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Perron's Formula and the Prime Number Theorem for Automorphic L-Functions

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Abstract: In this paper the classical Perron's formula is modified so that it now depends no longer on sizes of individual terms but on a sum over a short interval. When applied to automorphic L-functions, this new Perron's formula may allow one to avoid estimation of individual Fourier coefficients, without assuming the Generalized Ramanujan Conjecture (GRC). As an application, a prime number theorem for Rankin-Selberg L-functions $L(s, \pi \times \tilde{\pi}')$ is proved unconditionally without assuming GRC, where π and π' are automorphic irreducible cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively.
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1. Introduction

The classical Perron's formula gives a formula for a sum of complex numbers a_n, $1 \leq n \leq x$, in terms of their Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.1}$$

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and bounds for individual terms a_n, where here and throughout $s = \sigma + it \in \mathbb{C}$. Let $A(x) > 0$ be non-decreasing such that $a_n \ll A(n)$, and let

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \tag{1.2}$$

for $\sigma > \sigma_a$, the abscissa of absolute convergence of (1.1). Then the classical Perron's formula (see e.g. Heath-Brown's notes on Titchmarsh [31], p.70) states that, for $b > \sigma_a$,

$$\begin{aligned} \sum_{n \le x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{A(2x)x \log x}{T}\right) \\ + O\left(\frac{x^b B(b)}{T}\right) + O\left\{A(N) \min\left(\frac{x}{T|x-N|}, 1\right)\right\}, \end{aligned} \tag{1.3}$$

where N is the integer nearest to x.

When applying (1.3) to the Riemann zeta-function or Dirichlet L-functions, bounds for a_n pose no problem. When applying this formula to other automorphic L-functions, however, bounds for a_n often require an assumption of the Generalized Ramanujan Conjecture (GRC). Examples include a prime number theorem for Rankin-Selberg L-functions (Theorem 2.3 below) recently proved by the authors in [18] under the GRC.

In this paper, we will prove a revised version of Perron's formula (Theorem 2.1 and Corollary 2.2 below). Different from the classical (1.3), the new Perron's formula produces a formula for $\sum_{n \le x} a_n$ in terms of a sum of $|a_n|$ over a short interval. While bounding individual Fourier coefficients $|a_\pi(n)|$ of an automorphic cuspidal representation is hard and may require GRC, estimation of a sum of $|a_\pi(n)|$ can usually be done by the Rankin-Selberg method. The new Perron's formula thus allows us to prove certain results for automorphic L-functions without assuming the GRC.

As an application, we are now able to prove a prime number theorem (Theorem 2.3) unconditionally for Rankin-Selberg L-functions $L(s, \pi \times \tilde{\pi}')$, by removing the assumption of GRC in [18]. This prime number theorem has a remainder term of a size which reflects our current knowledge of zero-free regions of $L(s, \pi)$ as in (4.3) and (4.4). We will see that the new Perron's formula allows us to deduce the prime number theorem for $\pi \not\cong \pi'$ from the diagonal case of $\pi \cong \pi'$.

Several authors have already addressed the question of prime number theorem for Rankin-Selberg L-functions in the GL_2 context, and they all faced the problem of bounding Fourier coefficients. Moreno [20] avoided GRC by an averaging technique, while others restricted themselves to the case of holomorphic cusp forms where GRC is known (Ichihara [5]), or to the Selberg class where GRC is assumed (Kaczorowski and Perelli [11]).

We remark that using these prime number theorems one can count primes weighted by Fourier coefficients of automorphic cuspidal representations. This can be regarded as a direct connection between representation theory and prime distribution.

2. Main theorems

The following is a modification of (1.3).

Theorem 2.1. *Let $f(s)$ be as in (1.1) and absolutely convergent for $\sigma > \sigma_a$. Let $B(\sigma)$ be as in (1.2). Then, for $b > \sigma_a, x \geq 2$, $T \geq 2$, and $H \geq 2$,*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\Big\{ \sum_{x - x/H < n \leq x + x/H} |a_n| \Big\} + O\Big\{ \frac{x^b H B(b)}{T} \Big\}. \tag{2.1}$$

Taking $H = \sqrt{T}$ in Theorem 1.1, we deduce the following

Corollary 2.2. *With the same notation as in Theorem 2.1,*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\Big\{ \sum_{x - x/\sqrt{T} < n \leq x + x/\sqrt{T}} |a_n| \Big\} + O\Big\{ \frac{x^b B(b)}{\sqrt{T}} \Big\}. \tag{2.2}$$

