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Divergent Orbits on $\mathcal{S}$-adic Homogeneous Spaces

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Dedicated to Grisha Margulis on the occasion of his sixtieth birthday

Abstract: Let $\mathbf{G}$ be a semisimple algebraic group defined over a number field K and let $\mathcal{S}$ be a finite set of non-equivalent valuations of K containing the archimedean ones. Set $G = \prod_{v\in\mathcal{S}} \mathbf{G}(K_v)$ and $\Gamma = \mathbf{G}(\mathcal{O})$ where $\mathcal{O}$ is the ring of $\mathcal{S}$-integers of K. Fix $v \in \mathcal{S}$ and a K_v-split algebraic torus T_v of $\mathbf{G}(K_v)$. In this paper, in complement to results from [To], we prove results about the divergent orbits for the action of T_v on G/Γ by left translation.

1. Introduction

In this paper $\mathbf{G}$ denotes a semisimple algebraic group defined over a number field K and $\mathcal{S}$ denotes a finite set of non-equivalent valuations of K containing all archimedean ones. For every $v \in \mathcal{S}$ we let $G_v = \mathbf{G}(K_v)$, where K_v is the completion of K with respect to v. Let $\mathcal{O}$ be the ring of $\mathcal{S}$-integers of K. Set $G = \prod_{v\in\mathcal{S}} G_v$ and $\Gamma = \mathbf{G}(\mathcal{O})$. The group Γ is identified with its diagonal imbedding in G. It is well known that Γ is a lattice in G, i.e., Γ is a discrete subgroup of finite co-volume in G. We are interested of the action of algebraic tori T on G/Γ by left translations:

$$t\pi(g) = \pi(tg),$$

where $\pi : G \to G/\Gamma$ is the quotient map. The study of this action is especially important for the Diophantine approximations of numbers. For instance, the notable Littlewood conjecture would follow from a conjecture of G.Margulis which states that if D is the group of all diagonal matrices in $\mathrm{SL}_3(\mathbb{R})$ then every relatively compact D-orbit on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ is compact. (Actually, in [Ma1] Margulis formulated his conjecture in the context of all real Lie groups.) Recently, M.Einsiedler, A.Katok and E.Lindenstrauss proved, using the dynamical

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approach, that the Littlewood conjecture fails at most on a set of Hausdorff dimension zero [Ei-Ka-Li]. A similar result in p-adic setting has been subsequently proved by M.Einsiedler and D.Kleinbock [Ei-Kl].

Another recent application of the tori actions on G/Γ is related to the characterization of the rational decomposable homogeneous forms in terms of their values at the integer points. The present paper completes some of the results in [To] where this application has been obtained. For reader's convenience we will give the formulations of the main results from [To]. Denote by $K_{\mathcal{S}}$ the direct product of the topological fields K_v, $v \in \mathcal{S}$. Then $K_{\mathcal{S}}$ is a topological ring and the ring of $\mathcal{S}$-integers $\mathcal{O}$ is discrete in $K_{\mathcal{S}}$. For every $v \in \mathcal{S}$, let $f_v = l_1^{(v)} \dots l_m^{(v)} \in K_v[\vec{x}]$, where $l_1^{(v)}, \dots, l_m^{(v)}$ are linearly independent over K_v linear forms in n variables $\vec{x} = (x_1, \dots, x_n)$. We have

Theorem 1.1. ([To, Theorem 1.7]) *With the above notation, assume that* $\{(f_v(\vec{z}))_{v\in\mathcal{S}} \in K_{\mathcal{S}} | \vec{z} \in \mathcal{O}^n\}$ *is a discrete subset of* $K_{\mathcal{S}}$. *Then there exist a homogeneous form* g *with coefficients from* $\mathcal{O}$ *and an element* $(\alpha_v)_{v\in\mathcal{S}} \in K_{\mathcal{S}}^*$ *such that* $f_v = \alpha_v g$ *for all* $v \in \mathcal{S}$.

In the classical cases $K = \mathbb{Q}$ and $K = \mathbb{Q}(\,i\,)$ (= the field of Gaussian numbers) Theorem 1.4 immediately implies the following result which, to the best of our knowledge, is new:

Corollary 1.2. *Let* $f(\vec{x}) = l_1(\vec{x}) \dots l_m(\vec{x})$, *where* $l_1(\vec{x}), \dots, l_m(\vec{x})$ *are linear forms with real* (*respectively, with complex*) *coefficients. Suppose that* $l_1(\vec{x}), \dots, l_m(\vec{x})$ *are linearly independent over* $\mathbb{R}$ (*respectively, over* $\mathbb{C}$) *and that the set* $f(\mathbb{Z}^n)$ (*respectively, the set* $f(\mathbb{Z}[\,i\,]^n)$) *is discrete in* $\mathbb{R}$ (*respectively, in* $\mathbb{C}$). *Then there exists* $\alpha \in \mathbb{R}^*$ (*respectively,* $\alpha \in \mathbb{C}^*$) *such that* $\alpha f(\vec{x}) \in \mathbb{Z}[\vec{x}]$ (*respectively,* $\alpha f(\vec{x}) \in \mathbb{Z}[\,i\,][\vec{x}]$).

Theorem 1.1 is deduced from Theorem 1.3(a) below which classifies the closed orbits under the action of maximal split tori. Recall that if F is a field containing K then the *F-rank* of $\mathbf{G}$, denoted by $\mathrm{rank}_F\mathbf{G}$, is the dimension of any maximal F-split torus of $\mathbf{G}$. (It is well known that the maximal F-split tori are conjugated under $\mathbf{G}(F)$ [Bo, Theorem 15.14].) Further on we fix a maximal K-split torus $\mathbf{S}$ of $\mathbf{G}$ and for every $v \in \mathcal{S}$ we fix a maximal K_v-split torus $\mathbf{T}_v$ of $\mathbf{G}$ containing $\mathbf{S}$. We denote $T_v = \mathbf{T}_v(K_v)$ and for every non-empty $\mathcal{R} \subset \mathcal{S}$, we set $T_{\mathcal{R}} = \prod_{v\in\mathcal{R}} T_v$. An orbit $T_{\mathcal{R}}\pi(g)$ in G/Γ is called *divergent* if $\{t_i\pi(g)\}$ leaves compacts of G/Γ whenever $\{t_i\}$ leaves compacts of $T_{\mathcal{R}}$. (The group $T_{\mathcal{R}}$ is identified with its projection in G.) With the above notation we have the following:

Theorem 1.3. (*cf.*[To, Theorems 1.1 and 1.4]) *Let* $\sum_{v\in\mathcal{R}} \mathrm{rank}_{K_v}\mathbf{G} > 0$. *Then the following assertions hold:*

(a) *An orbit* $T_{\mathcal{R}}\pi(g)$ *is closed if and only if* $\mathcal{R} = \{v\}$ *and* $T_v\pi(g)$ *is divergent, or* $\mathcal{R} = \mathcal{S}$ *and there exists a* K*-torus* $\mathbf{L}$ *of* $\mathbf{G}$ *such that*

$$g^{-1}T_{\mathcal{S}}g = L_{\mathcal{S}},$$

where $L_{\mathcal{S}} = \prod_{v\in\mathcal{S}} \mathbf{L}(K_v)$*. Moreover, if* $\#\mathcal{S} > 1$ *then* $T_{\mathcal{S}}\pi(g)$ *is never divergent;*

(b) *An orbit* $T_v\pi(g)$*,* $v \in \mathcal{S}$*, is divergent if and only if* $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G}$ *and*

$$g \in \mathcal{Z}_G(T_v)\mathbf{G}(K),$$

where $\mathcal{Z}_G(T_v)$ *is the centralizer of* T_v *in* G*;*

(c) *An orbit* $T_{\mathcal{S}}\pi(g)$ *is closed and* $T_v\pi(g)$ *is divergent for every* $v \in \mathcal{S}$ *if and only if* $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G}$ *for every* $v \in \mathcal{S}$ *and*

$$g \in \mathcal{N}_G(T_{\mathcal{S}})\mathbf{G}(K),$$

where $\mathcal{N}_G(T_{\mathcal{S}})$ *is the normalizer of* $T_{\mathcal{S}}$ *in* G*.*

The theorem has been proved for $G = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ by Margulis (unpublished) and it generalizes and strengthens results for the *real* $\mathbb{Q}$-algebraic groups proved by Barak Weiss and the author in [To-We].

