Geometric Langlands Correspondence for $\mathcal{D}$-modules in Prime Characteristic: the $GL(n)$ Case

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To Robert MacPherson on the occasion of his 60th birthday

Abstract: Let X be a smooth projective algebraic curve of genus > 1 over and algebraically closed field k of characteristic $p > 0$. Denote by Bun_n (resp. Loc_n) the moduli stack of vector bundles of rank n on X (resp. the moduli stack of vector bundles of rank n endowed with a connection). Let also $\mathcal{D}_{\mathrm{Bun}_n}$ denote the sheaf of crystalline differential operators on Bun_n (cf. e.g. [3]). In this paper we construct an equivalence Φ_n between the bounded derived category $D^b(\mathcal{M}(\mathcal{O}_{\mathrm{Loc}_n^0}))$ of quasi-coherent sheaves on some open subset $\mathrm{Loc}_n^0 \subset \mathrm{Loc}_n$ and the bounded derived category $D^b(\mathcal{M}(\mathcal{D}_{\mathrm{Bun}_n}^0))$ of the category of modules over some localization $\mathcal{D}_{\mathrm{Bun}_n}^0$ of $\mathcal{D}_{\mathrm{Bun}_n}$. We show that this equivalence satisfies the *Hecke eigen-value property* in the manner predicted by the *geometric Langlands conjecture*. In particular, for any $\mathcal{E} \in \mathrm{Loc}_n^0$ we construct a "Hecke eigen-module" $\mathrm{Aut}_{\mathcal{E}}$.
The main tools used in the construction are the Azumaya property of $\mathcal{D}_{\mathrm{Bun}_n}$ (cf. [3]) and the geometry of the Hitchin integrable system. The functor Φ_n is defined via a twisted version of the Fourier-Mukai transform.

1. Introduction

1.1. **Geometric Langlands conjecture.** Let X be a smooth projective curve over $\mathbb{C}$ and let $\mathcal{E}$ be a local system of rank n on X. Let also Bun_n denote the moduli stack of rank n vector bundles on X.

The notion of an automorphic $\mathcal{D}$-module with respect to $\mathcal{E}$ on X has been defined by Beilinson and Drinfeld. For irreducible $\mathcal{E}$ the existence of such a $\mathcal{D}$-module has been shown by Frenkel, Gaitsgory and Vilonen (cf. [7], [8]; cf. also [6] for a review of the subject and [9] for a recent perspective coming from physics). Belinson and Drinfeld conjectured also the existence of a canonical equivalence

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(or "almost equivalence", see [2] for details) between the derived category of $\mathcal{D}$-modules on Bun_n and the derived category of quasi-coherent sheaves on the (appropriately understood) moduli space Loc_n of local systems on X (under this equivalence automorphic $\mathcal{D}$-modules correspond to sky-scraper sheaves).

1.2. **The case of characteristic** p**.** The purpose of this paper is to partially establish the above equivalence in a somewhat different (in fact, much easier) context. Namely, we assume that X is defined over an algebraically closed field k of characteristic $p > 0$. By $\mathcal{D}$-modules in this situation we mean quasi-coherent sheaves with a connection (i.e. we work with crystalline differential operators in the terminology of [3] and we *do not* consider differential operators with divided powers). In this case we define a certain dense open subset Loc_n^0 of Loc_n and explain a very simple construction, which attaches to any $\mathcal{E} \in \mathrm{Loc}_n^0$ an automorphic $\mathcal{D}$-module $\mathcal{S}_\mathcal{E}$ on Bun_n. We also show that such a $\mathcal{D}$-module is unique. As a byproduct we establish the equivalence between the derived category of quasi-coherent sheaves on Loc_n^0 and the derived category of a certain localization of the category of $\mathcal{D}$-modules on Bun_n (this equivalence is given by a twisted version of the Fourier-Mukai transform).