We remark that Corollary 2.2 can be used to derive the classical prime number theorem. In fact, taking $a_n = \Lambda(n)$, we have

$$\sum_{x - x/\sqrt{T} < n \leq x + x/\sqrt{T}} |a_n| \ll \log x \sum_{x - x/\sqrt{T} < n \leq x + x/\sqrt{T}} 1 \ll \frac{x \log x}{\sqrt{T}}.$$

By (3.10.6) in Titchmarsh [31], for $\sigma > \sigma_a = 1$,

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \ll \frac{1}{\sigma - 1}.$$

Therefore, Corollary 2.2 with $b = 1 + 1/\log x$ gives

$$\sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{\zeta'(s)}{\zeta(s)} \right\} \frac{x^s}{s} ds + O\left\{ \frac{x \log x}{\sqrt{T}} \right\}.$$

We can take $T = \exp(\sqrt{\log x})$. The prime number theorem

$$\sum_{n\leq x}\Lambda(n) = x + O\{x\exp(-c\sqrt{\log x})\}$$

now follows from the zero-free region of the Riemann zeta-function and a standard contour-integration argument; here and throughout c denotes a positive constant not necessarily the same at different occurrences.

In order to describe applications of this new Perron's formula to automorphic L-functions, let us recall that for an irreducible unitary cuspidal representation π of $GL_m(\mathbb{Q}_{\mathbb{A}})$, the global L-function attached to π is given by products of local factors for $\sigma > 1$ (Godement and Jacquet [4]):

$$L(s,\pi) = \prod_p L_p(s,\pi_p),$$

$$\Phi(s,\pi) = L_\infty(s,\pi_\infty)L(s,\pi),$$

where

$$L_p(s,\pi_p) = \prod_{j=1}^{m}\left(1 - \frac{\alpha_\pi(p,j)}{p^s}\right)^{-1},$$

and

$$L_\infty(s,\pi_\infty) = \prod_{j=1}^{m}\Gamma_{\mathbb{R}}(s+\mu_\pi(j)).$$

Here $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and $\alpha_\pi(p,j)$ and $\mu_\pi(j)$, $j = 1,\ldots,m$, are complex numbers associated with π_p and π_∞, respectively, according to the Langlands correspondence. Denote by

$$a_\pi(p^k) = \sum_{1\leq j\leq m}\alpha_\pi(p,j)^k$$

the Fourier coefficients of π. Then for $\sigma > 1$, we have

$$\frac{d}{ds}\log L(s,\pi) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)a_\pi(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function. If π' is an automorphic irreducible cuspidal representation of $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, we define $L(s,\pi')$, $\alpha_{\pi'}(p,i)$, $\mu_{\pi'}(i)$, and $a_{\pi'}(p^k)$ likewise, for $i = 1,\ldots,m'$. If π and π' are equivalent, then $m = m'$ and $\{\alpha_\pi(p,j)\} = \{\alpha_{\pi'}(p,i)\}$ for any p. Hence $a_\pi(n) = a_{\pi'}(n)$ for any $n = p^k$, when $\pi \cong \pi'$.

The prime number theorem for Rankin-Selberg L-functions has two different cases.

Theorem 2.3. *Let π and π' be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_\mathbb{A})$ and $GL_{m'}(\mathbb{Q}_\mathbb{A})$, respectively. Assume that at least one of π and π' is self-contragredient: $\pi \cong \tilde{\pi}$ or $\pi' \cong \tilde{\pi}'$. Then*

$$\sum_{n\le x} \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)$$

$$= \begin{cases} \dfrac{x^{1+i\tau_0}}{1+i\tau_0} + O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \not\cong \pi \otimes |\det|^{i\tau} \text{ for any } \tau \in \mathbb{R}. \end{cases} \tag{2.3}$$

Note that Theorem 2.3 is now an unconditional result, improved upon [18]. Previously known unconditional prime number theorems for Rankin-Selberg L-functions include a weighted version $\sum_{n\le x}(1-n/x)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)$ and a special case $\sum_{n\le x}\Lambda(n)|a_\pi(n)|^2$, both in Liu, Wang, and Ye [16]. By a standard argument of partial summation, we can deduce from Theorem 2.3 a Mertens theorem for Rankin-Selberg L-functions which is a version of Selberg's orthogonality (Selberg [26] and Ram Murty [22] [23]).

