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On the Fibres of a Toroidal Resolution

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To R. MacPherson

Introduction

The quotient $M = \Gamma\backslash D$ of a Hermitian symmetric domain by an arithmetic group of automorphisms is a quasiprojective algebraic variety. Varieties like this, variously called *modular* or *locally symmetric* or *arithmetic varieties*, play an important role in representation theory and arithmetic. Many naturally occurring arithmetic varieties are noncompact, and the study of their compactifications has a long history. The variety M has a canonical embedding in a projective space given by certain automorphic forms for Γ (essentially sections of a power of the canonical bundle); the closure M^* in this embedding is the *minimal compactification* of Satake and Baily-Borel. It is a normal variety which usually has complicated singularities at the boundary and any smooth compactification in which M is the complement of a normal-crossings divisor dominates it. There is no canonical smooth compactification of M in general, but in [1], Mumford et al. showed how to desingularize M^* using an extra choice. Given a suitable Γ-admissible rational polyhedral cone decomposition Σ (the notion is recalled in §§2, 4), the method produces a smooth projective *toroidal compactification* M^Σ in which the complement of M is a divisor with simple normal crossings. There is a morphism

$$\pi : M^\Sigma \to M^*$$

extending the identity on M. It is of interest in various questions to study the fibres of π. In this paper I want to describe a homological property of these fibres when the $\mathbb{Q}$-rank of M is one (so that $M^* - M$ is the quotient of a smooth variety by a finite group). I shall assume always that Γ is neat (which can always be achieved by passing to a subgroup of finite index), so that $M^* - M$ is smooth. In this introduction it will be further assumed that $M^* - M$ consists of points (i.e. D is a $\mathbb{Q}$-rank one tube domain with cusps). Examples include Hilbert modular varieties and arithmetic varieties associated to $\mathbb{Q}$-rank one forms of $\mathrm{SO}(2, n)$.

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A fibre of $\pi : M^{\Sigma} \to M^*$ at a cusp $s \in M^* - M$ is the geometric quotient

$$\pi^{-1}(s) = \Lambda \backslash \mathcal{D}$$

of a certain scheme $\mathcal{D}$ by a free action of an infinite discrete group Λ. The scheme $\mathcal{D}$ is not usually of finite type, but each irreducible component is smooth and of finite type, and $\mathcal{D}$ has normal crossings. There is a natural ind-variety structure on $\mathcal{D}$ (as we show here). The simplest example is that of Hilbert modular surfaces, where the preimage of a cusp of M^* is a cycle of rational curves: the quotient of an infinite chain of projective lines joined "end-to-end" (our $\mathcal{D}$) by an infinite cyclic group (our Λ). In general, there is a morphism $\mathcal{D} \to A$ to an abelian variety which on each irreducible component of $\mathcal{D}$ restricts to a smooth projective toric fibration over A. The group Λ is an arithmetic subgroup of the automorphism group of a self-adjoint homogeneous cone C (e.g. Λ could be $\mathbb{Z}^d$ or $\mathrm{SL}(n, \mathbb{Z})$ or an arithmetic group in $\mathrm{SO}(n, 1)$).

The ind-variety $\mathcal{D}$ comes with an embedding in a certain smooth scheme $\mathcal{Y} \to A$ locally of finite type; indeed, $\mathcal{Y}$ is a relative torus embedding of a relative torus $\mathcal{T} \to A$ using the cone system Σ, and $\mathcal{D}$ is the complement of the dense torus $\mathcal{T}$ in $\mathcal{Y}$. The action of Λ comes from one on $\mathcal{Y}$, but it is not proper, so that the quotient $\Lambda \backslash \mathcal{Y}$ exists only as an analytic space and not as a scheme. There is an analytic map $\Lambda \backslash \mathcal{Y} \to M^{\Sigma}$ identifying $\Lambda \backslash \mathcal{D}$ with the fibre and it is an analytic isomorphism locally along $\Lambda \backslash \mathcal{D}$. (Indeed, M^{Σ} is actually constructed in [1] by gluing together certain analytic neighbourhoods of $\Lambda \backslash \mathcal{D}$ in $\Lambda \backslash \mathcal{Y}$ for the various cusps of M^*.) The main result of this note is that $\mathcal{D} \hookrightarrow \mathcal{Y}$ is a homology isomorphism (cf. Thm 3.2 when A is trivial, Thm 4.1 in general), i.e.

$$\mathrm{H}_*(\mathcal{D}) = \mathrm{H}_*(\mathcal{Y}).$$

Since Λ acts freely on $\mathcal{D}$, it follows that there is a natural isomorphism

$$\mathrm{H}_*(\Lambda \backslash \mathcal{D}) = \mathrm{H}_*^{\Lambda}(\mathcal{Y})$$

giving a description of the homology of the fibre as the Λ-equivariant homology of the smooth scheme $\mathcal{Y}$. This suggests that the "quotient" of the smooth scheme $\mathcal{Y}$ by Λ should be thought of as an algebraic version of a regular neighbourhood of the fibre in M^{Σ}. A consequence of the isomorphism $\mathrm{H}_i(\mathcal{D}) \cong \mathrm{H}_i(\mathcal{Y})$ is that this group is pure of weight $-i$, i.e. the ind-variety $\mathcal{D}$ is pure. (This is Corollary 3.2 when A is trivial; note that this is only obvious in the simplest case of Hilbert modular surfaces.) Roughly speaking, $\mathcal{D}$ satisfies the valuative criterion for properness, so that $\mathrm{H}_i(\mathcal{D})$ has weights $\geq -i$, while $\mathcal{Y}$ is smooth, so that $\mathrm{H}_i(\mathcal{Y})$ has weights $\leq -i$. This purity has pleasant consequences: the spectral sequence computing $\mathrm{H}_*(\Lambda \backslash \mathcal{D}) = \mathrm{H}_*^{\Lambda}(\mathcal{Y})$ in terms of Λ-homology of $\mathrm{H}_*(\mathcal{D}) \cong \mathrm{H}_*(\mathcal{Y})$ degenerates at E_2 and the limit filtration is (with a shift) the weight filtration (Cor. 3.4, Thm 4.1). So one has an expression for the graded pieces of the weight filtration on the fibre homology in terms of group homology and $\mathcal{Y}$. ([1])

[1]There is a formal resemblance between the fibres studied here and the varieties appearing in [5] in the context of the "fundamental lemma": In each case one has a quotient of a ind-variety by a discrete group. Here, the discrete group Λ (which is typically like $\mathrm{GL}(n, \mathbb{Z})$) is complicated, while the ind-variety $\mathcal{D}$ is

Another consequence of purity (to which attention was drawn in [4]) is that the torus action on $\mathcal{D}$ is equivariantly formal. It follows that the equivariant cohomology is localized to the fixed-point locus and has a description in terms of the fixed point locus and one-dimensional (over A) orbits. This is mentioned in 3.6 and 4.5, a fuller treatment being left for another occasion. The purity statement also has consequences for the Hodge theory of the exceptional divisor $\pi^{-1}(M^* - M)$, which we mention in 4.6.

The essential tool in proving these results is an enumeration of the top-dimensional cones in Σ. The cone system Σ is a decomposition of the open self-adjoint homogeneous cone C into rational polyhedral cones. For example, if M is a Hilbert modular surface associated to a real quadratic field F, the cone C is the convex hull in $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^2$ of the set of totally positive elements of F and Σ is a decomposition of $C \cong \mathbb{R}^2_+$ into rational sectors, invariant under the action of a subgroup $\Lambda \cong \mathbb{Z}$ of the group of totally positive units of F. In general, the fan Σ consists of cones on the faces of an unbounded locally polyhedral convex set P (a Λ-polyhedral cocore in the terminology of [1, Chp. II]), the convexity of P being responsible for the projectivity of M^{Σ}. The idea of line shellings from the theory of convex polytopes allows us to enumerate the facets of P, and hence the top-dimensional cones of Σ, in a nice way (Prop. 2.5). This gives the ind-variety structure of $\mathcal{D}$ and defines filtrations of $\mathcal{D}$ and $\mathcal{Y}$ which allow for an inductive proof that $\mathrm{H}_*(\mathcal{D}) \cong \mathrm{H}_*(\mathcal{Y})$ using elementary facts about the topology of torus embeddings.

The methods used here to study the fibres of π are applicable if the $\mathbb{Q}$-rank is ≥ 2, but the results have to be reformulated (cf. 3.10 for some comments).

The contents of this article are as follows: In §1 the necessary properties of self-adjoint homogeneous cones are recalled from [1]. In §2 the construction of nice polyhedral decompositions Σ of such cones is recalled, following [1], and the enumeration of top-dimensional cones of such Σ mentioned above is proved. (No assumption is made on the $\mathbb{Q}$-rank in these sections, but the simplifications in the case of $\mathbb{Q}$-rank one are indicated.) From §3 we assume that the $\mathbb{Q}$-rank is one. In §3 the enumeration from §2 is used to prove the main theorem when the abelian variety A is trivial. Then weights are brought in and various consequences of purity indicated, under the same assumption. General $\mathbb{Q}$-rank one arithmetic varieties (where A is nontrivial) are treated in §4.

It is a pleasure to dedicate this article to R. MacPherson on his sixtieth birthday. I hope he finds the mix of topics – convexity, toric geometry, arithmetic groups – appealing.

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simple (its homology is pure). In [5], the discrete group is simple ($\cong \mathbb{Z}^d$), but the ind-variety, an affine Springer fibre, is complicated. In particular, the conjecture that they are pure appears to be difficult.

1. Self-adjoint homogeneous cones

This section contains background material on cones and their geometry. The key fact used later is Lemma 1.2.

1.1. **Self-adjoint homogeneous cones.** We start a real vector space V of finite dimension N and a nondegenerate open cone

$$C \subset V$$

i.e. an open subset invariant under dilations by $\mathbb{R}_+$ which contains no line in its closure. The (connected) automorphism group of the cone is

$$\mathrm{Aut}(C)^0 := \{g \in \mathrm{GL}(V) : gC = C\}^0.$$

The cone is *homogeneous* if $\mathrm{Aut}(C)^0$ acts transitively on C. It is *self-adjoint* if there is an inner product $\langle\, ,\rangle$ on V such that C is identified with its dual cone $C^* = \{v \in V : \langle v, w\rangle > 0 \ \forall w \in \overline{C}, w \neq 0\}$. A self-adjoint cone is convex.

Let $C \subset V$ be self-adjoint and homogeneous. The automorphism group is a Lie subgroup of $\mathrm{GL}(V)$, stable under transpose, and therefore reductive. The isotropy subgroup of any point in C is a maximal compact subgroup, and so C is identified with the symmetric space of $\mathrm{Aut}(C)^0$. (Conversely, if a connected reductive Lie group acts on a vector space with an open orbit at points of which the isotropy subgroups are maximal compact, then the orbit is a self-adjoint homogeneous cone.)

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{Aut}(C)^0$. Fix a basepoint $e \in C$ and let K be the isotropy subgroup at e. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ given by K fixes an isomorphism

$$\mathfrak{p} \to V \qquad \text{by } X \mapsto Xe$$

(here $X \in \mathfrak{g}$ acts on V via $\mathfrak{g} \subset \mathrm{End}(V)$). For a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ let $A = e^{\mathfrak{a}}$. Then $C = KAe$.

