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Motivic Decomposition and Intersection Chow Groups II

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This paper is a continuation of [CH], where we formulated the motivic analogue of the decomposition theorem in [BBD]. The decomposition theorem says, if X, S are quasi-projective complex algebraic varieties with X smooth, and $p : X \to S$ is a projective map, then the direct image of the constant sheaf $Rp_*\mathbb{Q}_X$ is a direct sum of intersection complexes (of local systems on smooth locally closed subvarieties of S) with shifts. The motivic analogue is a conjectural statement that the decomposition be lifted to a decomposition in a suitably defined motivic category. In [CH] we defined the category of Chow motives over S, and showed that the existence of the motivic decomposition in this category is a consequence of the conjectures of Grothendieck and of Bloch-Beilinson-Murre.

If the map $p : X \to S$ is a resolution of singularities, one of the direct summands of $Rp_*\mathbb{Q}_X$ is the intersection complex $IC_S = IC_S(\mathbb{Q})$ of S. The motivic decomposition has a direct summand corresponding to the intersection complex. We will call it the *motivic intersection complex* of S. The Chow group of this object we call the *intersection Chow group* of S, and denote it by $\mathrm{ICH}^r(S)$.

The content of this paper is as follows.

(1) In §3 we give an account of this theory, *under the conjectures of Grothendieck and of Bloch-Beilinson-Murre*. The definition of intersection Chow group $\mathrm{ICH}^r(S)$ of a quasi-projective variety S rests on the existence of the motivic decomposition for a desingularization $p : X \to S$. The group $\mathrm{ICH}^r(S)$ is a canonical subquotient of the Chow group $\mathrm{CH}^r(X)$. We then derive a formula (3.9) for the intersection Chow group in terms of the Chow groups of X and of the exceptional loci of p. These Chow groups have filtrations denoted $F_S^\bullet$, which appear in the formula. The filtration has to do with the perverse Leray filtration on objects in the motivic category, which is defined using the motivic decomposition.

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(2) In §4 we give an unconditional definition of intersection Chow group. Here "unconditional" means independent of any conjectures. For this we take the formula mentioned above, and turn it into the definition. We need to define the filtrations $F_S^\bullet$ on Chow groups, without assuming the existence of motivic decomposition. This can be done, using cohomology realizations, as Shuji Saito did for the case $S = \operatorname{Spec} k$. We can show the intersection Chow group is well-defined, independent of the choice of a resolution.

§4 was inspired by §3, but it is logically independent. We do not even need the category of Chow motives in §4.

(3) There is an analogous formula for the intersection cohomology of S, in terms of the cohomology of the exceptional loci of a desingularization, see (2.4). This is discussed in §2.

In the summer of 1996 the second author had a chance to communicate the present work to Bob MacPherson. On that occasion we had conversations on the motivic analogue of the lifting theorem [BBFGK], which later developed into [Ha-2]. We would like to take the present opportunity to thank Bob cordially for these discussions and for his profound influence on our work. We are also grateful to the referee for the useful suggestions on the manuscript.

Throughout this paper we consider quasi-projective varieties over $k = \mathbb{C}$. The Chow group of a quasi-projective variety X, *tensored with* $\mathbb{Q}$, is denoted $\mathrm{CH}(X)$.

Let $D_c^b(S) = D_c^b(S(\mathbb{C}), \mathbb{Q})$ denote the derived category of sheaves of $\mathbb{Q}$-vector spaces on $S(\mathbb{C})$ with cohomology sheaves bounded and constructible. The cohomology of X is always with $\mathbb{Q}$-coefficients: $H^i(X) = H^i(X, \mathbb{Q})$. For intersection complex and perverse sheaves, we always take the middle perversity and $\mathbb{Q}$ as coefficients: $IH^i(X) = IH^i(X, \mathbb{Q})$. We refer to [GM-1], [BBD] and [Bo] for expositions on intersection complexes and perverse sheaves; see [CH] for additional information. $Perv(S)$ denotes the category of perverse sheaves on $S(\mathbb{C})$.

§1. **Stratification of a projective map and the decomposition theorem.**

(1.1) **Definition.** Let S be an irreducible quasi-projective variety over $\mathbb{C}$. An algebraic *Whitney stratification* $\mathcal{S} = \{S_\alpha\}$ of S is a filtration of S by closed sets

$$S = S_0 \supset S_1 \supset \cdots \supset S_\alpha \supset \cdots \supset S_{\dim S}$$

such that $S_\alpha - S_{\alpha+1}$ are smooth of codimension α (or empty) satisfying Whitney's conditions A and B (see [GM-2, Chap.I] for details).

Let X be a quasi-projective variety and $p : X \to S$ be a projective map. $p : X \to S$ is a *stratified map* over $\mathcal{S}$ if there is a Whitney stratification Σ of

X such that p is a stratified map with respect to Σ and $\mathcal{S}$ ([GM-2, p.42]). In particular, p is a stratified fiber bundle over each stratum $S_\alpha^0 := S_\alpha - S_{\alpha+1}$.

Let $X_\alpha = p^{-1}(S_\alpha)$, $p_\alpha : X_\alpha \to S_\alpha$ the induced map, and $i_\alpha : S_\alpha \to S$, $k_\alpha : X_\alpha \to X$ be the closed immersions. Let $X_\alpha^0 = p^{-1}(S_\alpha^0)$ and $p_\alpha^0 : X_\alpha^0 \to S_\alpha^0$ be the induced map. For $\alpha = 0$, we will drop the subscript as follows: $S^0 = S_0^0 = S - S_1$, $X^0 = X_0^0 = X - X_1$, and $p^0 = p_0^0 : X^0 \to S^0$. We thus have a commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{k_\alpha} & X_\alpha & \hookleftarrow & X_\alpha^0 \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_\alpha}\downarrow & & {\scriptstyle p_\alpha^0}\downarrow \\ S & \xleftarrow{i_\alpha} & S_\alpha & \hookleftarrow & S_\alpha^0 \end{array}$$

Given a projective map $p : X \to S$, there is a Whitney stratification $\mathcal{S}$ on S over which p is stratified.

(1.2) **Definition.** Let X, S be quasi-projective varieties, with X *smooth*, and $p : X \to S$ a projective map. Let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of S over which p is stratified.

A *resolution* of $p : X \to S$ over $\mathcal{S}$ is a collection $\{\pi_\alpha : \tilde{X}_\alpha \to X_\alpha\}$ consisting of smooth quasi-projective varieties $\tilde{X}_\alpha$ and *projective surjective* maps π_α, for $\alpha \geq 1$.

Let $\iota_\alpha = k_\alpha \circ \pi_\alpha : \tilde{X}_\alpha \to X$ and $q_\alpha = p_\alpha \circ \pi_\alpha : \tilde{X}_\alpha \to S_\alpha$.

$$\begin{array}{ccc} & & \tilde{X}_\alpha \\ & \swarrow & \downarrow{\scriptstyle \pi_\alpha} \\ X & \hookleftarrow & X_\alpha \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ S & \hookleftarrow & S_\alpha \end{array}$$

Given a projective map $p : X \to S$ with smooth X, stratified over $\mathcal{S}$, there exists a resolution over $\mathcal{S}$.

Remark. We may require π_α be desingularizations, rather than projective surjective maps. For purposes of later sections, however, it is more convenient to allow projective surjective maps.

The following is known as the decomposition theorem, [BBD]. In the statement, ${}^pR^ip_*\mathbb{Q}_X = {}^p\mathcal{H}^iRp_*\mathbb{Q}_X$, where ${}^p\mathcal{H}^i$ is perverse cohomology, and $IC_{S_\alpha}(\mathcal{V})$ is the intersection complex of a local system $\mathcal{V}$.

(1.3) **Theorem.** *Let X be smooth and $p : X \to S$ a projective map, stratified over $\mathcal{S} = \{S_\alpha\}$. Then there is a non-canonical direct sum decomposition*

$$Rp_*\mathbb{Q}_X = \bigoplus_i {}^pR^ip_*\mathbb{Q}_X[-i]$$

and a unique direct sum decomposition

$${}^pR^ip_*\mathbb{Q}_X = \bigoplus_\alpha IC_{S_\alpha}(\mathcal{V}^i_\alpha)[\dim S_\alpha] \ ,$$

where $\mathcal{V}^i_\alpha$ is a local system on S^0_α. One thus has a direct sum decomposition

$$Rp_*\mathbb{Q}_X = \bigoplus_{i,\alpha} IC_{S_\alpha}(\mathcal{V}^i_\alpha)[-i + \dim S_\alpha] \ .$$

Remark. One has $\mathcal{V}^i_0 = R^{i-\dim S}p^0_*\mathbb{Q}_{X^0}$.