Corollary 2.4. *Let π and π' be as in Theorem 2.3. We have*

$$\sum_{n\le x} \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n}$$

$$= \begin{cases} \log x + c_1 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi; \\ \dfrac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)} + c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}^\times; \\ c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \not\cong \pi \otimes |\det|^{i\tau} \text{ for any } \tau \in \mathbb{R}. \end{cases} \tag{2.4}$$

Here c_1 and c_2 are constants depending on π and π':

$$c_1 = \lim_{s\to 0}\left(-\frac{L'}{L}(s+1, \pi\times\tilde{\pi}') - \frac{1}{s}\right) - 1, \qquad c_2 = -\frac{L'}{L}(1, \pi\times\tilde{\pi}').$$

3. Proof of Perron's summation formula

Proof of Theorem 2.1. We begin with the discontinuous integral

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{y^s}{s} ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ 1/2 & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases} \tag{3.1}$$

Denote the right-hand side by $\delta(y)$; the basic idea is to use $\delta(y)$ to pick up terms with $n \le x$ in the Dirichlet series (1.1). A more convenient form of (3.1) is (see e.g. [1], Lemma in Chapter 17)

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{y^s}{s}ds = \delta(y) + \begin{cases} O\{y^b \min(1, T^{-1}|\log y|^{-1})\} & \text{if } y \ne 1, \\ O(bT^{-1}) & \text{if } y = 1, \end{cases}$$

where the O-constant is absolute.

Let N be the integer nearest to x. Suppose first that

$$|x - N| \gg \frac{x}{T}, \tag{3.2}$$

so that x is not an integer. We take $y = x/n$, multiply both sides in (1.1) by a_n, and then sum over n, to get

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds = \sum_{n\le x} a_n + O(R), \tag{3.3}$$

where

$$R = x^b \sum_{n=1}^{\infty} \frac{|a_n|}{n^b} \min\left(1, \frac{1}{T|\log(x/n)|}\right).$$

For $H \ge 2$,

$$R = x^b \left\{ \sum_{n\le x-x/H} + \sum_{x-x/H<n\le x+x/H} + \sum_{n>x+x/H} \right\}. \tag{3.4}$$

In the first sum on the right, we have

$$\log\frac{x}{n} \ge \log\left(\frac{x}{x-x/H}\right) \gg \frac{1}{H}.$$

Therefore, the first sum is

$$\ll \frac{H}{T}\sum_{n\le x-x/H}\frac{|a_n|}{n^b} \ll \frac{H}{T}B(b).$$

The third sum in (3.4) has the same upper bound. The second sum in (3.4) is

$$\ll \sum_{x-x/H<n\le x+x/H}\frac{|a_n|}{n^b} \ll x^{-b}\sum_{x-x/H<n\le x+x/H}|a_n|,$$

and (3.4) becomes

$$R \ll \sum_{x-x/H<n\le x+x/H}|a_n| + \frac{x^b HB(b)}{T}.$$

This proves the theorem under (3.2).

Now suppose (3.2) is not true, that is $|x-N| \ll x/T$. All goes as before except for the contribution from the term with $n = N$, which can be estimated as

$$\begin{aligned}\int_{b-iT}^{b+iT} a_N \left(\frac{x}{N}\right)^s \frac{ds}{s} &= a_N \int_{b-iT}^{b+iT} \left\{1 + O\left(\frac{1}{T}\right)\right\}^s \frac{ds}{s} \\ &= a_N \int_{b-iT}^{b+iT} \left\{1 + O\left(\frac{|s|}{T}\right)\right\} \frac{ds}{s} \ll |a_N|.\end{aligned}$$

This proves the theorem. □

4. A weighted diagonal prime number theorem

We will use the Rankin-Selberg L-functions $L(s,\pi\times\tilde{\pi}')$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [8], Shahidi [27], and Moeglin and Waldspurger [19], where π and π' are automorphic irreducible cuspidal representations of GL_m and $GL_{m'}$, respectively, over $\mathbb{Q}$ with unitary central characters. This L-function is given by local factors:

$$L(s,\pi \times \tilde{\pi}') = \prod_p L_p(s,\pi_p \times \tilde{\pi}'_p) \tag{4.1}$$

where

$$L_p(s,\pi_p \times \tilde{\pi}'_p) = \prod_{j=1}^{m}\prod_{k=1}^{m'}\left(1 - \frac{\alpha_\pi(p,j)\bar{\alpha}_{\pi'}(p,k)}{p^s}\right)^{-1}.$$