Let $V_{\mathbb{Q}} \subset V$ be a rational structure. The cone C is *rational* if there is a $\mathbb{Q}$-algebraic group

$$G \subset \mathrm{GL}(V_{\mathbb{Q}}) \qquad \text{with } G(\mathbb{R})^0 = \mathrm{Aut}(C)^0,$$

i.e. the automorphism group is, up to connected components, the real points of a $\mathbb{Q}$-algebraic group. This will be assumed to be the case from now on, and we will fix, once and for all, a rational basepoint $e \in C \cap V_{\mathbb{Q}}$. It is known that the inner product on V (with respect to which C is self-dual) may be chosen to be rational.

For $k = \mathbb{Q}$ or $\mathbb{R}$, a cone is *k-irreducible* if it cannot be written as a sum $C = C_1 + C_2$ for a k-rational decomposition $V = V_1 \oplus V_2$ and k-rational self-adjoint homogeneous cones $C_i \subset V_i$. An equivalent condition is that V is irreducible for G over k (p. 87 of [1]). In this situation the k-split centre of G is one-dimensional and acts by dilations on V. Any k-rational cone C is a sum of irreducible ones. The $\mathbb{R}$-irreducible cones have been classified (cf. [1, p. 63] for the list, which includes four infinite families plus one exceptional cone).

1.2. **Jordan algebra.** A Jordan algebra is a finite-dimensional commutative algebra in which the product $a \cdot b$ satisfies $a^2 \cdot (b \cdot a) = (a^2 \cdot b) \cdot a$. It is Euclidean if $a^2 + b^2 = 0$ implies that $a = b = 0$.

It is a basic fact ([1, p. 76]) that for a rational self-adjoint homogeneous cone $C \subset V$ as in the previous paragraph, there is a Euclidean Jordan algebra structure on V, with e as unit, which is rational over $\mathbb{Q}$. Let $T_a \in \mathfrak{p}$ be the element mapped to $a \in V$ under $\mathfrak{p} \cong V$, i.e. $T_a e = a$. The Jordan algebra structure satisfies $T_a(b) = a \cdot b$ for all $b \in V$. The cone is $C = \{x^2 : x \in V \text{ is invertible}\}$ and its closure is $\overline{C} = \{x^2 : x \in V\}$. The inner product in V can be taken to be $\langle x, y\rangle$= trace of left multiplication by $x \cdot y$ on V. (Note that this is $\mathbb{Q}$-rational.)

Two idempotents $e_1, e_2 \in V$ (for the Jordan algebra structure), are *orthogonal* if $e_1 \cdot e_2 = 0$. An idempotent is *minimal* if it cannot be written as the sum of mutually orthogonal idempotents. A collection of idempotents $e_1, \ldots, e_r$ is *complete* if $\sum_i e_i = e$. A maximal collection of mutually orthogonal idempotents is complete. Let $\{e_i\}$ be such a collection. Let $x_i \in \mathfrak{p}$ be defined by $x_i = 2T_{e_i}$, (i.e. $x_i e = 2e_i$ in the action of $\mathfrak{p}$ on V). Then $\mathfrak{a} = \sum_i \mathbb{R}x_i$ is a maximal abelian subspace of $\mathfrak{p}$, and every such subspace arises from a complete set of mutually orthogonal minimal idempotents in this way (cf. p. 89 of [1]). From the description of the cone as invertible squares, one sees that $C \cap \sum_i \mathbb{R}e_i = Ae = \sum_i \mathbb{R}_+ e_i$. In other words, the orbit Ae is open in its linear span, which is $\mathfrak{a}e \subset V$, and $\mathfrak{a}e \cap C = Ae$ (cf. Prop. 14 on p. 104 of [1]).

Fix a maximal collection of mutually orthogonal idempotents $e_1, \ldots, e_r$. Let $x_1^*, \ldots, x_r^*$ be the basis of $\mathfrak{a}^*$ dual to the corresponding basis $x_1, \ldots, x_r$ of $\mathfrak{a}$. They will be considered as characters on A via the logarithm isomorphism $\log : A \to \mathfrak{a}$, i.e. $x_i^*(e^y) = x_i^*(y)$ for $y \in \mathfrak{a}$.

Lemma 1.1. $\det|_A = (x_1^*)^{\lambda_1}(x_2^*)^{\lambda_2} \ldots (x_r^*)^{\lambda_r}$ *where* $\lambda_i > 0$ *for all* i.

Proof. First assume that C is $\mathbb{R}$-irreducible. The space V has a (Peirce) decomposition

$$V = \bigoplus_{1 \le i \le j \le r} V_{ij}$$

where

$$V_{ii} = \{v \in V : e_i \cdot v = v\}$$
$$V_{ij} = \{v \in V : e_i \cdot v = e_j \cdot v = v/2\}.$$

If $k \neq i, j$ then $e_k \cdot v = 0$ for $v \in V_{ij}$. For $i \neq j$, the dimension of V_{ij} is a number d which depends only on the cone, i.e. it is independent of i and j; the dimension of V_{ii} is one. (These facts are proved in pp. 92–97 of [1].) This gives $N = r + dr(r-1)/2$. The trace of left multiplication by e_i on V is then $1 + d(r-1)/2 = N/r$. Since $x_i = 2T_{e_i}$ we have $\det(e^{x_i}) = e^{2\mathrm{tr}(T_{e_i})} = e^{2N/r}$. Then $\det|_A = (x_1^* x_2^* \dots x_r^*)^{2N/r}$. The general case follows by writing the cone as a sum of irreducible cones and applying this calculation to each one. □

Let

$$G_1 := \ker(\det|_G).$$

This is a rational algebraic group, and the symmetric space of $G_1(\mathbb{R})^0$ is the quotient $C/\mathbb{R}_+$ of C by dilations. The subgroup $A_1 \subset G_1(\mathbb{R})^0$ (i.e. the connected real points of a maximal $\mathbb{R}$-split torus in G_1) is the subgroup $\ker(\det|_A)$ of A, for which the lemma gives the explicit equation $\prod_i t_i^{\lambda_i} = 1$ (in the preferred coordinates on A). Similarly, $\mathfrak{a}_1 \subset \mathfrak{a}$ is given by $\Sigma_i \lambda_i t_i = 0$.

1.3. **Sublevel sets of the characteristic function.** Fix a translation-invariant measure dy on V. The *characteristic function* $\varphi : C \to \mathbb{R}_+$ of the cone is defined by

$$\varphi(x) := \int_C e^{-\langle x,y\rangle}\, dy.$$

It is canonical up to the choice of measure dy, i.e. up to a constant. Let us normalize it by requiring $\varphi(e) = 1$. The following properties of φ are found in [1, p. 57ff]:

(i) $\varphi(gx) = \det(g)^{-1}\varphi(x)$ for $g \in G(\mathbb{R})^0, x \in C$. In particular, $\varphi(\lambda x) = \lambda^{-N}\varphi(x)$ for $\lambda \in \mathbb{R}_+$. It follows that $\varphi(x)\,dx$ is a $G(\mathbb{R})^0$-invariant measure on C, φ is $G_1(\mathbb{R})^0$-invariant, and its level sets are $G_1(\mathbb{R})^0$-orbits. By our normalization we have $\varphi(ge) = \det(g)^{-1}$ for $g \in G_1(\mathbb{R})^0$.

(ii) $\varphi : C \to \mathbb{R}_+$ is strictly convex, i.e. $\varphi(t\,x + (1-t)\,y) < t\,\varphi(x) + (1-t)\,\varphi(y)$ for $x, y \in C, t \in (0,1)$. In particular its sublevel sets are convex.

Fix a point $y \in C$ and consider its $G_1(\mathbb{R})^0$-orbit:

$$S := G_1(\mathbb{R})^0\, y$$

It is a level set of φ (by (i)) and $\mathbb{R}_{\geq 1}S$ is a sublevel set of φ. By property (ii), $\mathbb{R}_{\geq 1}S$ is a closed convex set in V with boundary S and interior $\mathbb{R}_{>1}S$.

Let $x \in C - \mathbb{R}_{\geq 1}S$. A point $s \in S$ is *visible from* x if the tangent hyperplane T_sS separates x from $\mathbb{R}_{>1}S$. (The tangent hyperplane is being thought of as an affine hyperplane in V.) The set $\Omega_x := \{s \in S : s \text{ is visible from } x\}$ is open in S and invariant by $K(x) = \mathrm{Stab}_{G(\mathbb{R})^0}(x)$. If $x = gy$ for $g \in G(\mathbb{R})^0$, then $\Omega_x = g\Omega_y$.

Lemma 1.2. *The set of points in S visible from $x \in C - \mathbb{R}_{\geq 1}S$ is relatively compact.*

Proof. Let $s = \mathbb{R}x \cap S$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition given by $s \in C$, let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, and let $A = e^{\mathfrak{a}}$. Then $x \in (0, s] \subset \mathfrak{a}s$. The set of points visible from $x \in (0, s]$ is $K(x)\,B$ where B is the set of points in $\mathfrak{a}s \cap \mathbb{R}_{\geq 1}S$ visible from x inside the subspace $\mathfrak{a}s$. It suffices to show that B is relatively compact. Using the coordinates on $\mathfrak{a}s$ given by $\mathfrak{a} \cong \mathfrak{a}s$ and the basis $x_1, \dots, x_r$ of $\mathfrak{a}$, plus the fact that $\mathfrak{a}s \cap S = e^{\mathfrak{a}_1} s$, this reduces (by (i) and Lemma 1.1) to the following assertion in $\mathbb{R}^r$: If $B = \{(t_1, \dots, t_r) \in (\mathbb{R}_+)^r : t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_r^{\lambda_r} = c\}$ and all λ_i are > 0, then for any $t \in (\mathbb{R}_+)^r, t \notin \mathbb{R}_{\geq 1} B$, the set of points on B visible from t is relatively compact. This is easily checked by a direct calculation. □

(It is easy to see from the proof that the set of points visible from x is geodesically starlike around the point $\mathbb{R}x \cap S$ and has nonempty interior, and is in fact homeomorphic to a ball.)

1.4. **Examples.** (i) Let $F/\mathbb{Q}$ be totally real of degree d and $G = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$, acting on $V_\mathbb{Q} = F$. Consider the embedding $F \hookrightarrow V = F \otimes_\mathbb{Q} \mathbb{R} = \mathbb{R}^d$ given by the various real embeddings of F. The cone C is the convex hull of the set of totally positive elements of F in this embedding. The subgroup G_1 is the norm torus $G_1 = \ker(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \stackrel{\mathrm{Nm}}{\to} \mathbb{G}_m)$, which is $\mathbb{Q}$-anisotropic of rank $d - 1$. In suitable coordinates $C \cong \mathbb{R}_+^d$ and $\varphi(t_1, \dots, t_d) = (t_1 \dots t_d)^{-1}$.

