

A Short Proof for the Convergence of the Yamabe Flow on S^n

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1. INTRODUCTION

Let M be a Riemannian manifold of dimension $n \geq 3$ with Riemannian metric g . Consider the following evolution equation for the Riemannian metric:

$$(1) \quad \frac{\partial g}{\partial t} = -(R_g - r_g)g.$$

Here, R_g denotes the scalar curvature associated with the metric g and

$$(2) \quad r_g = \frac{\int_M R_g \, d\text{vol}_g}{\int_M d\text{vol}_g}$$

is the mean value of the scalar curvature on M . The evolution equation (1) is known as the Yamabe flow. It is well known that (1) can be reduced to a nonlinear partial differential equation of parabolic type. To see this, we choose a background metric g_0 which is conformal to the initial metric. Since the Yamabe flow preserves the conformal structure, the metric g can be written in the form $g = u^{\frac{4}{n-2}} g_0$, where u is a positive function. Then the Yamabe flow is equivalent to the equation

$$(3) \quad \frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = \frac{n+2}{4} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u + r_g u^{\frac{n+2}{n-2}} \right),$$

where R_{g_0} denotes the scalar curvature of the background metric g_0 and Δ_{g_0} is the Laplace operator associated with g_0 .

The Yamabe flow was proposed by Richard Hamilton in the 1980s as a tool for constructing metrics of constant scalar curvature in a given conformal class [7].

Hamilton proved that the Yamabe flow has a global solution for every initial metric. The asymptotic behavior of the Yamabe flow was first studied by Bennett Chow [5]. Chow proved that the flow approaches a metric of constant scalar curvature provided that the initial metric is locally conformally flat and has positive Ricci curvature.

Rugang Ye proved the convergence of the Yamabe flow assuming only that the initial metric is locally conformally flat [14]. Ye's proof is based on an estimate for the gradient of the function $\log u$ which is established using the method of moving planes.

A different approach was developed by Hartmut Schwetlick and Michael Struwe [11]. Schwetlick and Struwe established new integral estimates for the scalar curvature, which allowed them to perform a blow-up analysis. Moreover, they were able to rule out the formation of a singularity provided that the dimension is less than 6 and the initial energy is below a certain threshold. Unlike the method of moving planes, the techniques used in [11] are not limited to the locally conformally flat case.

In a previous paper, it was shown that the condition $n < 6$ suffices to rule out the formation of a singularity as $t \rightarrow \infty$ [4]. The proof of this result uses the positive mass theorem. This is perhaps not surprising, as the positive mass theorem plays a key role in Richard Schoen's solution of the Yamabe problem (see [8],[10]). Another tool used in [4] is the Lojasiewicz inequality. The use of this tool in geometric analysis was pioneered by Leon Simon in an influential paper on the asymptotic behavior of gradient flows [12].

The purpose of this note is to provide a short proof for the convergence of the Yamabe flow on S^n . This is a special case of Ye's theorem [14]. The proof given in this paper is based on variational techniques, and does not use the method of moving planes.

Theorem. *Consider an initial metric on S^n which is conformally equivalent to the standard metric. Then the Yamabe flow has a global solution, which converges exponentially to a metric of constant sectional curvature.*

2. THE YAMABE FLOW ON S^n

Throughout this paper, we assume that the initial metric is conformally equivalent to the standard metric g_{S^n} on S^n . It follows from a result of Hamilton that the Yamabe flow has a global solution $\{g(t) : t \geq 0\}$. Moreover, the solution

satisfies

$$(4) \quad \lim_{t \rightarrow \infty} \int_{S^n} |R_{g(t)} - r_{g(t)}|^s \, d\text{vol}_{g(t)} = 0$$

for $s < \frac{n+2}{2}$. If the initial metric has positive scalar curvature, this follows from a result of Schwetlick and Struwe (see [11], Lemma 3.3). For the general case, see [4], Proposition 3.1.