1.3. **Azumaya algebras and the Hitchin system.** The main tool in the construction is the observation from [3] saying that for a variety Y over a field k as above the sheaf $\mathcal{D}_Y$ of crystalline differential operators is an Azumaya algebra on $T^*Y^{(1)}$ – the cotangent bundle of the Frobenius twist of Y. In the case $Y = \mathrm{Bun}_n$ [1] we consider the Hitchin map $p : T^*\mathrm{Bun}_n^{(1)} \to \oplus_{i=0}^n H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes i})$ and observe that the Azumaya algebra $\mathcal{D}_{\mathrm{Bun}_n}$ splits on the generic fiber of p. We show that every splitting as above gives rise to a rank n vector bundle $\mathcal{E}$ with connection on X and that the corresponding splitting $\mathcal{D}_{\mathrm{Bun}_n}$-module is automorphic with respect to $\mathcal{E}$.

Let us now describe the contents of this paper in more detail. In Section 2 we recall some facts about duality and Fourier-Mukai transforms on commutative group-stacks and torosors over them following [1]. In Section 3 we recall the basic facts about differential operators in characteristic p and generalize some of them to the case of algebraic stacks (such a generalization is more or less straightforward but we couldn't find it in the literature). In Section 4 we introduce the Hitchin system and prove our main results about it; these results allow us to establish a certain geometric Langlands-type equivalence of categories. In Section 5 we prove that this equivalence of cateogries satisfies the Hecke eigen-value property.

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[1]Of course Bun_n is not an algebraic variety. However, a generalization of the above result to the case of "good" algebraic stacks is rather straightforward – cf. Section 3.13.

We believe that it should not be very difficult to generalize our constructions to the case of an arbitrary reductive group G.

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2. FOURIER-MUKAI TRANSFORMS ON COMMUTATIVE GROUP STACKS

The content of this section is mostly due to D. Arinkin (cf. [1]). We include it for completeness. In what follows we fix an algebraically closed field k and an irreducible scheme $\mathcal{W}$ of finite type over k. The word "scheme" (or "stack") will mean a scheme (stack) over $\mathcal{W}$. For example $B\mathbb{G}_m$ will denote the classifying stack of $\mathbb{G}_m$ over $\mathcal{W}$ (that is to say $\mathcal{W}/\mathbb{G}_m$).

We refer the reader to [4] for the basic definition about group-stacks.

2.1. **Coherent sheaves on gerbes.** Let $\mathcal{Y}$ be stack which is locally of finite type over $\mathcal{W}$. Recall that a $\mathbb{G}_m$-gerbe over a stack $\mathcal{Y}$ is a stack $\widetilde{\mathcal{Y}}$ endowed with an action of $B\mathbb{G}_m$ and with a map $\widetilde{\mathcal{Y}} \to \mathcal{Y}$ such that locally (in the smooth topology) on $\mathcal{Y}$ one has $\widetilde{\mathcal{Y}} = \mathcal{Y} \times B\mathbb{G}_m$. In other words, $\widetilde{\mathcal{Y}}$ corresponds to a sheaf of groupoids on $\mathcal{Y}_{sm}$ endowed with the natural action of the sheaf $\mathrm{Pic}(\mathcal{Y})_{sm}$ which locally is simply transitive. The gerbe is called *split* if it is globally isomorphic to $\mathcal{Y} \times B\mathbb{G}_m$. Let $D^b(\widetilde{\mathcal{Y}})$ denote the bounded derived category of coherent sheaves on $\widetilde{\mathcal{Y}}$.

Assume that $\widetilde{\mathcal{Y}}$ is split. Then the category $D^b(\widetilde{\mathcal{Y}})$ is equivalent to the bounded derived category of coherent sheaves on $\mathcal{Y}$ endowed with a $\mathbb{G}_m$-action; thus in this case we have the natural decomposition

$$(2.1) \qquad D^b(\widetilde{\mathcal{Y}}) = \bigoplus_{n \in \mathbb{Z}} D^b(\widetilde{\mathcal{Y}})_n$$

of $D^b(\widetilde{\mathcal{Y}})$ according to characters of $\mathbb{G}_m$. Each of the categories $D^b(\widetilde{\mathcal{Y}})_n$ is equivalent to $D^b(\mathcal{Y})$.