(ii) Let $G = \mathrm{GO}(q)$ be the similitude group of a nondegenerate quadratic form $q : V_\mathbb{Q} \to \mathbb{Q}$ such that $q_\mathbb{R}$ is of signature $(n, 1)$. The cone C is one component of $\{v \in V : q(v) > 0\}$, $G_1 = \mathrm{SO}(q)$, and $\varphi(v) = q(v)^{-(n+1)/2}$ for $v \in C$.

(iii) If $G = \mathrm{GL}(n)$ and $V = \{X \in \mathrm{M}(n, \mathbb{R}) : X^t = X\}$ with the action $(g, X) \mapsto g^t X g$, then the orbit of the identity matrix is the cone C of positive definite symmetric matrices, equivalently, the set of positive definite quadratic forms in n variables. Here $G_1 = \mathrm{SL}(n)$ and $\varphi(X) = \det(X)^{-(n+1)/2}$.

2. Polyhedral cone decompositions

In this section polyhedral cone decompositions associated to cocores are introduced (following Ash's Chapter I in [1]) and their top-dimensional cones are enumerated using the notion of visibility.

2.1. Let V, G, C etc. be as in §1. Let $L \subset V_\mathbb{Q}$ be a free $\mathbb{Z}$-module with $L \otimes_\mathbb{Z} \mathbb{R} = V$ and let $\Lambda_0 = \{g \in G(\mathbb{R}) : gL = L\}$. A subgroup of $G(\mathbb{R})$ is *arithmetic* (for the given $\mathbb{Q}$-structure) if it is commensurable with Λ_0. Let Λ be an arithmetic subgroup of $G(\mathbb{R})^0$ which is neat (i.e. the subgroup of $\mathbb{C}^*$ generated by eigenvalues of elements of Λ is torsion-free). Let $L^\times = L - \{0\}$.

A *convex polytope* in V is the convex hull of a finite set of points. It is *rational* if the points are in $V_\mathbb{Q}$. A *polyhedral cone* is a closed convex cone in V of the form $\sigma = \{x \in V : \xi_i(x) \geq 0 \text{ for } i = 1, \dots, r\}$ for some linear functionals $\xi_1, \dots, \xi_r \in V^*$. It is *rational* if the linear functionals may be chosen in $V_\mathbb{Q}^*$.

2.2. **Boundary components.** The relation in 1.2 between complete sets of mutually orthogonal Jordan idempotents in V and maximal split tori in G works over $\mathbb{Q}$: a complete set of mutually orthogonal rational idempotents determines a maximal $\mathbb{Q}$-split torus in G and vice versa (cf. [1, p. 90]). The maximal cardinality of a set of mutually orthogonal rational idempotents, called the $\mathbb{Q}$-rank of C, is therefore the same as the $\mathbb{Q}$-rank of G. So C has $\mathbb{Q}$-rank one if and only if $\Lambda\mathbb{R}_+\backslash C$ is compact.

Let C_+ be the convex hull of $\overline{C} \cap V_{\mathbb{Q}}$. For a rational idempotent e_1, let $V(e_1) = \{x \in V : x \cdot e_1 = x\}$; this is a rational Jordan subalgebra of V. The cone of invertible squares $C(e_1) = \{x^2 : x \in V(e_1) \text{ invertible}\}$ is a self-adjoint homogeneous cone in $V(e_1)$ and $C(e_1) \subset C_+$. The cones $C(e_1) \subset C_+$ as e_1 varies over rational idempotents are the *rational boundary components* of C. Distinct rational boundary components are disjoint, and the union of all rational boundary components is $C_+ - \{0\}$ (cf. Remark 3 on p. 133 of [1]). Note that $C_+ = C \cup \{0\}$ if and only if C has $\mathbb{Q}$-rank one.

2.3. **Examples.** Consider the examples in 1.4:

(i) $G = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$, $C \cong \mathbb{R}_+^d$ the convex hull in $V = F \otimes \mathbb{R}$ of the totally positive elements of F. This cone is of $\mathbb{Q}$-rank one and $C_+ = C \cup \{0\}$. An arithmetic group is a subgroup of the group of totally positive units in the ring of integers of F, so if it neat then it is free abelian of rank $d - 1$.

(ii) $G = \mathrm{GO}(q)$ and C is a component of $\{q(v) > 0\}$. The rational boundary components are half-lines $\mathbb{R}_+$, one for each $\mathbb{Q}$-rational q-isotropic line in V. If there are no such lines C is of $\mathbb{Q}$-rank one; otherwise it is of $\mathbb{Q}$-rank two.

(iii) $G = \mathrm{GL}(n)$ and C the cone of positive definite quadratic forms in n variables. The cone is of $\mathbb{Q}$-rank n (for the standard $\mathbb{Q}$-structure) and C_+ is the set of positive semidefinite quadratic forms in n variables with rational nullspace. An arithmetic group in $\mathrm{GL}(n, \mathbb{R})$ is a group commensurable with $\mathrm{GL}(n, \mathbb{Z})$.

2.4. **Kernels, cores, cocores and polarization functions.** A closed convex subset $K \subset \overline{C}$ is a *kernel* if $0 \notin K, C \subset \mathbb{R}_+K$, and $\mathbb{R}_{\geq 1}K \subset K$. (So K does not contain 0 and every ray in C is eventually in K.) (This is a slight departure from [1], where neither closure nor convexity is required of a kernel.) Two kernels K_1 and K_2 are *comparable* if for some $\lambda_1, \lambda_2 \in \mathbb{R}_+$ we have $\lambda_1 K_1 \subset K_2 \subset \lambda_2 K_1$.

The closed convex hull of $C \cap L$ is a kernel. Any kernel comparable with it is called a *core*. (For some other comparable kernels cf. p. 120 of [1].)

The closed convex hull of $\overline{C} \cap L^\times$ is also a kernel. Any kernel comparable with it is called a *cocore*.

(Note that if $\mathbb{Q}$-rank$(C) = 1$ there is no difference between cores and cocores since $\overline{C} \cap L^\times = C \cap L$.)

There is a duality on kernels which exchanges cores and cocores [1, p. 128]. For a set $K \subset V$ define

$$K^\vee := \{x \in V : \langle x, y\rangle \geq 1 \text{ for all } y \in K\}.$$

If K is a kernel for C then so is $K^\vee$, and $K^{\vee\vee} = K$. If K is a core then $K^\vee$ is a cocore.

A kernel K is Λ-*polyhedral* if

(i) $\gamma K = K$ for all $\gamma \in \Lambda$
(ii) K is *locally rationally polyhedral (l.r.p.)*, i.e. for any rational polyhedral cone Π with vertices in C_+, $\Pi \cap K$ is cut out of Π by a finite number of supporting hyperplanes of K.

The basic examples of such kernels are the closed convex hulls of $C \cap L$ and $\overline{C} \cap L^\times$ mentioned above (which are shown to be l.r.p. on p. 142 of [1]).

If K is a Λ-polyhedral kernel then $K^\vee$ is a Λ^t-polyhedral kernel (cf. p. 141 of [1]). (Here $\Lambda^t = \{\gamma^t : \gamma \in \Lambda\}$, an arithmetic group in $G^t = G$).

A Λ-*invariant polarization function* on C_+ is a continuous piecewise-linear function $\phi : C_+ \to \mathbb{R}_{\geq 0}$ such that:

(i) ϕ is convex: $\phi(x + y) \geq \phi(x) + \phi(y)$ for $x, y \in C_+$
(ii) $\phi(x) > 0$ if $x \neq 0$
(iii) ϕ takes integer values on $C_+ \cap L$
(iv) ϕ is Λ-invariant

(The notion of convex in (i) is the opposite of the notion of convex used in §1 for the characteristic function φ; (i) is the standard convention in the torus embedding game.) Such a function determines a Λ-polyhedral cocore $P := \{x \in C_+ : \phi(x) \geq 1\}$.

Conversely, given a Λ-polyhedral cocore P, there is a Λ-invariant polarization function ϕ such that $P = \{\phi \geq k\}$ for some large integer k. (Define ϕ to be the unique convex piecewise-linear function which takes the value k on each face of the cocore and is linear on the cone over each face (cf. p. 310 of [1]). Properties (i), (ii) and (iv) are easy. To get (iii) note that the extreme points of P lie in $\frac{1}{M}L$ for some integer M [1, p. 136]. The lattice generated by these points is of finite index in $\frac{1}{M}L$, so by choosing k suitably we may assume that ϕ is integral on $C_+ \cap \frac{1}{M}L$. Then (iii) holds.) Note that the maximal cones on which ϕ is linear are the cones over the faces of P.

(For a picture of a cocore P in the case of Example (i) of 1.4 with $F = \mathbb{Q}(\sqrt{3})$ see p. 52 of [1].)

2.5. **Facets and visibility.** Let P be a closed convex l.r.p. subset of V of dimension N. Recall that a *face* of P is a subset $f \subset P$ such that every closed line segment whose relative interior meets f lies entirely in f. A point $p \in P$ is an *extreme point* if $\{p\}$ is a face. Let $E(P)$ denote the set of extreme points. A *facet* of P is a face of dimension

$N-1$ (facets exist since P is l.r.p.). For a facet F denote by $\mathrm{aff}(F)$ its affine hull; then $\mathrm{aff}(F) \cap P = F$.

Lemma 2.1. *Let P be an l.r.p. kernel. The unit inward normal to a facet belongs to $\overline{C}$.*

Proof. Let v_F be the unit inward (i.e. pointing into P) normal vector to the facet F of P. Suppose that $v_F \notin \overline{C}$. Let $\xi : V \to \mathbb{R}$ be defined by $\xi(x) = \langle x, v_F \rangle$. Then $C - v_F^{\perp}$ has two connected components C_1 and C_2, where $\xi|_{C_1} < 0$ and $\xi|_{C_2} > 0$. Let $w \in F$. Then $w + v_F^{\perp} = \mathrm{aff}(F)$ is a supporting hyperplane of P, so $\xi(x) \geq \xi(w)$ for $x \in P$. On the other hand, $\xi|_{C_1}$ is unbounded below, so that there exists $y \in C_1$ such that $\xi(y) < \xi(w)$. Since $\xi(y) < 0$, one has $\xi(\lambda y) = \lambda \xi(y) \leq \xi(y) < \xi(w)$ for $\lambda \geq 1$. Thus $\mathbb{R}_{\geq 1} y \cap P = \phi$. But P is a kernel, so any ray in C must eventually be contained in P. This is a contradiction, showing that $v_F \in \overline{C}$. □

For a facet F let $\mathrm{aff}(F)_+$ be the closed affine halfspace containing P and bounded by $\mathrm{aff}(F)$. For $x \in V - P$, a facet F of P is *visible from* x if $\mathrm{aff}(F)$ separates x from the interior of P, equivalently if $x \notin \mathrm{aff}(F)_+$. A facet is *relevant* if it is visible from 0. The set of relevant facets is Λ-invariant. (When the $\mathbb{Q}$-rank of C is one every facet is relevant, cf. Remark 2.7(i) below.)