The part (a) of Theorem 1.3 follows from its parts (b) and (c) about the divergent orbits. Apart from the number-theoretical applications, our interest in the divergent orbits of split tori is also motivated by the classical result of Margulis [Ma3] (see also [Da1]) which implies that no subgroup which contains a nontrivial unipotent element can have divergent orbits. Further on our discussion concerns only the divergent orbits for split algebraic tori.

According to Theorem 1.3(b) the divergent orbits for the action of any *maximal* K_v-split torus T_v of G_v are always "standard" if $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G}$. On the other hand, if $\mathrm{rank}_{K_v}\mathbf{G} > \mathrm{rank}_K\mathbf{G}$ there are no divergent orbits for the action of T_v. In fact, the following more general result holds:

Theorem 1.4. *Let* G *and* Γ *be as in the formulation of Theorem 1.3,* $x \in G/\Gamma$*,* $v \in \mathcal{S}$ *and* D_v *be a* K_v*-split algebraic torus in* G_v*. Assume that* $\dim D_v > \mathrm{rank}_K\mathbf{G}$*. Then there are no divergent orbits for the action of* D_v *on* G/Γ*.*

Theorem 1.4 is due to Pralay Chatterjee and Dave Morris for the $\mathbb{Q}$-rank two real semisimple $\mathbb{Q}$-algebraic groups [Ch-Mo] and to Barak Weiss for all real semisimple $\mathbb{Q}$-algebraic groups [We1]. Our proof in §4 of the general case uses ideas from [We1], [To-We] and [To]. Note that if v is a non-archimedean valuation then the connected component of D_v is trivial. This is a reason for additional difficulties in proving Theorem 1.4 in the general case.

Theorem 1.4 is an *existence* theorem: by contrast with Theorem 1.3, it says nothing about the set of all $g \in G$ for which $D_v\pi(g)$ is a divergent orbit. Note that if $\dim D_v < \mathrm{rank}_K \mathbf{G}$ a simple description of the divergent orbits seems not plausible. Strong evidence in this sense is provided by the paper [Da2], where the study of atypical trajectories is related to properties of singular systems of linear forms, and by the paper [We2], where divergent trajectories on real homogeneous spaces are systematically studied.

In view of the above discussion, it is important to describe the set of all $g \in G$ for which $D_v\pi(g)$ is a divergent orbit when $\dim D_v = \mathrm{rank}_K \mathbf{G} < \mathrm{rank}_{K_v} \mathbf{G}$. Using Theorem 1.3 this problem can be solved for certain classes of real algebraic groups which can not be tackled with the results of [To-We] where the $\mathbb{R}$-split tori are always supposed to be maximal. Indeed, let K be a totally real number field and $\mathbf{H}$ be a semisimple K-split algebraic group. Set $\mathbf{G} = \mathrm{R}_{K/\mathbb{Q}}(\mathbf{H})$, where $\mathrm{R}_{K/\mathbb{Q}}$ is the restriction of scalars functor. Let $G = \mathbf{G}(\mathbb{R})$ and $H = \mathbf{H}(\mathbb{R})$. The group G is a real $\mathbb{Q}$-algebraic group and, in view of the standard properties of $\mathrm{R}_{K/\mathbb{Q}}$ (cf. [Weil]), G is naturally identified with the direct product of $m = [K : \mathbb{Q}]$ copies of H, $G(\mathbb{Z})$ is identified with $\mathbf{H}(\mathcal{O})$ and $\mathrm{rank}_{\mathbb{R}} G = m.\mathrm{rank}_{\mathbb{Q}} G$. Theorem 1.3(b) immediately implies:

Theorem 1.5. *With the above notation and assumptions, if D is a maximal $\mathbb{R}$-split torus in some of the factors H then $D\pi(g)$ is divergent if and only if $g \in \mathcal{Z}_G(D)G(\mathbb{Q})$.*

In general, the determination of all g for which $D\pi(g)$ is divergent might be quite complicated. In §5 we describe this set for the so-called Hilbert modular forms, that is, when $\mathbf{G} = \mathrm{R}_{K/\mathbb{Q}}(\mathbf{SL}_2)$. (We refer to [Fe] and [To, Corollary 1.7] for more results in this case.)

The following conjecture of B.Weiss characterizes the divergent orbits in terms of $\mathbb{Q}$-representations, cf. [To-We, §8]:

Conjecture 1. ([We2, Conjecture 4.10 B]) Let G be a real semisimple $\mathbb{Q}$-algebraic group and D be a split torus of dimension $\mathrm{rank}_{\mathbb{Q}} G$. The orbit $D\pi(g)$ is divergent if and only if for any unbounded sequence $\{d_i\} \subset D$ there exist a subsequence $\{d'_i\}$, a $\mathbb{Q}$-representation $\rho : G \to \mathrm{GL}(V)$ and a nonzero $v \in V(\mathbb{Q})$ such that $\lim_{i\to\infty} \rho(d'_i g) = 0$.

In [We2] the divergent orbits for which there exist representations ρ as in the formulation of the conjecture are called *obvious*. In this terminology the conjecture says that if $\dim D = \mathrm{rank}_{\mathbb{Q}} G$ then all divergent orbits are obvious. On the other hand, it is also conjectured [We2, Conjecture 4.10 C] that if $\dim D < \mathrm{rank}_{\mathbb{Q}} G$ then there are non-obvious divergent orbits.

For the real semisimple $\mathbb{Q}$-algebraic groups of $\mathbb{Q}$-rank 1 Conjecture 1 follows from [Da2, Theorem 6.1] which also deals with non-arithmetic lattices. The method of the proof of Theorem 1.4 allows to prove an $\mathcal{S}$-adic version of this result which we are going to formulate now. With the notation of Theorem 1.3, we also denote $\mathbf{g} = \mathrm{Lie}(\mathbf{G})$ and $\mathfrak{g} = \mathbf{g}(K_{\mathcal{S}})$. Let d be the dimension of the maximal unipotent K-subgroups of $\mathbf{G}$. For every $\mathbf{x} \in \wedge^d \mathfrak{g}$ we denote by $\mathbf{c}(\mathbf{x})$ the content of $\mathbf{x}$, i.e., $\mathbf{c}(\mathbf{x})$ is the product of the norms of the v-components, $v \in \mathcal{S}$, of $\mathbf{x}$ (see §2.2).

Theorem 1.6. *Let* $\mathbf{G}$ *be a semisimple* K*-algebraic group of* K*-rank 1,* $v \in \mathcal{S}$*,* D_v *be a 1-dimensional* K_v*-split torus of* G_v *and* $| \,.\, |_v$ *be the norm on* D_v *induced by the* v*-adic norm on* K_v *via an isomorphism* $D_v \cong K_v^*$*. An orbit* $D_v\pi(g)$ *is divergent if and only if the following holds: there exist maximal opposite to each other unipotent* K*-subalgebras* $\mathbf{u}^+$ *and* $\mathbf{u}^-$ *of* $\mathfrak{g}$ *such that if* $\mathbf{x}^+$ *and* $\mathbf{x}^-$ *are* K*-rational vectors in* $\wedge^d \mathfrak{g}$ *which span the 1-dimensional subspaces corresponding to* $\mathbf{u}^+$ *and* $\mathbf{u}^-$*, respectively, then*

(1) $$\lim_{|t|_v \to 0} \mathbf{c}(\wedge^d \mathrm{Ad}(tg)\mathbf{x}^+) = 0$$

and

(2) $$\lim_{|t|_v \to \infty} \mathbf{c}(\wedge^d \mathrm{Ad}(tg)\mathbf{x}^-) = 0.$$

The paper is organized as follows. The notation and the terminology are introduced in a systematical way in §2. In §3 we recall some preliminary results from [To], [To-We] and [We1]. The proofs of Theorems 1.4 and 1.6 are given in §4. The description of the divergent orbits for the Hilbert modular forms is presented in §5.