If $\widetilde{\mathcal{Y}}$ is not split we still have a decomposition of $D^b(\widetilde{\mathcal{Y}})$ into a direct sum as in (2.1). This decomposition is defined as follows: let us denote by a the canonical map $B\mathbb{G}_m \times \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{Y}}$. Then $\mathcal{F} \in D^b(\widetilde{\mathcal{Y}})_n$ if and only if $a^*\mathcal{F} \in D^b(B\mathbb{G}_m \times \widetilde{\mathcal{Y}})_n$. Note that for $n = 0$ the category $D^b(\widetilde{\mathcal{Y}})_0$ is still equivalent to $D^b(\mathcal{Y})$; however for $n \neq 0$ this is no longer true.

2.2. **Azumaya algebras and modules over them.** Let $\mathcal{Y}$ be a stack as above. Recall that an Azumaya algebra $\mathcal{A}$ on $\mathcal{Y}$ is a coherent sheaf of algebras which is locally in the smooth topology isomorphic to a matrix algebra (i.e. to the algebra of endomorphisms of a vector bundle). It follows that $\mathcal{A}$ is a locally free sheaf on $\mathcal{Y}$. We denote its rank by $\operatorname{rk}\mathcal{A}$.

Given an Azumaya algebra $\mathcal{A}$ we say that *a splitting* of $\mathcal{A}$ is a vector bundle $\mathcal{E}$ on $\mathcal{Y}$ and an isomorphism $\mathcal{A} \simeq \operatorname{End}(\mathcal{E})$. If such a splitting exists we say that $\mathcal{A}$ is split (or trivial). In general splittings of a given algebra $\mathcal{A}$ form an $\mathbb{G}_m$-gerbe $\mathcal{Y}_\mathcal{A}$ in the smooth topology.

For an Azumaya algebra $\mathcal{A}$ we denote by $\mathcal{A}^{op}$ the opposite algebra (clearly, it is again an Azumaya algebra). We also denote by $\mathcal{M}(\mathcal{A})$ the category of coherent sheaves of $\mathcal{A}$-modules. Every splitting of $\mathcal{A}$ gives rise to an equivalence $\mathcal{M}(\mathcal{A}) \simeq \mathcal{M}(\mathcal{O}_\mathcal{Y})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Azumaya algebras on Y. By an equivalence of $\mathcal{A}$ and $\mathcal{B}$ we mean a splitting of the algebra $\mathcal{A} \otimes \mathcal{B}^{op}$. Such a splitting gives rise to an equivalence of categories $\mathcal{M}(\mathcal{A}) \simeq \mathcal{M}(\mathcal{B})$. The category of equivalences between $\mathcal{A}$ and $\mathcal{B}$ is equivalent to the category of equivalences between the $\mathbb{G}_m$-gerbes $\mathcal{Y}_\mathcal{A}$ and $\mathcal{Y}_\mathcal{B}$.

Let $D^b(\mathcal{M}(\mathcal{A}))$ denote the bounded derived category of sheaves of coherent $\mathcal{A}$-modules.

Lemma 2.3. *The category $D^b(\mathcal{M}(\mathcal{A}))$ is canonically equivalent to $D^b(\mathcal{Y}_\mathcal{A})_1$.*

This follows easily from the definitions (cf. [5] for a detailed proof).

2.4. **Duality for commutative group-stacks.** Let $\mathcal{Y}$ be a commutative group stack which is locally of finite type over $\mathcal{W}$. The dual stack $\mathcal{Y}^\vee$ is the stack which classifies extensions of commutative group-stacks

$$0 \to \mathbb{G}_m \to \mathcal{X} \to \mathcal{Y} \to 0.$$

In other words $\mathcal{Y}^\vee$ classifies 1-morphisms of commutative group stacks $\mathcal{Y} \to B\mathbb{G}_m$. Note (this follows from the examples below) that if $\mathcal{Y}$ is algebraic then the stack $\mathcal{Y}^\vee$ need not be algebraic.

Examples.

1. Let $\mathcal{Y} = \mathbb{Z}$ (that is $\mathcal{W} \times \mathbb{Z}$). Then it is clear that $\mathcal{Y}^\vee = B\mathbb{G}_m$. More generally, if $\mathcal{Y} = \Gamma \times \mathcal{W}$ where Γ is a finitely generated abelian group then $\mathcal{Y}^\vee = \mathcal{W}/\Gamma^\vee$ where $\Gamma^\vee = \operatorname{Hom}(\Gamma, \mathbb{G}_m)$.