Lemma 2.2. *Let P be a Λ-polyhedral cocore, F a facet of P, and v_F its unit inward normal. Then F is relevant if and only if $v_F \in C$.*

Proof. For any affine hyperplane H meeting C, $\overline{C} - H \cap \overline{C}$ has two components. If either normal to H is in C then exactly one component is bounded, the one containing 0. Suppose F is irrelevant. Then $\overline{C} - \mathrm{aff}(F)$ has two components, one of which contains 0 in its closure and also contains the interior of P. If $v_F \in C$ then this component is bounded. But P is unbounded. So $v_F \notin C$.

By Remark 1 on p. 132 of [1], if $y \in \overline{C} - C$ then $\inf_{x \in \overline{C} \cap L^{\times}} \langle y, x \rangle = 0$. Since P is a cocore, there exists $\lambda \in \mathbb{R}_+$ such that $P \supset \lambda(\overline{C} \cap L^{\times})$, and so $\inf_{x \in P} \langle y, x \rangle = 0$. It follows that a facet F with $v_F \in \overline{C} - C$ must actually lie in $v_F^{\perp}$. But then $0 \in v_F^{\perp} = \mathrm{aff}(F)$, so F is irrelevant. □

(The lemma fails for cores if $\mathbb{Q}$-rank$(C) \geq 2$, as the example on p. 135 of [1] shows. Let $V_{\mathbb{Q}} = \mathbb{Q}^2, C = \mathbb{R}_+^2$ and P the closed convex hull of $\mathbb{Z}^2 \cap C = \mathbb{Z}_+^2$. There are facets which are visible from 0 but have normal vector in $\overline{C} - C$.)

Lemma 2.3. *Let H_+ be an affine halfspace in V with bounding hyperplane H such that $H \cap C \neq \phi$. Suppose that the inward normal vector v of H_+ belongs to C. Then for any $x \in C - H_+$ the set $\{\gamma \in \Lambda : x \notin \gamma H_+\}$ is finite.*

Proof. If a sequence of points x_i in C tends to a limit in $\overline{C} - C$ then the sequence of values $\varphi(x_i)$ of the characteristic function tends to infinity (Prop. 3 on p. 60 of [1]). Therefore the function $\varphi|_{C \cap H}$ extends continuously to $\varphi : \overline{C} \cap H \to (0, \infty]$ by setting $\varphi(x) = \infty$ for $x \in (\overline{C} - C) \cap H$. Since $v \in C$ and H is a translate of $v^\perp$, $\overline{C} \cap H$ is compact. So φ takes its minimum value at some point $s \in C \cap H$. Since φ is strictly convex, this point is unique. The orbit $S = G_1(\mathbb{R})^0 \cdot s = \varphi^{-1}(\varphi(s))$ has tangent hyperplane H at s. For $g \in G_1(\mathbb{R})^0$, $x \notin gH_+$ if and only if $gs \in S$ is visible from x. The set of such gs is relatively compact by Lemma 1.2, and so the set of such g is relatively compact, i.e. $\{g \in G_1(\mathbb{R})^0 : x \notin gH_+\}$ is relatively compact. Since $\Lambda \subset G_1(\mathbb{R})^0$ is discrete, the intersection $\Lambda \cap \{g : x \notin gH_+\}$ is finite. □

The following is a key fact:

Lemma 2.4. *Let P be a Λ-polyhedral cocore and $x \in C - P$. The number of facets of P visible from x is finite.*

Proof. First let us see that facets visible from $x \in C$ are relevant, i.e. visible from 0. Let F be visible from x, i.e. $x \notin \operatorname{aff}(F)_+$. If F is not visible from 0 we have $0 \in \operatorname{aff}(F)_+$, in which case $\mathbb{R}_{\geq 1}x \cap \operatorname{aff}(F)_+ = \phi$ and hence $\mathbb{R}_{\geq 1}x \cap P = \phi$. But P is a kernel, so that every ray in C must eventually be in P. So F is visible from 0 (and in fact from any point in the line segment $[0, x]$).

Since P is a Λ-polyhedral cocore, it has finitely many relevant facets modulo the action of Λ (Prop. 8 on p. 137 of [1]). Choose representatives $F_1, \ldots, F_r$ for the Λ-orbits among these facets. For each i, the unit normal vector to F_i belongs to C by Lemma 2.2. By Lemma 2.3 the set $\{\gamma \in \Lambda : x \notin \gamma \operatorname{aff}(F_i)_+\}$ is finite, i.e. only finitely many Λ-translates of F_i are visible from x. □

2.6. Lemma 2.4 lets us enumerate the relevant facets of P in a nice way using the idea of line shellings from the theory of convex polytopes:

Proposition 2.5. *Let P be a Λ-polyhedral cocore. There is an enumeration $F_1, F_2, \ldots$ of the relevant facets of P such that for each k, the intersection of F_k with $\cup_{j<k} F_j$ is the union of facets of F_k visible from a point in* $\operatorname{aff}(F_k)$.

Proof. Choose a starting relevant facet F_1 and a point x in its relative interior. The idea is to move towards 0 along the line segment $(0, x]$ and enumerate the facets as they become visible. From points sufficiently close to x the only visible facet is F_1. A relevant facet $F \neq F_1$, is invisible from x but as one moves towards 0 along the line segment $[x, 0]$ it becomes visible at the point $x_F = [x, 0] \cap \operatorname{aff}(F)$. By the previous lemma, the set of points $\{x_F\}_F$ is discrete in $[x, 0)$ and accumulates at 0. Enumerate the facets in the order they become visible, i.e. the order in which one meets the various points x_F. There is ambiguity only if $x_F = x_{F'}$ for distinct facets F and F'. By choosing $x \in F_1$ carefully we can ensure that this does not happen, i.e. the points $x_F \in [0, x]$ are all distinct as F runs over all facets of P. (Indeed, for each pair F, F' of facets, the set $\{x \in F_1 : x_F = x_{F'}\}$ is the set of x for which $[x, 0]$ meets $\operatorname{aff}(F) \cap \operatorname{aff}(F')$. This is the projection (centred at 0) to F_1 of a set of dimension $N - 2$, hence is of dimension $N - 2$.

As the pair F, F' varies, this gives a countable family of $N-2$-dimensional subsets of F_1, which has dimension $N-1$. Take x outside the union of this family.) This gives an enumeration of the relevant facets. The intersection of F_k with $\cup_{j<k} F_j$ consists of those facets of F_k visible in $\mathrm{aff}(F_k)$ from the point $x_{F_k} = [0, x] \cap \mathrm{aff}(F_k)$. □

(Note that for each k the union $\cup_{j\leq k} F_k$ is homeomorphic to a closed $N-1$-ball: F_1 is a closed ball, and to go from $\cup_{j\leq k-1} F_j$ to $\cup_{j\leq k} F_j$ one attaches a closed $N-1$ ball along a subset which is a closed $N-2$-ball in the boundary.)

2.7. **Remarks.** (i) In the $\mathbb{Q}$-rank one case, every facet of P is relevant. Indeed, in the proof of Lemma 2.2 it was shown that any irrelevant facet F is contained in $\overline{C} - C$. Since F is closed and convex and contains no line, it has at least one extreme point, which is then an extreme point of P. The extreme points of P are rational (Prop. 7 on p. 136 of [1]) and nonzero. But $\overline{C} - C$ contains no nonzero rational points if the $\mathbb{Q}$-rank is one.

(ii) In the $\mathbb{Q}$-rank one case, $P^\vee$ is a Λ^t-polyhedral (co)core, so that there is an enumeration of its facets as in the proposition.

2.8. **Polyhedral cone decompositions.** A cocore can be used to construct certain decompositions of C_+. A Λ-*admissible rational polyhedral decomposition (rpd)* of C_+ is a collection Σ of rational polyhedral cones in C_+ satisfying

(i) if τ is a face of $\sigma \in \Sigma$ then $\tau \in \Sigma$
(ii) if $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face of σ and τ
(iii) if $\sigma \in \Sigma$ and $\gamma \in \Lambda$ then $\gamma\sigma \in \Sigma$
(iv) there are only finitely many cones in Σ modulo Λ
(v) $C = \bigcup_{\sigma\in\Sigma} \sigma \cap C$.

It is *simplicial* if each $\sigma \in \Sigma$ is a simplicial cone, i.e. the cone on a simplex. It is *smooth* if

(vi) each $\sigma \in \Sigma$ is generated by a subset of a basis of L

and *projective* if it admits a polarization function, i.e. if

(vii) there is a Λ-invariant polarization function ϕ such that maximal cones on which ϕ is linear are precisely the top-dimensional cones of Σ.

Let us see how a Λ-polyhedral cocore gives a Λ-admissible rpd, following Ash. If F is relevant then $v_F \in C$, so that $F = \mathrm{aff}(F) \cap P$ is compact and convex, hence equal to the convex hull of its set $E(F)$ of extreme points. Now $E(F) \subset E(P)$, while $E(P) \subset \frac{1}{M} L$ for some $M \in \mathbb{Z}$ (cf. [1, p. 136]), so $E(P)$ is discrete. Then $E(F)$ is finite, F is a convex polytope, and the cone over F is a closed polyhedral cone of dimension N. (In contrast the cone over an irrelevant facet has dimension $< N$ and may not be polyhedral.) As F runs over the relevant facets of P, the resulting N-dimensional polyhedral cones, together with their faces and the cone $\{0\}$, give a Λ-admissible rpd of

C_+, i.e. a collection Σ of cones satisfying (i)–(v) of 2.8 (cf. Prop. 8 on p. 137 of [1]). Since the Λ-polyhedral cocore admits a polarization function, such a Λ-admissible rpd is necessarily projective, i.e. (vii) holds too.

The Λ-admissible rpd can be further refined so that (i)–(vii) hold. This is done using barycentric subdivision as follows. It is a consequence of the Siegel property for polyhedral cones (Corollary (i) on p. 116 of [1]) and the neatness of Λ that if $\sigma \in \Sigma, \gamma \in \Lambda$ such that $\sigma \cap \gamma\sigma \neq \phi$ then γ fixes σ pointwise. Fix a set $\{\sigma_i\}$ of representatives for the cones of Σ modulo Λ. For each σ_i choose a ray ρ_i in its relative interior (the barycentre). By the previous observation $\cup_i \Lambda\rho_i$ is a Λ-invariant set of barycentres for all cones in Σ. If the ρ_i are chosen (as they can be) so that every cone in the barycentric subdivision of σ_i with these barycentres is smooth, then the barycentric subdivision of Σ using the Λ-invariant family of barycentres is smooth. Conditions (i)–(v) are unaffected by this procedure and it is easy to see that there is a new polarization function ϕ for which (vii) holds. In other words, one can find a Λ-polyhedral cocore P for which the associated Λ-admissible rpd Σ is smooth and projective. Prop. 2.5 gives an enumeration of the relevant facets of P and hence of the maximal-dimensional cones of Σ.

A collection of cones Σ satisfying (i)–(v) above is locally finite at points of C, i.e. each point of C belongs to finitely many cones. (If $x \in C$ belongs to infinitely many cones then by (iv) there exist $\gamma \neq e$ and σ with $x \in \sigma \cap \gamma\sigma$. By neatness, γ fixes σ pointwise, hence fixes x. But Λ acts freely on C.) It is not usually locally finite at points of $C_+ - C$.