2. Notation and terminology

2.1. **Algebraic numbers.** As usual $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the real, rational and integer numbers, respectively.

In this paper K denotes a number field, that is, a finite extension of $\mathbb{Q}$. All valuations of K which we consider are supposed to be *normalized* (see [Ca-F, ch.2, §7]) and, therefore, pairwise non-equivalent. If v is a valuation of K then K_v is the completion of K with respect to v and $| \,.\, |_v$ is the corresponding norm on K_v. If v is non-archimedean then $\mathcal{O}_v = \{x \in K_v : | \, x \, |_v \leq 1\}$ is the ring of integers of K_v.

As in the introduction, $\mathcal{S}$ will denote a finite set of valuations of K containing all archimedean ones and $K_{\mathcal{S}} = \prod_{v \in \mathcal{S}} K_v$.

The ring of $\mathcal{S}$-integers of K is defined by $\mathcal{O} = K \bigcap (\bigcap_{v \notin \mathcal{S}} \mathcal{O}_v)$.

As usual, given a ring A, we denote by A^* the multiplicative group of all invertible elements in A.

2.2. **Norms and content.** Let $\mathbf{V}$ be a m-dimensional vector space defined over K. Let $V = \mathbf{V}(K_{\mathcal{S}})$ and $V_v = \mathbf{V}(K_v)$ for every $v \in \mathcal{S}$. There exists a natural isomorphism $V \cong \prod_{v\in\mathcal{S}} V_v$. Elements $\mathbf{x} \in V$ will be denoted as $\mathbf{x} = (\mathbf{x}_v)$, where $\mathbf{x}_v \in V_v$. We denote by $\|\cdot\|_v$ a norm on the K_v-vector space V_v, $v \in \mathcal{S}$, and define a *norm* $\|\cdot\|$ on V by

$$\|\mathbf{x}\| = \max_{v\in\mathcal{S}} \|\mathbf{x}_v\|_v.$$

The product

$$\mathbf{c}(\mathbf{x}) = \prod_{v\in\mathcal{S}} \|\mathbf{x}_v\|_v$$

is called *content* of $\mathbf{x}$. Since $\prod_{v\in\mathcal{S}} |\xi|_v = 1$ for every $\xi \in \mathcal{O}^*$ [Ca-F, ch.2, Theorem 12.1] and $\|a\mathbf{x}_v\|_v = |a|_v\|\mathbf{x}_v\|_v$ for every $a \in K_v$, we have that

$$\mathbf{c}(\mathbf{x}) = \mathbf{c}(\xi\mathbf{x}), \forall\xi \in \mathcal{O}^*. \tag{3}$$

By a *pseudo-ball* in V of radius $r > 0$ centered at 0 we mean the set $\mathcal{B}(r) = \{\mathbf{x} \in V | \mathbf{c}(\mathbf{x}) < r\}$. In view of (3), $\mathcal{B}(r)$ is invariant under multiplication by the elements from $\mathcal{O}^*$. We preserve the notation $B(r)$ to denote the usual ball in V of radius r centered at 0 with respect to $\|\cdot\|$.

2.3. ***K*-algebraic groups and their Lie algebras.** We use boldface upper case letters to denote the algebraic groups and boldface lower case Gothic letters to denote their Lie algebras.

In this paper $\mathbf{G}$ is a semisimple algebraic group defined over K (or shortly, K-group). The Lie algebra $\mathfrak{g}$ is equipped with a K-structure compatible with the K-structure of $\mathbf{G}$ [Bo, Theorem 3.4].

Given a K-subgroup $\mathbf{H}$ of $\mathbf{G}$ we denote $H \overset{def}{=} \mathbf{H}(K_{\mathcal{S}})$ and $\mathfrak{h} \overset{def}{=} \boldsymbol{\mathfrak{h}}(K_{\mathcal{S}})$. The group H (respectively, its Lie algebra $\mathfrak{h}$) is identified with the direct product $\prod_{v\in\mathcal{S}} H_v$ (respectively, $\prod_{v\in\mathcal{S}} \mathfrak{h}_v$), where $H_v \overset{def}{=} \mathbf{H}(K_v)$ (respectively, $\mathfrak{h}_v \overset{def}{=} \boldsymbol{\mathfrak{h}}(K_v)$). We let $R_u(\mathbf{H})$ be the unipotent radical of $\mathbf{H}$. The unipotent radical of $\boldsymbol{\mathfrak{h}}$ is by definition $\mathrm{Lie}(R_u(\mathbf{H}))$.

When $K = \mathbb{Q}$ and $\mathcal{S}$ contains only the archimedean valuation of $\mathbb{Q}$, the group $G = \mathbf{G}(\mathbb{R})$ is called real $\mathbb{Q}$-algebraic group and we write $G(\mathbb{Z})$ and $G(\mathbb{Q})$ instead of $\mathbf{G}(\mathbb{Z})$ and $\mathbf{G}(\mathbb{Q})$, respectively.

On every G_v we have a *Zariski topology* induced by the Zariski topology on $\mathbf{G}$ and a *Hausdorff topology* induced by the locally compact topology on K_v. The formal product of the Zariski (respectively, Hausdorff) topologies on G_v, $v \in \mathcal{S}$, is the Zariski (respectively, Hausdorff) topology on G. (Recall that $G = \prod_{v\in\mathcal{S}} G_v$.) In order to distinguish the two topologies, all topological notions connected with the first one will be used with the prefix "Zariski".

By a K_v-split torus of G_v we mean a Zariski closed Zariski connected subgroup of G_v which is diagonalizable over K_v. If T_v is a K_v-split torus of G_v we denote by $X(T_v)$ the multiplicative group of all rational characters of T_v.

A subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ is *unipotent* if it corresponds to a Zariski closed unipotent subgroup U of G, i.e., if there exists a subgroup $U \subset G$ such that $U = \prod_{v\in\mathcal{S}} U_v$, each U_v is Zariski closed in G_v, and $\mathfrak{u} = \prod_{v\in\mathcal{S}} \mathfrak{u}_v$ where $\mathfrak{u}_v = \mathrm{Lie}(U_v)$.

The adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the direct product of the adjoint representations $\mathrm{Ad}_v : G_v \to \mathrm{GL}(\mathfrak{g}_v)$, $v \in \mathcal{S}$.

2.4. **$\mathcal{S}$-arithmetic subgroups.** We fix some imbedding of $\mathbf{G}$ in $\mathbf{SL}_n$ such that $\mathbf{G}(\mathcal{O}) = \mathbf{SL}_n(\mathcal{O}) \cap \mathbf{G}$ and $\mathfrak{g}(\mathcal{O}) = \mathfrak{sl}_n(\mathcal{O}) \cap \mathfrak{g}$. So, $\mathfrak{g}(\mathcal{O})$ is invariant under the adjoint action of $\mathbf{G}(\mathcal{O})$.

