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A Comparison Theorem for f-vectors of Simplicial Polytopes

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Dedicated to Robert MacPherson on the occasion of his 60th birthday

Abstract: Let $f_i(P)$ denote the number of i-dimensional faces of a convex polytope P. Furthermore, let $S(n, d)$ and $C(n, d)$ denote, respectively, the stacked and the cyclic d-dimensional polytopes on n vertices. Our main result is that for every simplicial d-polytope P, if

$$f_r(S(n_1, d)) \le f_r(P) \le f_r(C(n_2, d))$$

for some integers n_1, n_2 and r, then

$$f_s(S(n_1, d)) \le f_s(P) \le f_s(C(n_2, d))$$

for all s such that $r < s$.
For $r = 0$ these inequalities are the well-known lower and upper bound theorems for simplicial polytopes.
The result is implied by a certain "comparison theorem" for f-vectors, formulated in Section 4. Among its other consequences is a similar lower bound theorem for centrally-symmetric simplicial polytopes.

1. Introduction

The following extremal problem and its ramifications have a long tradition in the theory of convex polytopes: among all d-dimensional polytopes P with n vertices determine the maximum (or, minimum) of $f_i(P)$. The answers were given around 1970 by McMullen [5] and Barnette [1], who proved that (as had been conjectured) the upper bound is attained in all dimensions by the cyclic polytope $C(n, d)$ and the lower bound is attained in all dimensions by the stacked polytope $S(n, d)$.

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What if we specify the number of r-dimensional faces of P, for some $r > 0$, and pose the analogous extremal problem? The following can be said in general.

Theorem 1. *Let P be a d-dimensional simplicial polytope. Suppose that*

$$f_r(S(n_1, d)) \leq f_r(P) \leq f_r(C(n_2, d))$$

for some integers n_1, n_2 and $0 \leq r \leq d-2$. Then,

$$f_s(S(n_1, d)) \leq f_s(P) \leq f_s(C(n_2, d))$$

for all s such that $r < s < d$.

For $r = 0$ these inequalities are the lower and upper bound theorems of Barnette and McMullen [1], [5], [9, Ch. 8]. The $s = d-1$ case of the upper bound part is also known; it is covered by the "generalized upper bound theorem" of Kalai [4, Theorem 2].

The proof of Theorem 1 relies on a comparison theorem for f-vectors of simplicial homology spheres (Theorem 4 in Section 4) together with Stanley's proof of necessity for the g-theorem [7]. By the same technique we obtain the following. Here $CS(2n, d)$ denotes the centrally-symmetric stacked d-dimensional polytopes on $2n$ vertices.

Theorem 2. *Let P be a d-dimensional centrally-symmetric simplicial polytope. Suppose that*

$$f_r(CS(2n, d)) \leq f_r(P)$$

for some integers n and $0 \leq r \leq d-2$. Then,

$$f_s(CS(2n, d)) \leq f_s(P)$$

for all s such that $r < s < d$.

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2. Preliminaries

For the standard notions concerning convex polytopes and simplicial complexes we refer to the literature, see e.g. [9]. In this section we gather some basic definitions and recall some core results.

The *cyclic polytope* $C(n, d)$ is defined and extensively discussed in [9]. The *stacked polytope* $S(n, d)$, $n > d$, is obtained from the d-simplex by performing an arbitrary sequence of $n - d - 1$ stellar subdivisions of facets. Similarly, the *centrally-symmetric stacked polytope* $CS(2n, d)$, $2n \geq 2d$, is obtained from the d-dimensional cross-polytope by performing an arbitrary sequence of $n - d$ pairs of centrally-symmetric stellar subdivisions of facets. For $n > d + 1 > 3$ the combinatorial types of the resulting polytopes depend on choices made during the construction, but their f-vectors are well-defined.