(1.4) **Proposition.** *Let X be smooth and $p : X \to S$ a projective map, stratified over $\{S_\alpha\}$. Keep the notation in (1.1) and (1.3).*

(1) Let

$$k^*_\alpha : Rp_*\mathbb{Q}_X \to i_{\alpha *}Rp_{\alpha *}\mathbb{Q}_{X_\alpha}$$

be the map induced by k_α. (Specifically, it is the composition of the adjunction map $Rp_*\mathbb{Q}_X \to i_{\alpha *}i_\alpha{}^*Rp_*\mathbb{Q}_X$ and the base change isomorphism $i_\alpha{}^*Rp_*\mathbb{Q}_X \cong Rp_{\alpha *}k^*_\alpha\mathbb{Q}_X = Rp_{\alpha *}\mathbb{Q}_{X_\alpha}$.) Upon applying the functor ${}^p\mathcal{H}^i$, one has a map

$${}^p\mathcal{H}^i(k^*_\alpha) : {}^pR^ip_*\mathbb{Q}_X \to i_{\alpha *}{}^pR^ip_{\alpha *}\mathbb{Q}_{X_\alpha} \ .$$

The restriction of this to $\bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}^i_\beta)[\dim S_\beta]$,

$$\bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}^i_\beta)[\dim S_\beta] \to i_{\alpha *}{}^pR^ip_{\alpha *}\mathbb{Q}_{X_\alpha} \ ,$$

is a split injection.

(2) Let

$$k_{\alpha *} : i_{\alpha *}Rp_{\alpha *}D_{X_\alpha}[-2d] \to Rp_*\mathbb{Q}_X$$

be the map induced by k_α. Here D_{X_α} is the dualizing complex of X_α; one has $D_X = \mathbb{Q}_X[2\dim X]$ and $D_{X_\alpha} = Rk^!_\alpha D_X$. Upon applying ${}^p\mathcal{H}^i$, one has

$${}^p\mathcal{H}^i(k_{\alpha *}) : i_{\alpha *}{}^p\mathcal{H}^i\big(Rp_{\alpha *}D_{X_\alpha}[-2d]\big) \to {}^pR^ip_*\mathbb{Q}_X \ .$$

Composition of this with the quotient map to $\bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}^i_\beta)[\dim S_\beta]$,

$${}^p\mathcal{H}^i(k_{\alpha *}) : i_{\alpha *}{}^p\mathcal{H}^i\big(Rp_{\alpha *}D_{X_\alpha}[-2d]\big) \to \bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}^i_\beta)[\dim S_\beta] \ ,$$

is a split surjection.

Proof. (1) Take a decomposition $Rp_*\mathbb{Q}_X = \bigoplus IC_{S_\alpha}(\mathcal{V}^i_\alpha)[-i + \dim S_\alpha]$. Examine the adjunction map on each summand to obtain the proof.

(2) is dual to (1).

(1.5) **Proposition.** *Let X be smooth and $p : X \to S$ a projective map, stratified over $\{S_\alpha\}$, and $\{\pi_\alpha : \tilde{X}_\alpha \to X_\alpha\}$ its resolution over $\{S_\alpha\}$. Keep the notation in (1.2) and (1.3).*

(1) Let $\iota_\alpha^* : Rp_*\mathbb{Q}_X \to i_{\alpha *}Rq_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha}$ be the map ι_α induces; applying ${}^p\mathcal{H}^i$, one has a map

$$ {}^p\mathcal{H}^i(\iota_\alpha^*) : {}^pR^ip_*\mathbb{Q}_X \to i_{\alpha *}{}^pR^iq_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha} \ . $$

The restriction of this map to the direct summand $IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha]$,

$$ {}^p\mathcal{H}^i(\iota_\alpha^*) : IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha] \to i_{\alpha *}{}^pR^iq_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha} \ , $$

is a split injection.

(2) Let $i_{\alpha *} : i_{\alpha *}Rq_{\alpha *}D_{\tilde{X}_\alpha}[-2d] \to Rp_*\mathbb{Q}_X$ be the map ι_α induces; applying ${}^p\mathcal{H}^i$ one has

$$ {}^p\mathcal{H}^i\iota_{\alpha *} : i_{\alpha *}{}^p\mathcal{H}^iRq_{\alpha *}D_{\tilde{X}_\alpha}[-2d] \to {}^pR^ip_*\mathbb{Q}_X \ . $$

The composition of this with the quotient map to $IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha]$,

$$ {}^p\mathcal{H}^i(\iota_{\alpha *}) : i_{\alpha *}{}^p\mathcal{H}^iR\tilde{p}_{\alpha *}D_{\tilde{X}_\alpha}[-2d] \to IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha] \ , $$

is a split surjection.

Proof. (1) The maps $\tilde{X}_\alpha \to X_\alpha \to X$ induce the maps

$$ Rp_*\mathbb{Q}_X \to i_{\alpha *}Rp_{\alpha *}\mathbb{Q}_{X_\alpha} \to i_{\alpha *}Rq_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha} \ . $$

Applying ${}^p\mathcal{H}^i$, one has

$$ {}^pR^ip_*\mathbb{Q}_X \to i_{\alpha *}{}^pR^ip_{\alpha *}\mathbb{Q}_{X_\alpha} \to i_{\alpha *}{}^pR^iq_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha} \ . $$

According to the theory of weights [BBD],[SaM], there is the category of mixed perverse sheaves on S with the following properties.

(i) An object of the category has as an underlying structure a perverse sheaf with weight filtration.

(ii) Pure perverse sheaves (objects with pure weight) form a semi-simple abelian category. A pure perverse sheaf is a direct sum of $IC_{S'_\alpha}(\mathcal{V}_\alpha)[\dim S'_\alpha]$, where (S'_α) is a stratification and $\mathcal{V}_\alpha$ are local systems of pure weight on $S'_\alpha - S'_{\alpha+1}$.

(iii) Perverse sheaves of "geometric origin", such as ${}^pR^ip_*\mathbb{Q}_X$ are mixed perverse sheaves. Additionally, ${}^pR^ip_*\mathbb{Q}_X$ and $R^iq_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha}$ are of pure weight i, and $R^ip_{\alpha *}\mathbb{Q}_{X_\alpha}$ is of weight $\leq i$ since p_α is proper.

Taking Gr_i^W, one has maps of pure perverse sheaves

$$ {}^pR^ip_*\mathbb{Q}_X \xrightarrow{\ \delta\ } i_{\alpha *}\,\mathrm{Gr}_i^W\,{}^pR^ip_{\alpha *}\mathbb{Q}_{X_\alpha} \xrightarrow{\ \gamma\ } i_{\alpha *}{}^pR^iq_{\alpha *}\mathbb{Q}_{\tilde{X}_\alpha} \ . $$

Our claim is the composition map $\gamma\circ\delta$, restricted to $IC_{S_\alpha}(\mathcal{V}^i_\alpha)[\dim S_\alpha]$, is a split injection. Since split injectivity and injectivity are equivalent in a semi-simple abelian category, we will just say injective from now.

By Proposition (1.4), the restriction of δ, $IC_{S_\alpha}(\mathcal{V}^i_\alpha)[\dim S_\alpha]\to i_{\alpha *}\operatorname{Gr}^W_i {}^pR^ip_{\alpha *}\mathbb{Q}_{X_\alpha}$, is a split injection. Let

$$i_{\alpha *}\operatorname{Gr}^W_i {}^pR^ip_{\alpha *}\mathbb{Q}_{X_\alpha}=IC_{S_\alpha}(\mathcal{W})[\dim S_\alpha]\oplus P$$

and

$$i_{\alpha *}{}^pR^iq_{\alpha *}\mathbb{Q}_{\tilde X_\alpha}=IC_{S_\alpha}(\mathcal{W}')[\dim S_\alpha]\oplus P'$$

be decompositions as in (ii), where $\mathcal{W}$ and $\mathcal{W}'$ are local systems on an open set of S_α, and P, P' are objects supported on a proper closed subset of S_α. The map obtained from δ,

$$\delta: IC_{S_\alpha}(\mathcal{V}^i_\alpha)[\dim S_\alpha]\to IC_{S_\alpha}(\mathcal{W})[\dim S_\alpha]\ ,$$

is injective. So one has only to show the map obtained from γ,

$$\gamma: IC_{S_\alpha}(\mathcal{W})[\dim S_\alpha]\to IC_{S_\alpha}(\mathcal{W}')[\dim S_\alpha]\ ,$$

is an injection. Since $\mathcal{W}\mapsto IC_{S_\alpha}(\mathcal{W})[\dim S_\alpha]$ is an exact functor from local systems to perverse sheaves, the injectivity is equivalent to the injectivity of the corresponding map of local systems $\mathcal{W}\to\mathcal{W}'$. To examine, look at the map fiberwise. Letting $(X_\alpha)_s={p_\alpha}^{-1}(s)$ and $(\tilde X_\alpha)_s={q_\alpha}^{-1}(s)$ for a general point $s\in S^0_\alpha$, one must show

$$Gr^W_i H^{i-\dim S_\alpha}((X_\alpha)_s)\to H^{i-\dim S_\alpha}((\tilde X_\alpha)_s)$$

is an injection. This is [De, part II, Proposition 8.2.5].