The time-dependent metric $g(t)$ can be written in the form $g(t) = u(t)^{\frac{4}{n-2}} g_{S^n}$ for some positive function $u(t)$. Without loss of generality, we may assume that the volume is normalized to 1. This implies

$$(5) \quad \int_{S^n} u(t)^{\frac{2n}{n-2}} \, d\text{vol}_{S^n} = 1$$

for all $t \geq 0$. For abbreviation, let

$$(6) \quad r_\infty = \lim_{t \rightarrow \infty} r_{g(t)}.$$

Since the function $t \mapsto r_{g(t)}$ is decreasing, the limit always exists.

For each point $p \in B^{n+1}$, we define a function $\delta_p : S^n \rightarrow \mathbb{R}$ by

$$(7) \quad \delta_p(x) = \left(\frac{n(n-1)}{r_\infty} \right)^{\frac{n-2}{4}} \left(\frac{1 - |p|^2}{1 + 2 \langle p, x \rangle + |p|^2} \right)^{\frac{n-2}{2}}$$

for $x \in S^n$. The conformal metric $\delta_p^{\frac{4}{n-2}} g_{S^n}$ has constant scalar curvature r_∞ . Therefore, the function δ_p satisfies the nonlinear elliptic partial differential equation

$$(8) \quad \frac{4(n-1)}{n-2} \Delta_{S^n} \delta_p - n(n-1) \delta_p + r_\infty \delta_p^{\frac{n+2}{n-2}} = 0.$$

For later use, we define a function $\xi_p : S^n \rightarrow \mathbb{R}^{n+1}$ by

$$(9) \quad \xi_p(x) = \delta_p(x) \left[p + \frac{1 - |p|^2}{1 + 2 \langle p, x \rangle + |p|^2} (x + p) \right] \in \mathbb{R}^{n+1}$$

for $x \in S^n$. For every point $x \in S^n$, we have

$$\frac{\partial}{\partial p} \delta_p(x) = -\frac{n-2}{1 - |p|^2} \xi_p(x) \in \mathbb{R}^{n=1}$$

Hence, the function ξ_p is a solution of the linearized equation.

$$(10) \quad \frac{4(n-1)}{n-2} \Delta_{S^n} \xi_p - n(n-1) \xi_p + \frac{n+2}{n-2} r_\infty \delta_p^{\frac{4}{n-2}} \xi_p = 0.$$

Our goal is to show that the Yamabe flow approaches a metric of constant scalar curvature as $t \rightarrow \infty$. To this end, we consider a sequence of times $\{t_\nu : \nu \in \mathbb{N}\}$

such that $t_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$. For abbreviation, let $u_\nu = u(t_\nu)$ for all $\nu \in \mathbb{N}$.

Lemma 1. *After passing to a subsequence if necessary, we can find a positive integer m and a family of points $\{p_{k,\nu} : 1 \leq k \leq m, \nu \in \mathbb{N}\} \subset B^{n+1}$ with the following properties:*

(i) *We have*

$$(11) \quad \left\| u_\nu - \sum_{k=1}^m \delta_{p_{k,\nu}} \right\|_{H^1(S^n)} \rightarrow 0$$

as $\nu \rightarrow \infty$.

(ii) *For every pair $i \neq j$ we have*

$$(12) \quad \frac{|p_{i,\nu} - p_{j,\nu}|^2}{(1 - |p_{i,\nu}|^2)(1 - |p_{j,\nu}|^2)} \rightarrow \infty$$

as $\nu \rightarrow \infty$.

(iii) *The integer m is given by*

$$(13) \quad r_\infty = (m Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}}.$$