3. Torus embedding

In this section the cone C will always have $\mathbb{Q}$-rank one. We consider the homological properties of torus embeddings associated to the decompositions of the previous section.

3.1. **Torus embeddings.** Let us recall some elementary facts and fix some notation about torus embeddings and their topology (cf. e.g. [3]). For the moment T is a torus of dimension d, Σ is a fan in $X_*(T)_\mathbb{R} \cong \mathbb{R}^d$ which is locally finite (i.e. every $x \in |\Sigma| - \{0\}$ belongs to finitely many cones of σ.) We will assume that $|\Sigma|$ has nonempty interior. Assume that Σ is smooth, i.e. each cone in Σ has a set of generators which extends to a basis for $X_*(T)$. For $\sigma \in \Sigma$ let $\mathrm{Star}(\sigma)$ be the union of the relative interiors of cones which intersect σ. Then $\mathrm{Star}(\sigma)$ is an open neighborhood of σ in $|\Sigma|$. Since Σ is simplicial, one has $\mathrm{Star}(\sigma) \cap \mathrm{Star}(\tau) = \mathrm{Star}(\sigma \cap \tau)$. The collection $\{\mathrm{Star}(\sigma)\}_{\sigma\in\Sigma}$ is an open cover of $|\Sigma|$. The $\mathrm{Star}(\sigma)$ for σ of maximal dimension already cover $|\Sigma|$.

Let $Y := T_\Sigma$ be the torus embedding of T with fan Σ. The T-orbits in Y correspond to cones $\sigma \in \Sigma$. Let O_σ be the orbit corresponding to σ and let T_σ be the isotropy subgroup along this orbit. If $\bar{T}_\sigma$ is the closure of T_σ in $Y = T_\Sigma$ then $(\bar{T}_\sigma, T_\sigma) \cong (\mathbb{C}^k, (\mathbb{C}^*)^k)$ for $k = \dim\sigma$. For $\sigma \in \Sigma$ let $Y(\sigma)$ be the interior of the union of T-orbits corresponding to cones in $\mathrm{Star}(\sigma)$. This is an open neighborhood of O_σ in Y and the collection $\{Y(\sigma)\}_{\sigma\in\Sigma}$ is an open cover of Y with the property that $Y(\sigma)\cap Y(\tau) = Y(\sigma\cap\tau)$. The $Y(\sigma)$ for σ maximal cover Y. For σ maximal, the orbit O_σ is a fixed point, and $Y(\sigma)$ is the union of orbits which have O_σ in their closure. It is smooth and affine with a single T-fixed point and is contractible. The same is true of $Y(\sigma) - T$.

For $\sigma \in \Sigma$, the closure $\overline{O}_\sigma$ of the orbit is smooth and projective. Let Σ_σ be the subfan consisting of cones τ with $\tau\cap\sigma \neq \phi$ and all their faces. The torus embedding $Y_\sigma := T_{\Sigma_\sigma}$ is identified with an open neighborhood of $\overline{O}_\sigma$ in $Y = T_\Sigma$. Choose $\mu : \mathbb{G}_m \to T$ in the relative interior of σ and consider $r(y) = \lim_{t\to 0} \mu(t)\, y$. Then $Y_\sigma = \{y \in Y : r(y) \in \overline{O}_\sigma\}$ and the morphism $r : Y_\sigma \to \overline{O}_\sigma$ is an affine fibration of relative dimension $\dim\sigma$. In particular, it is a homology isomorphism. For any T-stable subvariety $Z \subset \overline{O}_\sigma$, the restrictions $r : r^{-1}(Z) \to Z$ and $r : r^{-1}(Z) - T \to Z$ are both homology isomorphisms.

(When σ is of maximal dimension, Σ_σ is the fan consisting of faces of σ and $Y_\sigma = Y(\sigma)$.)

3.2. **Torus embedding using a polyhedral cone decomposition.** Now suppose we are in the following situation: $C \subset V$ is a self-adjoint homogeneous cone of $\mathbb{Q}$-rank one in a vector space V with an integral lattice L. The quotient $T := V_\mathbb{C}/L$ is a torus of dimension N with $X_*(T) = L$. Let $\phi : C \to \mathbb{R}_+$ be a Λ-invariant polarization function, $P = \{\phi \geq 1\}$ the associated Λ-polyhedral cocore, and Σ the fan consisting of cones on the facets of P and their faces; we assume (cf. 2.8) that ϕ is such that Σ is smooth.

Let

$$Y := \text{ torus embedding of } T \text{ using the fan } \Sigma \text{ in } X_*(T)_\mathbb{R} = V$$
$$D := Y - T$$

These schemes are separated and locally of finite type (by the local finiteness of Σ at points of C mentioned at the end of 2.8). The scheme Y is smooth while D has normal crossings and each irreducible component is a smooth projective toric variety. There is an action of the arithmetic group Λ on each. Since Λ is assumed neat it acts freely on D (Indeed, there are Λ-equivariant homeomorphisms $D/T_c \cong P/\mathbb{R}_{\geq 1} \cong C/\mathbb{R}_+$ and the action on $C/\mathbb{R}_+$ is free since it is the symmetric space of $G_1(\mathbb{R})^0$.) The quotient $\Lambda\backslash D$ is projective (an ample line bundle on $\Lambda\backslash D$ can be constructed from the polarization function in the usual way [3, Ch. 3]). The action on Y is not free (every element of Λ fixes the identity of T).

The enumeration $F_1, F_2, \ldots$ of the relevant facets of P in Prop. 2.5 defines filtrations of D and Y by subvarieties of finite type. Let $\sigma_1, \sigma_2, \ldots$ be the enumeration of top-dimensional cones (i.e. σ_k is the cone on F_k). For $k \geq 1$ let

$$\Sigma_k = \text{ fan consisting of cones } \sigma_1, \ldots, \sigma_k \text{ and their faces}$$
$$Y_k = T_{\Sigma_k} \text{ the torus embedding associated to } \Sigma_k$$
$$D_k = Y_k - T.$$

The variety Y_k is smooth and of finite type and is identified with an open complex submanifold of Y. This defines a filtration $Y_1 \subset Y_2 \subset \dots$ of Y by open subsets with $Y = \bigcup_k Y_k$. The divisor $D_k \subset Y_k$ has normal crossing and smooth irreducible components. Each D_k is open in D and $D = \bigcup_k D_k$. Denote the closure of D_k in D by $\overline{D}_k$; this is a divisor with normal crossings in Y_r for r large. The morphisms $\overline{D}_k \hookrightarrow \overline{D}_{k+1}$ are closed immersions and $D = \bigcup_k \overline{D}_k$. Each $\overline{D}_k$ is projective (indeed, it admits a finite morphism to the projective variety $\Lambda\backslash D$ which is surjective for k large enough). This gives D the structure of an ind-variety.

Theorem 3.1. *The inclusion $D \hookrightarrow Y$ is a homology isomorphism.*

Proof. We show by induction on k that for each k the inclusion $D_k \hookrightarrow Y_k$ is a homology isomorphism. Since $\mathrm{H}_*(D) = \varinjlim \mathrm{H}_*(D_k)$ and $\mathrm{H}_*(Y) = \varinjlim \mathrm{H}_*(Y_k)$, this will imply the theorem. For $k = 1$, the variety Y_1 is an affine toric variety with a single fixed point, hence is contractible. The same is true of D_1. Assume that $D_{k-1} \subset Y_{k-1}$ is known to be a homology isomorphism.

For simplicity write σ for σ_k and F for F_k, the corresponding facet of P. By the way in which facets were enumerated, the intersection $F \cap |\Sigma_{k-1}|$ is the union of facets of F visible from a point in $\mathrm{aff}(F)$. Let $f_1, \dots, f_r$ be these facets of F. Since F is a simplex, the intersection $f_0 = f_1 \cap \dots \cap f_r$ is a face of F of dimension $\dim F - r$, and, moreover, $f_1, \dots, f_r$ are exactly the facets of F containing f_0. Let $\tau_0 \in \Sigma$ be the cone on f_0.

Consider the Mayer-Vietoris sequences associated to the coverings $Y_k = Y_{k-1} \cup Y_k(\sigma)$ and $D_k = D_{k-1} \cup D_k(\sigma)$ (where $Y_k(\sigma)$ is defined as in 3.1 and $D_k(\sigma) = Y_k(\sigma) \cap D = Y_k(\sigma) - T$):

$$\begin{array}{ccccccccc} \longrightarrow & \mathrm{H}_i(D_{k-1} \cap D_k(\sigma)) & \longrightarrow & \mathrm{H}_i(D_{k-1} \sqcup D_k(\sigma)) & \longrightarrow & \mathrm{H}_i(D_k) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \mathrm{H}_i(Y_{k-1} \cap Y_k(\sigma)) & \longrightarrow & \mathrm{H}_i(Y_{k-1} \sqcup Y_k(\sigma)) & \longrightarrow & \mathrm{H}_i(Y_k) & \longrightarrow \end{array}$$

Since σ has maximal dimension, $Y_k(\sigma)$ and $D_k(\sigma)$ are both contractible (a retraction to the fixed point O_σ being given by any $\mathbb{G}_m \to T$ in the relative interior of σ.) To complete the inductive step, it suffices to show that $D_k(\sigma) \cap D_{k-1} \hookrightarrow Y_k(\sigma) \cap Y_{k-1}$ is a homology isomorphism. Let Z be the union of T-orbits corresponding to cones on faces f of F such that $f \supset f_0$ and $f \subset f_i$ for some $i \geq 1$. This is a subset of $\overline{O}_{\tau_0}$. Choose $\mu : \mathbb{G}_m \to T$ lying in the relative interior of τ_0 and consider the retraction $r : Y_{\tau_0} \to \overline{O}_{\tau_0}$. Then $r^{-1}(Z) = Y_k(\sigma) \cap Y_{k-1}$ while $r^{-1}(Z) - T = D_k(\sigma) \cap D_{k-1}$, so that r gives a retraction of each onto Z which is a homology isomorphism. This completes the inductive step. □

3.3. **Weights and purity.** From now until 3.6 all (co)homology groups and Chow groups have rational coefficients. The rational cohomology of an algebraic variety X carries an increasing weight filtration $\dots \subset W_j\mathrm{H}^i(X) \subset W_{j+1}\mathrm{H}^i(X) \subset \dots$ with $\mathrm{Gr}_j\mathrm{H}^i(X) \neq$

$0 \Rightarrow j \in [0, 2i]$ (cf. [2]). The weight filtration on the rational homology $\mathrm{H}_i(X) = \mathrm{H}^i(X)^*$ is the increasing filtration defined by $W_k\mathrm{H}_i(X) = (\mathrm{H}^i(X)/W_{-1-k}\mathrm{H}^i(X))^*$; then $\mathrm{Gr}_j\mathrm{H}_i(X) \neq 0 \Rightarrow j \in [-2i, 0]$. If X is proper then $\mathrm{Gr}_j\mathrm{H}_i(X) \neq 0$ only for $j \geq -i$; if X is smooth then $\mathrm{Gr}_j\mathrm{H}_i(X) \neq 0$ only for $j \leq -i$. The weight filtration is strictly compatible with the homomorphisms induced by morphisms of varieties, so that one has weight filtrations on $\mathrm{H}_*(Y)$ and $\mathrm{H}_*(D)$ from the previous subsection.