(2) Dual to (1).

§2. Intersection cohomology of projective maps.

In this section X, S are quasi-projective varieties, X is smooth, and $p: X\to S$ a projective map.

(2.1) Recall $D^b_c(S)$ is the bounded derived category of constructible sheaves. There is the perverse t-structure on this, in particular the functors ${}^p\tau_{\le}$ and ${}^p\tau_{\ge}$. For simplicity, denote them by $\tau_{\le}$ and $\tau_{\ge}$.

For the object $Rp_*\mathbb{Q}_X$, Theorem (1.3) implies $\tau_{\le k}Rp_*\mathbb{Q}_X$ is a subobject, and non-canonically a direct summand. One thus has a filtration by subobjects.

This filtration induces a filtration on $H^a(X)$, as follows. Let

$$F^\nu_S H^a(X):=\operatorname{Hom}_{D^b_c(S)}(\mathbb{Q}_S,\tau_{\le -\nu}\big(Rp_*\mathbb{Q}_X[a]\big)\ )\ ,$$

it is a subspace of $\mathrm{Hom}_{D^b_c(S)}(\mathbb{Q}_S, Rp_*\mathbb{Q}_X[a]) = H^a(X)$, with a non-canonical splitting. The decreasing filtration $F_S^\bullet$ on $H^a(X)$ thus defined has the following properties.

(1) $F_S^{-\dim S} H^a(X) = H^a(X)$.

(2) For ν large enough, $F_S^\nu H^a(X) = 0$.

(3) The graded pieces in the filtration are

$$\mathrm{Gr}^\nu_{F_S} H^a(X) = F_S^\nu / F_S^{\nu+1} H^a(X) = \mathrm{Hom}(\mathbb{Q}_S, {}^pR^{a-\nu}p_*\mathbb{Q}_X[\nu]) \ .$$

If p is stratified over $\{S_\alpha\}$, with the notation in Theorem (1.3) this is equal to

$$\mathrm{Hom}(\mathbb{Q}_S, \bigoplus_\alpha IC_{S_\alpha}(\mathcal{V}_\alpha^{a-\nu})[\dim S_\alpha][\nu]) = \bigoplus_\alpha IH^{\nu+\dim S_\alpha}(S_\alpha, \mathcal{V}_\alpha^{a-\nu}) \ .$$

(2.2) Assume p is stratified over $\{S_\alpha\}$, and $\{\pi_\alpha : \tilde{X}_\alpha \to X_\alpha\}$ its resolution over $\{S_\alpha\}$. For each $\alpha \geq 1$ the map ι_α induces maps

$$\iota_\alpha^* : H^a(X) \to H^a(\tilde{X}_\alpha)$$

and

$$\iota_{\alpha *} : H^{BM}_{2\dim X - a}(\tilde{X}_\alpha) \to H^a(X) \ .$$

Here H_*^{BM} denotes Borel-Moore homology. Taking graded pieces for $F_S^\bullet$, one has maps

$$\iota_\alpha^* : \mathrm{Gr}^\nu_{F_S} H^a(X) \to \mathrm{Gr}^\nu_{F_S} H^a(\tilde{X}_\alpha) \text{ and } \iota_{\alpha *} : \mathrm{Gr}^\nu_{F_S} H^{BM}_{2\dim X - a}(\tilde{X}_\alpha) \to \mathrm{Gr}^\nu_{F_S} H^a(X) \ .$$

The next Proposition follows from Proposition (1.5).

(2.3) **Proposition.** *(1) The kernel of the map* $\sum_{\alpha\geq 1} \iota_\alpha^* : \mathrm{Gr}^\nu_{F_S} H^a(X) \to \bigoplus_{\alpha \geq 1} \mathrm{Gr}^\nu_{F_S} H^a(\tilde{X}_\alpha)$ *is equal to* $IH^{\nu+\dim S}(S, \mathcal{V}_0^{a-\nu})$.

(2) The image of the map $\sum_{\alpha\geq 1} \iota_{\alpha *} : \bigoplus_{\alpha\geq 1} \mathrm{Gr}^\nu_{F_S} H^{BM}_{2\dim X - a}(\tilde{X}_\alpha) \to \mathrm{Gr}^\nu_{F_S} H^a(X)$ *is equal to* $\bigoplus_{\alpha \geq 1} IH^{\nu + \dim S_\alpha}(S_\alpha, \mathcal{V}_\alpha^{a-\nu})$.

In the case p is birational, we can describe the intersection cohomology of S in terms of the cohomology of X and $\{\tilde{X}_\alpha\}$, and the filtrations $F_S^\bullet$.

(2.4) **Theorem.** *If* $p : X \to S$ *is a birational map,* $d = \dim S$*,*

$$IH^a(S) = \frac{\bigcap_{\alpha\geq 1} (\iota_\alpha^*)^{-1} F_S^{a-d+1} H^a(\tilde{X}_\alpha)}{\sum_{\alpha \geq 1} \iota_{\alpha *} F_S^{a-d+1} H^{BM}_{2d-a}(\tilde{X}_\alpha)} \ .$$

We omit the proof, which is similar to the proof of an analogous formula for the intersection Chow group, Theorem (3.9).

§3. The intersection Chow group (under hypotheses).

(3.1) Let S be a quasi-projective variety over $k = \mathbb{C}$. Denote by $(Smooth/k, Proj/S)$ the category of smooth quasi-projective varieties X equipped with projective maps to S, $p : X \to S$.

For X, Y in $(Smooth/k, Proj/S)$, $\mathrm{CH}_a(X \times_S Y)$ denotes the rational Chow group of dimension a of the variety $X \times_S Y$. An element of this group is a *relative correspondence* from X to Y.

If X, Y, Z are in $(Smooth/k, Proj/S)$, with Y equi-dimensional, we have a map, the *composition* of correspondences,

$$\mathrm{CH}_a(X \times_S Y) \otimes \mathrm{CH}_b(Y \times_S Z) \to \mathrm{CH}_{a+b-\dim Y}(X \times_S Z)$$

which sends $u \otimes v$ to $v \circ u$, see [CH] for the definition. The composition is associative. In particular if X has connected components X_i, $\bigoplus_i \mathrm{CH}_{\dim X_i}(X \times_S X_i)$ is a ring with the composition as multiplication. The identity element is the class of the diagonal $\Delta_X = id$.

Let $\mathcal{CHM}(S)$ be the pseudo-abelian category of Chow motives over S, defined in [CH]. It has the following properties.

(1) An object of $\mathcal{CHM}(S)$ is of the form

$$(X, r, P) = (X/S, r, P)$$

where X is a smooth variety over k with a projective (not necessarily smooth) map $p : X \to S$, $r \in \mathbb{Z}$, and if X has connected components X_i,

$$P \in \bigoplus_i \mathrm{CH}_{\dim X_i}(X \times_S X_i)$$

such that $P \circ P = P$. If (Y, s, Q) is another object, Y_j the components of Y, then

$$\mathrm{Hom}((X, r, P), (Y, s, Q)) = Q \circ (\bigoplus_j \mathrm{CH}_{\dim Y_j - s + r}(X \times_S Y_j)) \circ P \ .$$

Composition of morphisms is induced from the composition of relative correspondences.

Denote by $M = (X, r, P) \mapsto M(n) = (X, r + n, P)$ the "Tate twist" functor.

(2) There is a functor $h : (Smooth/k, Proj/S)^{opp} \to \mathcal{CHM}(S)$, which sends $p : X \to S$ to the object $h(X/S) = (X/S, 0, id)$. Note $h(X/S)(n) = (X/S, n, id)$.

If X and Y are objects of $(Smooth/k, Proj/S)$ and $f : X \to Y$ is a map over S, there corresponds a morphism

$$f^* : h(Y/S) \to h(X/S) \ .$$

If X, Y are equidimensional, there corresponds

$$f_* : h(X/S) \to h(Y/S)(\dim Y - \dim X) \ .$$

(3) There is a functor

$$\mathrm{CH}^0(S,-) : \mathcal{CHM}(S) \to Vect_{\mathbb{Q}}$$

(the target is the category of $\mathbb{Q}$-vector spaces) such that $\mathrm{CH}^0(S,(X,r,P)) = \mathrm{CH}^0((X,r,P)) = P_* \,\mathrm{CH}^r(X)$.

Define $\mathrm{CH}^t(S,-) : \mathcal{CHM}(S) \to Vect_{\mathbb{Q}}$ by $\mathrm{CH}^t(S,K) = \mathrm{CH}^t(K) = \mathrm{CH}^0(K(t))$. Note $\mathrm{CH}^r(h(X/S)) = \mathrm{CH}^0(h(X/S)(r)) = \mathrm{CH}^r(X)$.

