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Free Subgroups of Linear Groups

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To Gregory Margulis in his birthday

In the celebrated paper [T] J. Tits proved the following fundamental dichotomy for a finitely generated linear group :

Let G be a finitely generated linear group over an arbitrary field. Then either G is virtually solvable or G contains a free non-abelian subgroup.

His proof of this alternative based on geometrical ideas came form Schottky groups. Recall that a Schottky group G is a group of fractional linear transformations of the hyperbolic plane $\mathbb{H}^2$ generated by a set of hyperbolic elements $S = \{g_i, i \in I\}$ and has the following property : there exist disjoint subsets $D_i^{\pm}, i \in I$ and D^0 of $\mathbb{H}^2$ such that for every $i \in I$:

1. $g_i^n(\cup_{j\in I, j\neq i} D_j^{\pm} \cup D^0) \subseteq D_i^+$ for $n > 0$
2. $g_i^n(\cup_{j\in I, j\neq i} D_j^{\pm} \cup D^0) \subseteq D_i^-$ for $n < 0$.

Note that from the definition immediately follows that the group G is a free group with free generators $g_i, i \in I$. Indeed, let $g = g_{i_1}^{m_1} \dots g_{i_k}^{m_k}$ be any reduced word. Take $p \in D^0$, then $gp \in D_{i_1}^{\pm} \subseteq \mathbb{H}^2 \setminus D^0$. Therefore $g \neq 1$.

One of the main purposes of the present work is to show how the beautiful ideas of Tits were developed in our joint works with G. Margulis [MS1], [MS2], [MS3]. Our interest to free subgroups of linear groups was initiated by the following
Problem 1 (V. Platonov) Does there exist a maximal subgroup of infinite index in $SL_n(\mathbb{Z})$ for $n \geq 3$?

We proved in [MS1] that the answer is positive. Actually we proved that

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Let G be a finitely generated linear group over an arbitrary field. Then either G is polycyclic or G contains a maximal subgroup of infinite index.
We proved this as a corollary of the following theorem:

Let G be a finitely generated linear group over an arbitrary field. If G is not virtually solvable then G contains a pro-finitely dense free subgroup.
We remark that questions about existence of a dense free subgroup in topological groups are very important and leads to a many deep consequences (see for example [BG2]).
Conjecture ([DPSS], [P]). Let G be a finitely generated linear group and let $\widehat{G}$ be a pro-finite completion of the group G. Assume that G contains a free subgroup. Then does $\widehat{G}$ contain a finitely generated pro-finitely dense free subgroup?

E. Breuillard and T. Gelander proved this conjecture in [BG2]

A. Shalev asked me a slightly different question. Namely
Problem 2 (A. Shalev). Let G be a finitely generated linear group and let $\widehat{G}$ be the pro-finite completion of the group G. Assume that G contains a free subgroup. Does G contain a finitely generated free subgroup which is pro-finitely dense in $\widehat{G}$?

In general an answer to this question is negative. For example it is not true if G is a lattice in a semisimple Lie group of real rank 1. But for lattices in higher rank semisimple groups the answer is positive [SV].

Several resent very interesting results [BG1],[BG2],[GGl] were proved under the influence of [MS3]. Therefore we decided to use this opportunity to explain some modifications of the concept, ideas and proofs from this paper together with further development done in our works [AMS1], [AMS2]. Based on this, we will give a short proof of Platonov's problem, construct a new counterexample to a Prasad-Tits conjecture and state some new and recall some old problems.

In the last section we will complete a proof of the following:
Conjecture *(G. Margulis) Let S be a crystallographic semigroup, then S is a group.*

Our proof of the above conjecture is based on existence of a free subgroup in the Zariski closure of S with some additional geometric properties. This step was done in [S].

We will use standard definitions of algebraic group theory, Lie group theory and group theory (see [B], [H], [R]). The letters $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ denotes respectively the set of integers, rational, real, complex and p–adic numbers. The index of a subgroup H of a group G will be denoted by $|G/H|$. If G is a group and $S \subseteq G$, then by $N_S(G)$ (resp. $C_S(G)$) we denote the normalizer (resp. centralizer) of S in G. By $\langle S\rangle$ we denote the subgroup of G generated by S. We denote as usual by GL_n and SL_n the group of invertible and unimodular n by n matrices. An algebraic subgroup of GL_n defined over a field k is called an algebraic k–group or a k–group. The set of all k–points of an algebraic variety W will be denoted by $W(k)$. If $\varphi : G \longrightarrow H$ is a k–rational homomorphism of k-groups G and H and the field l is an extension of k then the natural homomorphism $\varphi(l) : G(l) \longrightarrow H(l)$ will be denote by φ as well. Let V be a finite dimensional vector space. A subgroup G of $GL(V)$ is called irreducible if there is no proper G invariant subspace of V. Accordingly, we call a representation $\rho : G \longrightarrow GL(V)$ irreducible if the image $\rho(G)$ is a irreducible subgroup of $GL(V)$. A representation $\rho : G \longrightarrow GL(V)$ is called strongly irreducible if for every subgroup H of finite index of G the group $\rho(H)$ is irreducible. If k is a local field and W is an algebraic k–variety then $W(k)$ has two natural topologies, namely induced by the topology of k and by the Zariski topology of W. In case it will be necessary to avoid confusion the second topology will be distinguished by the prefix "k" i.e., k–open, k–dense etc.

G. Margulis became my official adviser in the mid 70's At that time in the former USSR such a step was really non-trivial and might have had many repercussions. I want to express here my deep gratitude to Grisha Margulis for all he gave me as a teacher and as a friend.
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1. Ping -Pong and free subgroups.

We will start from the following proposition which actually comes from Schottky groups and the proof based on the same arguments.

Lemma 1.1. Let G be a group acting on a set X. Let $S = \{H_i, i \in I\}$ be a collection of subgroups of G and let $\{X_i\}_{i\in I}$ be a set of disjoint subsets of X and let x_0 be a point , $x_0 \in X$. Assume that

1. $x_0 \in X \setminus \cup_{i\in I} X_i$
2. $h_i\left(\cup_{j\in I, j\neq i} X_j\right) \subseteq X_i$ *for all* $h_i \in H_i$ *and* $i \in I$,
3. $h_i x_0 \in X_i$ *for all* $h_i \in H_i$ *and* $i \in I$.

Then the group $\langle S\rangle$ is a free product of groups $H_i, i \in I$.

Corollary 1.2 (Ping-Pong Lemma) . Let G, X, $S = \{H_i, i \in I\}$, $\{X_i\}_{i\in I}$ and x_0

be as in Lemma 1.1. Assume that for every $i \in I$, H_i is an infinite cyclic group generated by some element h_i. Then $\langle S\rangle$ is a free group and $h_i, i \in I$ are free generators.

It is easy to see that the following statement is true.
Lemma 1.3. Let G be a group and $\rho : G \longrightarrow H$ be a homomorphism of a group. Let $S = \{h_i, i \in I\}$ be free generators of a free group $\langle S\rangle$. Assume that for every $i \in I$ we choose $g_i \in G$ such that $\rho(g_i) = h_i$. Then the group generated by $\{g_i, i \in I\}$ is free and $\{g_i, i \in I\}$ are free generators.

