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# Global Existence and Uniqueness Theorem for $3D$ – Navier-Stokes System on $\mathbb{T}^3$ for Small Initial Conditions in the Spaces $\Phi(\alpha)$.

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*Dedicated to G.A. Margulis on the occasion of his sixtieth birthday.*

**Abstract:**We consider Cauchy problem for three-dimensional Navier-Stokes system with periodic boundary conditions with initial data from the space of pseudo-measures $\Phi(\alpha)$. We provide global existence and uniqueness of the solution for sufficiently small initial data.

## 1. Introduction

Three-dimensional Navier-Stokes system with periodic boundary conditions after Fourier transform can be written in the form:

$$(1)\quad v(t,k)=\exp -t|k|^2 v_0(k)+2\pi i\int_0^t \exp\{-(t-s)|k|^2\}\sum_{l\in\mathbb{Z}^3}\langle k, v(s,k-l)\rangle P_k v(s,l)ds$$

Here $k\in\mathbb{Z}^3$, $t\in\mathbb{R}_+$, $v(t,k)\in\mathbb{C}^3$, $v(t,k)\perp k$ for any $k\neq 0$ and $v(t,0)=0$ for all $t>0$. $v_0(k)$ is the initial condition and $P_k$ denotes Leray projector to the subspace orthogonal to $k$ and has the form $P_k=\mathrm{Id}-\frac{\langle k,\cdot\rangle}{|k|^2}k$. Also (1) assumes that the viscosity $\nu=1$ and that the external forcing is absent.

T. Kato in [K] proved the local existence theorem for the $3D$-Navier-Stokes system on $\mathbb{R}^3$ and global existence and uniqueness theorem in the space $L^{\frac{3}{2}}(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$ for small initial conditions.

In this paper we consider Cauchy problem for the system (1) with initial data from the space $\Phi(\alpha)$ which is analogous to the subspace $\Phi(\alpha,\alpha)$ introduced in [S1], [S2] and consists of functions of the form

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$$\Phi(\alpha) = \left\{ f(k) = \frac{c(k)}{|k|^\alpha}, k \neq 0 \mid \sup_k |c(k)| < \infty \right\}, \qquad \|f(k)\|_\alpha = \sup_{k \in \mathbb{Z}^3} |k|^\alpha |f(k)|$$

We assume $\alpha > 2$ and shall write $\alpha = 2 + \varepsilon$. V. Kaloshin and Yu. Sannikov announced the global existence theorem in the spaces $\Phi(\alpha)$, $\alpha \geqslant 2$ for small initial data (see [KS]). In this paper we give a detailed proof of this result which shows also the character of decay of solutions in this case. It is worthwhile to mention that according to our point of view a similar result is not valid in the continuous case of $k \in \mathbb{R}^3$.

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## 2. Main result

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let* $0 < 3\varepsilon < 1$ *and* $\|v_0\|_\alpha \leqslant \delta$ *where* $v_0 = \dfrac{c_0(k)}{|k|^\alpha}$ *is the initial condition and* $\delta = \delta(\alpha)$ *is sufficiently small. Then the equation* (1) *has a global solution* $v(t,k) = \dfrac{c(t,k)}{|k|^\alpha}$ *such that* $c(t,k)$ *is a continuous mapping of* $[0,\infty)$ *into* $L^\infty(\mathbb{Z}^3 \setminus \{0\})$, $t > 0$.

The proof of the Theorem 1 goes by induction. Put $H_0^{(0)}(k) = \dfrac{c_0(k)}{|k|^\alpha}$, $k \neq 0$, $H_0^{(1)}(k) = G_0(k) = 0$ and assume that for some integer $m$ we constructed the solution $v(t,k)$, $0 \leqslant t \leqslant m$, such that

$$v(m,k) = H_m^{(0)}(k) + H_m^{(1)}(k) + G_m(k) \tag{2}$$

where

$$H_m^{(0)}(k) = \frac{\exp\{-m|k|^2\} c_0(k)}{|k|^\alpha},$$

$$H_m^{(1)}(k) = \sum_{j=1}^{m} \exp\{-(m-j)|k|^2\} h_j^{(1)}(k)$$

and

$$G_m(k) = \sum_{j=1}^{m} \exp\{-(m-j)|k|^2\} g_j(k)$$

Suppose that for all $j \leqslant m$ functions $h_j^{(1)}(k)$ satisfy the inequalities:

$$|h_j^{(1)}(k)| \leqslant \frac{D_1\delta^2 \exp\left\{-\frac{j}{2}|k|^2\right\}}{|k|^{2\varepsilon}}, \tag{3}$$

while the functions $g_j(k)$ satisfy the inequalities:

$$|g_j(k)| \leqslant \frac{D_2\delta^2 \exp\{-d_1|k|\sqrt{m}\}}{|k|^{\beta}} \tag{4}$$

Here $\beta > 3$ is a constant and $d$, $D$ with indices denote various absolute constants which appear during the proof, but their exact values play no role in the arguments.