Corollary 3.2. *The homology of D is pure, i.e. $\mathrm{H}_i(D)$ is pure of weight $-i$ for each i (and is entirely of Tate type).*

Proof. Since each Y_k is smooth, $\mathrm{H}_i(Y_k)$ has weights $\leq -i$, so $\mathrm{H}_i(Y) = \varinjlim_k \mathrm{H}_i(Y_k)$ has weights $\leq -i$. Since each $\overline{D}_k$ is proper, $\mathrm{H}_i(\overline{D}_k)$ has weights $\geq -i$, so $\mathrm{H}_i(D) = \varinjlim_k \mathrm{H}_i(\overline{D}_k)$ has weights $\geq -i$. The isomorphism $\mathrm{H}_i(Y) \cong \mathrm{H}_i(D)$ respects weights, so this group must be pure of weight $-i$. (That $\mathrm{H}_i(D)$ is purely Tate can be seen as follows: Since D has normal crossings, the map $\mathrm{H}_i(D^{[1]}) \to W_{-i}\mathrm{H}_i(D) = \mathrm{H}_i(D)$ is surjective, where $D^{[1]} \to D$ is the normalization (cf. [2, 8.2.5]). Since each component of $D^{[1]}$ is a smooth projective toric variety, $\mathrm{H}_*(D^{[1]})$ is entirely Tate.) □

(A slight refinement of this corollary is true, cf. 3.9 below.)

The theory of weights is not elementary, so it useful to remark that in our context weights have a nice concrete (and elementary) interpretation using natural endomorphisms of toric varieties which lift the various Frobenii, an idea of Totaro [7]. (In the setting of locally symmetric varieties these are the local Hecke operators of [6], cf. 4.1). I briefly sketch his argument to show how it applies here. For each positive integer $n \in \mathbb{Z}_+$, the nth power map $z \mapsto z^n$ on the torus T extends to a finite endomorphism of Y and D preserving each of $Y_k, D_k, \overline{D}_k$ and each irreducible component of each D_k. Their homology groups become representations of the monoid $\mathbb{Z}_+$. Let X be either Y_k or $\overline{D}_k$. It has a filtration by closed subsets defined by: X_i= union of torus orbits of codimension $\geq i$. The corresponding E^1 spectral sequence converging to $\mathrm{H}_*^{BM}(X)$ is natural with respect to the $\mathbb{Z}_+$-action and $n \in \mathbb{Z}_+$ acts on $\mathrm{E}^1_{p,q}$ by multiplication by n^p. The spectral sequence degenerates at E^1, the limit filtration is the weight filtration on $\mathrm{H}_*^{BM}(X)$, and the action of $n \in \mathbb{Z}_+$ on $\mathrm{Gr}_j\mathrm{H}_*^{BM}(X)$ is by n^j (cf. [7, §§5,6]). Now any extension of $\mathbb{Z}_+$-representations over $\mathbb{Q}$ of different weights (in the sense that n acts by different powers of n) has a unique splitting. So the actions of $\mathbb{Z}_+$ on $\mathrm{H}_i(Y_k) = \mathrm{H}^{BM}_{2\dim Y_k - i}(Y_k)^*$ (Y_k is smooth) and $\mathrm{H}_*(\overline{D}_k) = \mathrm{H}_*^{BM}(\overline{D}_k)$ ($\overline{D}_k$ is proper) are semisimple and split the weight filtration. The same then holds for $\mathrm{H}_*(Y) = \varinjlim \mathrm{H}_*(Y_k)$ and $\mathrm{H}_*(D) = \varinjlim \mathrm{H}_*(\overline{D}_k)$. So in fact we could define weights for all our cohomology groups using this action.

Let $D^{[k]}$ be the disjoint union of all k-fold intersections of irreducible components of D. There is a spectral sequence with

$$\mathrm{E}^1_{p,q} = \mathrm{H}_q(D^{[p+1]}) \Rightarrow \mathrm{H}_{p+q}(D).$$

Each $D^{[k]}$ is smooth with projective connected components, so $\mathrm{E}^1_{p,q}$ is pure of weight $-q$. The spectral sequence degenerates at E^2 and (by definition [2]) the $\mathrm{E}_2 = \mathrm{E}_\infty$ term is $\mathrm{E}^2_{p,q} = \mathrm{E}^\infty_{p,q} = \mathrm{Gr}^W_{-q}\mathrm{H}_{p+q}(D)$. By purity $\mathrm{E}^2_{p,q} = 0$ for $p > 0$, i.e. the rows of the spectral sequence are exact except at the last place. This implies the first part of:

Corollary 3.3. *For each q the sequence of Λ-modules*

$$0 \to \mathrm{H}_q(D^{[N]}) \to \cdots \to \mathrm{H}_q(D^{[2]}) \to \mathrm{H}_q(D^{[1]}) \to \mathrm{H}_q(D) \to 0$$

is a resolution of $\mathrm{H}_q(D)$ by finitely generated free $\mathbb{Q}[\Lambda]$-modules.

Proof. We have already seen that the sequence is exact. The connected components of $D^{[k]}$ are parametrized by cones of dimension k in Σ. Let $\sigma_1, \ldots, \sigma_r$ be representatives for the k-dimensional cones of Σ modulo Λ and $D_1, \ldots, D_r$ the corresponding components of $D^{[k]}$; each D_i is a smooth projective toric variety and hence has finite-dimensional homology. Then $\Lambda\left[\oplus_i \mathrm{H}_q(D_{\sigma_i})\right] = \mathrm{H}_q(D^{[k]})$, so that $\mathrm{H}_q(D^{[k]})$ is generated over $\mathbb{Q}[\Lambda]$ by a basis for $\oplus_i \mathrm{H}_q(D_{\sigma_i})$. The freeness of the Λ-action follows from the fact that if $\gamma\sigma_i \cap \sigma_i \neq \phi$ then $\gamma = e$ because Λ is neat. □

3.4. **Homology of $\Lambda\backslash D$.** For a space X with a Λ-action the Λ-equivariant homology is defined to be

$$\mathrm{H}^\Lambda_*(X) = \mathrm{H}_*(X \times_\Lambda E\Lambda)$$

where $E\Lambda$ is any contractible space with a free Λ-action (in particular, one could take $E\Lambda = C$ or $E\Lambda = V + iC$). The Leray spectral sequence for the fibration $X \times_\Lambda E\Lambda \to B\Lambda$ has

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(\Lambda, \mathrm{H}_q(X)) \Rightarrow \mathrm{H}^\Lambda_{p+q}(X).$$

In the situation at hand, the Λ-invariance of Σ gives an action of Λ on the torus embedding Y, preserving D. Since $D \hookrightarrow Y$ is Λ-equivariant, the homology isomorphism $\mathrm{H}_*(D) \cong \mathrm{H}_*(Y)$ of Theorem 3.1 is one of Λ-modules. By the spectral sequence the induced map $\mathrm{H}^\Lambda_*(D) \to \mathrm{H}^\Lambda_*(Y)$ is also an isomorphism. The differentials in the spectral sequence are natural, in particular they respect weights. By purity, the spectral sequence degenerates at E^2. Since Λ acts freely on D, $\mathrm{H}^\Lambda_*(D) = \mathrm{H}_*(\Lambda\backslash D)$, giving a natural isomorphism $\mathrm{H}_*(\Lambda\backslash D) = \mathrm{H}^\Lambda_*(Y)$. We arrive at:

Corollary 3.4. *There is a natural isomorphism $\mathrm{H}_*(\Lambda\backslash D) = \mathrm{H}^\Lambda_*(Y)$. The spectral sequence $\mathrm{E}^2_{p,q} = \mathrm{H}_p(\Lambda, \mathrm{H}_q(D)) \Rightarrow \mathrm{H}^\Lambda_{p+q}(D)$ degenerates at E^2. The limit filtration (suitably shifted) is the weight filtration on $\mathrm{H}_i(\Lambda\backslash D) = \mathrm{H}^\Lambda_i(D)$, so that $\mathrm{Gr}^W_{-j}\mathrm{H}_i(\Lambda\backslash D) = \mathrm{H}_{i-j}(\Lambda, \mathrm{H}_j(D)) = \mathrm{H}_{i-j}(\Lambda, \mathrm{H}_j(Y))$.*

3.5. **Remarks.** (i) The analogous results for singular cohomology follow from these results for singular homology since $\mathrm{H}^*(Y) = \varprojlim_k \mathrm{H}^*(Y_k)$. (Our main reason for working in homology rather than cohomology is that it is a little simpler to work with direct limits.)

(ii) The results, suitably rephrased, hold in ℓ-adic cohomology. For a discrete group Λ acting on a k-scheme X the Λ-equivariant ℓ-adic cohomology is defined to be

$$\mathrm{RHom}_{\bar{\mathbb{Q}}_\ell[\Lambda]}(\bar{\mathbb{Q}}_\ell, \mathrm{R}\Gamma(X_{\bar{k}}, \bar{\mathbb{Q}}_\ell)).$$

If the action is free, so that $\Lambda\backslash X$ exists as a k-scheme, then this is presumably the cohomology $\mathrm{H}^*((\Lambda\backslash X)_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ of the quotient. (In our situation, i.e. $X = D$ and $k = \mathbb{Q}$ this can be deduced from the corresponding statement in singular cohomology using the comparison theorem.) With these definitions, the analogues of the results above in $\mathbb{Q}_\ell$-cohomology hold (and follow from the results for singular cohomology by comparison theorems).

(iii) The results so far also hold if the fan Σ is simplicial, provided we work everywhere with rational coefficients.

3.6. **Torus-equivariant homology.** Another consequence of purity is that the torus action on D is equivariantly formal (cf. [4]). By considerations of weights the spectral sequence for the fibration $D \times_T ET \to BT$ with

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(D) \otimes \mathrm{H}_q(BT) \Rightarrow \mathrm{H}^T_{p+q}(D)$$

degenerates at E^2 to give $\mathrm{H}^T_*(D) \cong \mathrm{H}_*(D) \otimes \mathrm{H}_*(BT)$ (noncanonically). The same holds with D replaced by Y, showing, in particular, that $\mathrm{H}^T_*(D) \cong \mathrm{H}^T_*(Y)$. The cohomology of the classifying space $\mathrm{H}^*(BT) \cong \mathrm{Sym}\,\mathrm{H}^2(BT) \cong \mathrm{Sym}\,X^*(T)$ acts on $\mathrm{H}^T_*(D)$ by cap product, and $\mathrm{H}_*(D)$ can be recovered from $\mathrm{H}^T_*(D)$ with its $\mathrm{H}^*(BT)$-module structure. Since T is a torus, there is a localization of the equivariant cohomology to the T-fixed points, and a recipe for computing $\mathrm{H}^T_*(D)$ in terms of the fixed points and the 1-dimensional torus orbits (the Chang-Skjelbred lemma [4]). All this is natural with respect to Λ, so that we have, in principle, a description of $\mathrm{H}_*(D)$ as a Λ-module, and hence (by Corollary 3.4) a description of (the associated graded of) $\mathrm{H}_*(D_\Lambda)$.