Let V be a finite dimensional vector space over a local field k with absolute value $|\cdot|$ and let $P = \mathbb{P}(V)$ be the projective space based on V. Let $g \in GL(V)$ and let $\chi_g(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i) \in k[\lambda]$ be the characteristic polynomial of the linear transformation g. Set $\Omega(g) = \{\lambda_i : |\lambda_i| = \max_{1\le j\le n} |\lambda_j|\}$. Put $\chi_1(\lambda) = \prod_{\lambda_i\in\Omega(g)}(\lambda - \lambda_i)$ and $\chi_2(\lambda) = \prod_{\lambda_i\notin\Omega(g)}(\lambda - \lambda_i)$. Since the absolute value of an element is invariant under Galois automorphism then χ_1 and χ_2 belong to $k[\lambda]$. Therefore $\chi_1(g) \in GL(V)$ and $\chi_2(g) \in GL(V)$. Let us define by $A(g)$ (resp. $B(g)$) the subspace of P corresponding to $\ker(\chi_1(g))$ (resp. $\ker(\chi_2(g))$). Put $Cr(g) = B(g) \cup B(g^{-1})$. Recall that $g \in GL(V)$ is called *proximal* if $A(g)$ is a point. A proximal element g has a unique eigenvalue of maximal absolute value hence this eigenvalue has algebraic and geometric multiplicity one. For $S \subseteq GL(V)$ set $\Omega_0(S) = \{g \in S : g \text{ and } \mathrm{g}^{-1} \text{ are proximal}\}$. We will often use for an element $g \in \Omega_0(GL(V))$ the following notation $A(g) = A^+(g)$, $B(g) = B^+(g)$, $A(g^{-1}) = A^-(g)$ and $B(g^{-1}) = B^-(g)$.

For an element $g \in \Omega_0(GL(V))$ the dynamics of the group $\langle g\rangle$ are very transparent. Let us formulate their properties in terms of the projective map $\widehat{g}$ induced by g on the projective space P. Namely, the sequence of maps $\widehat{g}^n$ where n is positive integer converges to a map sending all points in $P \setminus B(g)$ to the point $A(g)$ and the sequence $(\widehat{g}^{-1})^n$ where n is positive integer converges to a map sending all points in $P \setminus B(g^{-1})$ to the point $A(g^{-1})$. This easily follows from the next
Lemma 1.4. Let $g \in \Omega_0(GL(V))$ and let K be a compact subset of $P \setminus B(g)$. Let d be the distance between the two compact subsets K and $B(g)$. Assume that U is an open subset in P such that $A(g) \in U$. Then there exits a positive integer $N = N(U, d)$ such that $\widehat{g}^n K \subseteq U$ for all $n \ge N$.

To illustrate the dynamics of a subgroup let us conceder the following
Example 1 . Let $G = SL_3(\mathbb{R})$. Let g_1 and g_2 be two diagonal matrixes where

$g_1 = diag(\alpha, \alpha^{-1}, 1), \alpha > 1$ and $g_2 = diag(1, \alpha, \alpha^{-1})$, $\alpha > 1$. Let A be the abelian group generated by $\{g_1, g_2\}$. Clearly A is a free abelian group of a rank 2 . Let $A_N = \{g_1^n g_2^m, |n| \ge N, |m| \ge N\}$. Put $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$.

Consider the set $\Sigma = \{t_1e_1 + t_2e_2 + t_3e_3, \sum_1^3 t_i = 1, t_i \geq 0, i = 1,2,3\}$. Let $P(\Sigma)$ be the projectivization of Σ and let $\partial P(\Sigma)$ be the boundary of $P(\Sigma)$. Let q (correspondingly, $\widehat{e}_i$ $i = 1,2,3$) be the point of $P(\Sigma)$ which corresponds to the line span by the vector $1/3e_1+1/3e_2+1/3e_3$ (correspondingly, e_i $i = 1,2,3$) . The boundary ∂L of the orbit $L = \widehat{A}q = \{\widehat{a}q, a \in A\}$ is a subset in $\partial P(\Sigma)$. Let us give some explanation how to understand what will be the set of a limit points. Let $\{g\}_{i\in\mathbb{N}}$ be a sequence of elements from A, then $g_i = g_1^{n_i} g_2^{m_i}$. Since we are looking for a limit points of the set $\{\widehat{g}_i q\}_{i\in\mathbb{N}}$ in the projective space we can assume that $g = diag\{\alpha^{n_i+m_i}, \alpha^{-n_i+2m_i}, 1\}$. Assume that $n_i > 0, m_i > 0$. Let k be an integer such that $m_i = 2n_i + k_i$. There are three possibilities : $k_i \to \infty$ (1), $k_i \to -\infty$ (2) and $k_i \to \kappa$, (3) when $i \to \infty$. Then the sequence of projective transformations $\widehat{g}_i$ converge to the quasi projective transformation (see the definition in H.Abels in this volume) $\widetilde{g}_1 = diag\{1,0,0\}$ in the case (1), $\widetilde{g}_2 = diag\{0,1,0\}$ in the case (2) and $\widetilde{g}_3 = diag\{\alpha^k, 1, 0\}$ in the case (3). Conceder this maps as a projections $\pi_i, i =$ 1, 2, 3 of the space $\mathbb{R}^3$. Then the sequence $\{\widehat{g}_i q\}_{i\in\mathbb{N}}$ can converge to the points $\pi_i q, i =$ 1, 2, 3. It is not difficult to see that for every ε there exists $N = N(\varepsilon)$ such that if $a \in A_N$ then $d(\widehat{a}q, \partial L) \leq \varepsilon$ and if U_1 (respectively U_2, U_3 be a neighborhood of the point $\widehat{e_1}$, (respectively $\widehat{e}_2$, $\widehat{e}_3$) then the set $\partial L \backslash (U_1 \cup U_2 \cup U_3)$ is finite.

We will say that two elements $g \in \Omega_0(GL(V))$ and $h \in \Omega_0(GL(V))$ are *transversal* if $A(g) \cup A(g^{-1}) \subseteq P \setminus Cr(h)$ and $A(h) \cup A(h^{-1}) \subseteq P \setminus Cr(g)$. The simple consequence of Lemma 1.4 is the following

Lemma 1.5. *Let $S = \{g_1, \ldots, g_m\}$ be a subset of $\Omega_0(GL(V))$ such that g_i and g_j are transversal for every i,j $1 \leq i,j \leq m$. Then there exists a positive integer N such that for every sequence of positive integers $k_1, \ldots, k_m$, such that $k_t \geq N$ for all t with $1 \leq t \leq m$ the set $S(k) = \{g_1^{k_1}, \ldots g_m^{k_m}\}$ is freely generates of the free group $\langle S(k) \rangle$.*

Definition 1.6. *Let G be a subgroup of $GL(V)$ and $g_0 \in \Omega_0(G)$. We say that the set $F = \{g_i \in G, i \in I\}$ is a g_0–free system for G (or simply g_0–free system if it is clear which group is considered) if $g_i \in \Omega_0(G)$ for all $i \in I$ and there exists a set of open subsets $O = \{O_i = O_i(F), O_i \subseteq P, i \in I\}$, a set of disjoint compact sets $K = \{K_i = K_i(F), K_i \subseteq P, i \in I\}$, an open $U_0 = U(g_0)$ and a compact subset K_0 of P such that*

1. $A(g_i) \cup A(g_i^{-1}) \subseteq O_i \subseteq K_i$ *for all* $i \in I$,
2. $A(g_0) \cup A(g_0^{-1}) \subseteq U_0 \subseteq K$
3. $\inf_{i\in I} d(K_i, Cr(g_0)) > 0$,
4, $K \subseteq P \setminus \overline{\cup_{i\in I} K_i}$
5. $\widehat{g}_i^n K_j \subseteq O_i$ *for every* $i,j \in I, i \neq j$ *and non-zero* $z \in \mathbb{Z}$.
6. $\widehat{g}_i^n K \subseteq O_i$ *for every* $i \in I$ *and non-zero* $z \in \mathbb{Z}$.