(4) There is the realization functor

$$\rho : \mathcal{CHM}(S) \to D^b_c(S)$$

such that on objects

$$(X,r,P) \mapsto P_* Rp_* \mathbb{Q}_X[2r] \ .$$

Here $P_* := \rho(P) \in \mathrm{End}_{D^b_c(S)}(Rp_*\mathbb{Q}_X)$ is a projector, and $P_*Rp_*\mathbb{Q}_X$ is its image, which exists since $Rp_*\mathbb{Q}_X$ is a direct sum of perverse sheaves with shifts.

(3.2) For $p : X \to S$ in $(Smooth/k, Proj/S)$ and $r \in \mathbb{Z}$, let

$${}^p\mathcal{H}^*(X/S,r) := \bigoplus_i {}^pR^{i+2r}p_*\mathbb{Q}_X \ ,$$

called the *total perverse cohomology*, a graded perverse sheaf (grading by i). Denote the category of graded perverse sheaves by $gr\,Perv(S)$. One has a map

$$\mathrm{Hom}_{\mathcal{CHM}(S)}((X/S,r,id),(Y/S,s,id)) \to \mathrm{Hom}_{gr\,Perv(S)}({}^p\mathcal{H}^*(X/S,r), {}^p\mathcal{H}^*(Y/S,s)) \ ,$$

obtained using the functor ρ and perverse cohomology. The image of this map is denoted by $\mathrm{Hom}_{gr\,Perv(S)}({}^p\mathcal{H}^*(X/S,r), {}^p\mathcal{H}^*(Y/S,s))_{alg}$. It is proved in [CH] that this group is closed under composition.

The pseudo-abelian category of Grothendieck motives over S, denoted by $\mathcal{M}(S)$ has objects $(X/S,r,p)$ where X/S is in $(Smooth/k, Proj/S)$, and $p \in \mathrm{End}({}^p\mathcal{H}^*(X/S,r))_{alg}$ is an idempotent. Morphisms are defined by

$$\mathrm{Hom}((X,r,p),(Y,s,q)) = q \circ \mathrm{Hom}({}^p\mathcal{H}^*(X/S,r), {}^p\mathcal{H}^*(Y/S,s))_{alg} \circ p \ .$$

There is a canonical full functor $\mathrm{cano} : \mathcal{CHM}(S) \to \mathcal{M}(S)$ and a faithful realization functor $\rho : \mathcal{M}(S) \to gr\,Perv(S)$. The following diagram commutes.

$$\begin{array}{ccc} \mathcal{CHM}(S) & \xrightarrow{\mathrm{cano}} & \mathcal{M}(S) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ D^b_c(S) & \xrightarrow{{}^p\mathcal{H}^*} & gr\,Perv(S) \end{array}$$

Here ${}^p\mathcal{H}^* = \bigoplus_i {}^p\mathcal{H}^i$ is the total perverse cohomology functor.

(3.3) **Theorem.** [CH, §7] *Assume the conjecture of Grothendieck and the conjecture of Bloch-Beilinson-Murre (recalled later). Let $p : X \to S$ be as before. Let $\{S_\alpha\}$ be a Whitney stratification of S over which p is stratified. Then:*

(1) There are local systems $\mathcal{V}_\alpha^j$ on $S_\alpha - S_{\alpha+1}$, non-canonical direct sum decomposition in $CH\mathcal{M}(S)$

$$h(X/S) = \bigoplus_{j,\alpha} h_\alpha^j(X/S)$$

and isomorphisms

$$\rho(h_\alpha^j(X/S)) \cong IC_{S_\alpha}(\mathcal{V}_\alpha^j)[-j + \dim S_\alpha]$$

in $D_c^b(S)$.

(2) For each i, the sum $\bigoplus_{j\leq i,\, \alpha} h_\alpha^j(X/S)$ is a well-defined subobject of $h(X/S)$ (independent of the decomposition).

(3) The category $\mathcal{M}(S)$ is semi-simple abelian, and the functor $\rho : \mathcal{M}(S) \to Perv(S)$ is exact and faithful.

(4) For $i \in \mathbb{Z}$, let $CH\mathcal{M}(S)^i$ (resp. $\mathcal{M}(S)^i$) be the full subcategory of $CH\mathcal{M}(S)$ (resp. $\mathcal{M}(S)$) consisting of objects with realizations of pure perverse degree i. Then the canonical functor $cano : CH\mathcal{M}(S)^i \to \mathcal{M}(S)^i$ is an equivalence of categories.

We recall the conjectures mentioned in the theorem.

1. Grothendieck's Standard conjecture.

This concerns the functorial behavior of cycle classes in (singular or étale) cohomology. It has two components, the Lefschetz type conjecture and the Hodge type conjecture. For $k = \mathbb{C}$, the latter holds true (Hodge index theorem). The Lefschetz type conjecture itself consists of three statements, Conjecture (A), (B) and (C). Conjecture (C) says: the Künneth components of the diagonal class of a smooth projective variety are algebraic.

The standard conjecture implies the semi-simplicity of the category of pure homological motives (Grothendieck).

2. Bloch-Beilinson-Murre conjecture.

This conjecture on the existence of a filtration on the Chow group is originally due to S. Bloch, and studied by A. Beilinson, J.P. Murre, U. Jannsen, and Shuji Saito among others.

A formulation due to Murre, which is closely related to the statement of the above theorem, consists of the existence of an orthogonal decomposition to projectors of the diagonal class Δ_X in $\mathrm{CH}(X \times X)$. To be precise, the conjecture states:

(A) Let X be a smooth projective variety. There exists a decomposition $\Delta_X = \sum \Pi^i$ to orthogonal projectors in the Chow ring such that the cohomology class of Π^i is the Künneth component $\Delta(2\dim X - i, i)$. The decomposition is called the *Chow-Künneth decomposition*.

(B) Π^i with $i = 0, \cdots, r-1$ or $i = 2d, \cdots, 2r+1$ acts as zero on $\mathrm{CH}^r(X)$.

(C) Put $F^0 = \mathrm{CH}^r(X)$, $F^1 = \operatorname{Ker}\Pi^{2r}$, $F^2 = \operatorname{Ker}(\Pi^{2r-1}|F^1)$, $\cdots$, $F^r = \operatorname{Ker}(\Pi^{r+1}|F^{r-1})$,$F^{r+1} = 0$. This is independent of the choice of the decomposition in (A).

(D) $F^1 = \mathrm{CH}^r(X)_{hom}$, the homologically trivial part.

For the rest of this section, we assume the conjecture of Grothendieck and the conjecture of Bloch-Beilinson-Murre.

(3.4) Let $p : X \to S$ be as above. Define a subobject ${}^p\tau_{\leq i}$ of $h(X/S)(r)$ by:

$${}^p\tau_{\leq i}(h(X/S)(r)) := \bigoplus_{j \leq i+2r,\, \alpha} h^j_\alpha(X/S)(r)$$

the sum over (j, α) with $j \leq i + 2r$. This is a subobject with a non-canonical splitting. ${}^p\tau_{\leq i}$ gives an increasing filtration by subobjects. From now we write $\tau_{\leq i}$ for ${}^p\tau_{\leq i}$. The subquotients are

$$\tau_{\leq i}/\tau_{\leq i-1}(h(X/S)(r)) = \bigoplus_\alpha h^{i+2r}_\alpha(X/S)(r) \ .$$

This decomposition is uniquely determined, independent of $\{S_\alpha\}$ (this follows from (3.3), (4)).

Correspondingly $\mathrm{CH}^r(X) = \mathrm{CH}^0(S, h(X/S)(r))$ has a decreasing filtration $F^\bullet_S$ defined by

$$F^\nu_S \mathrm{CH}^r(X) = \mathrm{CH}^0\big(S, \tau_{\leq -\nu}(h(X/S)(r))\big) \subset \mathrm{CH}^r(X) \ .$$

Note $F^\nu_S \mathrm{CH}^r(X) = \mathrm{CH}^r(X)$ for ν small enough, and $F^\nu_S \mathrm{CH}^r(X) = 0$ for ν large enough. We conjecture $\mathrm{CH}^r(X) = F_S^{-\dim S} \mathrm{CH}^r(X)$. The graded quotients are

$$\mathrm{Gr}^\nu_{F_S} \mathrm{CH}^r(X) = \mathrm{CH}^0(S, \bigoplus_\alpha h^{2r-\nu}_\alpha(X/S)(r)) = \mathrm{CH}^r(S, \bigoplus_\alpha h^{2r-\nu}_\alpha(X/S)) \ .$$

Each piece $\mathrm{CH}^r(S, h^{2r-\nu}_\alpha(X/S))$ is a direct summand of $\mathrm{Gr}^\nu_{F_S} \mathrm{CH}^r(X)$, in particular a subgroup. Thus one can write

$$\mathrm{CH}^r(S, h^{2r-\nu}_\alpha(X/S)) = A/F^{\nu+1}_S$$

for a subgroup $A \subset F^\nu_S$. We write $A = F^{\nu+1}_S + \mathrm{CH}^r(S, h^{2r-\nu}_\alpha(X/S))$ with a slight abuse of notation.