Proof. Using (4) with $\gamma = \frac{2n}{n+2}$, we obtain

$$\int_{S^n} \left| \frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dvol_{S^n} \rightarrow 0$$

as $\nu \rightarrow \infty$. Hence, we may apply a general compactness result due to Michael Struwe [13]. After passing to a subsequence if necessary, we obtain

$$(14) \quad \left\| u_\nu - u_\infty - \sum_{k=1}^l \left(\frac{n(n-1)}{r_\infty} \right)^{\frac{n-2}{4}} \cdot \left(\frac{2\varepsilon_{k,\nu}}{\varepsilon_{k,\nu}^2 + (1 - \frac{1}{4} \varepsilon_{k,\nu}^2) |x - a_{k,\nu}|^2} \right)^{\frac{n-2}{2}} \right\|_{H^1(S^n)} \rightarrow 0$$

as $\nu \rightarrow \infty$. Here, l is a non-negative integer, and $\{a_{k,\nu} : 1 \leq k \leq l, \nu \in \mathbb{N}\}$ is a family of points on S^n . Moreover, $\{\varepsilon_{k,\nu} : 1 \leq k \leq l, \nu \in \mathbb{N}\}$ is a family of positive real numbers such that $\varepsilon_{k,\nu} \rightarrow 0$ as $\nu \rightarrow \infty$. For every pair $i \neq j$, we have

$$(15) \quad \frac{(\varepsilon_{i,\nu} - \varepsilon_{j,\nu})^2 + |a_{i,\nu} - a_{j,\nu}|^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \rightarrow \infty$$

as $\nu \rightarrow \infty$. Finally, the function u_∞ is a non-negative smooth solution of the partial differential equation

$$\frac{4(n-1)}{n-2} \Delta_{S^n} u_\infty - n(n-1) u_\infty + r_\infty u_\infty^{\frac{n+2}{n-2}} = 0.$$

Using a theorem of Obata, we conclude that either $u_\infty = 0$ or $u_\infty = \delta_p$ for some point $p \in B^{n+1}$ (see [9], Proposition 6.1). Hence, we need to distinguish two cases:

Case 1: Suppose that $u_\infty = 0$. In this case, we put $m = l$, and

$$p_{k,\nu} = -\frac{2 - \varepsilon_{k,\nu}}{2 + \varepsilon_{k,\nu}} a_{k,\nu}$$

for $1 \leq k \leq l$.

Case 2: Suppose that $u_\infty = \delta_p$ for some $p \in B^{n+1}$. In this case, we put $m = l+1$. Moreover, we define

$$p_{k,\nu} = \begin{cases} -\frac{2 - \varepsilon_{k,\nu}}{2 + \varepsilon_{k,\nu}} a_{k,\nu} & \text{for } 1 \leq k \leq l \\ p & \text{for } k = l+1 \end{cases}.$$

The assertion follows now from (14) and (15).

The condition (12) has an interesting interpretation in terms of hyperbolic geometry. Note that

$$\left(\sinh \frac{d_{hyp}(p_{i,\nu}, p_{j,\nu})}{2} \right)^2 = \frac{|p_{i,\nu} - p_{j,\nu}|^2}{(1 - |p_{i,\nu}|^2)(1 - |p_{j,\nu}|^2)},$$

where $d_{hyp}(\cdot, \cdot)$ denotes the geodesic distance of two points in B^{n+1} relative to the hyperbolic metric on B^{n+1} . (This formula is a consequence of Corollary A.5.8 in [3].) Therefore, (12) is equivalent to $d_{hyp}(p_{i,\nu}, p_{j,\nu}) \rightarrow \infty$ as $\nu \rightarrow \infty$.

For each $\nu \in \mathbb{N}$, we minimize the expression

$$(16) \quad \int_{S^n} \left[\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \sum_{k=1}^m \alpha_k \delta_{q_k} \right) \right|_{S^n}^2 + n(n-1) \left(u_\nu - \sum_{k=1}^m \alpha_k \delta_{q_k} \right)^2 \right] dvol_{S^n}$$

subject to the constraints

$$(17) \quad \frac{1}{2} \leq \alpha_k \leq 2$$

and

$$(18) \quad |q_k - p_{k,\nu}| \leq \frac{1}{2} (1 - |p_{k,\nu}|)$$

for $1 \leq k \leq m$.

Proposition 2. *Suppose that, for each $\nu \in \mathbb{N}$, $\{(\alpha_{k,\nu}, q_{k,\nu}) : 1 \leq k \leq m\}$ is an m -tuple which minimizes the expression (16) subject to the constraints (17), (18). Then the following statements hold:*

(i) *We have*

$$(19) \quad \left\| u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \delta_{q_{k,\nu}} \right\|_{H^1(S^n)} \rightarrow 0$$

as $\nu \rightarrow \infty$.