3.7. **A refinement of purity.** Corollary 3.2 can be improved using the enumeration of the facets of the dual core $P^\vee$ mentioned in 2.7(ii). The result, Theorem 3.5 below, will not be used later. We first recall the description of the homology of a smooth projective toric variety using a $\mathbb{G}_m$-action (à la Bialynicki-Birula). From now until 3.9 Chow groups and homology groups have integer coefficients.

3.8. Let T be a torus and let F be a convex polytope in $X_*(T)_\mathbb{R}$ with nonempty interior containing zero. Let Σ_F be the fan consisting of cones on its facets and their faces, and let X be the torus embedding of T using Σ_F. Assume that X is smooth. Choose $\mu : \mathbb{G}_m \to T$ which lies in the relative interior of a top-dimensional cone of Σ_F. Then the set $X^{\mu(\mathbb{G}_m)} = X^T$ of fixed points is finite, in bijection with the top-dimensional cones of Σ_F or, equivalently, with the facets of F. Let us assume that the fixed points have been enumerated $x_1, x_2, \ldots$ according to visibility of the corresponding facets along the ray through μ. For each fixed point $x_i \in X^{\mu(\mathbb{G}_m)}$ define $X_i := \{x \in X : \lim_{t\to 0} \mu(t)x =$

$x_i\}$. Each X_i is an affine space and $X_{\leq k} := \sqcup_{i\leq k} X_i$ is closed in X for each k. Let $A_i(-)$ denote the Chow group of dimension i algebraic cycles on a variety modulo rational equivalence; there is a cycle class map $A_i(-) \to \mathrm{H}_{2i}^{BM}(-)$ to Borel-Moore homology (i.e. homology with locally finite supports). Let us show by induction that $A_*(X_{\leq k}) \cong \mathrm{H}_*^{BM}(X_{\leq k})$ for all k, the case $k = 1$ being trivial. Consider the diagram

$$\begin{array}{ccccccccc} & & A_i(X_{\leq k-1}) & \longrightarrow & A_i(X_{\leq k}) & \longrightarrow & A_i(X_k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_{2i}^{BM}(X_{\leq k-1}) & \longrightarrow & \mathrm{H}_{2i}^{BM}(X_{\leq k}) & \longrightarrow & \mathrm{H}_{2i}^{BM}(X_k) & \longrightarrow & 0. \end{array}$$

The top row is the usual exact sequence of Chow groups for a closed subscheme and its complement. The bottom row is part of the long exact sequence for Borel-Moore homology; it is short exact since $\mathrm{H}_i^{BM}(X_k)$ vanishes in odd degrees. The first and third vertical maps are isomorphisms (by the induction hypothesis and since $A_*(X_k) \cong \mathrm{H}_*^{BM}(X_k)$ respectively). So $A_*(X_{\leq k}) \cong \mathrm{H}_*^{BM}(X_{\leq k})$ for all k, and hence $A_*(X) \cong \mathrm{H}_*(X)$ since X is proper.

3.9. Suppose now that we are in the situation of §§1, 2 and that C has $\mathbb{Q}$-rank one. Let $T_c \subset T(\mathbb{C})$ be the compact torus. The quotient Y/T_c is naturally identified with $P^\vee$; denote the quotient map by $q : Y \to P^\vee$. Let $F_1', F_2', \ldots$ be the enumeration of the facets of $P^\vee$ given by visibility along a line segment $(0, x)$ for $x \in F_1'$ (cf. Remark 2.7(ii)). For each k, let $Z_k = \cup_{j\leq k} q^{-1}(F_j')$. Then $Z_1 \subset Z_2 \subset \ldots$ is a filtration of D by closed subsets and $D = \cup_k Z_k$.

Theorem 3.5. *The homology of D is concentrated in even degrees, is torsion-free, and is spanned by classes of algebraic cycles.*

Proof. It is enough, since $\mathrm{H}_i(D) = \varinjlim \mathrm{H}_i(Z_k)$, to prove the corresponding statements for each Z_k. We will do this by induction. For $k = 1$, Z_1 is a smooth projective toric variety, so that the cycle class map $A_*(Z_1) \to \mathrm{H}_*(Z_1)$ is an isomorphism (by 3.8). The induction hypothesis is that $A_*(Z_{k-1}) \cong \mathrm{H}_*(Z_{k-1})$. Suppose we knew that

$(*)$ $A_*(Z_k - Z_{k-1}) \cong \mathrm{H}_*^{BM}(Z_k - Z_{k-1})$.

Then there is a diagram

$$\begin{array}{ccccccccc} & & A_i(Z_{k-1}) & \longrightarrow & A_i(Z_k) & \longrightarrow & A_i(Z_k - Z_{k-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_{2i}(Z_{k-1}) & \longrightarrow & \mathrm{H}_{2i}(Z_k) & \longrightarrow & \mathrm{H}_{2i}^{BM}(Z_k - Z_{k-1}) & \longrightarrow & 0. \end{array}$$

The first row is the usual exact sequence of Chow groups. The second row (part of the long exact sequence in Borel-Moore homology) is short-exact by the induction hypothesis and $(*)$. The first and third vertical maps are isomorphisms, so the second one is too, showing that $A_*(Z_k) \cong \mathrm{H}_*(Z_k)$.

The argument to prove $(*)$ uses a suitable $\mathbb{G}_m$-action as in 3.8. To simplify notation let $Z' := Z_k - Z_{k-1}$ and let $F' := F'_k$. Let $\sigma \in \Sigma$ be the ray through the vertex of P dual to F'. The variety Z' is open and dense in the smooth projective torus embedding X of the torus $T' = T/T_\sigma$ associated to the convex polytope $(F')^\vee$. Let $q : X \to F'$ be the quotient by T'_c. Then $X - Z'$ is the preimage under q of those facets of F' visible from some point $x \in X_*(T')_\mathbb{R}$, which may be assumed to be rational. Let $\mu \in X_*(T')$ be a generator of the ray through x; consider the $\mathbb{G}_m$-action via $\mu : \mathbb{G}_m \to T'$. It gives a filtration of X by closed subsets $X_{\leq 1} \subset X_{\leq 2} \subset \ldots$ as in 3.8, with each $X_k = X_{\leq k} - X_{\leq k-1}$ an affine space. Since $X - Z'$ is closed and $\mu(\mathbb{G}_m)$-stable, we have $X - Z' = \bigsqcup_{x_i \in X-Z'} X_i$ and hence $Z' = \bigsqcup_{x_i \in Z'} X_i$. Then $Z'_{\leq k} := Z' \cap X_{\leq k}$ defines a filtration of Z' by closed subsets such that $Z'_{\leq k} - Z'_{\leq k-1}$ is either empty or an affine space. By the same argument as in 3.8 above, an induction gives $A_*(Z') \cong \mathrm{H}^{BM}_*(Z')$. □

3.10. **Remarks on the higher-rank situation.** We indicate how (some of) the results of this section extend to the situation when C has $\mathbb{Q}$-rank ≥ 2. No assumption was made in §2, so the results of that section are general. Thus if P is a Λ-polyhedral cocore in C_+ and Σ the associated rpd. The arguments of §2 give an enumeration of the facets of the top-dimensional cones in Σ. As in 3.2 we consider the torus embedding Y along Σ, and $D = Y - T$. These are no longer locally of finite type, but the schemes Y_k, D_k are of finite type and the proof of Theorem 3.1 goes through without change, so that $\mathrm{H}_*(Y) = \mathrm{H}_*(D)$. The statements involving weights also remain valid if we *define* weights using the action of $\mathbb{Z}_+$ mentioned after Cor. 3.2, and the same proofs work.

The spectral sequence computing $\mathrm{H}^\Lambda_*(Y)$ or $\mathrm{H}^\Lambda_*(D)$ degenerates as before (using weights), but this no longer converges to $\mathrm{H}_*(\Lambda\backslash D)$ as the action of Λ is not free. So the remaining results have to be suitably reformulated.

4. Arithmetic varieties of $\mathbb{Q}$-rank one

In this section we apply the previous results to arithmetic varieties. The notation here differs from that of previous sections. Let G be an algebraic group over $\mathbb{Q}$ such that the symmetric space D of $G(\mathbb{R})^0$ is Hermitian. Let $\Gamma \subset G(\mathbb{Q})$ be a neat arithmetic subgroup and $M := \Gamma\backslash D$ the associated *arithmetic variety*. Assume that G has $\mathbb{Q}$-rank one, so that M is noncompact. As a notational convention, for subgroups $J \triangleleft H \subset G$, let $\Gamma_H := \Gamma \cap H(\mathbb{R})$ and $\Gamma_{H/J} := \Gamma_H/\Gamma_J \subset H/J$.

4.1. **Minimal compactification.** The compact dual symmetric space $D^\vee$ is a flag variety of $G(\mathbb{C})$ with a $G(\mathbb{R})^0$-equivariant holomorphic embedding $D \hookrightarrow D^\vee$. The *boundary components* of D are the maximal connected analytic submanifolds of the closure of D in $D^\vee$. The stabilizer of a boundary component is a parabolic subgroup whose projection to each simple factor of $G_\mathbb{R}$ is a maximal $\mathbb{R}$-parabolic subgroup. The boundary component is *rational* if the stabilizer is $\mathbb{Q}$-rational, in which case it is a maximal $\mathbb{Q}$-parabolic subgroup. The union of rational boundary components D^* is $G(\mathbb{Q})$-invariant and carries a natural topology, the Satake topology, for which the $G(\mathbb{Q})$-action is continous. The quotient $M^* = \Gamma\backslash D^*$ is compact Hausdorff and contains M as an open dense subset. The decomposition of D^* into rational boundary components descends to a stratification of M^* by complex manifolds, each an arithmetic variety of smaller dimension. There is a unique structure of a normal analytic space on M^* restricting to the given complex structure on each stratum. Moreover, according to Baily and Borel, M^* is a normal projective variety. Since the $\mathbb{Q}$-rank of M is one and Γ is neat, the boundary $M^* - M$ is smooth.

4.2. **Stabilizer of a boundary component.** The stabilizer P_F of a rational boundary component F has the following structure (cf. [1, III.4.1–III.4.2]). Its unipotent radical W is two-step unipotent: its centre U is abelian and the quotient $V := W/U$ is abelian. The quotient P/W splits as an almost-direct product $P/W = M_\ell \cdot M_h$. (The groups M_h and M_ℓ are denoted $G_{h,F}$ and $G_{\ell,F}$ in [1].) The symmetric space of $M_h(\mathbb{R})^0$ is the rational boundary component F. The adjoint action of $M_\ell(\mathbb{R})^0$ on $\mathfrak{u} := \operatorname{Lie} U(\mathbb{R})$ has an open orbit $C \subset \mathfrak{u}$ and the stabilizer of a point is maximal compact. Thus C is a self-adjoint homogeneous cone, of $\mathbb{Q}$-rank one since G is of $\mathbb{Q}$-rank one and Γ_{M_ℓ} is an arithmetic group of its automorphisms (as in §§1,2).