2. Let $\mathcal{Y} = B\mathbb{G}_m$. We claim that $\mathcal{Y}^\vee = \mathbb{Z}$. Indeed, a group one-morphism $B\mathbb{G}_m \to B\mathbb{G}_m$ by definition corresponds to a tensor functor $\alpha_S : \operatorname{Pic}(S) \to \operatorname{Pic}(S)$ defined for every scheme S over $\mathcal{W}$ and satisfying some obvious compatibility

conditions (with respect to inverse images). We claim that there exists (unique) $n \in \mathbb{Z}$ such that the above functors are canonically isomorphic to $\mathcal{L} \mapsto \mathcal{L}^{\otimes n}$ (this establishes the desired isomorphism $\mathcal{Y}^\vee = \mathbb{Z}$). Since the trivial bundle on S is the identity object of $\mathrm{Pic}(S)$ it must go to itself under any tensor functor. Thus, since $\mathrm{Aut}(\mathcal{O}_S) = \Gamma(S, \mathcal{O}_S^*)$ we see that any functor as above gives rise to a group homomorphism $\eta_S : \Gamma(S, \mathcal{O}_S^*) \to \Gamma(S, \mathcal{O}_S^*)$. These homomorphisms must be compatible with pull-backs, i.e. for any morphism $f : S' \to S$ we must have $f^* \circ \eta_S = \eta_{S'} \circ f^*$; in addition each η_S must be equal to identity on $\Gamma(\mathcal{W}, \mathcal{O}_\mathcal{W}^*) \subset \Gamma(S, \mathcal{O}_S^*)$. Moreover, it is easy to see that the above morphism $B\mathbb{G}_m \to B\mathbb{G}_m$ is uniquely determined (i.e. up to canonical isomorphism) by all η_S's (since every line bundle is locally isomorphic to the trivial bundle). In other words, we see that a homomorphism $B\mathbb{G}_m \to B\mathbb{G}_m$ is given by a homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ of group-schemes over $\mathcal{W}$. Since $\mathcal{W}$ was assumed irreducible, it is easy to see that every such homomorphism is given by the formula $t \mapsto t^n$ for some $n \in \mathbb{Z}$. Hence, for every S as above the homomorphism η_S is given by $\eta_S(f) = f^n$. This implies that the functor α_S is canonically isomorphic to the functor $\mathcal{L} \mapsto \mathcal{L}^n$.

3. Let A be an abelian scheme over $\mathcal{W}$. Then $A^\vee$ is isomorphic to the dual abelian scheme in the usual sense.

4. Let now $\pi : \mathcal{C} \to \mathcal{W}$ be a smooth projective morphism of relative dimension one. Assume also that all the geometric fibers of π are irreducible. In this case one can form the *Picard scheme* of $\mathcal{C}$ over $\mathcal{W}$ which we shall denote by $\underline{\mathrm{Pic}}(\mathcal{C}/\mathcal{W})$ as well as the corresponding *Picard stack* $\mathrm{Pic}(\mathcal{C}/\mathcal{W})$. Both $\mathrm{Pic}(\mathcal{C}/\mathcal{W})$ and $\underline{\mathrm{Pic}}(\mathcal{C}/\mathcal{W})$ have infinitely many connected components naturally parametrised by $\mathbb{Z}$; for any $d \in \mathbb{Z}$ we shall denote by $\mathrm{Pic}^d(\mathcal{C}/\mathcal{W})$ (resp. $\underline{\mathrm{Pic}}^d(\mathcal{C}/\mathcal{W})$) the corresponding component. We have the natural morphism $\kappa : \mathrm{Pic}(\mathcal{C}/\mathcal{W}) \to \underline{\mathrm{Pic}}(\mathcal{C}/\mathcal{W})$. Note that there is no natural morphism in the opposite direction. Assume, however, that π has a section s. Then we can use it to construct an identification $\mathrm{Pic}(\mathcal{C}/\mathcal{W}) \simeq \underline{\mathrm{Pic}}(\mathcal{C}/\mathcal{W})/\mathbb{G}_m$ (where the action of $\mathbb{G}_m$ on $\underline{\mathrm{Pic}}(\mathcal{C}/\mathcal{W})$ is trivial). Indeed, in this case the scheme $\underline{\mathrm{Pic}}(\mathcal{C}/\mathcal{W})$ represents the functor sending a scheme S to the set of isomorphism classes of the following data [2]:

1) a morphism $f : S \to \mathcal{W}$;

2) a line bundle $\mathcal{L}$ on $S \underset{\mathcal{W}}{\times} \mathcal{C}$;

3) a trivialization of $(\mathrm{id} \times s)^* \mathcal{L}$.

Note that the data of 1 and 2 is the same a (one)-morphism $S \to \mathrm{Pic}(\mathcal{C}/\mathcal{W})$ and forgetting 3 corresponds to taking the quotient by the trivial action of $\mathbb{G}_m$.

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[2]In section Section 4.6 we give a slightly different (but equivalent) definition of the functor represented by $\underline{\mathrm{Pic}}(\mathcal{C}/\mathcal{W})$ which does not use a choice of a section s.

Hence $\text{Pic}(\mathcal{C}/\mathcal{W})$ is a $\mathbb{G}_m$-gerbe over $\overline{\text{Pic}}(\mathcal{C}/\mathcal{W})$. Also every section s of π gives to a morphism $\eta_s : \underline{\text{Pic}}(\mathcal{C}/\mathcal{W}) \to \text{Pic}(\mathcal{C}/\mathcal{W})$ such that the composition $\kappa \circ \eta_s = \text{id}$. However, it is easy to see that for a different choice of s we shall get a different morphism η_s (though the functors given by 1,2,3 above are canonically isomorphic).

Note also that a choice of s identifies all the schemes $\underline{\text{Pic}}^d(\mathcal{C}/\mathcal{W})$ (for different d). The same is true for all the stacks $\text{Pic}^d(\mathcal{C}/\mathcal{W})$.

The stack $\text{Pic}(\mathcal{C}/\mathcal{W}) \underset{\mathcal{W}}{\times} \text{Pic}(\mathcal{C}/\mathcal{W})$ is endowed with the natural Poincaré line bundle $\mathcal{P}$. Note that $\mathcal{P}$ is not a pull-back of any line on $\underline{\text{Pic}}(\mathcal{C}/\mathcal{W}) \underset{\mathcal{W}}{\times} \underline{\text{Pic}}(\mathcal{C}/\mathcal{W})$; thus there is no natural Poincaré bundle on $\underline{\text{Pic}}(\mathcal{C}/\mathcal{W}) \underset{\mathcal{W}}{\times} \underline{\text{Pic}}(\mathcal{C}/\mathcal{W})$. This can be seen in the following way. Assume that a section s of π is chosen as above. Then we can use the morphism η_s discussed above to pull-back the Poincaré bundle $\mathcal{P}$ to $\underline{\text{Pic}}(\mathcal{C}/\mathcal{W}) \underset{\mathcal{W}}{\times} \underline{\text{Pic}}(\mathcal{C}/\mathcal{W})$. We denote the resulting line bundle by $\overline{\mathcal{P}}_s$. By the construction $\overline{\mathcal{P}}_s$ is endowed with a natural $\mathbb{G}_m \times \mathbb{G}_m$-action. It is easy to see that this action takes the following form: each $(t_1, t_2) \in \mathbb{G}_m$ acts on $\overline{\mathcal{P}}_s|_{\underline{\text{Pic}}^{d_1}(\mathcal{C}/\mathcal{W}) \underset{\mathcal{W}}{\times} \underline{\text{Pic}}^{d_2}(\mathcal{C}/\mathcal{W})}$ by $t_1^{d_2} t_2^{d_1}$. Hence the action is non-trivial (unless $d_1 = d_2 = 0$).