Let Δ be a $(d-1)$-dimensional simplicial complex, and let f_i be the number of i-dimensional faces of Δ. The sequence $\mathbf{f} = (f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. We put $f_{-1} = 1$. The *h-vector* $\mathbf{h} = (h_0, \ldots, h_d)$ of Δ is defined by the equation

$$\sum_{i=0}^{d} f_{i-1} x^{d-i} = \sum_{i=0}^{d} h_i (x + 1)^{d-i}.$$

From now on we fix the integer $d \geq 3$, and let $\delta = \lfloor \frac{d}{2} \rfloor$. The *$g$-vector* of Δ is the integer sequence $\mathbf{g} = (g_0, g_1, \ldots, g_\delta)$ defined by $g_0 = 1$ and

$$g_i = h_i - h_{i-1}, \qquad i = 1, \ldots, \delta.$$

The f-vector, h-vector and g-vector of a simplicial d-polytope are those of its boundary complex.

In the case when Δ is a homology sphere (or, more generally, a psedomanifold such that the complex itself as well as the link of every face has the Euler characteristic of a sphere of the same dimension) we have the *Dehn-Sommerville equations* $h_i = h_{d-i}$, which show that the f-vector of Δ is completely determined by its g-vector. The linear relation can be expressed as a matrix product (see e.g. [2] or [9, p. 269])

$$\mathbf{f} = \mathbf{g} \cdot M_d,$$

where the $(\delta + 1) \times d$-matrix $M_d = (m_{ij})$ is defined by

$$m_{i,j} = \binom{d+1-i}{d-j} - \binom{i}{d-j}, \qquad \text{for } 0 \leq i \leq \delta,\ 0 \leq j \leq d-1.$$

Thus, the set of f-vectors of homology $(d-1)$-spheres coincides with the g-vector weighted linear span of the row vectors of M_d.

For instance, we have that

$$M_{10} = \begin{pmatrix} 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 & 11 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 9 \\ 0 & 1 & 9 & 36 & 84 & 126 & 126 & 84 & 35 & 7 \\ 0 & 0 & 1 & 8 & 28 & 56 & 70 & 55 & 25 & 5 \\ 0 & 0 & 0 & 1 & 7 & 21 & 34 & 31 & 15 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}$$

3. Nonnegativity of the M_d matrix

We need the following technical property of the matrix M_d.

Lemma 3. *All 2×2 minors of the matrix M_d are nonnegative.*

Proof. For $0 \le a < b \le \delta$ and $0 \le r < s \le d-1$, let

$$\Phi^{a,b}_{r,s} \stackrel{\text{def}}{=} m_{a,r} m_{b,s} - m_{a,s} m_{b,r}.$$

We want to show that $\Phi^{a,b}_{r,s} \ge 0$.

Let $\bar{r} \stackrel{\text{def}}{=} d-r$, $\bar{s} \stackrel{\text{def}}{=} d-s$, $\tilde{a} \stackrel{\text{def}}{=} d+1-a$ and $\tilde{b} \stackrel{\text{def}}{=} d+1-b$. Then, by definition

$$\Phi^{a,b}_{r,s} = \left[\binom{\tilde{a}}{\bar{r}} - \binom{a}{\bar{r}}\right]\left[\binom{\tilde{b}}{\bar{s}} - \binom{b}{\bar{s}}\right] - \left[\binom{\tilde{a}}{\bar{s}} - \binom{a}{\bar{s}}\right]\left[\binom{\tilde{b}}{\bar{r}} - \binom{b}{\bar{r}}\right]$$

Rearranging terms, and letting $B^{p,q}_{t,u}$ denote the binomial determinant

$$B^{p,q}_{t,u} \stackrel{\text{def}}{=} \det \begin{pmatrix} \binom{p}{t} & \binom{p}{u} \\ \binom{q}{t} & \binom{q}{u} \end{pmatrix}$$

we can write

$$\Phi^{a,b}_{r,s} = B^{a,\tilde{b}}_{\bar{s},\bar{r}} + B^{\tilde{b},\tilde{a}}_{\bar{s},\bar{r}} - B^{a,b}_{\bar{s},\bar{r}} - B^{b,\tilde{a}}_{\bar{s},\bar{r}} \tag{1}$$

Step 1. Note that

$$\det \begin{pmatrix} m_{i,t} & m_{i,u} \\ m_{j,t} & m_{j,u} \end{pmatrix} \ge 0 \quad \Leftrightarrow \quad \frac{m_{i,t}}{m_{i,u}} \ge \frac{m_{j,t}}{m_{j,u}}, \tag{2}$$

if $i < j$, $t < u$ and $m_{j,u} > 0$.

