameter subgroup of G. Then for every $x \in G/\Gamma$ there exists a compact subset K of G/Γ such that $\{t \geq 0 \mid u_t x \in K\}$ is unbounded; (in other words, the trajectory $\{u_t x\}_{t\geq 0}$ does not "go off to infinity").*

Developing upon Margulis's original proof I strengthened the result to the following [8]:

5.2 Theorem *Let G and Γ be as in Theorem 5.1. Then for every $\epsilon > 0$ there exists a compact subset K of G/Γ such that for any $x = g\Gamma \in G/\Gamma$ and any unipotent one-parameter subgroup $\{u_t\}$ of G one of the following holds:*

i) $l(\{t \geq 0 \mid u_t x \notin K\}) < \epsilon T$ for all large T, or

ii) $\{g^{-1} u_t g\}$ leaves invariant a proper nonzero rational subspace of $\mathbb{R}^n$.

Analogous results hold also for general Lie groups G and lattices Γ. One of the consequences of these results is that every locally finite ergodic invariant measure of a unipotent flow on G/Γ is necessarily finite; this turned out to be useful in Ratner's work on Raghunathan's conjecture. From the theorem I deduced also that every closed nonempty subset invariant under a unipotent one-parameter subgroup contains a minimal closed invariant subset, and the minimal sets are compact; this was used in our proofs of Theorems 1.3 and 2.1. The result was extended by Margulis to actions of general connected unipotent Lie subgroups acting on G/Γ [39].

The following quantitative version of Theorem 5.1 was proved by Kleinbock and Margulis, in the course of their study [28] of Diophantine approximation on manifolds.

5.3 Theorem (Kleinbock and Margulis): *Let Λ be a lattice in $\mathbb{R}^n$, $n \geq 2$. Then there exists $\rho > 0$ such that for any unipotent one-parameter subgroup $\{u_t\}$ of $SL(n, \mathbb{R})$, $T > 0$ and $\epsilon \in (0, \rho)$,*

$$l(\{t \in [0, T] \mid u_t \Lambda \cap B(\epsilon) \neq 0\}) \leq c_n (\epsilon/\rho)^{1/n^2} T,$$

where $B(\epsilon)$ denotes the open ball of radius ϵ with center at 0, and c_n is an explicitly described constant depending only on n.

(We note that the parenthetical set on the left hand side represents a neighbourhood of infinity in the space of lattices in $\mathbb{R}^n$, depending on ϵ.)

Actually the results in [28] apply also to a large class of curves, and also higher dimensional submanifolds, in the place of orbits of unipotent groups involved in the above theorem. The general results along the theme are involved in dealing with questions in Diophantine approximation on manifolds which we discuss briefly in the next section. The method involved has been further sharpened in recent years by Kleinbock; see [26].

It may also be mentioned here that results somewhat similar in flavour as the above theorems, but in different direction and concerning local behaviour, may be found in [23]; these were proved by the authors in preparation for their results on asymptotics of lattice points on homogeneous varieties [24].

In [20] Eskin and Margulis prove a random walk analogue of the recurrence properties as above, proving in particular that given a connected Lie group G which is generated as a closed subgroup by the unipotent elements in it, a lattice Γ in G, and a probability measure μ on G satisfying a certain moment condition, for every $\epsilon > 0$ there exists a compact subset K of G/Γ such that for every $x \in G/\Gamma$ there exists $N \in \mathbb{N}$, such that for all $n > N$, $(\mu^{*n} * \delta_x)(K) > 1 - \epsilon$; here μ^{*n} denotes the n th convolution power of μ and δ_x is the point measure at x. The authors also discuss other variations on the theme. Using the result the authors deduce the conjecture proposed by Nimish Shah, on the finiteness of countable orbit closures, mentioned in § 2.

Theorem 5.2 was applied by Margulis to give a new proof of the theorem of Borel and Harish-Chandra on arithmetic subgroups of semisimple groups being lattices [37]. The study of recurrence properties of random walks on homogeneous spaces in [20], discussed above, was also applied in a similar way, and recently a self-contained proof of the Borel-Harish Chandra theorem was also given by Margulis [44], via a simplified version of the approach from [20].

In the mid-eighties I observed that the notions of singular systems of linear forms and badly approximable systems, studied by W.M. Schmidt (see [50]), correspond to trajectories of points of $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ under the action of certain diagonal one-parameter subgroups of $SL(n,\mathbb{R})$ being divergent (tending to infinity) or being bounded respectively [7]. It was shown that in certain cases the orbit being bounded, while certainly not generic, was quite prevalent, if one took into account the Hausdorff dimension of the set of such points [10]. Margulis followed up the theme and formulated a conjecture on the issue in his ICM address at Kyoto. The conjecture as presented there needs some modifications. However, the underlying question was completely solved in a paper of Kleinbock and Margulis [27], proving the following.