Let $\Gamma = \mathbf{G}(\mathcal{O})$ and $\pi : G \to G/\Gamma$ be the natural projection. For every $x = \pi(g), g \in G$, we introduce the notation:

$$\mathfrak{g}_x \overset{def}{=} \mathrm{Ad}(g)(\mathfrak{g}(\mathcal{O})).$$

Since $\mathfrak{g}(\mathcal{O})$ is $\mathrm{Ad}(\Gamma)$-invariant, $\mathfrak{g}_x$ does not depend on the choice of the element g.

Let Γ' be an $\mathcal{S}$-*arithmetic* subgroup of G, that is, $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. Let $\pi' : G \to G/\Gamma'$, $\phi : G/\Gamma \to G/\Gamma \cap \Gamma'$ and $\psi : G/\Gamma' \to G/\Gamma \cap \Gamma'$ be the natural maps. Note that ϕ and ψ are proper maps. Therefore if H is an arbitrary closed subgroup of G and $g \in G$ then the orbit $H\pi(g)$ is closed (respectively, divergent) if and only if $H\pi'(g)$ is closed (respectively, divergent). Using this remark one can easily see that the results of this paper remain valid for an *arbitrary* $\mathcal{S}$-arithmetic subgroup Γ instead of $\Gamma = \mathbf{G}(\mathcal{O})$.

3. Preliminary results

3.1. **$\mathcal{S}$-adic Mahler's criterion.** The group $\mathrm{SL}_n(K_\mathcal{S})$ is acting naturally on $K_\mathcal{S}^n$ and $\mathrm{SL}_n(\mathcal{O})$ is the stabilizer of $\mathcal{O}^n$ in $\mathrm{SL}_n(K_\mathcal{S})$. If $r > 0$ then $B(r)$ (resp., $\mathcal{B}(r)$) is the ball (resp. pseudoball) in $K_\mathcal{S}^n$ centered in 0 and with radius r (see §2.3).

Let $\pi : \mathrm{SL}_n(K_\mathcal{S}) \to \mathrm{SL}_n(K_\mathcal{S})/\mathrm{SL}_n(\mathcal{O})$ be the natural projection. We have the following analog of Mahler's criterion (cf.[To, Theorem 3.1]):

Theorem 3.1. *Given a subset $M \subset \mathrm{SL}_n(K_\mathcal{S})$ the following conditions are equivalent:*

(i) *$\pi(M)$ is relatively compact in $\mathrm{SL}_n(K_\mathcal{S})/\mathrm{SL}_n(\mathcal{O})$;*
(ii) *There exists $r > 0$ such that $g\mathcal{O}^n \cap \mathcal{B}(r) = \{0\}$ for all $g \in M$;*
(iii) *There exists $r > 0$ such that $g\mathcal{O}^n \cap B(r) = \{0\}$ for all $g \in M$.*

The following lemma will be also needed:

Lemma 3.2. (*cf.*[To, Lemma 3.2]) *There exists a constant* $\kappa > 1$ *with the following property. Let* $\boldsymbol{x} = (\boldsymbol{x}_v)_{v\in\mathcal{S}} \in K_{\mathcal{S}}^n$ *be such that* $\boldsymbol{x}_v \neq 0$ *for all* $v \in \mathcal{S}$. *For each* $v \in \mathcal{S}$ *we choose a positive real number* a_v *in such a way that* $\mathbf{c}_{\mathcal{S}}(\boldsymbol{x}) = \prod_{v\in\mathcal{S}} a_v$. *Then there exists* $\xi \in \mathcal{O}^*$ *such that*

$$\frac{a_v}{\kappa} \leq \|\xi \boldsymbol{x}_v\|_v \leq \kappa a_v$$

for all $v \in \mathcal{S}$. *In particular, for every* $\boldsymbol{x}$ *as above there exists* $\xi \in \mathcal{O}^*$ *such that*

$$\frac{\mathbf{c}_{\mathcal{S}}(\boldsymbol{x})^{1/m}}{\kappa} \leq \|\xi \boldsymbol{x}\|_{\mathcal{S}} \leq \kappa \mathbf{c}_{\mathcal{S}}(\boldsymbol{x})^{1/m}, \tag{4}$$

where $m = \#\mathcal{S}$.

3.2. **Horospherical subsets.** Let $\mathbf{G}$ be an arbitrary semisimple K-algebraic group. Fix a minimal parabolic K-subgroup $\mathbf{P}$ of $\mathbf{G}$ and denote by $\mathbf{P}_1, \ldots, \mathbf{P}_l$ the maximal parabolic K-subgroups of $\mathbf{G}$ containing $\mathbf{P}$. Recall that $l = \operatorname{rank}_K \mathbf{G}$, cf.[Bo, ch.7]. Put $\mathfrak{u}_i = \operatorname{Lie}(R_u(\mathbf{P}_i))$, $i = 1, \ldots, l$.

The following definition differs slightly from [To-We, Definition 3.4] and [To, Definition 3.3].

Definition 3.3. A subset (finite or infinite) $\mathcal{M}$ of $\mathfrak{g}$ is called horospherical of type i (or i-type horospherical) if for some $g \in G$ and $\mathfrak{u}_i$ as above, $\operatorname{Ad}(g)\mathcal{M} \subset \mathfrak{u}(\mathcal{O})$ and $\operatorname{Ad}(\gamma)\mathcal{M}$ spans linearly $\mathfrak{u}_i$. When the specification of $\mathfrak{u}_i$ is not needed, we say simply that $\mathcal{M}$ is a horospherical subset.

The next proposition follows immediately from [Bo, Propositions 14.22 and 21.13]:

Proposition 3.4. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be horospherical subsets of the same type. Assume additionally that the set* $\mathcal{M}_1 \cup \mathcal{M}_2$ *is contained in a unipotent subalgebra of* $\mathfrak{g}$. *Then* $\langle \mathcal{M}_1 \rangle = \langle \mathcal{M}_2 \rangle$. (Here and further on $\langle \mathcal{M} \rangle$, where $\mathcal{M} \subset \mathfrak{g}$, denotes the linear span over $K_{\mathcal{S}}$ of $\mathcal{M}$ in $\mathfrak{g}$.)

The next proposition provides a compactness criterion in terms of horospherical subsets of $\mathfrak{g}_x$, $x \in G/\Gamma$ (see 2.4 for the notation). It generalizes [To-We, Propositions 3.3 and 3.5] and [To, Proposition 3.4].

Proposition 3.5. *We have:*

(a) *There exists* $r > 0$ (*respectively,* $t > 0$) *such that for any* $x = \pi(g)$ *the subalgebra of* $\mathfrak{g}$ *spanned by* $\mathcal{B}(r) \cap \mathfrak{g}_x$ (*respectively,* $B_{\mathcal{S}}(t) \cap \mathfrak{g}_x$) *is unipotent;*

(b) (**Compactness Criterion**) *A subset* M *of* G/Γ *is relatively compact if and only if there exists* $r > 0$ (*respectively,* $t > 0$) *such that* $\mathcal{B}(r) \cap \mathfrak{g}_x$ (*respectively,* $B(t) \cap \mathfrak{g}_x$) *does not contain a horospherical subset for any* $x \in M$.