An elementary argument based on this observation shows that it suffices to prove nonnegativity of $\Phi^{a,b}_{r,s}$ for the special case when $b = a+1$.

(*Remark:* We could also reduce to the case $s = r+1$; however, this leads to no simplification in what follows.)

Step 2.

In order to show that $\Phi^{a,a+1}_{r,s} \geq 0$ we put to use the lattice-path interpretation of binomial determinants, due to Gessel and Viennot [3].

Let $L^{p,q}_{t,u}$ denote the set of pairs (P, Q) of vertex-disjoint NE-lattice paths in $\mathbb{Z}^2$, such that P leads from $(0, -p)$ to $(t, -t)$ and Q from $(0, -q)$ to $(u, -u)$. By a NE-*lattice path* we mean a path taking steps N=$(0, 1)$ to the *north* and steps E=$(1, 0)$ to the *east*.

The formula of Gessel and Viennot [3, Theorem 1] states that

$$B^{p,q}_{t,u} = \#L^{p,q}_{t,u}$$

Thus, from equation (1) we have

$$\Phi^{a,a+1}_{r,s} = \#L^{a,\tilde{a}-1}_{\overline{s},\overline{r}} + \#L^{\tilde{a}-1,\tilde{a}}_{\overline{s},\overline{r}} - \#L^{a,a+1}_{\overline{s},\overline{r}} - \#L^{a+1,\tilde{a}}_{\overline{s},\overline{r}}$$

For ease of notation we from now let $L^{p,q} \stackrel{\text{def}}{=} L^{p,q}_{\overline{s},\overline{r}}$. The proof will be concluded by producing an injective mapping

$$\varphi : L^{a,a+1} \cup L^{a+1,\tilde{a}} \to L^{a,\tilde{a}-1} \cup L^{\tilde{a}-1,\tilde{a}}$$

The construction of the mapping φ proceeds by cases.

Case 1: $(P, Q) \in L^{a,a+1}$. Then $\varphi(P, Q) \in L^{a,\tilde{a}-1}$ is constructed by keeping the path P and extending the path Q by an intitial vertical segment (a sequence of North steps) so that it begins at the point $(0, -(\tilde{a} - 1))$.

Case 2: $(P, Q) \in L^{a+1,\tilde{a}}$.

Subcase 2a: Both Q and P begin with N steps. Then $\varphi(P, Q) \in L^{a,\tilde{a}-1}$ is constructed by removing the first step from both paths.

Subcase 2b: Q begins with an E step. Then $\varphi(P, Q) \in L^{\tilde{a}-1,\tilde{a}}$ is constructed by keeping the path Q and extending the path P by an intitial vertical segment so that it originates in $(0, -(\tilde{a} - 1))$.

Subcase 2c: Q begins with an N step, and P begins with an E step. Then $\varphi(P, Q) \in L^{\tilde{a}-1,\tilde{a}}$ is constructed as follows. We may assume that $a \geq \overline{s}$, since otherwise some binomial coefficients are zero and the situation simplifies. Thus, the path P begins with a sequence of E steps, say k of them, followed by a N step. Denoting the rest of P by P' we can write: $P = E^k N P'$. Similarly, Q has the factorization $Q = NREN^vEQ'$, where the two E:s designate the k-th and $(k+1)$-st occurrences of the letter "E" in Q. See Figure 1 for the geometric idea.

The integers k and v are determined by the definition of the paths P and Q. Let h be the number of occurrences of the letter "N" in R. Let $\overline{P}$ and $\overline{Q}$ be the paths

$$\overline{P} = N^{\tilde{a}-a-h-3}ERN^2P' \quad \text{and} \quad \overline{Q} = E^kN^vEN^{h+1}Q',$$

originating in the points $(0, -\tilde{a}+1)$ and $(0, -\tilde{a})$, respectively. A straightforward inspection of the construction shows that these paths are disjoint. Namely, the lowest point on $\overline{P}$ and the highest point on $\overline{Q}$ with first coordinate k are, respectively, $(k, -a-h-2)$ and $(k, -\tilde{a}+v)$. Their distance is $\tilde{a}-a-h-v-2>0$. Let $\varphi(P,Q) = (\overline{P}, \overline{Q}) \in L^{\tilde{a}-1,\tilde{a}}$.