Consider $0 \leqslant t \leqslant 1$ and write down the solution of (1) in the form:

$$\begin{aligned} v(t+m,k) = {} & \frac{\exp\{-(m+t)|k|^2\}c_0(k)}{|k|^{\alpha}} + \sum_{j=1}^{m} \frac{\exp\{-(m-j+t)|k|^2\}h_j^{(1)}(k)}{|k|^{2\varepsilon}} + \\ & + \frac{h_{m+1}^{(1)}(t,k)}{|k|^{2\varepsilon}} + \sum_{j=1}^{m} \exp\{-(m-j+t)|k|^2\}g_j(k) + g_{m+1}(t,k) \end{aligned} \tag{5}$$

We show that the inequalities (3), (4) holds for $h_{m+1}^{(1)}(1,k)$ and $g_{m+1}(1,k)$ respectively.

## 3. Proof of the main result

Denote

$$H_{m+1}^{(0)}(t,k) = \frac{\exp\{-(m+t)|k|^2\}c_0(k)}{|k|^{\alpha}}, \tag{6}$$

$$H_{m+1}^{(1)}(t,k) = \sum_{j=1}^{m} \frac{\exp\{-(m-j+t)|k|^2\}h_j^{(1)}(t,k)}{|k|^{2\varepsilon}} + \frac{h_{m+1}^{(1)}(t,k)}{|k|^{2\varepsilon}}, \tag{7}$$

$$G_{m+1}(t,k) = \sum_{j=1}^{m} \exp\{-(m-j+t)|k|^2\}g_j(t,k) \tag{8}$$

and

$$\left(H' \circledast H''\right)(t,k) = i \int_0^t \exp\{-(t-s)|k|^2\} \sum_{\substack{k\in\mathbb{Z}^3\setminus\{0\}\\ k-l\neq 0}} \frac{\langle k, H'(s,k-l)\rangle P_k H''(s,l)}{|k-l|^\alpha |l|^\alpha}$$

If we substitute (5) into (1) we can write the expression for $h^{(1)}_{m+1}(t,k)$:

$$h^{(1)}_{m+1}(t,k) = |k|^{2\varepsilon}\left(H^{(0)}_{m+1} \circledast H^{(0)}_{m+1}\right)(t,k) \tag{9}$$

and the expression for $g_{m+1}(t,k)$:

$$g_{m+1}(t,k) = \sum_{j_1=1}^{8} I^{(1,j_1)}_{m+1}(t,k) + \sum_{j_2=1}^{3} I^{(2,j_2)}_{m+1}(t,k) + I^{(3)}_{m+1}(t,k) \tag{10}$$

where

$$I^{(1,j_1)}_{m+1}(t,k) = \left(H' \circledast H''\right)(t,k),$$
$$I^{(2,j_2)}_{m+1}(t,k) = \left(H' \circledast g_{m+1}\right)(t,k) + \left(g_{m+1} \circledast H'\right)(t,k)$$

and $H'$, $H''$ are either $H^{(0)}_{m+1}(t,k)$, or $H^{(1)}_{m+1}(t,k)$ or $G_{m+1}(t,k)$ except the case $H' = H'' = H^{(0)}_{m+1}$ which corresponds to the $h^{(1)}_{m+1}$ according to (9). Therefore $j_1$ changes from 1 to 8 and $j_2$ changes from 1 to 3. Also

$$I^{(3)}_{m+1}(t,k) = g_{m+1} \circledast g_{m+1}.$$

We see that $I^{(1)}_{m+1}$ does not depend on $g_{m+1}(t,k)$, $I^{(2)}_{m+1}$ is a linear function of $g_{m+1}(t,k)$ and $I^{(3)}_{m+1}$ is a quadratic function of $g_{m+1}(t,k)$. Therefore (10) is a typical equation which can be solved by iterations if the coefficients are small enough. Below we provide necessary estimates and later we come back to the analysis of (8).