(ii) *For every pair $i \neq j$ we have*

$$(20) \quad \frac{|q_{i,\nu} - q_{j,\nu}|^2}{(1 - |q_{i,\nu}|^2)(1 - |q_{j,\nu}|^2)} \rightarrow \infty$$

as $\nu \rightarrow \infty$.

(iii) *For every $1 \leq k \leq m$, we have*

$$(21) \quad \alpha_{k,\nu} \rightarrow 1$$

and

$$(22) \quad \frac{|q_{k,\nu} - p_{k,\nu}|}{1 - |p_{k,\nu}|} \rightarrow 0$$

as $\nu \rightarrow \infty$.

Proof. Since the expression (16) attains its minimum at $\{(\alpha_{k,\nu}, q_{k,\nu}) : 1 \leq k \leq m\}$, the estimate (19) is an immediate consequence of (11). Similarly, (20) follows from (12). To see this, consider a pair $i \neq j$. Using the triangle inequality, we obtain

$$|p_{i,\nu} - p_{j,\nu}| \leq |q_{i,\nu} - q_{j,\nu}| + (1 - |q_{i,\nu}|) + (1 - |q_{j,\nu}|),$$

hence

$$\begin{aligned} & |p_{i,\nu} - p_{j,\nu}|^2 \\ & \leq 3|q_{i,\nu} - q_{j,\nu}|^2 + 3(1 - |q_{i,\nu}|)^2 + 3(1 - |q_{j,\nu}|)^2 \\ & = 3|q_{i,\nu} - q_{j,\nu}|^2 + 3(|q_{i,\nu}| - |q_{j,\nu}|)^2 + 6(1 - |q_{i,\nu}|)(1 - |q_{j,\nu}|) \\ & \leq 6|q_{i,\nu} - q_{j,\nu}|^2 + 6(1 - |q_{i,\nu}|)(1 - |q_{j,\nu}|). \end{aligned}$$

Moreover, we have

$$\begin{aligned} (1 - |p_{i,\nu}|^2)(1 - |p_{j,\nu}|^2) &\geq (1 - |p_{i,\nu}|)(1 - |p_{j,\nu}|) \\ &\geq \frac{4}{9}(1 - |q_{i,\nu}|)(1 - |q_{j,\nu}|) \\ &\geq \frac{1}{9}(1 - |q_{i,\nu}|^2)(1 - |q_{j,\nu}|^2). \end{aligned}$$

Thus, we conclude that

$$\frac{|p_{i,\nu} - p_{j,\nu}|^2}{(1 - |p_{i,\nu}|^2)(1 - |p_{j,\nu}|^2)} \leq 54 \frac{|q_{i,\nu} - q_{j,\nu}|^2}{(1 - |q_{i,\nu}|^2)(1 - |q_{j,\nu}|^2)} + \frac{27}{2}.$$

Hence, it follows from Lemma 1 that

$$\frac{|q_{i,\nu} - q_{j,\nu}|^2}{(1 - |q_{i,\nu}|^2)(1 - |q_{j,\nu}|^2)} \rightarrow \infty$$

as $\nu \rightarrow \infty$. This completes the proof of (20). Finally, it follows from (11) and (19) that

$$\left\| \sum_{k=1}^m \alpha_{k,\nu} \delta_{q_{k,\nu}} - \sum_{k=1}^m \delta_{p_{k,\nu}} \right\|_{H^1(S^n)} \rightarrow 0$$

as $\nu \rightarrow \infty$. From this, the relations (21) and (22) follow easily.

For each $\nu \in \mathbb{N}$, we define a function v_ν by

$$(23) \quad v_\nu = \sum_{k=1}^m \alpha_{k,\nu} \delta_{q_{k,\nu}},$$

where $\{(\alpha_{k,\nu}, q_{k,\nu}) : 1 \leq k \leq m\}$ is chosen so as to minimize the expression (16). Moreover, we define a function w_ν by

$$(24) \quad w_\nu = u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \delta_{q_{k,\nu}}$$

Clearly, $\|w_\nu\|_{H^1(S^n)} \rightarrow 0$ as $\nu \rightarrow \infty$.