This defines the mapping φ in all cases. Each case separately is clearly injective. That there is no interference among the four cases, and hence that φ is injective globally, is most easily seen from following properties of the construction:

- $\varphi(P,Q) \in L^{a,\tilde{a}-1}$ in cases 1 and 2a
- $\varphi(P,Q) \in L^{\tilde{a}-1,\tilde{a}}$ in cases 2b and 2c
- $(0, -a-1) \in \varphi(Q)$ in cases 1 and 2b
- $(0, -a-1) \notin \varphi(Q)$ in cases 2a and 2c

This completes the proof. □

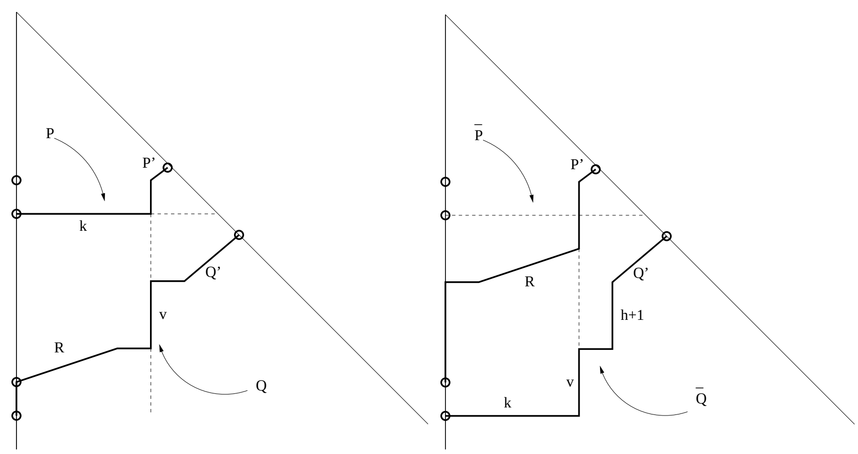

Figure 1: A sketch of subcase 2c.

Remark: We conjecture that the matrix M_d is *totally nonnegative*, meaning that all minors of all orders are nonnegative. This has been verified for all $d \leq 13$ by A. Hultman.

4. Homology spheres

A key role for this paper is played by the following comparison theorem for f-vectors of homology spheres.

Theorem 4. *Let* Δ *and* Γ *be* $(d-1)$*-dimensional simplicial homology spheres whose g-vectors for some* t $(0 \le t \le \delta)$ *satisfy*

- $g_i(\Delta) \ge g_i(\Gamma)$ *for* $i = 1, \ldots, t$
- $g_i(\Delta) \le g_i(\Gamma)$ *for* $i = t+1, \ldots, \delta$.

Suppose that

$$f_r(\Delta) \le f_r(\Gamma)$$

for some $0 \le r \le d-2$. *Then*

$$f_s(\Delta) \le f_s(\Gamma)$$

for all s *such that* $r < s < d$.

Proof. Let $v_i = g_i(\Delta) - g_i(\Gamma)$. Now,

$$0 \ge f_r(\Delta) - f_r(\Gamma) = \sum_{i=0}^{\delta} v_i m_{i,r} = \sum_{i=0}^{\delta} v_i m_{i,s} \frac{m_{i,r}}{m_{i,s}} \tag{3}$$

Lemma 3 implies, in view of equivalence (2), that

$$\frac{m_{0,r}}{m_{0,s}} \ge \frac{m_{1,r}}{m_{1,s}} \ge \cdots \ge \frac{m_{\delta,r}}{m_{\delta,s}} \ge 0$$

(*Remark:* It is possible that $m_{i,s} = 0$ for $i = k, \ldots, \delta$. Then also $m_{i,r} = 0$ for $i = k-1, \ldots, \delta$ while $m_{i,s} > 0$ for all $i < v$. This requires notational adjustments in our argument, but no new ideas.)