The Archimedean local factor $L_\infty(s,\pi_\infty \times \tilde{\pi}'_\infty)$ is defined by

$$L_\infty(s,\pi_\infty \times \tilde{\pi}'_\infty) = \prod_{j=1}^{m}\prod_{k=1}^{m'} \Gamma_{\mathbb{R}}(s + \mu_{\pi\times\tilde{\pi}'}(j,k))$$

where the complex numbers $\mu_{\pi\times\tilde{\pi}'}(j,k)$ satisfy the trivial bound

$$\text{Re}\ (\mu_{\pi\times\tilde{\pi}'}(j,k)) > -1. \tag{4.2}$$

Denote

$$\Phi(s,\pi \times \tilde{\pi}') = L_\infty(s,\pi_\infty \times \tilde{\pi}'_\infty)L(s,\pi \times \tilde{\pi}').$$

We will need the following properties of the L-functions $L(s,\pi\times\tilde{\pi}')$ and $\Phi(s,\pi\times\tilde{\pi}')$.

RS1. The Euler product for $L(s,\pi\times\tilde{\pi}')$ in (4.1) converges absolutely for $\sigma > 1$ (Jacquet and Shalika [9]).

RS2. The complete L-function $\Phi(s,\pi \times \tilde{\pi}')$ has an analytic continuation to the entire complex plane and satisfies a functional equation

$$\Phi(s,\pi \times \tilde{\pi}') = \varepsilon(s,\pi \times \tilde{\pi}')\Phi(1-s,\tilde{\pi} \times \pi')$$

with

$$\varepsilon(s,\pi\times\tilde{\pi}')=\tau(\pi\times\tilde{\pi}')Q_{\pi\times\tilde{\pi}'}^{-s},$$

where $Q_{\pi\times\tilde{\pi}'}>0$ and $\tau(\pi\times\tilde{\pi}')=\pm Q_{\pi\times\tilde{\pi}'}^{1/2}$ (Shahidi [27], [28], [29], and [30]).

RS3. Denote $\alpha(g)=|\det(g)|$. When $\pi'\not\cong\pi\otimes\alpha^{i\tau}$ for any $\tau\in\mathbb{R}$, $\Phi(s,\pi\times\tilde{\pi}')$ is holomorphic. When $m=m'$ and $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ for some $\tau_0\in\mathbb{R}$, the only poles of $\Phi(s,\pi\times\tilde{\pi}')$ are simple poles at $s=i\tau_0$ and $1+i\tau_0$ coming from $L(s,\pi\times\tilde{\pi}')$ (Jacquet and Shalika [9] and [10], Moeglin and Waldspurger [19]).

RS4. $\Phi(s,\pi\times\tilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in vertical strips (Gelbart and Shahidi [3]).

RS5. $\Phi(s,\pi\times\tilde{\pi}')$ and $L(s,\pi\times\tilde{\pi}')$ are non-zero in $\sigma\geq 1$ (Shahidi [27]). Furthermore, it is zero-free in the region

$$\sigma\geq 1-\frac{c_3}{\log\big(Q_{\pi\times\tilde{\pi}'}(|t|+c_4)\big)},\quad |t|\geq 1, \tag{4.3}$$

and at most one exceptional zero in the region

$$\sigma\geq 1-\frac{c_3}{\log(Q_{\pi\times\tilde{\pi}'}c_4)},\quad |t|\leq 1, \tag{4.4}$$

for some effectively computable positive constants c_3 and c_4, if at least one of π and π' is self-contragredient (Moreno [20] [21], Sarnak [25], and Gelbart, Lapid, and Sarnak [2]).

Now we prove a weighted prime number theorem in the diagonal case.

Lemma 4.1. *Let π be a self-contragredient automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$. Then*

$$\sum_{n\leq x}\Big(1-\frac{n}{x}\Big)\Lambda(n)|a_\pi(n)|^2=\frac{x}{2}+O\{x\exp(-c\sqrt{\log x})\}.$$

Proof. By **RS1**, we have for $\sigma>1$,

$$J(s):=-\frac{d}{ds}\log L(s,\pi\times\tilde{\pi})=\sum_{n=1}^{\infty}\frac{\Lambda(n)a_\pi(n)\bar{a}_\pi(n)}{n^s}.$$