Proposition 3.5(b) easily implies:

Corollary 3.6. *Let $D_v \subset G_v$ be a K_v-split torus. Assume that $D_v x$, $x \in G/\Gamma$, is a divergent orbit. Let $\|\cdot\|$ be a norm on D_v. For every pseudo-ball $\mathcal{B}(r)$ there exists a constant $\tau > 0$ such that if $t \in D_v$ and $\|t\| \geq \tau$ then $\mathrm{Ad}(t)\mathfrak{g}_x \cap \mathcal{B}(r)$ contains a horospherical subset.*

Proof. According to Proposition 3.5(b) $M = \{y \in G/\Gamma | \mathcal{B}(r) \cap \mathfrak{g}_y$ does not contain a horospherical subset$\}$ is relatively compact. Since $D_v x$ is divergent, there exists $\tau > 0$ such that $tx \notin M$ if $\|t\| \geq \tau$. Therefore, if $\|t\| \geq \tau$ then $\mathrm{Ad}(t)\mathfrak{g}_x \cap \mathcal{B}(r)$ contains a horospherical subset. □

The following proposition plays an important role in the proof of Theorem 1.4:

Proposition 3.7. [We2, Proposition 7] *Let S^{n-1} be an $n-1$-dimensional sphere centered at 0 in $\mathbb{R}^n$. Suppose $\mathcal{V}$ is a cover of S^{n-1} by open sets such that for any $V \in \mathcal{V}$ there is a linear functional $\chi : \mathbb{R}^n \to \mathbb{R}$ such that $\chi(s) < 0$ for any $s \in V$. Then there is $s \in S$ such that*

$$\#\{V \in \mathcal{V} : s \in V\} \geq n,$$

i.e., the multiplicity of the cover $\mathcal{V}$ is at least n.

4. Proofs of Theorems 1.4 and 1.6

4.1. We keep the notation from the formulation of Theorem 1.4. First we prove the following proposition:

Proposition 4.1. *Let $v \in \mathcal{S}$, D_v be a K_v-split torus in G_v and $g \in G$. There exist a real $r > 0$ and finitely many characters $\chi_1, \ldots, \chi_m$ in $X(D_v)$ with the following property: For every $\mathfrak{u} = \mathrm{Lie}(R_u(\mathbf{P}))$, where $\mathbf{P}$ is a maximal parabolic K-subgroup of $\mathbf{G}$, there exists $1 \leq i \leq m$ such that if $t \in D_v$ and $\mathrm{Ad}(tg)(\mathfrak{u}(\mathcal{O})) \cap \mathcal{B}(r)$ contains a horospherical subset then $|\chi_i(t)|_v < 1$.*

Proof. The algebra $\mathfrak{u}$ in the formulation of the proposition is conjugated under $\mathbf{G}(K)$ to one of the algebras $\mathfrak{u}_i$ introduced in §3.2. For every i we denote $d_i = \dim \mathfrak{u}_i$. We let $\chi_1, \ldots, \chi_m \in X(D_v)$ be the set of weight-characters for the actions of $\wedge^{d_i}\mathrm{Ad}(D_v)$ on $\wedge^{d_i}\mathfrak{g}_v$ for all i.

For every i we fix a basis $\mathbf{e}_1^{(i)}, \ldots, \mathbf{e}_m^{(i)}$ of $\wedge^{d_i}\mathfrak{g}_v$ consisting of weight vectors for the adjoint action of D_v on $\wedge^{d_i}\mathfrak{g}_v$. We denote by $\|\cdot\|_v^{(i)}$ the norm sup on $\wedge^{d_i}\mathfrak{g}_v$ with respect to $\mathbf{e}_1^{(i)}, \ldots, \mathbf{e}_m^{(i)}$. If $w \in \mathcal{S}, w \neq v$, we denote by $\|\cdot\|_w^{(i)}$ any norm on $\wedge^{d_i}\mathfrak{g}_w$ compatible with the topology on $\wedge^{d_i}\mathfrak{g}_w$. We let $\mathbf{c}^{(i)}(\ \cdot\ )$ be the content on $\wedge^{d_i}\mathfrak{g}$ (as defined in §2.2) and for every $a > 0$ we put $\mathcal{B}^{(i)}(a) = \{\mathbf{x} \in \wedge^{d_i}\mathfrak{g} | \mathbf{c}^{(i)}(\mathbf{x}) < a\}$, $a > 0$. The notation $\|\cdot\|_w, w \in \mathcal{S}$, and $\mathbf{c}(\ \cdot\ )$ are preserved for the norms and the content on $\mathfrak{g}_w, w \in \mathcal{S}$, and $\mathfrak{g}$, respectively.

Let $x = \pi(g)$. Since $\wedge^{d_i}\mathfrak{g}_x$ is discrete in $\wedge^{d_i}\mathfrak{g}$, it follows from the formula (4) of Lemma 3.2 that there exists $r_0 > 0$ such that

$$\mathcal{B}^{(i)}(r_0) \cap \wedge^d \mathfrak{g}_x = \{0\} \tag{5}$$

for all i. It is easy to see that there exists r with $0 < r < r_0$ such that if $\mathbf{a}_1, \dots, \mathbf{a}_{d_i} \in \mathcal{B}(r)$ then

$$\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_{d_i} \in \mathcal{B}^{(i)}(r_0) \tag{6}$$

for all $i = 1, \dots, l$.

Assume that $\mathfrak{u} \subset \mathfrak{g}$ is conjugated to $\mathfrak{u}_i$. Fix a vector

$$\sum_j \alpha_j \mathbf{e}_j^{(i)} \in \wedge^{d_i}\mathfrak{g}_v, \ \alpha_j \in K_v,$$

which spans the 1-dimensional subspace corresponding to $\mathrm{Ad}(g_v)(\mathfrak{u}_v)$, where g_v is the v-component of g. Let χ be the weight character associated to some $\mathbf{e}_{j_0}^{(i)}$ with $|\alpha_{j_o}|_v \geq |\alpha_j|_v$ for all j. Let $\mathcal{M}$ be a horospherical subset of $\mathrm{Ad}(tg)(\mathbf{u}(\mathcal{O})) \cap \mathcal{B}(r)$. Choose a linearly independent (over $K_{\mathcal{S}}$) subset $\mathbf{m}_1, \dots, \mathbf{m}_{d_i}$ of $\mathrm{Ad}(t^{-1})(\mathcal{M})$. It follows from (5) that

$$\mathbf{c}^{(i)}(\mathbf{m}_1 \wedge \dots \wedge \mathbf{m}_{d_i}) \geq r_o.$$

Let $\mathbf{m}_j^{(v)} \in \mathfrak{g}_v$ be the v-component of $\mathbf{m}_j, j = 1, \dots, d_i$. Using (6) we get

$$\mathbf{c}^{(i)}(\wedge^{d_i}\mathrm{Ad}(t)(\mathbf{m}_1 \wedge \dots \wedge \mathbf{m}_{d_i})) =$$
$$\frac{\| \wedge^{d_i} \mathrm{Ad}(t)(\mathbf{m}_1^{(v)} \wedge \dots \wedge \mathbf{m}_{d_i}^{(v)})\|_v^{(i)}}{\|\mathbf{m}_1^{(v)} \wedge \dots \wedge \mathbf{m}_{d_i}^{(v)})\|_v^{(i)}} \mathbf{c}^{(i)}(\mathbf{m}_1 \wedge \dots \wedge \mathbf{m}_{d_i}) < r_0,$$

where $\mathbf{m}_1^{(v)} \wedge \dots \wedge \mathbf{m}_{d_i}^{(v)}$ is the v-component of $\mathbf{m}_1 \wedge \dots \wedge \mathbf{m}_{d_i}$. Therefore,

$$\| \wedge^{d_i} \mathrm{Ad}(t)(\mathbf{m}_1^{(v)} \wedge \dots \wedge \mathbf{m}_{d_i}^{(v)})\|_v^{(i)} < \|\mathbf{m}_1^{(v)} \wedge \dots \wedge \mathbf{m}_{d_i}^{(v)})\|_v^{(i)}.$$

Since $\mathbf{m}_1^{(v)} \wedge \dots \wedge \mathbf{m}_{d_i}^{(v)}$ and $\sum_j \alpha_j \mathbf{e}_j^{(i)}$ are colinear vectors, it follows from the choice of $\| \cdot \|_v^{(i)}$ and χ that $|\chi(t)|_v < 1$. □

4.2. Proof of Theorem 1.4. We will identify D_v with K_v^{*n}, $n = \dim D_v$ and we denote by log the real logarithmic function with base $a > 1$. Define

$$\varphi : D_v \to \mathbb{R}^n, \varphi((t_1, \dots, t_n)) = (\log |t_1|_v, \dots, \log |t_n|_v).$$

If v is non-archimedean we choose a in such a way that $\mathrm{Im}(\varphi) = \mathbb{Z}^{\mathrm{n}}$. Introduce a norm $\| \cdot \|^{\sim}$ on D_v as follows:

$$\| \ t \ \|^{\sim} = \|\varphi(t)\|_\infty, \forall t \in D_v,$$

where $\| \cdot \|_\infty$ is the Euclidian norm on $\mathbb{R}^n$.