From this definition immediately follows that the group generated by a g_0–free system F is free and elements of F are free generators.

Lemma 1.7. *Let a finite subset F of G be a g_0–free system. Let g be an element from Ω transversal to g_0 and to every element from F. Then there are two positive integers N such that for every $n > N$ there exists a positive integer $M = M(n)$ such that if $\widehat{g} = g_0^n g^m g_0^{-n}$, then $\widetilde{F} = F \cup \widetilde{g}$ is a g_0–free system for all $m > M$.*

Proof. Let $d_1 = d(A^+(g) \cup A^-(g), Cr(g_0))$ and let $d_2 = \inf_{i \in I} d(K_i, Cr(g_0))$. Put $d_0 = 1/4 \min(d_1, d_2)$ and let $B(A^-(g_0, d_0))$ be a ball of radius d_0 with center in $A^-(g_0)$. It follows from Lemma 1.4 that there exists a positive integer $N_1 = N_1(d_0, B(A^-(g_0), d_0))$ such that $\widehat{g_0}^{-n} K_i \subseteq B(A^-(g_0), d_0)$ for all positive $n > N_1$. Since $A^+(xgx^{-1}) = \widehat{x}A^+(g)$, $A^-(xgx^{-1}) = \widehat{x}A^+(g)$ and $Cr(xgx^{-1}) = \widehat{x}Cr(g)$, then $\overline{\cup_{i \in I} K_i} \subseteq P \setminus Cr(g_0^n g g_0^{-n})$ for all $n > N_1$. Set $d(n) = d(\overline{\cup_{i \in I} K_i}, Cr(g_0^n g g_0^{-n}))$. By Lemma 1.4 there exists a positive integer $N_2 = N_2(d_0, U_0)$ such that $\widehat{g_0}^n (A^+(g) \cup A^-(g)) \subseteq U_0$ for all $n > N_2$. Two elements g and g_0 are transversal, therefore two elements $g_0^n g g_0^{-n}$ and g_0 are transversal for all $n \geq \max\{N_1, N_2\}$. For $n \geq N$ set $h = g_0^n g g_0^{-n}$ and $d_h = d(n)$. There exists a compact subset K_h, an open subset O_h in P such that $A^+(h) \cap A^-(h) \subseteq O_h \subseteq K_h \subset U_0$ and $K_h \subseteq P \setminus Cr(g_0)$. From $K_h \subseteq U_0$ follows that $\widehat{g}_i^n K_h \subseteq O_i$ for all $i \in I$. It follows from Lemma 1.4 that there exists $M_1 = M_1(d_h)$ such that $\widehat{h}^m K_i \subseteq O_h$ for all $m, |m| \geq M_1$. Since $K_h \subseteq P \setminus Cr(g_0)$ there exists a compact set $K^* \subset U_0$ such that $A^+(g) \cup A^-(g) \subseteq K^*$. By Lemma 1.4 there exists a positive integer M_2 such that $\widehat{h}^m K^* \subseteq O_h$ for all all integers m, $|m| \geq M_2$. Put $M = \max\{M_1, M_2\}$. Set $\widetilde{g} = h^m$ then $\widetilde{F} = F \cup \{\widetilde{g}\}$ is a g_0–free system for $m \geq M$.

Then from Definition 1.6 no.5 follows that for every open subset U, $\widehat{g_0}^n (A^+(g) \cup A^-(g)) \subseteq U \subseteq U_0$ and $n > N_2$ for every $i \in I$ and positive integer m we have $\widehat{g}_i^m U \subseteq O_i$. Lemma 1.4 now shows that there exists $M = M(d(n), U)$ such that $\widehat{g}^m(K_i) \subseteq U$ for all $i \in I$ and $m > M$. Since elements g and g_0 are transversal elements g_0 and $g_0^m g^n g_0^{-m}$ are transversal for all integers m and n. Combining the above arguments, we conclude that $\widetilde{F} = F \cup \widetilde{g}$ is a g_0–free system. □

Therefore we have the following important

Corollary 1.8. *Let F, g_0, g be as in Lemma 1.7. Let H be a subgroup of finite index in G. Assume that $x \in G$ and $g \in xH$. Then there are two infinite sets of positive integers N and M such that $\widetilde{g} = g_0^n g^m g_0^{-n} \in xH$ and the set $\widetilde{F} = F \cup \widetilde{g}$ is a g_0–free system for all $n \in N$, $m \in M$.*

The proof is straightforward.

Assume now that :

1. There exists a proximal element in G.
2. The Zariski closure $\mathbb{G}$ of G is a semisimple group.
3. $\mathbb{G}^0 \cap G$ is an absolutely irreducible subgroup of $GL(V)$.

Then

Proposition 1.9. *Let H be a subgroup of G. Assume that H contains a proximal element and $\mathbb{G}^0 \cap H$ is Zariski dense in $\mathbb{G}^0$. Then for every $g \in G$ the set $gH \cap \Omega_0(G)$ is nonempty.*

Remark 1.10. This important proposition first was proved in [MS 3]. A different proof of this proposition can be deduced from the main theorem in [AMS].

The principal significance of the next theorem is in reduction to linear groups over local fields which allows us to use all above arguments. First this reduction was done in [T] for a finitely generated linear group where the Zariski closure is connected. A reduction to linear groups over local field in the general case when the Zariski closure is not necessarily connected is considerably more complicated.

Theorem 1.10. *Let G be a finitely generated non virtually solvable linear group, than there exist a local field k vector space W over k and irreducible representation $\rho : G \longrightarrow GL(W)$ such that*

1. *There exists a proximal element in $\rho(G)$.*
2. *The Zariski closure $\mathbb{G}$ of $\rho(G)$ is a semisimple group.*
3. *$\mathbb{G}^0 \cap \rho(G)$ is an absolutely irreducible subgroup of $GL(W)$.*

Since G is not virtually solvable, we can assume that the Zariski closure of G is semisimple. The proof splits naturally into a few steps.

Since we can reduce our group by taking a factor-group by non-trivial connected normal subgroup, it is easy to see that

Step 1. *It is enough to prove our statement under the following assumption: the Zariski closure of G is the wreath product $F \wr G_*$ where G_* is a simple (non-connected) algebraic group of an adjoint type and F is a finite group. The description of $F \wr G_*$ follows $F \wr G_* = G_*^F \rtimes F$ where F acts on G_*^F by shift.*

Step 2. *Observe that it is enough to prove the statement of Theorem 1.10 for a group G such that the Zariski closure of G is a simple algebraic (not necessarily connected) group of adjoint type*

Indeed by step 1 we can assume that our group is a wreath product $F \wr G_*$ where G_* is a simple (non-connected) algebraic group of an adjoint type and F is a finite group. In [AMS1, Theorem 5.17] we proved that if G is a direct product $G = \Pi_1^m G_i$ of semisimple groups such for every $1 \leq i \leq m$ there exists $g^{(i)} \in G$ such the projection $\pi_i(g^{(i)})$ of $g^{(i)}$ is a proximal element in G_i then there exists an element $g \in G$ such that $\pi_i(g)$ is a proximal element for every $1 \leq i \leq m$. Hence if there exists a representation we need for G_* taking the m-th tensor product of this representation we have an irreducible representation of G^F which we can and will extend it to $F \wr G_*$. This representation has all the necessary properties.