Note that

$$\frac{1}{2\pi i}\int_{(b)}\frac{y^s}{s(s+1)}ds=\begin{cases}1-1/y & \text{if } y\geq 1,\\ 0 & \text{if } 0<y<1,\end{cases}$$

where (b) means the line $\sigma = b > 0$. Taking $b = 1 + 1/\log x$, we have

$$\sum_{n\le x}\left(1-\frac{n}{x}\right)\Lambda(n)|a_\pi(n)|^2 = \frac{1}{2\pi i}\int_{(b)} J(s)\frac{x^s}{s(s+1)}\,ds$$
$$= \frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT} + \int_{b-i\infty}^{b-iT} + \int_{b+iT}^{b+i\infty}\right).$$

The last two integrals are clearly bounded by

$$\ll \int_T^\infty \frac{x}{t^2}dt \ll \frac{x}{T}.$$

Thus,

$$\sum_{n\le x}\left(1-\frac{n}{x}\right)\Lambda(n)|a_\pi(n)|^2 = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} J(s)\frac{x^s}{s(s+1)}\,ds + O\left(\frac{x}{T}\right).$$

By an argument as in [17], we may choose a real number a with $-2 < a < -1$ and a large $T > 0$, and consider the contour

$$\begin{aligned} C_1: &\quad b \ge \sigma \ge a, \quad t = -T;\\ C_2: &\quad \sigma = a, \quad -T \le t \le T;\\ C_3: &\quad a \le \sigma \le b, \quad t = T. \end{aligned}$$

Note that three poles $s = 1, 0, -1$, some trivial zeros, and certain nontrivial zeros $\rho = \beta + i\gamma$ of $L(s, \pi \times \tilde{\pi})$ are passed by the shifting of the contour. Also note that $s = 0$ is a double pole. The trivial zeros can be determined by **RS2** and (4.2): $s = -\mu_{\pi\times\tilde{\pi}}(j,k)$ with $a < -\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k)) < 1$ and $s = -2 - \mu_{\pi\times\tilde{\pi}}(j,k)$ with $a + 2 < -\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k)) < 1$. Here we have used $-2 < a < -1$. Then we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{b-iT}^{b+iT} J(s)\frac{x^s}{s(s+1)}ds\\ &= \frac{1}{2\pi i}\left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) + \operatorname*{Res}_{s=1,0,-1} J(s)\frac{x^s}{s(s+1)}\\ &\quad + \sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)}\\ &\quad + \sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)}\\ &\quad + \sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s(s+1)}. \end{aligned} \tag{4.5}$$

By Lemma 4.1(d) of [17], for any large $\tau > 0$ we can choose T in $\tau < T < \tau + 1$ such that, when $-1 \le \sigma \le 2$,

$$J(\sigma \pm iT) \ll \log^2(Q_{\pi\times\tilde{\pi}}T),$$

and hence,

$$\int_{C_1} \ll \int_a^b \log^2(Q_{\pi\times\tilde{\pi}}T)\frac{x^\sigma}{T^2}d\sigma \ll \frac{x\log^2(Q_{\pi\times\tilde{\pi}}T)}{T^2}.$$

The same upper bound also holds for the integral on C_3. By Lemma 4.2 in [17] we can choose a so that, when $|t| \le T$,

$$J(a+it) \ll 1,$$

and therefore,

$$\int_{C_2} \ll \int_{-T}^{T} \frac{x^a}{(|t|+1)^2}dt \ll \frac{1}{x}.$$

On taking $T \gg \exp(\sqrt{\log x})$, the three integrals on C_1, C_2, C_3 are

$$\ll x\exp(-c\sqrt{\log x}). \tag{4.6}$$

The function

$$J(s)\frac{x^s}{s(s+1)}$$

has simple poles at $s = 1, -1$, and a double pole at $s = 0$; the residues are $x/2$, $O(x^{-1})$, and $O(\log x)$ respectively. Therefore,

$$\operatorname*{Res}_{s=1,0,-1} J(s)\frac{x^s}{s(s+1)} = \frac{x}{2} + O(\log x). \tag{4.7}$$

Near a trivial zero $s = -\mu_{\pi\times\tilde{\pi}}(j,k)$ of order l, we can express $J(s)$ as $-l/(s+\mu_{\pi\times\tilde{\pi}}(j,k))$ plus an analytic function. The residues at these trivial zeros can therefore be computed similarly to what we have done in (4.7). By (4.2), we know that Re $(\mu_{\pi\times\tilde{\pi}}(j,k)) \ge 1-\delta$ for some $\delta > 0$. Consequently,

$$\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{1-\delta}, \tag{4.8}$$

$$\sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}}(j,k))<1} \operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{-1-\delta}. \tag{4.9}$$