We fix $r > 0$ such that the conclusions of Proposition 3.5(a) and Proposition 4.1 are satisfied. There exists a positive real $r_1 < r$ such that $\mathrm{Ad}(t)\mathcal{B}(r_1) \subset \mathcal{B}(r)$ for all $t \in D_v$ with $\| t \|^{\sim} \leq 5\sqrt{n}$. In view of Corollary 3.6, there exists $R > \sqrt{n}$ such that $\mathrm{Ad}(t)\mathfrak{g}_x \cap \mathcal{B}(r_1)$ contains a horospherical subset for all $\| t \|^{\sim} \geq R - \sqrt{n}$.

Denote by S^{n-1} the sphere of radius R centered at 0 in $\mathbb{R}^n$. Fix a finite covering of S^{n-1} by balls $B_1, \ldots, B_q$ in $\mathbb{R}^n$ of radii $\sqrt{n}$ and with centers on S^{n-1}. For every $1 \leq i \leq q$ there exists $t_i \in D_v$ such that $\varphi(t_i) \in B_i$. Since $\| t_i \|^{\sim} \geq R - \sqrt{n}$, we can associate with every B_i, $i = 1, \ldots, q$, a horospherical subset $\mathcal{M}_i \subset \mathrm{Ad}(t_i)\mathfrak{g}_x \cap \mathcal{B}(r_1)$.

Let $i \neq j$, $B_i \cap B_j \cap S^{n-1} \neq \emptyset$ and $\mathcal{M}_i$ and $\mathcal{M}_j$ be of the same type. Then

$$\|t_i t_j^{-1}\|^{\sim} < 4\sqrt{n}.$$

Therefore,

$$\mathrm{Ad}(t_i t_j^{-1})\mathcal{M}_j \subset \mathcal{B}(r).$$

In view of the choice of r, $\mathrm{Ad}(t_i t_j^{-1})\mathcal{M}_j$ and $\mathcal{M}_i$ belong to one and the same unipotent subalgebra of $\mathfrak{g}$. Using Proposition 3.4 we get that $\langle \mathrm{Ad}(t_i t_j^{-1})\mathcal{M}_j \rangle = \langle \mathcal{M}_i \rangle$. Therefore there exists a unique maximal parabolic K-subalgebra of $\mathfrak{g}$ with unipotent radical $\mathfrak{u}$ such that $\mathcal{M}_i \subset \mathrm{Ad}(t_i g)(\mathfrak{u}(\mathcal{O})) \cap \mathcal{B}(r)$ and $\mathcal{M}_j \subset \mathrm{Ad}(t_j g)(\mathfrak{u}(\mathcal{O})) \cap \mathcal{B}(r)$. It follows from Proposition 4.1 that there exists a character $\chi \in X(D_v)$ such that $|\chi(t_i)|_v < 1$ and $|\chi(t_j)|_v < 1$. The character χ yields a functional ρ_χ on $\mathbb{R}^n$ uniquely defined by the relation

$$\rho_\chi(\varphi(t)) = \log |\chi(t)|_v, \forall t \in D_v.$$

In particular, $\rho_\chi(\varphi(t_i)) < 0$ and $\rho_\chi(\varphi(t_j)) < 0$. Moreover, it is easy to see that

$$\rho_\chi(s) < 0, \quad \forall s \in (B_i \cap S^{n-1}) \cup (B_j \cap S^{n-1}). \tag{7}$$

Indeed, in view of the choice of r, r_1 and $\mathcal{M}_i$, if $\|t t_i^{-1}\|^{\sim} < 5\sqrt{n}$ then

$$\mathrm{Ad}(t t_i^{-1})\mathcal{M}_i \subset \mathcal{B}(r).$$

Applying again Proposition 4.1 we get that $|\chi(t)|_v < 1$. This implies that if B' is the open ball in $\mathbb{R}^n$ centered at $\varphi(t_i)$ and with radius $5\sqrt{n}$, then the restriction of ρ_χ to $B' \cap \mathbb{Z}^n$ takes negative values. But ρ_χ takes also negative values at the points of the minimal convex body containing $B' \cap \mathbb{Z}^n$. The latter contains $(B_i \cap S^{n-1}) \cup (B_j \cap S^{n-1})$ which complets the proof of (7).

We denote by $\mathcal{V}$ the cover of S^{n-1} defined by the following properties: each $V \in \mathcal{V}$ is connected and coincides with the union of a maximal number of subsets $B_i \cap S^{n-1}$ such that the horospherical subsets $\mathcal{M}_i$ associated with B_i are all of the same type. In view of the above discussion, for every $V \in \mathcal{V}$ there exists a functional ρ_V on $\mathbb{R}^n$ which takes only negative values on V. By Proposition 3.7 the multiplicity of $\mathcal{V}$ is $\geq n$. On the other hand, since the number of types of horospherical subsets is exactly equal to $\mathrm{rank}_K\mathbf{G}$, we have that the multiplicity of the cover $\mathcal{V}$ is $\leq \mathrm{rank}_K\mathbf{G}$. Therefore, $\mathrm{rank}_K\mathbf{G} \geq \dim D_v$. □

4.3. Proof of Theorem 1.6. Recall that $D_v \cong K_v^*$. We will use the map $\varphi : D_v \to \mathbb{R}$ and the norm $\| \cdot \|^\sim$ on D_v as defined in the proof of Theorem 1.4. Fix a constant $c > 1$ such that for every $t \in D_v$ with $\| t \|^\sim \leq 2$ and every $z \in \mathfrak{g}_v$

$$\|\mathrm{Ad}(t)z\|_v \leq c\|z\|_v. \tag{8}$$

In view of Proposition 3.5(a) there exists $r > 0$ such that $\mathcal{B}(cr) \cap \mathfrak{g}_y$ spans a unipotent algebra for all $y \in G/\Gamma$.

For every $n \in \mathbb{Z}$ we fix an element $t_n \in D_v$ with $\varphi(t_n) = n$. (In particular, $\|t_n\|^\sim = |n|$.) Since $D_v\pi(g)$ is a divergent orbit, it follows from Proposition 3.5(b) that there exists a positive integer ρ such that if $n \in \mathbb{Z}$ and $|n| \geq \rho$ then $\mathcal{B}(r) \cap \mathfrak{g}_{t_n\pi(g)}$ contains a horospherical subset. For every such n we choose a horospherical subset $\mathcal{M}_n \subset \mathcal{B}(r) \cap \mathfrak{g}_{t_n\pi(g)}$ such that the distance d_n (with respect to the norm on $\mathfrak{g}$) from $\mathcal{M}_n \setminus \{0\}$ to $\{0\}$ is minimal. Using again Proposition 3.5(b) we obtain that

$$\lim_{|n|\to\infty} d_n = 0. \tag{9}$$

Let

$$I_n = \{t \in D_v : \|tt_n^{-1}\|^\sim < 2\}.$$

In view of (8) and the choice of r

$$\mathrm{Ad}(tt_n^{-1})\mathcal{M}_n \subset \mathcal{B}(cr) \tag{10}$$

whenever $|n| > \rho$ and $t \in I_n$.