The filtration $\tau_{\leq i}$ is respected by morphisms in $\mathcal{CHM}(S)$, see [CH, Theorem (7.4),(1)]. If $u : h(X/S)(r) \to h(Y/S)(s)$ is a morphism, there is a unique morphism $\tau_{\leq i}u : \tau_{\leq i}h(X/S)(r) \to \tau_{\leq i}h(Y/S)(s)$ such that the following diagram commutes.

$$\begin{array}{ccc} h(X/S)(r) & \xrightarrow{u} & h(Y/S)(s) \\ \uparrow & & \uparrow \\ \tau_{\leq i}\, h(X/S)(r) & \xrightarrow{\tau_{\leq i} u} & \tau_{\leq i}\, h(Y/S)(s) \end{array}$$

Thus one has induced morphisms $\tau_{\leq i}/\tau_{\leq i-1}h(X/S)(r) \to \tau_{\leq i}/\tau_{\leq i-1}h(Y/S)(s)$. This is the direct sum, for α, of morphisms $h_\alpha^{i+2r}(X/S)(r) \to h_\alpha^{i+2s}(Y/S)(s)$.

(3.5) **Definition.** Let X^0 and S^0 be smooth quasi-projective, and $p^0 : X^0 \to S^0$ be a smooth projective map. Let S be a quasi-projective variety and $S^0 \to S$ an open immersion.

Take a smooth variety X, an open immersion $X^0 \to X$ and a projective map $p : X \to S$ which extends p^0. Let $h(X/S) = \bigoplus h_\alpha^j(X/S)$ be a decomposition as in Theorem (3.3). Define the *intersection Chow group* of the higher direct image $R^i p_*^0 \mathbb{Q}_{X^0}$ to be

$$\mathrm{ICH}^r(S, R^i p_*^0 \mathbb{Q}_{X^0}) := \mathrm{CH}^0(S, h_0^{i+\dim S}(X/S)(r)\,)\ .$$

The group depends on $p^0 : X^0 \to S^0$, S and i. As we show below, it does not depend on the choice of $p : X \to S$. One should take $R^i p_*^0 \mathbb{Q}_{X^0}$ as a notation; the intersection Chow group is not determined by S and the local system $R^i p_*^0 \mathbb{Q}_{X^0}$ alone.

(3.6) **Proposition.** *(1) The object $h_0^j(X/S)$ is independent of the choice of $p : X \to S$, up to canonical isomorphism. Hence $\mathrm{ICH}^r(S, R^i p_*^0 \mathbb{Q}_{X^0})$ is well-defined.*

(2) Let $S^1 \subset S^0$ be an open set, $X^1 = p^{-1}(S^1)$, and $p^1 : X^1 \to S^1$ the induced map. Then one has a canonical isomorphism $\mathrm{ICH}^r(S, R^i p_*^0 \mathbb{Q}_{X^0}) = \mathrm{ICH}^r(S, R^i p_*^1 \mathbb{Q}_{X^1})$.

Proof. (1) More precisely if $p' : X' \to S$ is another extension of p^0, there is an isomorphism

$$\iota(X, X') : h_0^j(X/S) \to h_0^j(X'/S)\ ;$$

if $X'' \to S$ is another such, the three isomorphisms satisfy the cocycle condition $\iota(X, X'') = \iota(X', X'')\iota(X, X')$.

To prove this, one is reduced to the case where there is a map $f : X' \to X$ over S, extending the identity on X^0. Then $f^* : h(X/S) \to h(X'/S)$ induces an isomorphism $f^* : h_0^j(X/S) \to h_0^j(X'/S)$.

(2) This is obvious.

(3.7) **Definition.** Let S be an irreducible quasi-projective variety of dimension d, and $p : X \to S$ its desingularization (proper birational map from a smooth variety). Define the *intersection Chow group* by

$$\mathrm{ICH}^r(S) := \mathrm{CH}^0(S, h_0^d(X/S)) \ .$$

This is a special case of Definition (3.5),where one takes a smooth open set $S^0 \subset S$, $p^0 = id : X^0 = S^0 \to S^0$, and $i = 0$. By Proposition (3.6), the intersection Chow group is well-defined.

The map $\iota_\alpha : \tilde{X}_\alpha \to X$ induces maps ${\iota_\alpha}^* : \mathrm{CH}^r(X) \to \mathrm{CH}^r(\tilde{X}_\alpha)$ and $\iota_{\alpha *} : \mathrm{CH}_{\dim X - r}(\tilde{X}_\alpha) \to \mathrm{CH}^r(X)$, thus also maps between the graded pieces

$$\mathrm{Gr}^\nu_{F_S} \mathrm{CH}_{\dim X-r}(\tilde{X}_\alpha) \xrightarrow{\iota_{\alpha *}} \mathrm{Gr}^\nu_{F_S} \mathrm{CH}^r(X) \xrightarrow{{\iota_\alpha}^*} \mathrm{Gr}^\nu_{F_S} \mathrm{CH}^r(\tilde{X}_\alpha) \ .$$

Taking the sum over $\alpha \geq 1$, one has maps

$$\sum_{\alpha\geq 1} \iota_\alpha^* : \mathrm{CH}^r(X) \longrightarrow \bigoplus_{\alpha \geq 1} \mathrm{CH}^r(\tilde{X}_\alpha) \ ,$$

$$\sum_{\alpha\geq 1} \iota_{\alpha *} : \bigoplus_{\alpha \geq 1} \mathrm{CH}_{\dim X-r}(\tilde{X}_\alpha) \to \mathrm{CH}^r(X) \ ,$$

as well as the maps on graded pieces.

Proposition (1.5) and Theorem (3.3) imply:

(3.8)**Proposition.** *(1) The kernel of the map* $\sum_{\alpha\geq 1} \iota_\alpha^* : \mathrm{Gr}^\nu_{F_S} \mathrm{CH}^r(X) \to \bigoplus_{\alpha\geq 1} \mathrm{Gr}^\nu_{F_S} \mathrm{CH}^r(\tilde{X}_\alpha)$ *is equal to* $\mathrm{CH}^r(h_0^{2r-\nu}(X/S))$.

(2) The image of the map $\sum_{\alpha\geq 1} \iota_{\alpha *} : \bigoplus_{\alpha\geq 1} \mathrm{Gr}^\nu_{F_S} \mathrm{CH}_{\dim X-r}(\tilde{X}_\alpha) \to \mathrm{Gr}^\nu_{F_S} \mathrm{CH}^r(X)$ *is equal to* $\mathrm{CH}^r(S, \bigoplus_{\alpha\geq 1} h_\alpha^{2r-\nu}(X/S))$.

(3.8.1) **Corollary.** *One has*

$$\begin{aligned}\mathrm{ICH}^r(S, R^i p^0_* \mathbb{Q}_X) &= \mathrm{Ker}[\, \mathrm{Gr}^{2r-i-\dim S}_{F_S} \mathrm{CH}^r(X) \to \bigoplus_{\alpha\geq 1} \mathrm{Gr}^{2r-i-\dim S}_{F_S} \mathrm{CH}^r(\tilde{X}_\alpha)\,] \\ &= \mathrm{Cok}[\, \bigoplus_{\alpha\geq 1} \mathrm{Gr}^{2r-i-\dim S}_{F_S} \mathrm{CH}_{\dim X-r}(\tilde{X}_\alpha) \\ &\quad \to \mathrm{Gr}^{2r-i-\dim S}_{F_S} \mathrm{CH}^r(X)\,] \ .\end{aligned}$$

(3.9) **Theorem.** *If* $p : X \to S$ *is a birational map,* $d = \dim S$,

$$\mathrm{ICH}^r(S) = \frac{\bigcap_{\alpha\geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \mathrm{CH}^r(\tilde{X}_\alpha)}{\sum_{\alpha\geq 1} \iota_{\alpha *} F_S^{2r-d+1} \mathrm{CH}_{d-r}(\tilde{X}_\alpha)} \ .$$

Proof. The map $\sum_{\alpha\geq 1}\iota_\alpha^*\ :\ \mathrm{Gr}^\nu_{F_S}\mathrm{CH}^r(X)\longrightarrow\bigoplus_{\alpha\geq 1}\mathrm{Gr}^\nu_{F_S}\mathrm{CH}^r(\tilde X_\alpha)$ is injective for $\nu\neq 2r-d$, and has kernel equal to $\mathrm{ICH}^r(S)$ if $\nu=2r-d$. The map $\sum_{\alpha\geq 1}\iota_{\alpha *}\ :\ \bigoplus_{\alpha\geq 1}\mathrm{Gr}^\nu_{F_S}\mathrm{CH}_{\dim X-r}(\tilde X_\alpha)\to\mathrm{Gr}^\nu_{F_S}\mathrm{CH}^r(X)$ is surjective for $\nu\neq 2r-d$ and has image equal to $\mathrm{CH}^r(S,\bigoplus_{\alpha\geq 1}h^d_\alpha(X/S)\,)$ for $\nu=2r-d$. So

$$\bigcap_{\alpha\geq 1}(\iota_\alpha^*)^{-1}F_S^{2r-d+1}\,\mathrm{CH}^r(\tilde X_\alpha)=F_S^{2r-d+1}\,\mathrm{CH}^r(X)+\mathrm{ICH}^r(S)$$

and

$$\sum_{\alpha\geq 1}\iota_{\alpha *}F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(\tilde X_\alpha)=F_S^{2r-d+1}\,\mathrm{CH}^r(X)\ ,$$

from which the claim follows.