Since the Zariski closure $\mathbb{G}$ is a simple group of adjoint type we will assume that $\mathbb{G} \leq \text{Aut } \mathbb{G}^0$

Step 3. *This step splits naturally into two cases:*

Case1. *The factor group $\mathbb{G}/\mathbb{G}^0$ is cyclic.*

Case 2. *The factor group $\mathbb{G}/\mathbb{G}^0$ is the symmetric group of degree* 3

Since the proof of the statement in the case 2 may be handle in much the same way as in case 1, for the sake of exposition we restrict ourselves to the case $\mathbb{G}/\mathbb{G}^0$ is a cyclic group. By using standard arguments, we conclude that there exists a non torsion element $g \in G$ such that $\mathbb{G}$ is generated by $\mathbb{G}^0 \cup \{g\}$. Since G is finitely generated, it follows from [T, Lemma 4.1] that there exists a local field k with absolute value $|\cdot|$ vector space W over k and absolutely irreducible representation $\rho : G \longrightarrow GL(W)$ such that $\rho(g)$ is proximal. We claim that the group $\rho(\mathbb{G}^0 \cap G) = G_1$ is absolutely irreducible. Suppose the contrary. Let $\widetilde{W} = W \otimes_k \widetilde{k}$ and W_0 be a minimal G_1 invariant proper subspace of W. Put $x = \rho(g)$. Then there exist integers $i_1, \dots, i_t$ such that $\widetilde{W} = W_0 \oplus W_{i_1} \oplus \cdots \oplus W_{i_t}$ where $W_{i_s} = x^{i_s} W_0$. Since x^n is a proximal element for all positive integer $n \in \mathbb{Z}$, the eigenvector v corresponding to the maximal eigenvalue belongs to some W_i. Suppose for instance $v \in W_0$. Then $x^{i_1} v \in W_{i_1}$. Therefore $x^{i_1} v$ and v are two different eigenvectors for a proximal element x and correspond to the maximal eigenvalue, a contradiction.
Remark 1.12. In the reduction done in [T], Tits used an easy but crucial lemma saying that for a finitely generated infinite group G there exists a local field k with a absolute value $|\cdot|$ such that for some element $g \in G$ at least one eigenvalue has an absolute value $\neq 1$. This fact can be deduced from a natural and useful generalization proved in [BG2, Lemma 2.1].

2. Dense free subgroups.

For a start let us prove the Platonov problem.

Proof Observe that there exists a proximal element in $SL_n(\mathbb{Z})$. Indeed, let g be a diagonal matrix $g = \text{diag } (a_1, \ldots, a_n)$, $a_i > 0$ $a_i \in \mathbb{R}$, $a_1 > a_2 \geq \cdots \geq a_n$. Since the action of the group generated by g on $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is ergodic, then by well-known arguments [R] for every neighborhood of identity $U \in SL_n(\mathbb{R})$ there exists an infinite set of positive integers M such that the intersection $Ug^mU \cap SL_n(\mathbb{Z}) \neq \emptyset, m \in M$. From $g \in \Omega_0$ $(SL_n(\mathbb{R}))$ follows that there are a positive integer M and a neighborhood of identity $U, U \subseteq SL_n(\mathbb{R})$ such that $Ug^mU \subseteq \Omega_0$ $(SL_n(\mathbb{R}))$ for $m \geq M$. Let $g_0 \in \Omega_0(SL_n(\mathbb{R})) \bigcap SL_n(\mathbb{Z})$. The group $SL_n(\mathbb{Z})$ is absolutely irreducible; therefore there exists $x \in SL_n(\mathbb{Z})$ such that two elements $h = xg_0x^{-1}$ and g_0 are transversal. Consequently, there exists a positive integer k such that $\{h^k\}$ is a g_0 free system. Therefore we will assume that $\{h\}$ is a g_0–free system. It is not difficult to show that there exists a g_0–free system $F = \{h_1, \ldots, h_s\}$ such that the group generated by F is Zariski dense in $SL_n(\mathbb{R})$. Then [W] for $n \geq 3$ the pro- finite closure F^* of F is subgroup of finite index in $SL_n(\mathbb{Z})$. Let $x_iF^*, i = 1, \ldots, t$ be all different classes $SL_n(\mathbb{Z})/F^*$. By Corollary 1.8, there exist elements h_{s+1}, h_{s+t} such that $h_{s+i} \in x_iF^*, i = 1, \ldots, t$ and $F_0 = \{h_1, \ldots, h_s, h_{s+1}, \ldots, h_{s+t}\}$ is a g_0 –free system. It is clear that the group generated by F_0 is pro-finitely dense in $SL_n(\mathbb{Z})$ for $n \geq 3$. Since the group $SL_n(\mathbb{Z})$ is finitely generated there exists a maximal proper subgroup H which contains F_0. Obviously H is a maximal subgroup of $SL_n(\mathbb{Z})$. Assume that the index $SL_n(\mathbb{Z})/H$ is finite. Then H is a proper pro-finitely dense open subgroup of $SL_n(\mathbb{Z})$ which is impossible and the proof is completed.

Since a subgroup of a finite index of a finitely generated group is a finitely generated group we reformulated [MS1] Platonov's problem as following:

Conjecture (G. Margulis, G. Soifer) Let $G = SL_n(\mathbb{Z})$ and H be a maximal subgroup of G. Assume that H is a finitely generated group, then the index G/H is finite.

This conjecture is true for $n = 2$. Furthermore it is true if G is a lattice in $SL_2(\mathbb{R})$ (see [SV]). Y. Glasner pointed out that $SL_n(\mathbb{Z})$ is a maximal subgroup of infinite index of a lattice $SL_n(\mathbb{Z}[1/p])$. Nevertheless the above conjecture is still open for $n \geq 3$.

When our results were announced [MS 1] we received a letter from G.Prasad with the following conjecture

Conjecture (G. Prasad, J. Tits) Every maximal subgroup of $SL_n(\mathbb{Z})$, $n \geq 3$ of infinite index is virtually free.

We show in [MS2] that there exists a not virtually free maximal subgroup of an infinite index in $SL_n(\mathbb{Z})$ for $n \geq 4$. Now we will show that for $n = 3$ this conjecture is also not true. Recall that the group $SL_2(\mathbb{Z})$ is virtually free.

Theorem 2.1. There exists a maximal subgroup of infinite index in $SL_3(\mathbb{Z})$ which is not virtually free.