Denote by $\mathfrak{u}^+$ and $\mathfrak{u}^-$ the maximal unipotent K-subalgebras of $\mathfrak{g}$ such that $\langle \mathcal{M}_\rho \rangle = \mathrm{Ad}(t_\rho g)\mathfrak{u}^+$ and $\langle \mathcal{M}_{-\rho} \rangle = \mathrm{Ad}(t_{-\rho} g)\mathfrak{u}^-$. Let $|n| > \rho$. It follows from (10) that

$$\mathrm{Ad}(t_{n+1}t_n^{-1})\mathcal{M}_n \cup \mathcal{M}_{n+1} \subset \mathcal{B}(cr).$$

Using Proposition 3.4, we get that

$$\langle \mathrm{Ad}(t_{n+1}t_n^{-1})\mathcal{M}_n \rangle = \langle \mathcal{M}_{n+1} \rangle.$$

A simple inductive argument shows that

$$\langle \mathcal{M}_n \rangle = \mathrm{Ad}(t_n g)\mathfrak{u}^+$$

for all $n > \rho$, and

$$\langle \mathcal{M}_{-n} \rangle = \mathrm{Ad}(t_{-n} g)\mathfrak{u}^-$$

for all $-n > \rho$.

Now let $t \in D_v$ and $\| t \|^\sim > \rho$. There exists $t_n, |n| \geq \rho$, such that

$$\|tt_n^{-1}\|^\sim < 2.$$

Put $\mathcal{M} = \mathcal{B}(r) \cap \mathfrak{g}_{t\pi(g)}$. (It is clear that $\mathcal{M}$ is a horospherical subset.) Then

$$\mathrm{Ad}(tt_n^{-1})\mathcal{M}_n \cup \mathcal{M}_{n+1} \subset \mathcal{B}(cr),$$

and, in view of Proposition 3.4,

$$\langle \mathrm{Ad}(tt_n^{-1})\mathcal{M}_n \rangle = \langle \mathcal{M} \rangle.$$

Therefore

$$\langle \mathcal{M} \rangle = \mathrm{Ad}(tg)\mathfrak{u}^+$$

if $n > \rho$, and

$$\langle \mathcal{M} \rangle = \mathrm{Ad}(tg)\mathfrak{u}^-$$

if $-n > \rho$. Now the formulas (1) and (2) follow immediately from (9).

It remains to show that $\mathfrak{u}^+$ and $\mathfrak{u}^-$ are opposite to each other. First, it follows easily from (1) and (2) that $\mathfrak{u}^+$ and $\mathfrak{u}^-$ are different. Let $\mathbf{P}^+$ (respectively, $\mathbf{P}^-$) be the parabolic K-subgroup of $\mathbf{G}$ such that $\mathfrak{u}^+ = \mathrm{Lie}(R_u(\mathbf{P}^+))$ (respectively, $\mathfrak{u}^- = \mathrm{Lie}(R_u(\mathbf{P}^-)))$. According to [Bo, Proposition 20.7] $\mathbf{P}^+ \cap \mathbf{P}^-$ contains the centralizer of a maximal K-split torus $\mathbf{S}$. But the Weyl group relative to $\mathbf{S}$ acts simply transitively on the set of minimal parabolic K-subgroups containing $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ [Bo, Corollary 21.3]. Since $\dim \mathbf{S} = 1$ the Weyl group is of order two and $\mathbf{P}^+$ and $\mathbf{P}^-$ are minimal K-parabolic subgroups. Therefore either $\mathbf{P}^+ = \mathbf{P}^-$ or $\mathbf{P}^+$ and $\mathbf{P}^-$ are opposite. Since $\mathfrak{u}^+ \neq \mathfrak{u}^-$ we get that $\mathfrak{u}^+$ and $\mathfrak{u}^-$ are opposite parabolic subalgebras. □

5. Hilbert modular lattices

5.1. Formulation of the result. In this section we describe the divergent orbits of the split tori on G/Γ, where $G = \mathrm{SL}_2(\mathbb{R}) \times \ldots \times \mathrm{SL}_2(\mathbb{R})$ is a direct product of $m > 1$ copies of $\mathrm{SL}_2(\mathbb{R})$ and Γ is an irreducible non-cocompact lattice in G. In view of the arithmeticity theorem up to conjugation by an element from G the group Γ is commensurable to a Hilbert modular lattice in G obtained in the following way. Let K be a totally real number field with $m = [K : \mathbb{Q}]$. Then G is naturally identified with the group of $\mathbb{R}$-points of $\mathrm{R}_{\mathrm{K}/\mathbb{Q}}(\mathbf{SL_2})$, where $\mathrm{R}_{\mathrm{K}/\mathbb{Q}}$ is the restriction of scalars functor. So, if $\sigma_1, \ldots, \sigma_m$ are the different imbeddings of K into $\mathbb{R}$ then $G(\mathbb{Z})$ is identified with the image of $\mathbf{SL_2}(\mathcal{O})$ in G via the map $g \to (g^{\sigma_1}, \ldots, g^{\sigma_m})$, where g^{σ_i} denotes the matrix obtained after applying σ_i to the entries of g. The group $G(\mathbb{Z})$ is called Hilbert modular lattice associated to K. Further on we will assume that $\Gamma = G(\mathbb{Z})$. (In view of 2.4 no loss of generality is entailed in this assumption.)

It is clear that $\mathrm{rank}_{\mathbb{Q}} G = 1$ and $\mathrm{rank}_{\mathbb{R}} G = m$. In view of [We1] the divergent orbits of $\mathbb{R}$-split tori exist only for the action of tori of dimension one, that is, for 1-parameter $\mathbb{R}$-split tori. So, let D be a 1-parameter $\mathbb{R}$-split algebraic torus of G. Up to conjugation by an element from G, $D = \{d(t) = (d_1(t), \ldots, d_m(t)) : t \in \mathbb{R}^*\}$, where

$$d_i(t) = \begin{pmatrix} t^{\alpha_i} & 0 \\ 0 & t^{-\alpha_i} \end{pmatrix}$$

and $\alpha_1, \ldots, \alpha_m$ are non-negative integers not all equal to 0. We will denote by B^+, respectively B^-, the group of upper, respectively lower, triangular matrices in $\mathrm{SL}_2(\mathbb{R})$.

For every $g = (g_1, \ldots, g_m)$ in G we denote

$$I^+(g) = \{i \in \{1, 2, \ldots, m\} : g_i \in B^+\}$$

and

$$I^-(g) = \{i \in \{1, 2, \ldots, m\} : g_i \in B^-\}.$$

The main result in this section is the following:

Theorem 5.1. *With the above notation, $D\pi(g)$ is a divergent orbit if and only if there exists a $q \in G(\mathbb{Q})$ such that*

$$\sum_{i \in I^+(gq)} \alpha_i > \sum_{j \notin I^+(gq)} \alpha_j$$

and

$$\sum_{i \in I^-(gq)} \alpha_i > \sum_{j \notin I^-(gq)} \alpha_j.$$

If $\alpha_i = 1$ and $\alpha_j = 0$ for all $i \neq j$, Theorem 5.1 follows from Theorem 1.5.

Let D be a group of $\mathbb{R}$-points of a $\mathbb{Q}$-split torus in G. In this case $\alpha_1 = \ldots = \alpha_m = 1$. Theorem 5.1 easily implies:

Corollary 5.2. *Let D be a group of $\mathbb{R}$-points of a $\mathbb{Q}$-split torus in G. Then $D\pi(g)$ is divergent if and only if there exists a $q \in G(\mathbb{Q})$ such that*

$$\#I^+(gq) > \frac{m}{2} \quad \text{and} \quad \#I^-(gq) > \frac{m}{2}.$$

In particular, if $r = 2$ then $D\pi(g)$ is divergent if and only if

$$g \in \mathcal{Z}_G(D)G(\mathbb{Q}).$$

5.2. Proof of Theorem 5.1. We will need the following notation: if $v(t) = (v_1(t), \ldots, v_n(t))$ and $w(t) = (w_1(t), \ldots, w_n(t))$, where $v_i(t)$ and $w_j(t)$ are real functions defined on a set A, we will write $v(t) \asymp w(t)$ to indicate that there exists a constant $c > 1$ such that $\frac{v_i(t)}{c} \leq w_i(t) \leq v_i(t)c$ for all i and $t \in A$.