3.1. **First estimates.** Here we show that all functions $h^{(1)}_{m+1}$ behaves like gaussian functions of $t|k|$ and then provide necessary estimates for coefficients in (10).

As in [S1], [S2], we use the identity:

$$a_1|k-l|^2 + a_2|l|^2 = \frac{a_1 a_2}{a_1+a_2}|k|^2 + (a_1+a_2)\left|l - \frac{a_1}{a_1+a_2}k\right|^2 \tag{11}$$

**An estimate of $H^{(1)}_{m+1}$.** At first we estimate $h^{(1)}_{m+1}(t,k)$. From (9) it follows that

$$h^{(1)}_{m+1}(t,k) =(H^{(0)}_{m+1} \circledast H^{(0)}_{m+1})(t,k) = 2\pi i \int\limits_0^t \exp\{-(t-s)|k|^2\}\cdot$$
$$\cdot \sum_{\substack{l\in\mathbb{Z}^3\setminus\{0\}\\ k-l\neq 0}} \frac{\langle k, c_0(k-l)\rangle P_k c_0(l)}{|k-l|^\alpha |l|^\alpha} \exp\{-(m+s)|k-l|^2-(m+s)|l|^2\}ds$$

Using (11) we can write

$$\begin{aligned}|h^{(1)}_{m+1}(t,k)| \leqslant &\exp\left\{-\frac{m|k|^2}{2}\right\} \int\limits_0^t \exp\{-(t-\frac{s}{2})|k|^2\}\cdot\\ &\cdot \sum_{\substack{l\in\mathbb{Z}^3\setminus\{0\}\\ k-l\neq 0}} \frac{\langle k, c_0(k-l)\rangle P_k c_0(l)}{|k-l|^\alpha |l|^\alpha} \exp\{-2m|l-\frac{1}{2}k|^2\}ds \leqslant\\ &\leqslant \delta^2 \exp\left\{-\frac{m|k|^2}{2}\right\} \frac{\exp\{-\frac{t}{2}|k|^2\}-\exp\{-t|k|^2\}}{|k|^2}\cdot\\ &\cdot \sum_{\substack{l\in\mathbb{Z}^3\setminus\{0\}\\ k-l\neq 0}} \frac{\exp\{-2m|l-\frac{1}{2}|k|^2\}}{|k-l|^\alpha |l|^\alpha} \leqslant\\ &\leqslant \frac{D_3\delta^2}{|k|^{2\varepsilon}} \exp\left\{-\frac{(m+t)|k|^2}{2}\right\} \frac{1-\exp\{-\frac{t}{2}|k|^2\}}{|k|^2}\end{aligned}$$

Substituting this inequality to (7) we conclude

$$\begin{aligned}\left|H^{(1)}_{m+1}(t,k)\right| &\leqslant \frac{D_3\delta^2}{|k|^{2\varepsilon}} \frac{(1-\exp\{-\frac{t}{2}|k|^2\})}{|k|^2} \sum_{j=1}^{m+1} \exp\left\{-(m+1-\frac{j}{2})|k|^2\right\} \leqslant\\ &\leqslant \frac{D_4\delta^2}{|k|^{2\varepsilon}} \frac{(1-\exp\{-\frac{t}{2}|k|^2\})}{|k|^2} \exp\left\{-\frac{(m+1)|k|^2}{2}\right\}\end{aligned} \tag{12}$$

**Estimates for $H^{(j_1)}_{m+1} \circledast H^{(j_2)}_{m+1}$.** We present detailed estimate only for $H^{(0)}_{m+1} \circledast H^{(1)}_{m+1}$ since all other terms can be estimated in the same manner. From (11) and

(12) we have

$$\left|(H^{(0)}_{m+1} \circledast H^{(1)}_{m+1})(t,k)\right| \leqslant |k|\delta^3 \int\limits_0^t \exp\{-(t-s)|k|^2\}\cdot$$

$$\cdot \sum_{\substack{l\in\mathbb{Z}^3\setminus\{0\}\\ k-l\neq 0}} \frac{\exp\{-(m+s)|l|^2 - \frac{m+s}{2}|k-l|^2\}}{|k-l|^{2\varepsilon}|l|^{\alpha}} ds \leqslant$$

$$\leqslant |k|\delta^3 \exp\left\{-\frac{m+1}{3}|k|^2\right\} \int\limits_0^t \exp\{-(t-s)|k|^2\} \sum_{\substack{l\in\mathbb{Z}^3\setminus\{0\}\\ k-l\neq 0}} \frac{\exp\{-\frac{3}{2}(m+s)|l-\frac{m+s}{3}k|^2\}}{|k-l|^{2\varepsilon}|l|^{\alpha}} ds$$