By assumption, the vector $v = (v_0, v_1, \ldots, v_\delta)$ satisfies

$$v_1, \ldots, v_t \ge 0 \quad \text{and} \quad v_{t+1}, \ldots, v_\delta \le 0.$$

Thus,

$$\sum_{i=0}^{\delta} v_i m_{i,s} \frac{m_{i,r}}{m_{i,s}} \ge \left(\sum_{i=0}^{t} v_i m_{i,s} \right) \frac{m_{t,r}}{m_{t,s}} + \left(\sum_{i=t+1}^{\delta} v_i m_{i,s} \right) \frac{m_{t,r}}{m_{t,s}}$$

which implies that

$$0 \ge f_r(\Delta) - f_r(\Gamma) \ge \frac{m_{t,r}}{m_{t,s}} \left(\sum_{i=0}^{\delta} v_i m_{i,s} \right) = \frac{m_{t,r}}{m_{t,s}} \left(f_s(\Delta) - f_s(\Gamma) \right)$$

It follows that

$$0 \ge f_s(\Delta) - f_s(\Gamma),$$

as desired. □

We will say that an integer vector $(n_0, \ldots, n_\delta)$ is an m-sequence if $n_0 = 1$ and $n_j \geq \binom{m}{j}$ implies that $n_{j-1} \geq \binom{m-1}{j-1}$, for all $m \geq j > 1$. In particular, if some entry in an m-sequence is positive then so are all earlier entries. The notion of m-sequence is less restrictive than the well-established concept of M-sequence, recalled in Section 5.

Corollary 5. (Upper bounds) *Let Δ be a $(d-1)$-dimensional homology sphere whose g-vector is an m-sequence. Suppose that*

$$f_r(\Delta) \leq f_r(C(n,d))$$

for some integers n and $0 \leq r \leq d-2$. Then

$$f_s(\Delta) \leq f_s(C(n,d))$$

for all s such that $r < s < d$.

Proof. The g-vector of the cyclic polytope $C(n,d)$ is

$$g_i(C(n,d)) = \binom{n-d-2+i}{i}$$

Thus, since $g(\Delta)$ is an m-sequence the conditions of Theorem 4 are satisfied. □

Stanley's upper bound theorem for homology spheres [6] shows that in the special case when $r = 0$ Corollary 5 is valid also without the assumption that $g(\Delta)$ is an m-sequence.

Corollary 6. (Lower bounds) *Let Γ be a $(d-1)$-dimensional homology sphere whose g-vector is nonnegative. Suppose that*

$$f_r(S(n,d)) \leq f_r(\Gamma)$$

for some integers n and $r \leq d-2$. Then

$$f_s(S(n,d)) \leq f_s(\Gamma)$$

for all s such that $r < s < d$.

Proof. The g-vector of the stacked polytope $S(n,d)$ is

$$g_i(S(n,d)) = \begin{cases} 1, & \text{for } i = 0 \\ n-d-1, & \text{for } i = 1 \\ 0, & \text{for } i > 1 \end{cases}$$

Thus, since $g(\Gamma)$ is nonnegative the conditions of Theorem 4 are satisfied. □

5. Polytopes

We recall the definition of an M-sequence. For any integers $k, n \geq 1$ there is a unique way of writing

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_i}{i},$$

so that $a_k > a_{k-1} > \ldots > a_i \geq i \geq 1$. Then define

$$\partial^k(n) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \ldots + \binom{a_i - 1}{i-1}.$$

Also let $\partial^k(0) = 0$.

A nonnegative integer sequence $(n_0, n_1, n_2, \ldots)$ such that $n_0 = 1$ and

$$\partial^k(n_k) \leq n_{k-1} \qquad \text{for all } k > 1$$

is called an *M-sequence*. Clearly, an M-sequence is an m-sequence (as defined in connection with Corollary 5), but not conversely.

Proof of Theorem 1. The g-vector of a simplicial polytope is an M-sequence, by the theorem of Stanley [7]. In particular, it is a nonnegative m-sequence, so both Corollaries 5 and 6 apply. □

Proof of Theorem 2. The g-vector of the centrally-symmetric stacked polytope $CS(2n, d)$ is

$$g_i(CS(n,d)) = \begin{cases} 1, & \text{for } i = 0 \\ 2n - d - 1, & \text{for } i = 1 \\ \binom{d}{i} - \binom{d}{i-1}, & \text{for } i > 1 \end{cases}$$

Stanley [8] has shown that

$$g_i(P) \geq \binom{d}{i} - \binom{d}{i-1}, \text{ for } i \geq 1$$

holds for every centrally-symmetric simplicial polytope P. Hence, Theorem 4 applies.

□

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