To compute the residues corresponding to nontrivial zeros, we recall **RS4** and **RS5**, to get

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} < \infty.$$

Consequently,

$$\begin{aligned}\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s(s+1)} &= -\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} \frac{1}{s-\rho}\frac{x^s}{s(s+1)}\\ &\ll \sum_{|\gamma|\le T}\left|\frac{x^\rho}{\rho(\rho+1)}\right|\\ &= \Big(\sum_{\substack{|\gamma|\le T\\ \rho\in E}} + \sum_{\substack{|\gamma|\le T\\ \rho\notin E}}\Big)\frac{x^\beta}{|\rho(\rho+1)|},\end{aligned} \tag{4.10}$$

where E is the set of exceptional zeros in (4.4). We have $|E|\le 1$, and hence the sum over $\rho\in E$ is clearly $\ll x^{1-\delta}$ for some $\delta>0$. By (4.3), the sum over $\rho\notin E$ is

$$\ll x\exp\left(-c_3\frac{\log x}{2\log(Q_{\pi\times\tilde\pi}T)}\right)\sum_\rho \frac{1}{|\rho(\rho+1)|} \ll x\exp(-c\sqrt{\log x}), \tag{4.11}$$

by taking $T=\exp(\sqrt{\log x})+d$ for some d with $0<d<1$. Hence (4.10) is bounded by $x\exp(-c\sqrt{\log x})$.

Lemma 4.1 then follows by applying (4.6)-(4.9) and (4.11) to (4.5). □

5. Weight removal

Lemma 5.1. *Let π be a self-contragredient automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$. Then*

$$\sum_{n\le x}\Lambda(n)|a_\pi(n)|^2 = x + O\{x\exp(-c\sqrt{\log x})\}. \tag{5.1}$$

Proof. The weight $1-n/x$ can be removed from Lemma 4.1 by a standard argument of de la Vallée Poussin. To this end, let $\Psi(x)$ denote the quantity on the left-hand side of (5.1); then Lemma 4.1 states that

$$\int_1^x \Psi(t)dt = \frac{x^2}{2} + O\{x^2\exp(-c\sqrt{\log x})\}.$$

From this,

$$\begin{aligned}\frac{1}{h}\int_x^{x+h}\Psi(t)dt &= x+\frac{h}{2}+O\left\{\frac{x^2}{h}\exp(-c\sqrt{\log x})\right\}\\ &= x+O\left\{x\exp\left(-\frac{c}{2}\sqrt{\log x}\right)\right\},\end{aligned} \tag{5.2}$$

where we have chosen

$$h = x\exp\left(-\frac{c}{2}\sqrt{\log x}\right);$$

and similarly,

$$\frac{1}{h}\int_{x-h}^{x}\Psi(t)dt = x + O\left\{x\exp\left(-\frac{c}{2}\sqrt{\log x}\right)\right\}. \tag{5.3}$$

Now the terms in $\Psi(t)$ are non-negative. Therefore,

$$\frac{1}{h}\int_{x-h}^{x}\Psi(t)dt \le \Psi(x) \le \frac{1}{h}\int_{x}^{x+h}\Psi(t)dt. \tag{5.4}$$

By (5.2)-(5.4),

$$\Psi(x) = x + O\left\{x\exp\left(-\frac{c}{2}\sqrt{\log x}\right)\right\},$$

which gives Lemma 5.1. □

Without assuming π to be self-contragredient, we can prove a prime number theorem in Lemma 5.2 by the Tauberian theorems of Landau [15] or Ikehara [6]. Note that the error term in Lemma 5.2 is not as good as that in Lemma 5.1.