Proof. Let $G = SL_3(\mathbb{R})$ and $\Gamma = SL_3(\mathbb{Z})$. Let g_1, g_2 be two commuting elements from $\Omega(\Gamma)$ which generate a free abelian group of rank 2 . Let e_1, e_2, e_3 be their eigenvectors corresponding to a three different eigenvalues. Consider the set $\Sigma = \{t_1e_1+t_2e_2+t_3e_3, \sum_1^3 t_i = 1, t_i \geq 0, i = 1,2,3\}$. Let $P(\Sigma)$ be the projectivization of Σ and let $\partial P(\Sigma)$ be the boundary of $P(\Sigma)$. Let q (correspondingly, $\widehat{e_i}$ $i = 1,2,3$) be the point of $P(\Sigma)$ which corresponds to the line span by the vector $1/3e_1 + 1/3e_2 + 1/3e_3$ (correspondingly, e_i $i = 1,2,3$). Analysis similar to that in Example 1 with a bit more routine calculations shows that elements g_1 and g_2 will fulfil following properties:

1. The boundary ∂L of the orbit $L = \widehat{A}q = \{\widehat{a}q, a \in A\}$ is a subset of $\partial P(\Sigma)$
2. Let U_i be a neighborhood of the point $\widehat{e_i}$, $i = 1,2,3$. Then the set $\partial L \backslash (U_1 \cup U_2 \cup U_3)$ is finite.

Hence there are two lines L_1 and L_2 in the projective space P such that $\partial L \cap L_i = \emptyset$ for $i = 1,2$ and $q = L_1 \cap L_2$. It is easy to see that there exists a positive integer N and neighborhood W of the point q and a compact $K_0, W \subseteq K_0$ such that $\widehat{a}^n K_0 \cap (L_1 \cup L_2) = \emptyset$ for all $a \in A$ and $|n| \geq N$. Let g_0 be a hyperbolic element of G such that $B^+(g_0) = L_1$, $B^-(g_0) = L_2$ and $A^+(g_0) \cup A^-(g_0) \subseteq W$. Put $A_0 =< g_1^N, g_2^N >$. Then there exists a positive d such that the distance $\min_{a \in A_0} d(\widehat{a}K_0, L_1), d(\widehat{a}K_0, L_2) > d$. Since the action of the subgroup group generated by g_0 on $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$ is ergodic, for every neighborhood U of the identity in the group G there exists an infinite set of positive integers M such that $Ug_0^rU \cap \Gamma \neq \emptyset$. Hence because for $m \in M$ $A^+(g^m) = A^+(g)$, $A^-(g^m) = A^-(g), B^+(g^m) = B^+(g)$ and $B^-(g^m) = B^-(g)$ for every positive ε there exist neighborhood U of the identity in the group G and positive integer M_0 such that if $m \geq M_0$ for every $g \in Ug_0^rU$ we have

1. $g \in \Omega_0(G)$,
2. $A^+(g) \cup A^-(g) \subseteq W$,
3. $\overline{\cup_{n\in\mathbb{Z}}\, \widehat{a}^n K_0} \subseteq P \setminus (B^+(g) \cup B^-(g))$.

Therefore we can and will assume that $g_0 \in \Gamma$. Let g be an element from Γ transversal to g_0. It follows from (2) and (3) since $A^+(g_0) \cup A^-(g_0) \subseteq P \backslash (B^+(g) \cup B^-(g))$, that there exists a positive integer N_0 such that for $n \geq N_0$ we have $g_0^{-n} \overline{\cup_{n\in\mathbb{Z}}\, \widehat{a}^n K_0} g(n)_0 \subseteq P \backslash (B^+(g) \cup B^-(g))$. Hence there exists a positive integer N_0 such that for $n \geq N_0$ we have $B^+(g_0^n g g_0^{-n}) \cup B^-(g_0^n g g_0^{-n}) \subseteq P \backslash \overline{\cup_{n\in\mathbb{Z}}\, \widehat{a}^n K_0}$. Repeated application of Corollary 1.8 enables us using an arguments from [SV] to claim that there exists a finite g_0 –free system $F = \{f_i, 1 \leq i \leq m\}$ such that

1. The group generated by F is pro-finitely dense in Γ.
2. $A^+(f_i) \cup A^-(f_i) \subseteq W$ for every $i, 1 \le i \le m$.
3. $d(\widehat{a}^n K_0, (B^+(f_i) \cup B^-(f_i)) > d/2$ for every $1 \le i \le m$ and $a \in A_0$.

Indeed, assume that $f \in \Omega(\Gamma)$ is an element transversal to g_0. Since for every neighborhood U of $A^-(g_0)$ there exists a positive integer N_0 such that for all $n \ge N_0$ we have $\widehat{g}^{-n_0}(\overline{\cup_{n\in\mathbb{Z}}\, \widehat{a}^n K_0}) \subseteq U$. Therefore if $U \in P \setminus (B^+(f) \cup B^-(f))$ then $\widehat{g}^{-n_0}(\overline{\cup_{n\in\mathbb{Z}}\, \widehat{a}^n K_0}) \subseteq P \setminus (B^+(f) \cup B^-(f))$ for $n \ge N_0$. Hence $\overline{\cup_{n\in\mathbb{Z}}\, \widehat{a}^n K_0} \subseteq P \setminus (B^+(g^n f g^{-n}) \cup B^-(g^n f g^{-n}))$ for $n \ge N_0$. As we proved in [SV] there exists a finite set of elements $f_1, \dots, f_k$ such that the Zariski closure of the group $< f_s >$ is connected for every $1\ leqs \le k$ and the pro-finite completion of the group $F_1 =< f_1, \dots, f_k >$ is a subgroup of a finite index in Γ. It follows from Lemma 1.7 that we can and will assume that $f_1, \dots, f_k$ is a g_0–free system and fulfil property no3. Let $T_1, ;T_r$ be all different co-sets Γ/F_1. Repeated arguments above enabled us to show that there are elements $f_{k+1}, \dots, f_{k+r}$ such that the set $f_1, \dots f_{k+r}$ is a g_0–free system which fulfil properties no1,2,3.
It follows from no.3 that there exists a compact K_N such that $\cup_{n\in\mathbb{Z}}\widehat{a}^n K_0 \subseteq K_N$ and $d(K_N, B^+(f_i) \cup B^-(f_i)) > d/4$. It follows from Lemma 1.4 that there exists a positive number N_1 such that $f_i^n K_N \subseteq W$ for all $n \ge N_1$. Set $n_i = 2N_1$ for $i = 1, \dots, k$ and $n_i \ge N_1, f_i^{n_i} \in M_i$ for $i = k+1, \dots, k+r$. Then the group $F =< f_1^{n_1}, \dots f_{k+r}^{n_{k+r}} >$ is a free subgroup which is pro-finitely dense in Γ such that $\widehat{f} K_N \subseteq W$ for every $f \in F$. Hence the group generated by A_0 and F is a free product $A_0 * F$. Therefore a maximal subgroup of Γ which contains the group generated by A_0 and $\widetilde{F}$ will be a maximal subgroup of Γ. This subgroup will be of infinite index since the group generated by $\widetilde{F}$ is pro-finitely dense. This subgroup contains a free abelian group A_0 of rank 2. Hence it is not virtually free. □

There are some other results in the spirit of the statement of Theorem 2.1, see for example [S], [V].