Since $\alpha + 2\varepsilon > 2$ the last sum is not more than some constant $D_5$. We get

$$\left|(H^{(0)}_{m+1} \circledast H^{(1)}_{m+1})(t,k)\right| \leqslant |k| D_5 \exp\left\{-\frac{m+1}{3}|k|^2\right\} \frac{1-\exp\{-t|k|^2\}}{|k|^2} \tag{13}$$

Similarly, for $(H^{(1)}_{m+1} \circledast H^{(1)}_{m+1})(t,k)$ we can write

$$\left|(H^{(1)}_{m+1} \circledast H^{(1)}_{m+1})(t,k)\right| \leqslant D_6 \exp\left\{-\frac{m+1}{4}|k|^2\right\} \frac{1-\exp\{-t|k|^2\}}{|k|} \tag{14}$$

3.2. **Spaces $\mathcal{F}_m(c)$.** Fix positive constant $\beta > 0$ and introduce functional space $\mathcal{F}_m(c)$

$$\mathcal{F}_m(c) = \left\{ f(k) \mid |f(k)| \leqslant \frac{D_7}{|k|^{\beta}} \exp\{-c\sqrt{m}|k|\},\ k\neq 0 \right\},\ \|f\|_{m,c} = \inf D_7$$

We show that functions $g_{m+1}(t,k)$ belong to the spaces $\mathcal{F}_m$ with uniform constant if only all coefficients in (10) are sufficiently small. First of all we show that $H^{(j_1)} \circledast H^{(j_2)}$, $j_1 + j_2 > 0$ belongs to the space $\mathcal{F}_m(d_2)$ for some constant $d_2$. It follows from previous estimates, that all of these functions decay as a Gaussian functions. For our purpose it is convenient to consider them as functions from the space $\mathcal{F}_m(d_2)$. Since $m|k| \geqslant 1$ we can write

$$\exp\left\{-\frac{m|k|^2}{3}\right\} \leqslant \frac{D_8}{|k|^{\beta}} \exp\left\{-\frac{\sqrt{m}|k|}{\sqrt{3}}\right\}$$

for some constant $D_8$. We see that $(H^{(0)}_{m+1} \circledast H^{(1)}_{m+1})(t,k) \in \mathcal{F}_{m+1}(\frac{1}{\sqrt{3}})$ and

$$\|H^{(0)}_{m+1} \circledast H^{(1)}_{m+1}\|_{m+1,\frac{1}{\sqrt{3}}} \leqslant D_9 \tag{15}$$

for some constant $D_9$, which does not depend on $t$.

Assuming that $G_{m+1} \in \mathcal{F}_{m+1}(d_2)$ we can write for $G_{m+1} \circledast H^{(0)}_{m+1}$

$$|(G_{m+1} \circledast H^{(0)}_{m+1})(t,k)| \leqslant \|G_{m+1}\|_{m+1,d_2} \delta |k| \int\limits_0^t \exp\{-(t-s)|k|^2\} \cdot$$

$$\cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-d_2\sqrt{m}|k-l| - m|l|^2\}}{|l|^{\alpha}|k-l|^{\beta_1}} ds$$

For the last expression we get

$$\exp\{-d_2\sqrt{m}|k-l| - m|l|^2\} \leqslant \exp\{-d_2\sqrt{m}|k|\} \exp\{d_2\sqrt{m}|l| - m|l|^2\} \leqslant$$

$$\leqslant D_{10} \exp\{-d_2\sqrt{m}|k|\} \exp\{|l - \frac{d_2}{2\sqrt{m}}|^2\}$$

So for $(G_{m+1} \circledast H^{(0)}_{m+1})(t,k)$ we obtain

$$|(G_{m+1} \circledast H^{(0)}_{m+1}(t,k))| \leqslant D_{11} \|G_{m+1}\|_{m+1,d_2} \delta \frac{\exp\{-d_2|k|\sqrt{m}\}}{|k|^{\beta_1}} \frac{1-\exp\{-t|k|^2\}}{|k|^2}$$

All other terms in $I^{(1)}_{m+1}(t,k)$ can be similarly estimated.

Thus we embed the first term in the representation of $g_{m+1}(t,k)$ given by (10) into the space $\mathcal{F}_{m+1}(d_2)$. Now we provide the necessary estimates for the terms $I^{(3)}_{m+1}$ and $I^{(2)}_{m+1}$.