Lemma 5.2. *For any automorphic irreducible cuspidal unitary representation π of GL_m over $\mathbb{Q}$, not necessarily self-contragredient, we have*

$$\sum_{n\le x}\Lambda(n)|a_\pi(n)|^2 \sim x.$$

Proof. A Tauberian theorem of Ikehara [6] says that, if $f(s)$ is given for $\sigma > 1$ by a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

with $a_n \ge 0$, and if

$$g(s) = f(s) - \frac{1}{s-1}$$

has analytic continuation to $\sigma \ge 1$, then

$$\sum_{n\le x} a_n \sim x.$$

By **RS1**, **RS3**, and **RS5**, we can apply this theorem to

$$f(s) = -\frac{L'}{L}(s, \pi\times\tilde{\pi}).$$

Lemma 5.2 then follows. □

6. Proof of the off-diagonal prime number theorem

Proof of Theorem 2.3. Without loss of generality, we suppose π is self-contragredient. When $\pi' \cong \pi$, the theorem reduces to Lemma 5.1. Therefore, it remains to consider two cases:

(i) $\pi' \cong \pi \otimes |\det|^{i\tau_0}$ for some $\tau_0 \in \mathbb{R}^\times$;
(ii) $\pi' \not\cong \pi \otimes |\det|^{i\tau}$ for any $\tau \in \mathbb{R}$.

We only treat case (i) in detail; the proof in case (ii) is exactly the same, except that all arguments below concerning τ_0 will disappear.

By Lemma 5.1, we obtain a bound for the short sum

$$\sum_{x<n\le x+y} \Lambda(n)|a_\pi(n)|^2 \ll y$$

for $y \gg x\exp(-c\sqrt{\log x})$. Remember that π' is not necessarily self-contragredient; nevertheless, Lemma 5.2 gives for $0<y\le x$ that

$$\sum_{x<n\le x+y} \Lambda(n)|a_{\pi'}(n)|^2 \ll \sum_{x<n\le 2x} \Lambda(n)|a_{\pi'}(n)|^2 \ll x.$$

Let $a_n = \Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)$; then for the above y,

$$\sum_{x<n\le x+y} |a_n| \ll \Big\{\sum_{x<n\le x+y} \Lambda(n)|a_\pi(n)|^2\Big\}^{1/2}\Big\{\sum_{x<n\le x+y} \Lambda(n)|a_{\pi'}(n)|^2\Big\}^{1/2}$$
$$\ll \sqrt{yx}.$$

Now let $T \gg \exp(\sqrt{\log x})$. Then

$$\sum_{x-x/\sqrt{T}<n\le x+x/\sqrt{T}} |a_n| \ll \sqrt{\Big(\frac{x}{\sqrt{T}}\Big)x} = \frac{x}{T^{1/4}}. \tag{6.1}$$

Still we need an upper bound estimate for $B(\sigma)$. We have

$$B(\sigma) = \sum_{n=1}^\infty \frac{|a_n|}{n^\sigma} \ll \Big\{\sum_{n=1}^\infty \frac{\Lambda(n)|a_\pi(n)|^2}{n^\sigma}\Big\}^{1/2}\Big\{\sum_{n=1}^\infty \frac{\Lambda(n)|a_{\pi'}(n)|^2}{n^\sigma}\Big\}^{1/2}. \tag{6.2}$$

But by Lemma 5.2, for $1<\sigma\le 2$,

$$\frac{1}{u^\sigma}\sum_{n\le u} \Lambda(n)|a_\pi(n)|^2 \ll u^{1-\sigma}$$

and tends to 0 when $u \to \infty$. Consequently,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Lambda(n)|a_\pi(n)|^2}{n^\sigma} &= \int_1^\infty \frac{1}{u^\sigma} d\Big\{\sum_{n\le u} \Lambda(n)|a_\pi(n)|^2\Big\} \\ &\ll 1 + \sigma \int_1^\infty \frac{du}{u^\sigma} \ll \frac{1}{\sigma - 1}. \end{aligned} \tag{6.3}$$

Note that (6.3) also holds for π'. Applying (6.3) to both sums on the right side of (6.2), we get for $1 < \sigma \le 2$ that

$$B(\sigma) \ll \frac{1}{\sigma - 1}. \tag{6.4}$$

The upper bound (6.4) holds for π, π' not necessarily self-contragredient, since it depends only on Lemma 5.2.