We note some properties of the filtration $F_S^\bullet$.

(3.10) **Proposition.** *Let X be smooth and $p:X\to S$ a projective map. Let $S\hookrightarrow S'$ be a closed immersion of quasi-projective varieties. Then the filtration $F_S^\bullet$ and $F_{S'}^\bullet$ on $\mathrm{CH}^r(X)$ coincide.*

(3.11) **Proposition.** *Let X and Y be smooth varieties, projective over S, and $f:X\to Y$ be a projective surjective map. Then:*

(1) The injection $f^*:\mathrm{CH}^r(Y)\to\mathrm{CH}^r(X)$ is strictly compatible with the filtrations F_S, namely $F_S^\nu\,\mathrm{CH}^r(Y)=(f^*)^{-1}F_S^\nu\,\mathrm{CH}^r(X)$.

(2) The surjection $f_*:\mathrm{CH}_s(X)\to\mathrm{CH}_s(Y)$ is strictly compatible with the filtrations F_S, namely $f_*F_S^\nu\,\mathrm{CH}^r(X)=F_S^\nu\,\mathrm{CH}^r(Y)$.

Proof. (1) Take a smooth subvariety $X'\subset X$ such that the restriction $f|_{X'}:X'\to Y$ is generically finite. Considering the composition of $f^*:\mathrm{CH}^r(Y)\to\mathrm{CH}^r(X)$ with the restriction $\mathrm{CH}^r(X)\to\mathrm{CH}^r(X')$, one is reduced to the case where f is generically finite.

In that case the map $f^*:h(Y/S)\to h(X/S)$ has a left inverse $(1/d)f_*:h(X/S)\to h(Y/S)$, where d is the degree of p. Twisting and taking $\tau_{\leq -\nu}$, one has

$$f^*:\tau_{\leq-\nu}\,h(Y/S)(r)\to\tau_{\leq-\nu}\,h(X/S)(r)$$

with left inverse $(1/d)f_*$. The claim follows.

(2) Similar to (1).

§4. Unconditional theory of the intersection Chow group.

(4.1) For a smooth projective variety X over k, Shuji Saito defined a filtration $F^\bullet$ on the Chow group $\mathrm{CH}^r(X)$, [SaS-1, 2]. In a similar way, if X is a smooth variety with a projective map to S, one can define a filtration $F_S^\bullet$ on the Chow group of X.

Let S be a quasi-projective variety, and X a smooth variety with a projective map $p : X \to S$. For another smooth variety W with a projective map $q : W \to S$, an element $\Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X)$ induces a map $\Gamma_* : \mathrm{CH}^{r-s}(W) \to \mathrm{CH}^r(X)$, see [CH]. The cycle class of Γ in Borel-Moore homology gives a map $\Gamma_* : Rq_*\mathbb{Q}_W[-2s] \to Rp_*\mathbb{Q}_X$; passing to perverse cohomology one has a map (for each ν)

$$ {}^p\mathcal{H}^{2r-\nu}\Gamma_* : {}^p\mathcal{H}^{2r-2s-\nu} Rq_*\mathbb{Q}_W \to {}^p\mathcal{H}^{2r-\nu} Rp_*\mathbb{Q}_X \ . $$

(Here ${}^p\mathcal{H}^*$ stands for perverse cohomology.)

We define a filtration $F_S^\bullet$ on $\mathrm{CH}^r(X)$ as follows. Let $\mathrm{CH}^r(X) = F_S^{-\dim S}\,\mathrm{CH}^r(X)$. Assume F_S^ν has been defined. Define

$$ F_S^{\nu+1}\,\mathrm{CH}^r(X) := \sum \mathrm{Image}[\Gamma_* : F_S^\nu\,\mathrm{CH}^{r-s}(W) \to \mathrm{CH}^r(X)\,] $$

where the sum is over $(q : W \to S, \Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X)\,)$ satisfying the following condition: the map ${}^p\mathcal{H}^{2r-\nu}\Gamma_* : {}^p\mathcal{H}^{2r-2s-\nu} Rq_*\mathbb{Q}_W \to {}^p\mathcal{H}^{2r-\nu} Rp_*\mathbb{Q}_X$ is zero. One can show:

(4.2) **Proposition.** *The filtration $F_S^\bullet$ on $\mathrm{CH}^r(X)$ has the following properties.*

(1) $\mathrm{CH}^r(X) = F_S^{-\dim S}\,\mathrm{CH}^r(X)$. For any $\Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X)$, the induced map $\Gamma_* : \mathrm{CH}^{r-s}(W) \to \mathrm{CH}^r(X)$ respects $F_S^\bullet$.

(2) If ${}^p\mathcal{H}^{2r-\nu}\Gamma_* : {}^p\mathcal{H}^{2r-2s-\nu} Rq_*\mathbb{Q}_W \to {}^p\mathcal{H}^{2r-\nu} Rp_*\mathbb{Q}_X$ is zero, then Γ_* sends $F_S^\nu\,\mathrm{CH}^{r-s}(W)$ to $F_S^{\nu+1}\,\mathrm{CH}^r(X)$.

(3) The filtration is the smallest one with properties (1) and (2).

(4.3) **Definition.** Let S be an irreducible quasi-projective variety of dimension d, and $p : X \to S$ a resolution of singularities. Take a Whitney stratification $\{S_\alpha\}$ of S and resolutions $\tilde{X}_\alpha \to X_\alpha$ so that $(p, \{\tilde{X}_\alpha \to X_\alpha\})$ is stratified over $\{S_\alpha\}$. Recall $\iota_\alpha : \tilde{X}_\alpha \to X$ are the induced maps, which give rise to maps $\iota_{\alpha *} : \mathrm{CH}_{d-r}(\tilde{X}_\alpha) \to \mathrm{CH}^r(X)$ and ${\iota_\alpha}^* : \mathrm{CH}^r(X) \to \mathrm{CH}^r(\tilde{X}_\alpha)$.

Define the *intersection Chow group* as a subquotient of the Chow group of X given by:

$$ \mathrm{ICH}^r(S) := \frac{\bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1}\,\mathrm{CH}^r(\tilde{X}_\alpha)}{\sum_{\alpha \geq 1} \iota_{\alpha *} F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(\tilde{X}_\alpha)} \ . $$

(4.4) **Theorem.** $\mathrm{ICH}^r(S)$ *is well-defined (up to canonical isomorphism) independent of the choice of a desingularization $p : X \to S$, a stratification and a resolution.*

(4.5) **Proposition.** *Let X and Y be smooth varieties, projective over S, and $f : X \to Y$ be a projective surjective map. Then:*

(1) The injection $f^* : \mathrm{CH}^r(Y) \to \mathrm{CH}^r(X)$ is strictly compatible with the filtrations F_S, namely $F_S^\nu \mathrm{CH}^r(Y) = (f^*)^{-1} F_S^\nu \mathrm{CH}^r(X)$.

(2) The surjection $f_* : \mathrm{CH}_s(X) \to \mathrm{CH}_s(Y)$ is strictly compatible with the filtrations F_S, namely $f_* F_S^\nu \mathrm{CH}^r(X) = F_S^\nu \mathrm{CH}^r(Y)$.

Proof. Take a smooth subvariety $X' \subset X$ that maps generically finitely onto Y. Let $i : X' \to X$ be the inclusion and $f' := f \circ i : X' \to Y$. For (1), suppose $\alpha \in \mathrm{CH}^r(Y)$ such that $f^*\alpha \in F_S^\nu \mathrm{CH}^r(X)$. Then $f'_* i^* f^* \alpha = d\,\alpha \in F_S^\nu \mathrm{CH}^r(Y)$ ($d = \deg f'$) by the functoriality of F_S with respect to pull-back and push-forward. The proof of (2) is similar.

In the rest of this section we give the proof of Theorem (4.4).