We next derive an upper bound for the Yamabe energy of the function v_ν . This estimate plays a key role in our analysis.

Proposition 3. *The function v_ν satisfies the estimate*

$$(25) \quad \begin{aligned} &\int_{S^n} \left(\frac{4(n-1)}{n-2} |dv_\nu|_{S^n}^2 + n(n-1) v_\nu^2 \right) dvol_{S^n} \\ &\leq r_\infty \left(\int_{S^n} v_\nu^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{n}} \end{aligned}$$

for all $\nu \in \mathbb{N}$.

Proof. Note that

$$\int_{S^n} \sum_{k=1}^m \delta_{q_{k,\nu}}^{\frac{2n}{n-2}} dvol_{S^n} = m \left(\frac{Y(S^n)}{r_\infty} \right)^{\frac{n}{2}} = 1.$$

Using (8) and Hölder’s inequality, we obtain

$$\begin{aligned} & \int_{S^n} \left(\frac{4(n-1)}{n-2} |dv_\nu|_{S^n}^2 + n(n-1)v_\nu^2 \right) dvol_{S^n} \\ &= \int_{S^n} \sum_{i,j=1}^m \alpha_{i,\nu} \alpha_{j,\nu} \left(\frac{4(n-1)}{n-2} \langle d\delta_{q_{i,\nu}}, d\delta_{q_{j,\nu}} \rangle_{S^n} + n(n-1) \delta_{q_{i,\nu}} \delta_{q_{j,\nu}} \right) dvol_{S^n} \\ &= r_\infty \int_{S^n} \sum_{i,j=1}^m \alpha_{i,\nu} \alpha_{j,\nu} \delta_{q_{i,\nu}}^{\frac{n+2}{n-2}} \delta_{q_{j,\nu}} dvol_{S^n} \\ &\leq r_\infty \int_{S^n} \left(\sum_{k=1}^m \delta_{q_{k,\nu}}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} v_\nu^2 dvol_{S^n} \\ &\leq r_\infty \left(\int_{S^n} \sum_{k=1}^m \delta_{q_{k,\nu}}^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{2}{n}} \left(\int_{S^n} v_\nu^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{n}} \\ &= r_\infty \left(\int_{S^n} v_\nu^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{n}}. \end{aligned}$$

This completes the proof.

Corollary 4. *The function v_ν satisfies*

$$\begin{aligned} & \int_{S^n} \left(\frac{4(n-1)}{n-2} |dv_\nu|_{S^n}^2 + n(n-1)v_\nu^2 \right) dvol_{S^n} \\ (26) \quad & \leq \frac{2}{n} r_\infty + \frac{n-2}{n} r_\infty \int_{S^n} v_\nu^{\frac{2n}{n-2}} dvol_{S^n}. \end{aligned}$$

Proof. Since the function $x \mapsto x^{\frac{n-2}{n}}$ is concave, we have

$$x^{\frac{n-2}{n}} \leq \frac{2}{n} + \frac{n-2}{n} x$$

for all $x \geq 0$. Hence, the assertion follows from Proposition 3.

Besides Corollary 4, we shall need an estimate for the second variation operator of the Yamabe functional at v_ν . Since this estimate is standard, we present only a brief sketch of the proof.

Proposition 5. *There exists a positive constant η such that*

$$(27) \quad \begin{aligned} & \frac{n+2}{n-2} r_\infty \int_{S^n} v_\nu^{\frac{4}{n-2}} w_\nu^2 dvol_{S^n} \\ & \leq (1-\eta) \int_{S^n} \left(\frac{4(n-1)}{n-2} |dw_\nu|_{S^n}^2 + n(n-1) w_\nu^2 \right) dvol_{S^n} \end{aligned}$$

if ν is sufficiently large.