Let G be a subgroup of $GL_n(k)$ where k is a local field. The full linear group $GL_n(k)$ and hence any subgroup of it is endowed with the standard topology that is the topology induced from the local field k. We will denote by $\|\cdot\|$ a norm on $GL_n(k)$ induced from the local field absolute value $|\cdot|$. Let $S = \{s_1, \dots, s_m, s_i \in GL_n(k)\}$ be a finite set. Put $S(\varepsilon) = \{(\widehat{s}_1, \dots, \widehat{s}_m)$ such that $\|\widehat{s}_i - s_i\| \le \varepsilon$ for all $1 \le i \le m$ and $\varepsilon > 0\}$. Assume that Γ is a finitely generated dense subgroup of a connected semisimple group G, $G \le GL_n(k)$. We claim that there exists ε_0 such that for every $\varepsilon < \varepsilon_0$ the group generated by the set $\{\widehat{s_1}, \dots, \widehat{s_m}\}$ where $(\widehat{s_1}, \dots, \widehat{s_m}) \in S(\varepsilon)$ is dense in G (see [BG2, 5.1]). Indeed, there exist a finite set $S_1 = \{g_1, \dots, g_l\}$ where $l = \dim G$ and ε_1 such that if $\|\widehat{g}_i - g_i\| \le \varepsilon_1$ for $i = 1, \dots, l$ then the group generated by $\widehat{S}_1 = \{\widehat{g}_1, \dots, \widehat{g}_l\}$ is dense in G. Since Γ is dense in G then there exist elements $\widehat{\gamma}_i \in \Gamma$ such that $\|\widehat{\gamma}_i - g_i\| \le \varepsilon_1/2$ for all $i =$

$1, \dots, l$. Then $\widehat{\gamma}_i = w_i(s_1, \dots, s_m)$. Consider maps $w_i : \underbrace{G \times \cdots \times G}_{m} \longrightarrow G$ where $i = 1, \dots, l$. There exists an ε such that if $\|\widehat{s}_i - s_i\| \leq \varepsilon$ for all $i = 1, \dots, l$ then $\|w_i(\widehat{s}_1, \dots, \widehat{s}_m) - w_i(s_1, \dots, s_m)\| \leq \varepsilon_1$ and therefore $\|w_i(\widehat{s}_1, \dots, \widehat{s}_m) - g_i\| \leq \varepsilon_1$ for all $i = 1, \dots, l$. Hence the group generated by the set $\{\widehat{s}_1, \dots, \widehat{s}_m\}$ will be dense in G.

E. Breuillard and T. Gelander proved in [BG2] the following topological Tits alternative.

Theorem 2.2 [BG2]. Let k be a local field and Γ a subgroup of $GL_n(k)$. Then Γ contains either open solvable subgroup or a dense free subgroup.

Note that for a non discrete subgroup Γ the two cases are mutually exclusive. Hence if Γ a dense subgroup of a semisimple connected Lie group G, then for any set
$S = \{\gamma_1, \dots, \gamma_m\}$ of generators of Γ there exists an ε_0 such that for every $\varepsilon \leq \varepsilon_0$ there exists $\widehat{S} = (\widehat{\gamma}_1, \dots, \widehat{\gamma}_m) \in S(\varepsilon) \cap \underbrace{\Gamma \times \cdots \times \Gamma}_{m}$ such that the group $< \widehat{S} >$ is free and dense in G. For a compact connected Lie group one can deduce this fact for a from [S, Proposition 4.5].

Let G be a connected compact group Lie. Then the set of torsion elements of G is dense in G. Therefore for every ε and any set of generators $S = \{\gamma_1, \dots, \gamma_m\}$ of a subgroup Γ there exists $(\widehat{\gamma}_1, \dots, \widehat{\gamma}_m) \in S(\varepsilon)$ such that the group generated by $\widehat{S} = \{\widehat{\gamma}_1, \dots, \widehat{\gamma}_m\}$ is not free because it contains torsion. On the other hand every finitely generated linear group has a subgroup of a finite index without torsion. Therefore we state the following

Conjecture. (G. Margulis, G. Soifer.) Let G be a non solvable connected Lie group. Assume that the subgroup of G generated by a set $S = \{s_1, \dots, s_m\}$ is a free dense subgroup. Then for every ε there exists $(s_1^*, \dots, s_m^*) \in S(\varepsilon)$ such that the group generated by the set $S^* = \{s_1^*, \dots, s_m^*\}$ is not virtually free.

This conjecture first was stated for a compact group and it was proved recently by T.Gelander. His results show that it will interesting to answer to the following

Problem. Let G be a connected Lie group let F be a free dense subgroup of G generated by a set $S = \{s_1, \dots s_n\}$. Is it true that for every dense subgroup Γ of G and every ε there exists a set $\{\gamma_1, \dots, \gamma_n\}$ such that $\{\gamma_1, \dots, \gamma_n\} \subseteq S(\varepsilon)$ and $< \gamma_1, \dots, \gamma_n >= \Gamma$.

3. Euclidean crystallographic semigroups.

Recall that a semigroup S acts properly discontinuously on a topological space X if for every compact subset $K \subseteq X$ the set $\{s \in S | sK \cap K \neq \varnothing\}$ is finite. A semigroup S is called crystallographic if it acts properly discontinuously and there exists a compact subset $K_0 \subseteq X$ such that $\cup_{s \in S} sK_0 = X$. In the case when $X = \mathbb{R}^n$ and $S \subseteq \text{Isom } X$, S is called a Euclidean crystallographic semigroup.

In this section we will prove the following conjecture due to G.Margulis.

Conjecture Let S be an Euclidean crystallographic semigroup, then S is a group.

Here is a scenario of our proof of the above conjecture. The main idea of the proof is to show that the Zariski closure G of a semigroup S does not contain a free subgroup. We prove this using our ideas and results from [S]. Therefore by the Tits' alternative G is a virtually solvable group. Hence the linear part of G is a compact virtually solvable group. Consequently the linear part of G is virtually abelian. Combining this with the fact that S acts properly discontinuously, we show that every element in S is invertible. Thus S is a group.

Let us recall some necessary definitions . Let $G = \text{Aff } \mathbb{R}^n$ be the group of all affine transformation of the n–dimensional real affine space $\mathbb{R}^n$. This group is the semidirect product of $GL_n(\mathbb{R})$ and the subgroup of all parallel translations which can be identified with $\mathbb{R}^n$, i.e,

$$\mathrm{Aff}\mathbb{R}^{\mathrm{n}} = \mathbb{R}^{\mathrm{n}} \rtimes \mathrm{GL_n}(\mathbb{R}).$$

We will consider the natural homomorphism

$$\ell : \mathrm{Aff}\mathbb{R}^{\mathrm{n}} \longrightarrow \mathrm{GL_n}(\mathbb{R}),$$

and because the group $\mathrm{Aff}\mathbb{R}^{\mathrm{n}}$ is semidirect product we have for every element $g \in \mathrm{Aff}\mathbb{R}^{\mathrm{n}}$ the decomposition

$$g = v_g \ \ \ell(g), \ \ v_g \in \mathbb{R}^n, \ \ \ell(g) \in GL_n(\mathbb{R}^n).$$

Let q be a positive definite quadratic form on $\mathbb{R}^n$. Then

$$\mathrm{Isom}\mathbb{R}^{\mathrm{n}} = \{\mathrm{g} \in \mathrm{Aff}\mathbb{R}^{\mathrm{n}} : \mathrm{q}(\ell(\mathrm{g})(\mathrm{x})) = \mathrm{q}(\mathrm{x})\}.$$