The definition depends on X, $\mathcal{S} = \{S_\alpha\}$ and $\{\pi_\alpha : \tilde{X}_\alpha \to X_\alpha\}$. Let

$$N = \bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \mathrm{CH}^r(\tilde{X}_\alpha) \quad \text{and} \quad D = \sum_{\alpha \geq 1} \iota_{\alpha *} F_S^{2r-d+1} \mathrm{CH}_{d-r}(\tilde{X}_\alpha)$$

be the subgroups of $\mathrm{CH}^r(X)$, which appear in (4.4). If $(X', \mathcal{S}', \pi'_\alpha : \tilde{X}'_\alpha \to X'_\alpha)$ is another choice, we have the similarly defined subgroups N', D' of $\mathrm{CH}^r(X')$. We must show there is a canonical isomorphism $N/D \cong N'/D'$.

(I) Assume $X = X'$, $\mathcal{S} = \mathcal{S}'$, and only $(\tilde{X}_\alpha \to X_\alpha)$ differs. One may assume there are projective surjective maps $g_\alpha : \tilde{X}'_\alpha \to \tilde{X}_\alpha$ over X_α. Then (4.5) shows $N = N'$ and $D = D'$.

(II) Assume $X = X'$ and $\mathcal{S}$ and $\mathcal{S}'$ differ. One may assume $\mathcal{S}'$ is a refinement of $\mathcal{S}$.

Let $(p : X \to S, \{\pi_\alpha : \tilde{X}_\alpha \to X_\alpha\})$ be a resolution of p over $\mathcal{S}$. Let $S_{\alpha i}$ be the irreducible components of S_α, $X_{\alpha i} = p^{-1}(S_{\alpha i})$, and $\tilde{X}_{\alpha i} = q_\alpha^{-1}(S_{\alpha i})$.

We construct a resolution of p over $\mathcal{S}'$ as follows. Let $S'_{\alpha j}$ be the irreducible components of S'_α, for $\alpha \geq 1$. Let $X'_{\alpha j} = p^{-1}(S'_{\alpha j})$.

If $S'_{\alpha j}$ is an irreducible component of S_α, say $S'_{\alpha j} = S_{\alpha i}$, let $\tilde{X}'_{\alpha j} = \tilde{X}_{\alpha i}$. If $S'_{\alpha j} \not\subset S_\alpha$ let S_β be such that $S'_{\alpha j} \subset S_\beta$ and $S'_{\alpha j} \not\subset S_{\beta+1}$. Take smooth $\tilde{X}'_{\alpha j}$ so that there are a projective surjective map $\tilde{X}'_{\alpha j} \to X'_{\alpha j}$ and a map $g : \tilde{X}'_{\alpha j} \to \tilde{X}_\beta$

over S, namely the following diagram commutes.

$$\begin{array}{ccc} \tilde{X}'_{\alpha j} & \xrightarrow{g} & \tilde{X}_\beta \\ \downarrow & & \downarrow \\ X'_{\alpha j} & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ S'_{\alpha j} & \hookrightarrow & S_\beta \end{array}$$

Let $\tilde{X}'_\alpha = \amalg_j \tilde{X}'_{\alpha j}$, and $\pi'_\alpha : \tilde{X}'_\alpha \to X'_\alpha$ the induced map.

We now show $N = N'$. Clearly $N' \subset N$. The inclusion $N \subset N'$ follows from the existence of the maps g. Similarly one shows $D = D'$.

(III) Assume now X and X' are not equal. In view of the weak factorization theorem of birational maps [AKMW], one may assume X' is the blow-up of X along a smooth center.

Let $\mu : X' \to X$ be the blow-up of a smooth center $Z \subset X$. Assume the maps $X' \to X \to S$ are stratified over $\mathcal{S}$. Let $D \subset N \subset \mathrm{CH}^r(X)$ and $D' \subset N' \subset \mathrm{CH}^r(X')$ be defined as above.

In the rest of this section we show: $\mu^*(N) \subset N'$, $\mu_*(N') \subset N$, $\mu^*(D) \subset D'$, $\mu_*(D') \subset D$. Letting $K = \operatorname{Ker} \mu_*$, $N' = N \bigoplus (K \cap N')$, $D' = D \bigoplus (K \cap D')$, and $K \cap N' = K \cap D'$. Hence

$$\mu^* : N/D \xrightarrow{\sim} N'/D' \ .$$

(4.6) Let S be a quasi-projective variety, Z a smooth variety with a projective map $Z \to S$, and $\pi : E \to Z$ a $\mathbb{P}^n$-bundle. Let $\xi \in \mathrm{CH}^1(E)$ be the first Chern class of $\mathcal{O}_E(1)$. One has $\mathrm{CH}^r(E) = \bigoplus_{0 \le i \le n} \mathrm{CH}^{r-i}(Z) \cdot \xi^i$. One easily shows the following proposition. (From now we will often not write ξ^i.)

Proposition. *The above decomposition is compatible with the filtrations $F_S^\bullet$, namely*

$$F_S^\bullet \mathrm{CH}^r(E) = \bigoplus_{0 \le i \le n} F_S^\bullet \mathrm{CH}^{r-i}(Z) \ .$$

(4.7) Let S be quasi-projective, X smooth, and $p : X \to S$ be a projective map. We do not assume p is birational, although we are mainly interested in that case. Let $Z \subset X$ be a smooth subvariety and $\mu : X' \to X$ be the blow-up of a smooth center $Z \subset X$. Let E be the exceptional divisor. One has a commutative diagram with maps as labeled.

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mu} \\ Z & \xrightarrow{i} & X \end{array}$$

If c is the codimension of Z, g is a $\mathbb{P}^n$-bundle with $n = c-1$.

The kernel of the map $g_* : \mathrm{CH}^{r-1}(E) \to \mathrm{CH}^{r-n-1}(Z)$ is a direct summand:

$$\operatorname{Ker} g_* = \bigoplus_{0 \le i \le n-1} \mathrm{CH}^{r-1-i}(Z) \subset \mathrm{CH}^{r-1}(E) = \bigoplus_{0 \le i \le n} \mathrm{CH}^{r-1-i}(Z) \ .$$

It has the filtration induced from $F_S^\bullet$ on $\mathrm{CH}^{r-1}(E)$. By Proposition (4.6),

$$F_S^\nu \operatorname{Ker} g_* = \bigoplus_{0 \le i \le n-1} F_S^\nu \, \mathrm{CH}^{r-1-i}(Z) \ ,$$

namely the filtration coincides with the one induced by F_S on the Chow groups of Z.

One has an isomorphism

$$\operatorname{Ker} g_* \oplus \mathrm{CH}^r(X) \xrightarrow{\sim} \mathrm{CH}^r(X') \ ,$$

which sends (α, x) to $j_*\alpha + \nu^* x$. Here j_* is the restriction of $j_* : \mathrm{CH}^{r-1}(E) \to \mathrm{CH}^r(X')$.

The following proposition concerning the composition of j_* with $j^* : \mathrm{CH}^r(X') \to \mathrm{CH}^r(E)$ will be used later.

(4.8) **Proposition.** *The map $j^* j_* : \operatorname{Ker} g_* \to \mathrm{CH}^r(E)$ is injective and strictly compatible with the filtrations $F_S^\bullet$.*

Proof. One has for $\alpha \in \mathrm{CH}^{r-1}(E)$

$$j^* j_*(\alpha) = -\xi \cdot \alpha \ .$$

So for $\alpha \in \operatorname{Ker} g_*$, $\alpha = \sum_{0 \le i \le n-1} g^*\alpha_i \cdot \xi^i$,

$$j^* j_* \Big(\sum_{0 \le i \le n-1} g^*\alpha_i \cdot \xi^i \Big) = \sum_{0 \le i \le n-1} g^*\alpha_i \cdot \xi^{i+1} \ .$$

The claim follows using Proposition (4.6).

We now assume $p : X \to S$ is birational. We have maps $\mu^* : \mathrm{CH}^r(X) \to \mathrm{CH}^r(X')$ and $\mu_* : \mathrm{CH}^r(X') \to \mathrm{CH}^r(X)$. One has $\mu_*\mu^* = id$, $\operatorname{Ker} \mu_* = \operatorname{Ker} g_*$, and $\mathrm{CH}^r(X) \oplus \operatorname{Ker} g_* \cong \mathrm{CH}^r(X')$. Recall we have subgroups $D \subset N \subset \mathrm{CH}^r(X)$ and $D' \subset N' \subset \mathrm{CH}^r(X')$, defined using a stratification (S_α) over which p and $p' = p \circ \mu$ are stratified. We may assume $p(Z)$ is contained in S_1.

Let $X_\alpha = p^{-1}(S_\alpha)$ as before, and $X_{\alpha\, i}$ its irreducible components. For $\alpha \geq 1$ we have either

(a) $X_{\alpha\, i} \not\subset Z$, or (b) $X_{\alpha\, i} \subset Z$.

Take a desingularization $\tilde{X}_{\alpha\, i} \to X_{\alpha\, i}$ such that, in case (a), the map $\tilde{X}_{\alpha\, i} \to X$ factors through X'. Let $\tilde{X}_\alpha = \amalg \tilde{X}_{\alpha\, i}$ and $\pi_\alpha : \tilde{X}_\alpha \to X_\alpha$ be the induced map. This gives a resolution of p over (S_α).