Proof. Recall that the expression (16) attains its minimum at $\{(\alpha_{k,\nu}, q_{k,\nu}) : 1 \leq k \leq m\}$. It follows from (21) and (22) that none of the constraints (17) and (18) is binding if ν is sufficiently large. The first order conditions for a minimum are

$$\begin{aligned} 0 &= \int_{S^n} \left(\frac{4(n-1)}{n-2} \Delta_{S^n} \delta_{q_{k,\nu}} - n(n-1) \delta_{q_{k,\nu}} \right) w_\nu dvol_{S^n} \\ &= -r_\infty \int_{S^n} \delta_{q_{k,\nu}}^{\frac{n+2}{n-2}} w_\nu dvol_{S^n} \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_{S^n} \left(\frac{4(n-1)}{n-2} \Delta_{S^n} \xi_{q_{k,\nu}} - n(n-1) \xi_{q_{k,\nu}} \right) w_\nu dvol_{S^n} \\ &= -\frac{n+2}{n-2} r_\infty \int_{S^n} \delta_{q_{k,\nu}}^{\frac{4}{n-2}} \xi_{q_{k,\nu}} w_\nu dvol_{S^n} \end{aligned}$$

for $1 \leq k \leq m$. The assertion follows now from standard arguments. A proof for $n = 3$ can be found in [2], Lemma A2. The general case is discussed in [1].

Proposition 6. *We have*

$$(28) \quad \begin{aligned} & \int_{S^n} \left(\frac{4(n-1)}{n-2} |du_\nu|_{S^n}^2 + n(n-1) u_\nu^2 \right) dvol_{S^n} \\ & \leq r_\infty + \frac{2}{\eta Y(S^n)} \left(\int_{S^n} \left| \frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu \right. \right. \\ & \quad \left. \left. - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dvol_{S^n} \right)^{\frac{n+2}{n}} \end{aligned}$$

if ν is sufficiently large.

Proof. Using the identity

$$\int_{S^n} u_\nu^{\frac{2n}{n-2}} dvol_{S^n} = 1$$

and the relation $u_\nu = v_\nu + w_\nu$, we obtain

$$\begin{aligned}
& \int_{S^n} \left(\frac{4(n-1)}{n-2} |du_\nu|_{S^n}^2 + n(n-1) u_\nu^2 \right) dvol_{S^n} \\
&= \frac{n-2}{n} r_\infty - \frac{n-2}{n} r_\infty \int_{S^n} v_\nu^{\frac{2n}{n-2}} dvol_{S^n} \\
&+ \int_{S^n} \left(\frac{4(n-1)}{n-2} |dv_\nu|_{S^n}^2 + n(n-1) v_\nu^2 \right) dvol_{S^n} \\
&- 2 \int_{S^n} \left(\frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right) w_\nu dvol_{S^n} \\
&- \int_{S^n} \left(\frac{4(n-1)}{n-2} |dw_\nu|_{S^n}^2 + n(n-1) w_\nu^2 - \frac{n+2}{n-2} r_\infty v_\nu^{\frac{4}{n-2}} w_\nu^2 \right) dvol_{S^n} \\
&+ r_\infty \int_{S^n} \left(\frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 \right. \\
&\quad \left. + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right) dvol_{S^n}.
\end{aligned}$$

This implies

$$\begin{aligned}
& \int_{S^n} \left(\frac{4(n-1)}{n-2} |du_\nu|_{S^n}^2 + n(n-1) u_\nu^2 \right) dvol_{S^n} \\
&\leq r_\infty - 2 \int_{S^n} \left(\frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right) w_\nu dvol_{S^n} \\
&- \int_{S^n} \left(\frac{4(n-1)}{n-2} |dw_\nu|_{S^n}^2 + n(n-1) w_\nu^2 - \frac{n+2}{n-2} r_\infty v_\nu^{\frac{4}{n-2}} w_\nu^2 \right) dvol_{S^n} \\
&+ r_\infty \int_{S^n} \left(\frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 \right. \\
&\quad \left. + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right) dvol_{S^n}
\end{aligned}$$