**Estimate for $I^{(3)}_{m+1}$.** We show, that for given functions $f_1$, $f_2 \in \mathcal{F}_{m+1}(d_2)$ $f_1 \circledast f_2$ also belongs to the space $\mathcal{F}_{m+1}(d_2)$ and $\|f_1, f_2\|_{m+1,d_2} \leqslant D_{12} \|f_1\|_{m+1,d_2} \|f_2\| m+1, d_2$ for some constant $D_{12}$.

Write down the estimate

$$|f_1 \circledast f_2| \leqslant \|f_1\|_{m+1,d_2} \|f_2\|_{m+1,d_2} |k| \int\limits_0^t \exp\{-(t-s)|k|^2\} \cdot$$

$$\cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-d_2\sqrt{m+1}(|l| - |k-l|\}}{|l|^{\beta}|k-l|^{\beta}} ds \leqslant$$

$$\leqslant \frac{D_{13}\|f_1\|_{m+1,d_2}\|f_2\|_{m+1,d_2}}{|k|^{2\beta-3}} \exp\{-d_2|k|\sqrt{m+1}\} \frac{1-\exp\{-t|k|^2\}}{|k|}$$

Since $\beta > 3$ and the last expression is not more than 1, we get

$$|f_1 \circledast f_2| \leqslant \frac{D_{14}\|f_1\|_{m+1,d_2}\|f_2\|_{m+1,d_2}}{|k|^{\beta}} \exp\{-d_2\sqrt{m+1}|k|\} \tag{16}$$

In particular, for $g_{m+1}(t,k) \in \mathcal{F}_{m+1}(d_2)$ it follows that

$$\|I^{(3)}_{m+1}(t,k)\|_{m+1,d_2} \leqslant D_{14}\|g_{m+1}(t,k)\|^2_{m+1,d_2} \tag{17}$$

Estimates for $I^{(2)}_{m+1}$. Here we produce the upper bound for $\|I^{(2)}(t,k)\|_{m+1,D_{14}} = \sum_{j_2=1}^{3} I^{(2,j_2)}_{m+1}(t,k)$ assuming, that $g_{m+1}(t,k) \in \mathcal{F}_{m+1}(d_2)$.

$$|(g_{m+1}(t,k) \circledast H^{(0)}_{m+1}(t,k))| \leqslant \|g_{m+1}\|_{m+1,d_2}\delta|k| \int_0^t \exp\{-(t-s)|k|^2\}\cdot$$

$$\cdot \sum_{\substack{l\in\mathbb{Z}^3\setminus\{0\}\\ k-l\neq 0}} \frac{\exp\{-d_2\sqrt{m}|k-l| - m|l|^2\}}{|k-l|^\beta |l|^\alpha} ds$$

Again for the last expression holds

$$\exp\{-d_2\sqrt{m}|k-l| - m|l|^2\} \leqslant D_{15}\exp\{-d_2\sqrt{m}|k|\}\exp\{|l - \frac{d_2}{2\sqrt{m}}|^2\}$$

So for $(g_{m+1}(t,k) \circledast H^{(0)}_{m+1}(t,k))$ we can write

$$|(g_{m+1}(t,k) \circledast H^{(0)}_{m+1}(t,k))| \leqslant D_{16}\|g_{m+1}\|_{m+1,d_2}\delta \frac{1-\exp\{-t|k|^2\}}{|k|} \frac{\exp\{-d_2\sqrt{m}|k|\}}{|k|^\beta}$$

For the terms $g_{m+1} \circledast G_{m+1}$ we can produce an appropriate estimate using (16). All other terms in $I^{(2)}_{m+1}(t,k)$ can be estimated in a similar way.

Collecting all present estimates we see that for some constant *newcon*

$$\|g_{m+1}(t,k)\|_{m+1,d_2} \leqslant D_{17}\delta^2 + D_{18}\delta\|g_{m+1}\|_{m+1,d_2} + D_{19}\|g_{m+1}\|^2_{m+1,d_2} \tag{18}$$

So for sufficiently small $\delta$ all coefficients in (18) are small and the equation (10) can be solved by iterations. The solution $g_{m+1}(t,k)$ belongs to $\mathcal{F}_{m+1}(d_2)$ and unique in this class of functions. Each function $g_{m+1}(t,k)$ provides the unique solution $v(m+t,k)$ of (5). The Theorem 1 is proven.

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