5.4 Theorem (Kleinbock and Margulis): *Let G be a connected Lie group and Γ be a lattice in G. Let $\{g_t\}$ be a one-parameter subgroup of G. Let W be the normal subgroup of G generated by the two opposite horospherical subgroups with respect to $\{g_t\}$. Suppose that $\overline{W\Gamma} = G$. Let B be the set of points x in G/Γ such that the orbit $\{g_t x\}$ of x is bounded (relatively compact). Then for every nonempty open subset Ω of G/Γ the intersection $B \cap \Omega$ is of Hausdorff dimension equal to the dimension of G.*

It may be noted that if $\overline{W\Gamma}$ is a proper subgroup then the set B as above is of Hausdorff dimension at most $\dim G - 1$, unless it is the whole of G/Γ; this is related to Ratner's work for unipotent flows and its extension to quasi-unipotent flows (see [52], § 21).

The correspondence between divergence or boundedness properties of trajectories of specific one-parameter subgroups on the one hand and issues in Diophantine approximation on the other hand was also extended by Kleinbock to broader classes of one-parameter subgroups (see [30] for details).

§6. Diophantine approximation on manifolds

We next come to yet another area of Diophantine approximation to which Margulis has made important contributions, which in some ways are continuation of the study of unipotent flows and their applications.

We recall that $v \in \mathbb{R}^n$ is said to be *very well approximable (VWA)* if for some $\epsilon > 0$ there exist infinitely many positive integers k such that $\operatorname{dist}(kv, \mathbb{Z}^n) \leq k^{-(\frac{1}{n}+\epsilon)}$. Also v is said to be *very well multiplicatively approximable (VWMA)* if for some $\epsilon > 0$ there exist infinitely many positive integers k such that $\inf_{p\in\mathbb{Z}^n} \Pi(kv+p) \leq k^{-(1+\epsilon)}$, where Π is the function on $\mathbb{R}^n$ defined by $\Pi(v) = |v_1 v_2 \cdots v_n|$ for $v = (v_1, v_2, \ldots, v_n)$. Clearly, if a vector is VWA then it is also VWMA.

These concepts, arise naturally in higher-dimensional extensions of the theory of approximation of irrationals by rationals in the one-dimensional case. A vector being VWA or, more generally, VWMA is atypical and in particular the set of points which are VWMA is of Lebesgue measure 0. In the higher-dimensional case this raises an interesting question whether given a (differentiable) submanifold of the ambient space the set of VWMA (or VWA) vectors contained in it has measure 0 as a subset of the submanifold (on a differential manifold there is a natural notion of a set being of measure zero, namely that the intersection of the set with each chart be of Lebesgue measure zero). Motivated by a 1932 conjecture due to Mahler that vectors on the curves $\{(t, t^2, \ldots, t^n) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^n$, $n \geq 2$ are not VWA for almost all t, this question, and certain generalisations, were studied by several number theorists, including Kasch, Volkmann, Sprindzuk, W.M. Schmidt, A. Baker, Bernik and also very recently by Beresnevich (see [43] for some references on the work).

Kleinbock and Margulis (1998) [28] proved the following result settling a conjecture of Sprindzuk (1980); the latter was a generalisation of a conjecture of A. Baker which corresponds to the special case of $d = 1$ and $f_k(t) = t^k$, $k = 1, \ldots, n$ in the statement below.

6.1 Theorem (Kleinbock and Margulis) *Let Ω be a domain in $\mathbb{R}^d$ for some $d \geq 1$ and let $f_1, f_2, \ldots, f_n$ be n real analytic functions on Ω such that $\Sigma a_i f_i$ is* *not a constant function for any $a_1, \dots, a_n$ in $\mathbb{R}$, not all zero. Then for almost all v in Ω the vector $(f_1(v), \dots, f_n(v))$ is not VWMA (and hence not VWA either).*

Actually the result is proved in [28] in greater generality, allowing $f_1, \dots, f_n$ to be C^r functions satisfying a certain "nondegeneracy" condition. The question is reduced to one of estimating measures of subsets of the parameter set Ω for which $u_{f_1(v),\dots,f_n(v)}\mathbb{Z}^{n+1}$ belongs to certain neighbourhoods of infinity (complements of compact subsets), where the neighbourhoods concerned depend on numerical values for size and also on certain diagonal matrices, by way of "shape"; for $w_1, \dots, w_n \in \mathbb{R}$, $u_{w_1,\dots,w_n}$ denotes the unipotent element of $SL(n+1, \mathbb{R})$ corresponding to the linear transformation given by $e_0 \mapsto e_0$ and $e_i \mapsto e_i + w_i e_0$ for $i = 1, \dots, n$, with $\{e_i\}_{i=0}^n$ as the standard basis of $\mathbb{R}^{n+1}$. From this point on, the ideas are akin to those in Theorems 5.1 and 5.2 but now appear in quantitative and highly intricate form. The reader is referred to [43] for an exposition of the ideas involved. A modification of the method was used in [3] to prove a part (the convergence part) of the Khintchine-Groshev theorem for nondegenerate smooth submanifolds of $\mathbb{R}^n$; see also [2].

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