It is easy to see that the line bundle $\mathcal{P}$ gives rise to an equivalence $\mathcal{Y} \simeq \mathcal{Y}^\vee$. In more down-to-earth terms this equivalence can be seen as follows. Locally in the smooth topology on $\mathcal{W}$ we can choose a section of $\mathcal{C}$; such a choice gives rise to a (local) isomorphism $\text{Pic}(\mathcal{C}/\mathcal{W}) = \underline{\text{Pic}}^0(C/\mathcal{W}) \times \mathbb{Z} \times B\mathbb{G}_m$. It is well known that the abelian scheme $\underline{\text{Pic}}^0(\mathcal{C}/\mathcal{W})$ is self dual; also our duality interchanges $\mathbb{Z}$ and $B\mathbb{G}_m$. Thus locally $\mathcal{Y}$ is self-dual and it is easy to see that this local equivalence $\mathcal{Y} \simeq \mathcal{Y}^\vee$ does not depend on the choice of a local section of $\mathcal{C}$ made above (more precisely, for any two choices there is a canonical isomorphism between the two equivalences) and hence it can be glued to a global equivalence.

It is easy to see that we always have a natural 1-morphism $\mathcal{Y} \to (\mathcal{Y}^\vee)^\vee$. We say that $\mathcal{Y}$ is *nice* [3] if this morphism is an equivalence of categories. Note all the stacks considered in examples 1-4 above are nice. We say that $\mathcal{Y}$ is *very nice* if locally in smooth topology on $\mathcal{W}$ is is isomorphic to a finite product of stacks considered in examples 1-3 above (note that the stack in example 4 is also very nice). It is also clear that if $\mathcal{Y}$ is very nice then so is $\mathcal{Y}^\vee$. For a very nice stalk we let $d(\mathcal{Y})$ denote the (relative over $\mathcal{W}$) dimension of the corresponding abelian scheme; more invariantly, one can say that $d(\mathcal{Y})$ is equal to the sum of the relative dimension of $\mathcal{Y}$ over $\mathcal{W}$ and the dimension of the group of automorphisms of any k-point of $\mathcal{Y}$. Clearly, one has $d(\mathcal{Y}) = d(\mathcal{Y}^\vee)$.

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[3]Arinkin in [1] calls it "good"; we prefer to use another word since we want to reserve the word "good" for a different property.

Remark. Note that the last equality does not hold if we replace $d(\mathcal{Y})$ by $\dim(\mathcal{Y}/\mathcal{W})$: indeed the latter number is equal to 0 for $\mathcal{Y} = \mathbb{Z}$ and to -1 for $\mathcal{Y} = B\mathbb{G}_m$.

The proof of the following lemma is left to the reader.

Lemma 2.5. *Let*
$$0 \to \mathcal{Y}_1 \to \mathcal{Y}_2 \to \mathcal{Y}_3 \to 0$$
be a short exact sequence of commutative group-stacks. Assume that any two of the above stacks are nice. Then the third one is nice too.

Note that Lemma 2.5 may fail if "nice" is replaced by "very nice".

2.6. **Fourier-Mukai transform for group-stacks.** Let $\mathcal{Y}$ be a very nice group stack. By the definition we have a universal $\mathbb{G}_m$-torsor on $\mathcal{Y} \underset{\mathcal{W}}{\times} \mathcal{Y}^\vee$ which gives rise to a natural line bundle $\mathcal{P}_\mathcal{Y}$ there. Let $D^b(\mathcal{Y})$ denote the bounded derived category of coherent sheaves on $\mathcal{Y}$. We define the Fourier-Mukai functor $\Phi_\mathcal{Y} : D^b(\mathcal{Y}) \to D^b(\mathcal{Y}^\vee)$ by setting
$$\Phi(F) = (p_2)_*(p_1^*(F) \otimes \mathcal{P}_\mathcal{Y}).$$
Here we let $p_1 : \mathcal{Y} \underset{\mathcal{W}}{\times} \mathcal{Y}^\vee \to \mathcal{Y}$ and $p_2 : \mathcal{Y} \underset{\mathcal{W}}{\times} \mathcal{Y}^\vee \to \mathcal{Y}^\vee$ denote the natural projections.

The following result is an easy corollary of the corresponding statement about the Fourier-Mukai transform on abelian varieties.