by virtue of Corollary 4. By Proposition 5, we can find a positive constant η such that

$$\begin{aligned}
& \int_{S^n} \left(\frac{4(n-1)}{n-2} |dw_\nu|_{S^n}^2 + n(n-1) w_\nu^2 - \frac{n+2}{n-2} r_\infty v_\nu^{\frac{4}{n-2}} w_\nu^2 \right) dvol_{S^n} \\
&\geq \eta \int_{S^n} \left(\frac{4(n-1)}{n-2} |dw_\nu|_{S^n}^2 + n(n-1) w_\nu^2 \right) dvol_{S^n} \\
&\geq \eta Y(S^n) \left(\int_{S^n} w_\nu^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{n}}
\end{aligned}$$

if ν is sufficiently large. Using Hölder's inequality, we obtain

$$\begin{aligned} & \int_{S^n} \left(\frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right) w_\nu \, dvol_{S^n} \\ & \geq - \left(\int_{S^n} \left| \frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dvol_{S^n} \right)^{\frac{n+2}{2n}} \\ & \quad \cdot \left(\int_{S^n} |w_\nu|^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{2n}}, \end{aligned}$$

hence

$$\begin{aligned} & \int_{S^n} \left(\frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right) w_\nu \, dvol_{S^n} \\ & \geq - \frac{1}{\eta Y(S^n)} \left(\int_{S^n} \left| \frac{4(n-1)}{n-2} \Delta_{S^n} u_\nu - n(n-1) u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dvol_{S^n} \right)^{\frac{n+2}{n}} \\ & \quad - \frac{\eta Y(S^n)}{4} \left(\int_{S^n} |w_\nu|^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{n}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \left| \int_{S^n} \left(\frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 \right. \right. \\ & \quad \left. \left. + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right) dvol_{S^n} \right| \\ & \leq C \int_{S^n} |w_\nu|^{\frac{2n}{n-2}} dvol_{S^n} \\ & \quad + C \left(\int_{S^n} |w_\nu|^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{n} \min\{\frac{n}{n-2}, \frac{3}{2}\}}. \end{aligned}$$

Since $\|w_\nu\|_{H^1(S^n)} \rightarrow 0$ as $\nu \rightarrow \infty$, this implies

$$\begin{aligned} & r_\infty \int_{S^n} \left(\frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 \right. \\ & \quad \left. + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right) dvol_{S^n} \\ & \leq \frac{\eta Y(S^n)}{2} \left(\int_{S^n} |w_\nu|^{\frac{2n}{n-2}} dvol_{S^n} \right)^{\frac{n-2}{n}} \end{aligned}$$

if ν is sufficiently large. Putting these facts together, the assertion follows.

3. EXPONENTIAL CONVERGENCE

In this section, we prove that the metric $g(t)$ converges exponentially to a metric of constant sectional curvature as $t \rightarrow \infty$. The following proposition plays a key role in our analysis.

Proposition 7. *We can find positive real numbers C and t_0 , depending only on the initial data, such that*

$$(29) \quad r_{g(t)} \leq r_\infty + C \left(\int_{S^n} |R_{g(t)} - r_\infty|^{\frac{2n}{n+2}} dvol_{g(t)} \right)^{\frac{n+2}{n}}$$

for all $t \geq t_0$.

Proof. Suppose this is not true. Then we can find a sequence of times $\{t_\nu : \nu \in \mathbb{N}\}$ such that $t_\nu \geq \nu$ and

$$r_{g(t_\nu)} > r_\infty + \nu \left(\int_{S^n} |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} dvol_{g(t_\nu)} \right)^{\frac{n+2}{n}}$$

for all $\nu \in \mathbb{N}$. By Proposition 6, there exists an infinite subset $I \subset \mathbb{N}$ and a positive constant η such that