Hence if $g \in \mathrm{Isom}\mathbb{R}^{\mathrm{n}}$ then $\ell(g) \in O(q)$. Let $g \in \mathrm{Isom}\mathbb{R}^{\mathrm{n}}$ and let $V^0(g) = \{v \in \mathbb{R}^n : \ell(g)v = v\}$. Recall that an element g of an algebraic group $G,\ G \subseteq GL_n(\mathbb{R})$ is called regular if and only if

$$\dim V^0(g) = \min_{x \in G} V^0(x).$$

Let $g \in \mathrm{Aff}\mathbb{R}^{\mathrm{n}}$ then there exists the maximal g –invariant affine space $A^0(g)$ of $\mathbb{R}^n$ such that g induces a translation on it. This translation can be zero and in that case all points in $A^0(g)$ are fixed points. It is easy to see that if we define the vector $v^0(g)$ as follows: consider a point $x \in A^0(g)$ put

$$v^0(g) = \frac{gx - x}{\|gx - x\|}$$

then this vector does not depend on x. We define $v^0(g) = 0$ if there exists a g–fixed point. For every set of vectors X in $\mathbb{R}^n$ there is a smallest convex cone C_X such that $X \subseteq C_X$. Therefore for every semigroup $S \subseteq \mathrm{Isom}\mathbb{R}^n$ there exists the convex cone $C(S)$ defined as $C(S) = C_X$, where $X = \{v^0(g),\, g \in S,\, \ell(g)$ is a regular element of the group $\overline{\ell(\mathrm{S})}\}$. Let $V^0(S)$ be the subspace generated by the set $\{V^0(g)\}_{g\in S}$. Assume that there is no non-trivial S–invariant *affine* subspaces of the affine space $\mathbb{R}^n$ then
Lemma 3.1. $V^0(S) = \mathbb{R}^n$.

Proof. It immediately follows from [S, Lemma 4.1] that $V^0(S)$ is an $\ell(S)$–invariant subspace of the vector space $\mathbb{R}^n$. This subspace is non-trivial because S acts properly discontinuously. Since the closure $\overline{\ell(S)}$ is a reductive group, there exists an $\ell(S)$–invariant subspace W of $\mathbb{R}^n$ such that $V^0(S) \oplus W = \mathbb{R}^n$. Assume that $W \neq \{0\}$. We have the natural projection of the affine space $\mathbb{R}^n$ onto to the affine space $A_1 = \mathbb{R}^n/V^0(S)$ along $V^0(S)$ and hence an induced homomorphism $\rho : S \to \mathrm{Isom}\ A_1$. Let $\ell(\rho(g))$ be the linear part of $\rho(g)$. Since $V^0(g) \subseteq V^0(S)$ there exists a fixed point for every element $\rho(g), g \in S$. Therefore the closure of the group generated by $\rho(g)$ is compact for every $g \in S$. Thus the group $\overline{\rho(S)}$ is compact. Hence there exists a $\overline{\rho(S)}$–fixed point p_0 in A_1 . Consequently $p_0 + V^0(S)$ is non-trivial proper S–invariant affine subspace. Contradiction which proves the lemma. □

Consider the closure $\overline{C(S)}$ of the cone $C(S)$ in $\mathbb{R}^n$. Our next goal is to prove
Lemma 3.2 Let S be a crystallographic semigroup. Then $C(S) = \mathbb{R}^n$.

Proof. By [S, Lemma 4.1] it is enough to prove that $\overline{C(S)} = \mathbb{R}^n$. To obtain a contradiction assume that $\overline{C(S)} \neq \mathbb{R}^n$. Then there exists a non-zero vector v_0 in $\mathbb{R}^n$ such that the scalar product $(v, v_0) \leq 0$ for all $v \in C(S)$ and there is a vector $\widetilde{v} \in C(S)$ such that $(\widetilde{v}, v_0) < 0$. It follows from our assumption and Lemma 3.1 that there exists an element $s_0 \in S$ such that $V^0(s_0) \subsetneq C(S)$. Let v^* be a vector from $V^0(s_0)$ such that $(v^*, v_0) > 0$.
Let K be a compact subset in $\mathbb{R}^n$ such that $\cup_{s\in S}\, sK = \mathbb{R}^n$. Fix a point p_0 in K. It is clear that if p_m is a point of $\mathbb{R}^n$, where $p_m = p_0 + mv_0$ and m is a positive number, then there exists $s_m \in S$ such $p_m \in s_m K$. Therefore for every point $p \in K$

$$\lim_{m\to\infty} \frac{s_m p - p}{\|s_m p - p\|} = v^*. \tag{3.1}$$

Consider the subset $\{\ell(s_m)\}_{m\in\mathbb{N}}$ of the compact group $O(q)$. We can and will assume that this sequence is converge to some element g_0 of the closure $\overline{\ell(S)}$ in $O(q)$. Put $g_1 = \ell(s_0)g_0^{-1}$. Since $g_1 \in \overline{\ell(S)}$ there is a sequence $\{\overline{s}_m\}_{m\in\mathbb{N}}$ of elements from S such that the sequence $\{\ell(\overline{s}_m)\}_{m\in\mathbb{N}}$ converges to g_1. We can assume that for every point $p \in K$ we have

$$\lim_{m\to\infty} \frac{\|\overline{s}_m(p) - p\|}{\|s_m p - p\|} = 0. \tag{3.2}$$

Define $t_m = \overline{s}_m s_m$ for all positive integers m. Let us show, that

$$\lim_{m\to\infty}(v^0(t_m), v_0) > 0. \qquad (3.3).$$

Indeed, let $q_m = \overline{s}_m^{-1} p_0$ and let v_m be a vector $v_m = t_m q_m - q_m/\|t_m q_m - q_m\|$. Since $t_m q_m - q_m = t_m p_0 - p_0 + \overline{s}_m q_m - q_m$ and $\|\overline{s}_m q_m - q_m\| = \|\overline{s}_m p_0 - p_0\|$ from (3.2) follows that

$$\lim_{m\to\infty} v_m = v^*. \qquad (3.4).$$

The sequence $\{\ell(\overline{s}_m)\}_{m\in\mathbb{N}}$ converges to g_1. Therefore the sequence $\{\ell(t_m)\}_{m\in\mathbb{N}}$ converges to $\ell(s_0)$. Thus the sequence $\{V^0(t_m)\}_{m\in\mathbb{N}}$ converges to the subspace $V^0(t)$. Now we use the following idea. Let g be an euclidian transformation and let $\pi_g : \mathbb{R}^n \longrightarrow A^0(g)$ be the orthogonal projection onto $A^0(g)$. Then for every point x of $\mathbb{R}^n$ we have $\pi_g(gx-x) = \alpha v^0(g)$ where $\alpha = \|\pi_g(gx-x)\|$. Consequently because the sequence $\{V^0(t_m)\}_{m\in\mathbb{N}}$ converges to the subspace $V^0(t)$ from (3.4) follows that $\lim_{m\to\infty} v_m^0 = v^*$. Hence $\lim_{m\to\infty}(v^0(t_m), v_0) > 0.$. On the other hand, $v^0(t_m) \in \overline{C(S)}$. Therefore $(v^0(t_m), v_0) \le 0$. Contradiction which proves the lemma. □

We will use the following fact proved in [S, Proposition 4.5].

Proposition 3.3. Let S be a subsemigroup of Isom $\mathbb{R}^n$. *Assume that $\overline{\ell(S)}$ is a connected non-solvable group. Then for every finite set $g_1, \ldots, g_m \subseteq S$ such that $v^0(g_i)$ non-zero for all $i, 1 \le i \le m$, and every positive $\varepsilon, \varepsilon < 1, \varepsilon \in \mathbb{R}$, there are elements $g_1^*, \ldots, g_m^* \subseteq S$ such that*

(1) *Semigroup generated by $g_1^*, \ldots, g_m^*$ is free and $g_1^*, \ldots, g_m^*$ are free generators,*
(2) *$(v^0(g_i^*), v^0(g_i)) > 1 - \varepsilon$ for all $i, 1 \le i \le m$.*

Proposition 3.4. Let S be a crystallographic semigroup, then the Zariski closure G of S is a virtually solvable group.