Theorem 2.7. *The composition $\Phi_{\mathcal{Y}^\vee} \circ \Phi_\mathcal{Y}$ is naturally isomorphic to $(-1)^*[-d(\mathcal{Y})]$ (here (-1) stands for the inverse morphism $\mathcal{Y} \to \mathcal{Y}$). In particular, $\Phi_\mathcal{Y}$ is an equivalence of categories.*

Remark. In [1] D. Arinkin claims that Theorem 2.7 actually holds for any nice stack. However we do not know a proof of this statement.

2.8. **Duality for torsors.** Let now $\mathcal{Y}'$ be a torsor over a very nice group-stack $\mathcal{Y}$. Such a torsor gives rise to a canonical extension of group-stacks
$$0 \to \mathcal{Y} \to \widetilde{\mathcal{Y}} \xrightarrow{\alpha} \mathbb{Z} \to 0$$
such that $\mathcal{Y}' \simeq \alpha^{-1}(1)$. We denote by $\widetilde{\mathcal{Y}}^\vee$ the corresponding dual stack. It fits into a short exact sequence
$$(2.2) \qquad\qquad 0 \to B\mathbb{G}_m \to \widetilde{\mathcal{Y}}^\vee \to \mathcal{Y}^\vee \to 0.$$

Note that since $\widetilde{\mathcal{Y}}'$ is smooth over $\mathcal{W}$ it follows that it splits locally in the smooth topology on $\mathcal{W}$. This implies that the stacks $\widetilde{\mathcal{Y}}$ and $\widetilde{\mathcal{Y}}^\vee$ are automatically very nice.

In fact we claim that the converse to the last statement is also true, i.e. we claim that any group-stack $\widetilde{\mathcal{Y}}^\vee$ which fits into a short exact sequence as in (2.2) comes from a torsor $\mathcal{Y}'$ as above. In other words we claim that any $\mathbb{G}_m$-gerbe over $\mathcal{Y}^\vee$ with a commutative group structure is very nice. For this it is enough to show that any sequence of the form (2.2) splits locally in the smooth topology in $\mathcal{W}$. Since the stacks $\mathcal{Y}^\vee$ and $B\mathbb{G}_m$ are nice it is obvious from (2.2) that $\widetilde{\mathcal{Y}}^\vee$ is nice. Thus to check the spitting of (2.2) it is enough to check the splitting of the dual sequence obtained by applying $^\vee$ to all the terms. However, the latter sequence takes the form

$$0 \to \mathcal{Y} \to \widetilde{\mathcal{Y}} \to \mathbb{Z} \to 0$$

which is obviously locally split.

In the future we shall need the following

Proposition 2.9. *The Fourier-Mukai functor $\Phi_{\widetilde{\mathcal{Y}}}$ restricts to an equivalence*

(2.3) $$\Phi_{\mathcal{Y}'} : D^b(\mathcal{Y}') \quad \xrightarrow{\sim} \quad D^b(\widetilde{\mathcal{Y}}^\vee)_1.$$

For the proof cf. [1].

2.10. **Azumaya algebras with a group structure.** Let now $\mathcal{Y}$ be any commutative group stack over $\mathcal{W}$. Let $\mathcal{A}$ be an Azumaya algebra on $\mathcal{Y}$. As was discussed before this algebra induces canonical $\mathbb{G}_m$-gerbe $\mathcal{Y}_\mathcal{A}$ over $\mathcal{Y}$. We want to investigate when this gerbe has a group structure. For this it is suufficient to define a *group structure* on $\mathcal{A}$. We now want to explain what this means.

Let $m : \mathcal{Y} \underset{\mathcal{W}}{\times} \mathcal{Y} \to \mathcal{Y}$, $i : \mathcal{Y} \to \mathcal{Y}$ and $e : \mathcal{W} \to \mathcal{Y}$ be respectively the multiplication morphism, the inversion morphism and the unit. We also denote by $p_1, p_2 : \mathcal{Y} \underset{\mathcal{W}}{\times} \mathcal{Y} \to \mathcal{Y}$ the two natural projections.