$$\begin{aligned} & \int_{S^n} \left(\frac{4(n-1)}{n-2} |du(t_\nu)|_{S^n}^2 + n(n-1)u(t_\nu)^2 \right) dvol_{S^n} \\ & \leq r_\infty + \frac{2}{\eta Y(S^n)} \left(\int_{S^n} \left| \frac{4(n-1)}{n-2} \Delta_{S^n} u(t_\nu) \right. \right. \\ & \quad \left. \left. - n(n-1)u(t_\nu) + r_\infty u(t_\nu)^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dvol_{S^n} \right)^{\frac{n+2}{n}} \end{aligned}$$

for all $\nu \in I$. Using the transformation law of the scalar curvature under a conformal change of the metric, we obtain

$$r_{g(t_\nu)} \leq r_\infty + \frac{2}{\eta Y(S^n)} \left(\int_{S^n} |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} dvol_{g(t_\nu)} \right)^{\frac{n+2}{n}}$$

for all $\nu \in I$. This is a contradiction.

Corollary 8. *We can find positive real numbers C and t_0 , depending only on the initial data, such that*

$$(30) \quad r_{g(t)} \leq r_\infty + C \int_{S^n} (R_{g(t)} - r_{g(t)})^2 dvol_{g(t)}$$

for all $t \geq t_0$.

Proof. By Proposition 7, we have

$$r_{g(t)} \leq r_\infty + C \left(\int_{S^n} |R_{g(t)} - r_\infty|^{\frac{2n}{n+2}} d\text{vol}_{g(t)} \right)^{\frac{n+2}{n}}$$

for $t \geq t_0$. This implies

$$\begin{aligned} r_{g(t)} &\leq r_\infty + 2C \left(\int_{S^n} |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} d\text{vol}_{g(t)} \right)^{\frac{n+2}{n}} \\ &\quad + 2C (r_{g(t)} - r_\infty)^2 \end{aligned}$$

for $t \geq t_0$. Thus, we conclude that

$$r_{g(t)} \leq r_\infty + 4C \left(\int_{S^n} |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} d\text{vol}_{g(t)} \right)^{\frac{n+2}{n}}$$

if t is sufficiently large. Hence, the assertion follows from Hölder's inequality.

Proposition 9. *There exist positive real numbers C and δ , depending only on the initial data, such that*

$$(31) \quad \int_T^\infty \left(\int_{S^n} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g(t)} \right)^{\frac{1}{2}} dt \leq C e^{-\delta T}$$

for all $t \geq 0$.

Proof. Using the identity

$$\frac{d}{dt} r_{g(t)} = -\frac{n-2}{2} \int_{S^n} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g(t)}$$

and Corollary 8, we obtain

$$r_{g(t)} - r_\infty \leq C e^{-2\delta t}$$

for all $t \geq 0$. (Here, C and δ are positive real numbers which depend only on the initial data.) This implies

$$\begin{aligned} &\int_T^{T+1} \left(\int_{S^n} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g(t)} \right)^{\frac{1}{2}} dt \\ &\leq \left(\int_T^{T+1} \int_{S^n} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g(t)} dt \right)^{\frac{1}{2}} \\ &\leq \left(\frac{2}{n-2} (r_{g(T)} - r_{g(T+1)}) \right)^{\frac{1}{2}} \\ &\leq \left(\frac{2}{n-2} (r_{g(T)} - r_\infty) \right)^{\frac{1}{2}} \\ &\leq C e^{-\delta T} \end{aligned}$$

for all $T \geq 0$. From this the assertion follows.

Proposition 9 is very useful because it rules out concentration of volume. In light of the results of Schwetlick and Struwe, this implies that the function $u(t)$ is uniformly bounded from above. Moreover, it is easy to see that the scalar curvature $R_{g(t)}$ is uniformly bounded from below. Hence, the Harnack inequality for linear elliptic equations (see [6], Theorem 8.18 on p. 194) implies that $u(t)$ is uniformly bounded away from 0. Finally, Proposition 9 implies that the function $u(t)$ converges exponentially in $L^2(S^n)$. The limiting metric is smooth and has constant scalar curvature. Therefore, the limiting metric must have constant sectional curvature by Obata's theorem (see [9], Proposition 6.1).

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