To construct a resolution of $p' : X' \to S$, let $\tilde{X}'_\alpha = \amalg \tilde{X}'_{\alpha\, i}$ where

$$\tilde{X}'_{\alpha\, i} = \begin{cases} \tilde{X}_{\alpha\, i} & \text{in case (a)} \\ \tilde{X}_{\alpha\, i} \times_Z E & \text{in case (b)} \end{cases}$$

In case (b), $\tilde{X}'_{\alpha\, i}$ is a $\mathbb{P}^n$-bundle over $\tilde{X}_{\alpha\, i}$. The natural maps $\tilde{X}'_\alpha \to X'_\alpha = (p')^{-1}(S_\alpha)$ give a resolution. Denote by $\iota'_\alpha : \tilde{X}'_\alpha \to X'$ the induced maps. It is now easy to show the following descriptions for N and N'.

(4.9) **Proposition.** *(1) One has*

$$N = \bigcap_{\textit{type (a)}} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \operatorname{CH}^r(\tilde{X}_{\alpha\, i}) \cap (i^*)^{-1} F_S^{2r-d+1} \operatorname{CH}^r(Z)\ .$$

Here the first intersection is over $\tilde{X}_{\alpha\, i}$ *of type (a). The restriction of* $\iota_\alpha : \tilde{X}_\alpha \to X$ *to each component* $\tilde{X}_{\alpha\, i}$ *is still denoted* ι_α.

(2) Similarly,

$$N' = \bigcap_{\textit{type (a)}} ({\iota'}_\alpha^*)^{-1} F_S^{2r-d+1} \operatorname{CH}^r(\tilde{X}_{\alpha\, i}) \cap (j^*)^{-1} F_S^{2r-d+1} \operatorname{CH}^r(E)\ .$$

Proof. (1) Let $z \in \operatorname{CH}^r(X)$ be an element in the right hand side of the equality, in particular $i^* z \in F_S^{2r-d+1} \operatorname{CH}^r(Z)$. For a component $X_{\alpha\, i}$ of type (b), it follows $\iota_\alpha^* z \in F_S^{2r-d+1} \operatorname{CH}^r(\tilde{X}_{\alpha\, i})$. Thus z is in the left hand side.

To show the converse note that there is some $X_{\alpha\, i}$, $\alpha \geq 1$, containing Z, so one can take a smooth variety Z' which fits into the following commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{f} & \tilde{X}_{\alpha\, i} \\ {\scriptstyle h}\downarrow & & \downarrow \\ Z & \hookrightarrow & X_{\alpha\, i} \end{array}$$

where $h : Z' \to Z$ is projective surjective. (Take Z' to be a desingularization of a component of the fiber product.)

Assume now z is in the left hand side. In particular for the above $\tilde{X}_{\alpha i}$, we have $\iota_\alpha^* z \in F_S^{2r-d+1}\operatorname{CH}^r(\tilde{X}_{\alpha i})$, so $f^*\iota_\alpha^* z \in F_S^{2r-d+1}\operatorname{CH}^r(Z')$. By Proposition (4.5) (1), we have $i^*z \in F_S^{2r-d+1}\operatorname{CH}^r(Z)$.

(2) Similar to (1).

(4.10) **Proposition.** *(1)* $\mu^*(N) \subset N'$. *(2)* $\mu_*(N') \subset N$. *(3)* $N' = N \oplus (K \cap N')$.

Proof. (1) Obvious.

(2) For $z' \in N'$,

$$\begin{aligned} i^*\mu_* z' &= g_* i^! z' \\ &= g_*(c(\mathcal{E}) \cdot j^* x') \end{aligned}$$

where $c(\mathcal{E})$ is the top Chern class of the excess bundle $\mathcal{E}$, see [Fu]. The excess bundle is the quotient of the pull-back of the normal bundle of Z by the normal bundle of E: $\mathcal{E} = g^* N_Z X / N_E X'$. Thus $i^*\mu_* z' \in F_S^{2r-d+1}\operatorname{CH}^r(Z)$.

(3) Follows from (1), (2) and $\mu_*\mu^* = id$ on N.

Dually, one has the following propositions for the groups D and D'.

(4.11) **Proposition.** *(1) One has*

$$D = \sum_{\text{type (a)}} \iota_{\alpha *} F_S^{2r-d+1}\operatorname{CH}_{d-r}(\tilde{X}_{\alpha i}) + i_* F_S^{2r-d+1}\operatorname{CH}_{d-r}(Z)\ .$$

The first sum is over $\tilde{X}_{\alpha i}$ *of type (a).*

(2) $D' = \sum_{\text{type (a)}} \iota'_{\alpha *} F_S^{2r-d+1}\operatorname{CH}_{d-r}(\tilde{X}_{\alpha i}) + j_* F_S^{2r-d+1}\operatorname{CH}_{d-r}(E)\ .$

Proof. (1) To show the inclusion ($\subset$), note for a component of type (b), the existence of the map $\tilde{X}_{\alpha i} \to Z$ implies: if $\alpha \in \iota_{\alpha *} F_S^{2r-d+1}\operatorname{CH}_{d-r}(\tilde{X}_{\alpha i})$ then $\alpha \in i_* F_S^{2r-d+1}\operatorname{CH}_{d-r}(Z)$.

For the other inclusion, take a component $X_{\alpha i}$ containing Z and consider the diagram as in the proof of (4.9):

$$\begin{array}{ccc} Z' & \xrightarrow{f} & \tilde{X}_{\alpha i} \\ {\scriptstyle h}\downarrow & & \downarrow \\ Z & \to & X_{\alpha i} \end{array}$$

If $z \in F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(Z)$, by (4.5)(2), there is $z' \in F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(Z')$ such that $z = h_* z'$. Thus

$$i_* z = \iota_{\alpha *} f_* z' \in \iota_{\alpha *} F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(\tilde{X}_{\alpha i}) \ .$$

(2) Similar to (1).

(4.12) **Proposition.** *(1) $\mu_*(D') \subset D$. (2) $\mu^*(D) \subset D'$. (3) $D' = D \oplus (K \cap D')$.*

Proof. (1) Obvious.

(2) For $\alpha \in F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(Z)$,

$$\mu^* i_* \alpha = j_*(c(\mathcal{E}) \cdot g^* \alpha) \in j_* F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(E) \ .$$

(3) Follows from (1) and (2).

(4.13) **Proposition.** *We have $K \cap N' = K \cap D'$. Thus μ^* induces an isomorphism $\mu^* : N/D \xrightarrow{\sim} N'/D'$, the inverse being the map induced by μ_*.*

Proof. By $K \cong \operatorname{Ker} g_*$, if $z \in K$ then $z = j_* w$ for an element $w \in \operatorname{Ker} g_*$. Assume further $z \in K \cap N'$. Then by (4.9)

$$j^* z = j^* j_* w \in F_S^{2r-d+1}\,\mathrm{CH}^r(E) \ .$$

By Proposition (4.8), $w \in F_S^{2r-d+1}\,\mathrm{CH}^{r-1}(E)$. Thus, using Proposition (4.12), $z \in K \cap D'$.

(4.14) Let X be a quasi-projective variety, with a quasi-projective map p to S. One can define a unique filtration $F_S^{\bullet}$ on the Chow group $\mathrm{CH}_s(X)$ satisfying the following properties. The definition of the filtration and the verification of the properties are similar to the case $S = \operatorname{Spec} k$, which was carried out in [CH].

(1) $\mathrm{CH}^r(X) = F_S^{-\dim S}\,\mathrm{CH}^r(X)$ and $F_S^{\nu}\,\mathrm{CH}^r(X) = 0$ for ν large enough.

(2) If $f : X \to Y$ be a projective map over S, $f_* : \mathrm{CH}_s(X) \to \mathrm{CH}_s(Y)$ respects the filtrations $F_S^{\bullet}$. If in addition f is surjective, f_* is strictly compatible with the filtrations.

(3) If $j : U \hookrightarrow X$ is an open immersion, then $j^* : \mathrm{CH}_s(X) \to \mathrm{CH}_s(U)$ is strictly compatible with $F_S^{\bullet}$.

(4) If $f : X \to Y$ be an lci map of codimension d and f is over S, and

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

a Cartesian square where $g : Y' \to Y$ is a quasi-projective map, then the refined Gysin map $f^! : \mathrm{CH}_s(Y') \to \mathrm{CH}_{s-d}(X')$ respects $F_S^\bullet$.

(4.15) Using the filtration (4.14), the group $D \subset \mathrm{CH}^r(X)$ can be identified with

$$\mathrm{Image}[(k_1)_* : F_S^{2r-d+1}\,\mathrm{CH}_{d-r}(X_1) \to \mathrm{CH}_{d-r}(X)]\ ,$$

which is clearly independent of $\tilde{X}_1$.

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