The proof will be deduced from Theorem 1.6. According to Theorem 1.6, $D\pi(g)$ is divergent if and only if there exist opposite unipotent K-subalgebras

$\mathfrak{u}^+$ and $\mathfrak{u}^-$ of the K-algebraic group $\mathbf{SL}_2$ such that if $\mathbf{x}^+$ and $\mathbf{x}^-$ are non-zero K-rational elements of $\mathfrak{u}^+$ and $\mathfrak{u}^-$, respectively, then

$$\lim_{t\to 0} \mathbf{c}(\mathrm{Ad}(d(t)g)\mathbf{x}^+) = 0 \tag{11}$$

and

$$\lim_{t\to +\infty} \mathbf{c}(\mathrm{Ad}(d(t)g)\mathbf{x}^-) = 0. \tag{12}$$

Since the non-trivial K-split tori of $\mathbf{SL}_2$ are conjugate by elements of $\mathrm{SL}_2(K)$ and the intersection of any two opposite Borel K-subgroups of $\mathbf{SL}_2$ coincides with a non-trivial K-split torus, there exists an element $q \in G(\mathbb{Q})$ such that $\mathrm{Ad}(q^{-1})\mathbf{x}^+$, respectively $\mathrm{Ad}(q^{-1})\mathbf{x}^-$, is an upper, respectively lower, triangular matrix. (Recall that $G(\mathbb{Q})$ is identified with $\mathrm{SL}_2(K)$ and $\mathfrak{g}(\mathbb{Q})$ with $\mathrm{sl}_2(K)$.

With $g = (g_1, \ldots, g_m)$ as in the formulation of the theorem, for every i we write the Bruhat decompositions of $g_i q$ with respect to B^+ and B^-:

$$g_i q = u_i w_i b_i \quad \text{and} \quad g_i q = u_i^- w_i^- b_i^-,$$

where $u_i \in \mathrm{R}_u(B^+)$, $u_i^- \in \mathrm{R}_u(B^-)$, $b_i \in B^+$, $b_i^- \in B^-$ and w_i and w_i^- are representatives of elements of the Weyl group.

Note that

$$\lim_{t\to 0} d_i(t) u_i d_i(t)^{-1} = e \tag{13}$$

and

$$w_i^{-1} d_i(t) w_i = d_i(ct^{\epsilon}), \tag{14}$$

where c is a constant depending on the choice of w_i and $\epsilon = 1$ or -1 depending on whether or not w_i represents the trivial element of the Weyl group. Using (13), (14) and

$$d_i(t) g_i q = (d_i(t) u_i d_i(t)^{-1}) w_i (w_i^{-1} d_i(t) w_i) b_i$$

one easily sees that if $0 < t < 1$ then

$$\mathrm{Ad}(d_i(t)g_i)\mathbf{x}^+ \asymp \begin{cases} t^{2\alpha_i}\mathrm{Ad}(q^{-1})\mathbf{x}^+ & \text{if } g_i q \in B^+, \\ t^{-2\alpha_i}\mathrm{Ad}(q^{-1})\mathbf{x}^- & \text{if } g_i q \notin B^+. \end{cases}$$

Therefore if $0 < t < 1$ then

$$\mathbf{c}(\mathrm{Ad}(d(t)g)\mathbf{x}^+) \asymp t^{\sum_{i\in I^+(gq)} \alpha_i - \sum_{j\notin I^+(gq)} \alpha_j}.$$

A similar argument shows that for $t > 1$ we have

$$\mathbf{c}(\mathrm{Ad}(d(t)g)\mathbf{x}^-) \asymp t^{\sum_{i\in I^-(gq)} \alpha_i - \sum_{j\notin I^-(gq)} \alpha_j}.$$

Now the theorem follows from (11) and (12). □

5.3. Remark. It is easy to see that Theorem 5.1 remains valid if the assumption in its formulation that D is an *algebraic* split torus is replaced by the weaker one that D is *any* split torus. (In the latter case $\alpha_1, \dots, \alpha_m$ are non-negative *real* rather than integer numbers.) A similar generalization of Theorem 1.6 when $v \in \mathcal{S}$ is an archimedean valuation is also possible. We gave preference to slightly more particular formulations for the sake of simplicity, because this allows us to treat both the archimedean and non-archimedean cases in Theorem 1.6 in a uniform way.

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References

[Bo] A. Borel, **Linear Algebraic Groups. Second Enlarged Edition** Springer, 1991.

[Ca-F] J.W.S.Cassels and A.Frölich, **Algebraic Number Theory**, Academic Press, New York and London, 1967.

[Ch-Mo] P.Chatterjee and Dave Morris, *Divergent torus orbits in homogeneous spaces of $\mathbb{Q}$-rank two*, Isr. J. Math. (2006).

[Da1] S.G.Dani, *On orbits of unipotent flows on homogeneous spaces* Ergod. Th. Dynam. Sys. **4** (1984) 25–34.

[Da2] S.G.Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J.Reine Angew. Math. **359** (1985) 55–89.

[Ei-Ka-Li] M.Einsiedler, A.Katok and E.Lindenstrauss, *Invariant measures and the set of exceptions of Littlewood conjecture*, Ann.Math., to appear.

[Ei-Kl] M.Einsiedler and D.Kleinbock, *Measure rigidity and p-adic Littlewood type problems*, Preprint (March 2005).

[Fe] D. Ferte, *Weyl chamber flow on irreducible quotients of products of $PSL(2, \mathbb{R})$*, Transf.Groups **11** (2006), no.1, 17-28.

[Kl-To] D.Kleinbock and G.Tomanov, *Flows of S-arithmetic homogeneous spaces and applications to metric Diophantine approximation*, to appear in Commentarii Mathematici Helvetici.

[Ma1] G. A. Margulis, *Problems and Conjectures in Rigidity Theory*, in **Mathematics: Frontiers and Perspectives** 161–174, Amer. Math. Soc. (2000).

[Ma2] G. A. Margulis, *Oppenheim Conjecture*, Fields Medalists' lectures, World Sci. Ser. 20th Century Math., World Sci. Publishing, River Edge, NJ, **5** (1997) 272-327.

[Ma3] G.A. Margulis, *On the actions of unipotent groups on the space of lattices*, in: **Lie groups and their representations** Proc. Summer School, Bolyai Janos Math. Soc., Budapest, 1971.

[Pl-R] V.P.Platonov and A.S.Rapinchuk, **Algebraic Groups and Number Theory**, Academic Press, 1994.

[To] G. Tomanov, *Values of decomposable forms at S-integral points and orbits of tori on homogeneous spaces*, to appear in Duke Math. Journal, 29 pages.

[To-We] G. Tomanov, B.Weiss *Closed Orbits for Actions of Maximal Tori on Homogeneous Spaces*, Duke Math. Journal **119** (2003) 367-392.

[Weil] A.Weil **Adels and Algebraic Groups**, Institut for Advanced Study, Princeton, NJ, 1961.

[We1] B.Weiss *Divergent trajectories and $\mathbb{Q}$-rank*, Isr. J. Math. **152** (2006) 221-227.

[We2] B.Weiss *Divergent trajectories on noncompact parameter spaces*, GAFA **14** no.1 (2004) 94-149.

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