4.3. **Relative torus embedding.** Fix a rational boundary component (r.b.c.) F. Consider the open domain $D(F) := U(\mathbb{C}) \cdot D$ in $D^\vee$, on which $P(\mathbb{R})U(\mathbb{C})$ acts. Let $T = \Gamma_U\backslash U(\mathbb{C})$, a torus of rank $\dim U$, and write $M_hW \subset P$ for the preimage of M_h by $P \to P/W$. Let

$$\begin{aligned}
\mathcal{T} &:= \Gamma_{M_hW}\backslash D(F)\\
\mathcal{A} &:= \Gamma_{M_hW}U(\mathbb{C})\backslash D(F)\\
S &:= \Gamma_{M_h}W(\mathbb{R})U(\mathbb{C})\backslash D(F) = \Gamma_{M_h}\backslash F
\end{aligned}$$

Each is a smooth algebraic variety with a Γ_{M_ℓ}-action, the action on S being trivial. There are Γ_{M_ℓ}-equivariant morphisms

$$\mathcal{T} \to \mathcal{A} \to S.$$

Here $\mathcal{T} \to \mathcal{A}$ is a torsor for $T = \Gamma_U\backslash U(\mathbb{C})$ and $\mathcal{A} \to S$ is an abelian scheme with fibres $\mathbb{R}$-isomorphic to $\Gamma_V\backslash V(\mathbb{R})$.

Let Σ_F be a Γ_{M_ℓ}-admissible rpd of C which is smooth and projective (as in 2.8). Performing a torus-torsor embedding along the fibres of $\mathcal{T} \to \mathcal{A}$ using Σ_F gives a scheme $\mathcal{Y} \to \mathcal{A}$ which is locally of finite type over $\mathcal{A}$. Let $\mathcal{D} := \mathcal{Y} - \mathcal{T}$. For $s \in S$, write $\mathcal{T}_s$ for the fibre of $\mathcal{T} \to S$ and $\mathcal{D}_s := \mathcal{T}_s \cap \mathcal{D}$. Applying Theorem 3.1 fibrewise one concludes that $\mathcal{D}_s \hookrightarrow \mathcal{Y}_s$ is a homology isomorphism. The action of Γ_{M_ℓ} on $\mathcal{D}$ is free, and preserves the fibres of $\mathcal{Y} \to S$. One gets a natural isomorphism $\mathrm{H}_*(\Gamma_{M_\ell}\backslash\mathcal{D}_s) = \mathrm{H}_*^{\Gamma_{M_\ell}}(\mathcal{Y}_s)$ for each s. The spectral sequence $\mathrm{E}^2_{p,q} = \mathrm{H}_p(\Gamma_{M_\ell}, \mathrm{H}_q(\mathcal{Y}_s)) \Rightarrow \mathrm{H}_*(\Gamma_{M_\ell}\backslash\mathcal{D}_s)$ degenerates at E^2 for weight reasons and the statements of 3.4 carry over.

4.4. **Toroidal compactifications.** Fix a Γ-admissible rational polyhedral cone decomposition Σ. Recall what this means: For each r.b.c. F one has a Γ_{M_ℓ}-admissible rpd (as in 2.8) Σ_F of the cone C and the whole collection $\Sigma = \sqcup_F \Sigma_F$ is required to be Γ-invariant. Given Σ, the theory of [1] constructs a *toroidal compactification* M^Σ. If each Σ_F is smooth and projective (as in 2.8) then M^Σ is a smooth projective variety containing M as a Zariski-dense open subvariety, and there is a projective morphism $\pi : M^\Sigma \to M^*$ such that $\pi^{-1}(M^* - M)$ is a divisor with simple normal crossings (cf. p. 312 of [1]).

The situation of 4.3 is a local model for the toroidal compactification along the preimage of the stratum S: Fix a stratum S of M^* and let $F \subset D^*$ be an r.b.c. covering S. Let P, M_ℓ, M_h etc. be the associated groups and $\mathcal{D}, \mathcal{Y}$ etc. the associated schemes. There exist:

(i) a Γ_{M_ℓ}-equivariant morphism $\mathcal{Y} \to M^\Sigma$ (where Γ_{M_ℓ} acts trivially on M^Σ);
(ii) a neighbourhood $\mathcal{U}$ of $\mathcal{D}$ in $\mathcal{Y}$ (in the classical topology) on which Γ_{M_ℓ} acts freely

such that the induced map $\Gamma_{M_\ell}\backslash\mathcal{U} \to M^\Sigma$ is a biholomorphism onto an analytic neighbourhood of $\pi^{-1}(S)$ and restricts to a natural isomorphism

$$\pi^{-1}(S) = \Gamma_{M_\ell}\backslash\mathcal{D}$$

of complex spaces over S. Algebraically speaking, the geometric quotient of the ind-scheme $\mathcal{D}$ by the action of the discrete group Γ_{M_ℓ} exists and is naturally identified with $\pi^{-1}(S)$ (as a scheme over S); the formal completion $\mathcal{Y}_\mathcal{D}$ of $\mathcal{Y}$ along $\mathcal{D}$ has a free action of the discrete group Γ_{M_ℓ}, and the geometric quotient is isomorphic to the formal completion $(M^\Sigma)_{\pi^{-1}(S)}$ as a scheme over S. The “quotient” $\Gamma_{M_\ell}\backslash\mathcal{Y}$ (which does not exist as a scheme) should be thought of as a formal neighbourhood of $\pi^{-1}(S)$ in M^Σ. Let $\mathcal{D}_s$ and $\mathcal{Y}_s$ denote the fibres over $s \in S$. Applying the remarks in 4.3 we arrive at:

Theorem 4.1. *For $s \in S$ there is a natural isomorphism*

$$\mathrm{H}_*(\pi^{-1}(s)) = \mathrm{H}_*(\Gamma_{M_\ell}\backslash\mathcal{D}_s) = \mathrm{H}_*^{\Gamma_{M_\ell}}(\mathcal{Y}_s).$$

The homology of $\mathcal{Y}_s$ (or $\mathcal{D}_s$) is pure. The spectral sequence

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(\Gamma_{M_\ell}, \mathrm{H}_q(\mathcal{Y}_s)) \Rightarrow \mathrm{H}^{\Gamma_{M_\ell}}_{p+q}(\mathcal{Y}_s)$$

degenerates at E^2 and the (shifted) limit filtration is the weight filtration on $\mathrm{H}_*(\pi^{-1}(s))$. The graded pieces are given by

$$\mathrm{Gr}^W_{-j}\mathrm{H}_i(\pi^{-1}(s)) = \mathrm{H}_{i-j}(\Gamma_{M_\ell}, \mathrm{H}_j(\mathcal{D}_s)) = \mathrm{H}_{i-j}(\Gamma_{M_\ell}, \mathrm{H}_j(\mathcal{Y}_s)).$$

4.5. **Remarks.** (i) The $\mathbb{Z}_+$-action of 3.3 is explicitly realized by Looijenga's local Hecke operators [6] (and the assertion that they split the weight filtration on $\mathrm{H}_*(\pi^{-1}(s))$ is contained there). One can replace $\mathcal{Y}_s$ by an open neighbourhood of $\mathcal{D}_s$ in $\mathcal{Y}_s$ in the classical topology stable under these operators and one can formulate an etale version of the theorem.

(ii) The story in 3.6 carries over to the current context: The T-action on $\mathcal{D}$ is equivariantly formal and $\mathrm{H}_*^T(\mathcal{D})$ is localized to the fixed point locus. The fixed point locus $\mathcal{D}^T$ is a disjoint union of copies of A, one for each top-dimensional cone. The equivariant homology can be computed from $\mathcal{D}^T$ and the locus corresponding to cones of one less dimension, each connected component of which is an $\mathbb{G}_m$-fibration over A.

4.6. **Remark on the Hodge theory of $\pi^{-1}(M^* - M)$.** Fix a connected component of $M^* - M$ and let $\mathcal{D}, \mathcal{Y}, \Gamma_{M_\ell}$ etc. be as in 4.3. The corresponding component of the exceptional divisor has cohomology

$$\mathrm{H}^k_{\Gamma_{M_\ell}}(\mathcal{Y}, \mathbb{C}) = \mathrm{H}^k(\mathcal{Y} \times_{\Gamma_{M_\ell}} E\Gamma_{M_\ell}, \mathbb{C}).$$

A natural complex computing this group is the total complex associated to the double complex $K^{\bullet,\bullet} = C^\bullet(\Gamma_{M_\ell}, \Gamma(\mathcal{Y}, \mathscr{E}^\bullet_{\mathcal{Y}}))$. (Here $C^\bullet(\Gamma_{M_\ell}, -)$ is the usual group cohomology complex of a Γ_{M_ℓ}-module.) This complex has two filtrations: An increasing filtration $W_\bullet$ can be defined by applying the canonical truncation functor $\tau_{\leq i}$ in the first index, a decreasing filtration $F^\bullet$ by applying the filtration F^p by type to $\mathscr{E}^\bullet_{\mathcal{Y}}$. It follows from theorems proved above that the spectral sequence for $W_\bullet$ degenerates at E_2 and the induced filtration on the graded pieces $\mathrm{Gr}_i^W \mathrm{H}^k_{\Gamma_{M_\ell}}(\mathcal{Y}) = \mathrm{H}^{k-i}(\Gamma_{M_\ell}, \mathrm{H}^i(\mathcal{Y}))$ is the Hodge filtration under the identification $\mathrm{Gr}_i^W \mathrm{H}^k_{\Gamma_{M_\ell}}(\mathcal{Y}) = \mathrm{Gr}_i^W \mathrm{H}^k(\Gamma_{M_\ell} \backslash \mathcal{D})$. (There is another way to define a naive Hodge filtration on $\mathrm{H}^k_{\Gamma_{M_\ell}}(\mathcal{Y}) = \mathrm{H}^k(\mathcal{Y} \times_{\Gamma_{M_\ell}} E\Gamma_{M_\ell})$, namely using the model $E\Gamma_{M_\ell} = C + i\mathfrak{u}$, which has a complex structure; this should give the same filtration.)

This suggests that it is possible to understand the mixed Hodge structure (MHS) on the exceptional divisor using the embedding in M^Σ, i.e. it should be possible to find a cohomological mixed Hodge complex (i.e. a complex with two filtrations which define a mixed Hodge structure in cohomology, cf. [2]) consisting of differential forms on a classical neighborhood of $\pi^{-1}(s)$ (of the form $\Gamma_{M_\ell} \backslash \mathcal{U}$ for $\mathcal{U}$ as in 4.4) which computes the MHS on $\mathrm{H}^*(\pi^{-1}(s), \mathbb{C})$. (The Hodge filtration should be the natural one on differential forms. Note that the MHS is defined in [2] without reference to the embedding, and a priori has no simple relation to it, so this is a special feature of arithmetic varieties.) This would then have interesting applications to the Hodge theory of M^Σ, and hence to M.

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