Proof. On the contrary assume that the group G is not virtually solvable. Without loss of generality we can assume that the group G is connected. It follows from Lemma 3.1 and Lemma 3.2 that there exists a finite subset $\{s_1, \ldots, s_m\}$ of

S such that convex hole of the set $\{v^0(s_1), \ldots, v^0(s_m)\}$ is $\mathbb{R}^n$. Hence there exists a positive real number ε, $\varepsilon < 1$ such that if vectors $w_1, \ldots, w_m$ are taken such that $(w_i, v^0(s_i)) \geq 1 - \varepsilon$ then the convex hole of the set $\{w_1, \ldots, w_m\}$ is $\mathbb{R}^n$. Let s_0 be any element of S such that $v^0(s_0) \neq 0$. It follows from Proposition 3.3 that there exists a subset $s_0^*, s_1^* \ldots, s_m^*$ such that

(1) subgroup generated by the subset $\{s_0^*, s_1^* \ldots, s_m^*\}$ is free and $s_0^*, s_1^* \ldots, s_m^*$ are free generators,
(2) convex hole of the set $\{v^0(s_1^*), \ldots, v^0(s_m^*)\}$ is $\mathbb{R}^n$.

Since the convex hole of the set $\{v^0(s_1^*), \ldots, v^0(s_m^*)\}$ is $\mathbb{R}^n$, there exists a positive real number $\delta = \delta(s_1^* \ldots, s_m^*)$ such that for arbitrary unite vector $v \in \mathbb{R}^n$ and some vector $v_{i_0}, 1 \leq i_0 \leq m$ we have $\cos\measuredangle(v, v_{i_0}) \leq -\delta$. Let $x_i \in A^0(s_i^*)$ be a point $i, 1 \leq i \leq m$. Put $l_i = \|s_i^* x_i - x_i\|$ for each $i, 1 \leq i \leq m$. Set $l = \max_{1 \leq i \leq m} l_i$. Fix a point $p_0 \in \mathbb{R}^n$. Let p be a point of $\mathbb{R}^n$. It is easy to see that there exist positive integers $j, 1 \leq j \leq m$ and k_j such that $d((s_j^*)^{k_j} p, p_0) \leq \sqrt{1-\delta^2} d(p, p_0) + \frac{1}{2} l$. Assume that the euclidian distance $d(p, p_0) > \frac{2l}{\delta^2}$. Then there is a positive number $\theta = \theta(\delta) \leq 1$ such that if $d(p, p_0) \geq \frac{2l}{\delta^2}$ we have

$$d((s_j^*)^{m_j} p, p_0) \leq \theta d(p, p_0). \tag{3.5}$$

Let us now define the following infinite set of disjoint subsets S_j of S. Let S^* be a subsemigroup generated by the set $\{s_1^* \ldots, s_m^*\}$. Put $S_j = S^* s_0^j$. Let $d_j = \min_{s \in S_j} d(sp_0, p_0)$. By (3.5) we have that $d_j \leq l$. Since subsets S_j are disjoint the intersection $B(p_0, 2l/\delta^2) \cap S_j$ is non-empty for every positive integer j. Therefore the semigroup S does not acts properly discontinuously on $\mathbb{R}^n$. Contradiction. □

Now we will prove the Margulis conjecture.
Proof. Let G be the Zariski closure of the semigroup S. Let us show that $G \cap \mathbb{R}^n$ is a finite index subgroup in G. It is enough to show that if the group G is connected then $G \subseteq \mathbb{R}^n$. From Proposition 3.4 follows that the group $\overline{\ell(S)}$ is solvable. This group is compact and connected, therefore it is an abelian group. It is well known that a finitely generated linear group contains torsion free subgroup of a finite index. Therefore by above arguments we can and will assume that S is torsion free. Thus for two different regular elements s_1 and s_2 from S we have $V^0(s_1) = V^0(s_2)$. Then by lemma 3.1 $V^0(s) = \mathbb{R}^n$ for every regular element of S. Consequently for every element $s \in S$ we have $\ell(s) = 1$. Therefore the connected component $G^0 \subseteq \mathbb{R}^n$.

Let $S_0 = G^0 \cap S$. It follows from Lemma 3.2 since G^0 is a subgroup of finite index in G that $C(S^0) = \mathbb{R}^n$. On the other hand the semigroup S^0 contains only translations and acts properly discontinuously. Therefore S^0 is a group. It is clear that if a subsemigroup of finite index in semigroup is a group then the semigroup is a group which proves the statement.

References

[A] H. Abels, *Proximal Linear Maps*, (in this volume)

[AMS 1] H. Abels, G.A. Margulis and G.A. Soifer, *Semigroups containing proximal linear maps*, Israel J. of Math. 91 (1995) 1–30

[AMS 2] —, *On the Zariski closure of the linear part of a properly discontinuous group of affine transformations*, J. Diff. Geom. 60, (2003),2, 314-335

[B] A. Borel, *Linear Algebraic Groups*, Springer Verlag.

[BG 1] E. Breuillard, T. Gelander, *On dense free subgroups of Lie groups*, J. Algebra 261 (2003), no. 2, 448-467.

[BG 2] E. Breuillard, T. Gelander, *Topological Tits alternative*, Ann of Math (to appear).

[GGl] T. Gelander, Y. Glasner, *Infinite Primitive Groups*, GAFA.

[DPSS] J.D. Dixon, L. Pyber, A. Seress, A. Shalev, *Residual properties of free groups and probabilistic methods*, J. Reine Angew. Math. 556 (2003), 159-172.

[MS 1] G.A. Margulis, G.A. Soifer, *The criterion of the existence of maximal subgroups of infinite index in a finitely generated linear group*, Soviet Math. Dokl. 18, N3 (1977), 847-851

[MS 2] G.A. Margulis, G.A. Soifer, *Nonfree maximal subgroups of infinite index in the group* $SL_n(\mathbb{Z})$, Uspekhi Math. Nauk, 34,4(208) (1979) 203-204

[MS 3] G.A. Margulis, G.A. Soifer, *Maximal subgroups of infinite index in finitely generated linear group*, J. Algebra 69, No 1., (1981), 1 -23

[P] L. Pyber, *Groups enumeration and where it leads us*, Prog. Math. Birkhauser, 169 (1996), 187-199

[R] M.S. Raghunathan, *Discrete subgroups of a Lie groups*, Springer -Verlag, 1972

[S] G.A. Soifer, *The linear part of a discontinuously acting euclidean semigroup*, J. Algebra, 250 (2002), 647- 663.

[SV] G. Soifer, T. Venkataramana, *Finitely generated pro-finitely dense free groups in higher rang semisimple groups*, Transform. Groups 5 (2000), no. 1, 93-100

[T] J. Tits, *Free subgroups in linear groups*, J. Algebra 20 (1972)

[V] T.N. Venkataramana, J. of Algebra 108, (1987), 325-339

[W] B. Weisffeiler, *Strong approximation for Zariski dense subgroups of semisimple algebraic groups*, Ann. of Math. 120 (1984), 271-315.

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