Next we apply Corollary 2.2 with $b = 1 + 1/\log x$ and $T \gg \exp(\sqrt{\log x})$ to $a_n = \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)$:

$$\begin{aligned} \sum_{n\le x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{L'}{L}(s, \pi \times \tilde{\pi}') \right\} \frac{x^s}{s} ds \\ + O\Big\{ \sum_{x - x/\sqrt{T} < n \le x + x/\sqrt{T}} |a_n| \Big\} + O\Big\{ \frac{x^b B(b)}{\sqrt{T}} \Big\}. \end{aligned}$$

By (6.1) and (6.4), we get

$$\sum_{n\le x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{L'}{L}(s, \pi \times \tilde{\pi}') \right\} \frac{x^s}{s} ds + O\{x \exp(-c\sqrt{\log x})\}. \tag{6.5}$$

The integral in (6.5) can be evaluated by shifting the contour to the left as in §4. Let a with $-2 < a < -1$ and $T > 0$ be as in §4, and define the new contour $C_1 \cup C_2 \cup C_3$ in the same way as in §4. Three poles $s = 1 + i\tau_0, i\tau_0, 0$, some trivial zeros, and certain nontrivial zeros $\rho = \beta + i\gamma$ of $L(s, \pi \times \tilde{\pi}')$ are passed by the shifting of the contour. The trivial zeros can also be determined similarly to what we have done in the proof of Lemma 4.1: $s = -\mu_{\pi\times\tilde{\pi}'}(j,k)$ with $a < -\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k)) < 1$ and $s = -2 - \mu_{\pi\times\tilde{\pi}'}(j,k)$ with $a + 2 < -\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k)) < 1$.

Similarly to (4.5), we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{b-iT}^{b+iT} J(s)\frac{x^s}{s}ds \\
&= \frac{1}{2\pi i}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right) + \operatorname*{Res}_{s=1+i\tau_0,i\tau_0,0} J(s)\frac{x^s}{s} \\
&\quad + \sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s} \\
&\quad + \sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s} \\
&\quad + \sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s}.
\end{aligned}
\tag{6.6}
$$

Applying Lemma 4.1(d) of [17], for any large $\tau > 0$ we can choose T in $\tau < T < \tau + 1$ such that

$$\int_{C_1} \ll \int_a^b \log^2(Q_{\pi\times\tilde{\pi}'}T)\frac{x^\sigma}{T}d\sigma \ll \frac{x\log^2(Q_{\pi\times\tilde{\pi}'}T)}{T}.$$

The same upper bound also holds for the integral on C_3. By Lemma 4.2 in [17] we can choose a so that

$$\int_{C_2} \ll \int_{-T}^{T} \frac{x^a}{|t|+1}dt \ll \frac{\log T}{x}.$$

Thus, on taking $T \gg \exp(\sqrt{\log x})$, all the three integrals on C_1, C_2, C_3 are

$$\ll x\exp(-c\sqrt{\log x}). \tag{6.7}$$

Computing the residues at $s = 1 + i\tau_0, i\tau_0$, and 0 respectively, we get

$$\operatorname*{Res}_{s=1+i\tau_0,i\tau_0,0} J(s)\frac{x^s}{s} = \frac{x^{1+i\tau_0}}{1+i\tau_0} + O(1). \tag{6.8}$$

The residues at the trivial zeros can be computed similarly to what we have done in (4.8) and (4.9), and the results are again

$$\ll x^{1-\delta}. \tag{6.9}$$

To compute the residues corresponding to nontrivial zeros, we recall that the number of zeros $\rho = \beta + i\gamma$ of $L(s, \pi\times\tilde{\pi}')$ with $|\gamma| \le t$ is $O(t\log t)$, and hence

$$\sum_{|\gamma|\le T} \frac{1}{|\rho|} \ll \log^2 T.$$

Consequently,

$$\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s} = -\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} \frac{1}{s-\rho}\frac{x^s}{s}$$

$$\ll \sum_{|\gamma|\le T}\left|\frac{x^\rho}{\rho}\right| = \Big(\sum_{\substack{|\gamma|\le T\\ \rho\in E}} + \sum_{\substack{|\gamma|\le T\\ \rho\notin E}}\Big)\frac{x^\beta}{|\rho|}, \tag{6.10}$$

where E is the set of exceptional zero in (4.4). Since $|E|\le 1$, the sum over $\rho\in E$ is again $\ll x^{1-\delta}$, which is the same as in §4. By (4.3), the sum over $\rho\notin E$ is

$$\ll x\exp\left(-c_3\frac{\log x}{2\log(Q_{\pi\times\tilde{\pi}'}T)}\right)\sum_{|\gamma|\le T}\frac{1}{|\rho|} \ll x\exp(-c\sqrt{\log x}), \tag{6.11}$$

by taking $T=\exp(\sqrt{\log x})+d$ for some d with $0<d<1$. Hence (6.10) is bounded by $x\exp(-c\sqrt{\log x})$. Collecting (6.6)-(6.11), we complete the proof of Theorem 2.3. □

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