By a *group structure* on $\mathcal{A}$ we shall mean the following structure:

1) An equivalence between $p_1^* \mathcal{A} \otimes p_2^* \mathcal{A}$ and $m^* \mathcal{A}$;

2) Let now $\pi_1, \pi_2, \pi_3, m : \mathcal{Y} \underset{\mathcal{W}}{\times} \mathcal{Y} \underset{\mathcal{W}}{\times} \mathcal{Y} \to \mathcal{W}$ denote the natural projections and the multiplication morphism. Then from 1) one gets two equivalences between $\pi_1^* \mathcal{A} \otimes \pi_2^* \mathcal{A} \otimes \pi_3^* \mathcal{A}$ and $m^* A$. Our second piece of structure is an isomorphism between these two equivalences.

This data must satisfy a cocycle condition (taking place on the 4th Cartesian power of $\mathcal{Y}$ over $\mathcal{W}$). The details can be found in [14].

The fact that the group structure on $\mathcal{A}$ induces a group structure on $\mathcal{Y}_\mathcal{A}$ is clear.

3. Azumaya algebras and differential operators

3.1. Frobenius twist of a k-scheme. Let Y be a scheme over an algebraically closed field k of characteristic $p > 0$. The Frobenius map of schemes $Y \to Y$ is defined as identity on topological spaces, but the pull-back of functions is the p-th power: $\mathrm{Fr}_Y^*(f) = f^p$ for $f \in \mathcal{O}_Y$. The Frobenius twist $Y^{(1)}$ of Y is the k-scheme that coincides with Y as a scheme (i.e. $Y^{(1)} = Y$ as a topological space and $\mathcal{O}_{Y^{(1)}} = \mathcal{O}_Y$ as a sheaf of rings), but with a different k-structure: $a \underset{(1)}{\cdot} f = a^{1/p} \cdot f$, $a \in k$, $f \in \mathcal{O}_{Y^{(1)}}$. It makes Frobenius map into a map of k-schemes $\mathrm{Fr}_Y : Y \to Y^{(1)}$. Since Fr_Y is a bijection on k-points, we will often identify k-points of Y and $Y^{(1)}$. Also, since Fr_Y is affine, we may identify sheaves on Y with their direct images under Fr_Y.

For a vector space V over k its Frobenius twist $V^{(1)}$ again has a natural structure of a vector space. Given two vector spaces V and W over k we say that a map $\alpha : V \to W$ is p-linear if it is additive and $\alpha(a \cdot v) = a^p \alpha(v)$ (for $a \in k$ and $v \in V$). This is the same as a map $V^{(1)} \to W$. For every V as above we have a natural isomorphism $(V^*)^{(1)} \simeq (V^{(1)})^*$.

Let Y be a smooth variety over k. Then it is easy to see that we have canonical isomorphisms

$$(TY)^{(1)} \simeq T(Y^{(1)}) \quad \text{and} \quad (T^*Y)^{(1)} \simeq T^*(Y^{(1)}).$$

We set $T^{*,1}Y = Y \underset{Y^{(1)}}{\times} (T^*Y)^{(1)}$. We have natural morphisms $\eta : T^{*,1}Y \to Y$ (corresponding to the projection on the first multiple) and $\rho : T^{*,1}Y \to (T^*Y)^{(1)}$ (corresponding the projection on the second multiple).

3.2. The sheaves D_Y and $\mathcal{D}_Y$. In what follows Y denotes a smooth variety over k. We let D_Y denote the quasi-coherent sheaf of algebras on Y generated by $\mathcal{O}_Y$ and TY with the following relations:

(3.1) $$\partial \cdot f - f \cdot \partial = \partial(f) \text{ and } \partial \cdot \partial' - \partial' \cdot \partial = [\partial, \partial']$$

where f, ∂ and ∂' are local sections of $\mathcal{O}_X$ and TX respectively.

The sheaf D_X acts on $\mathcal{O}_X$. This action, however, is not faithful. For example when $Y = \mathbb{A}^1$ (with coordinate y) the element $\left(\frac{d}{dy}\right)^p$ (which is non-zero in D_Y) clearly kills every function.

For any vector field $\partial \in T_Y$, the element $\partial^p \in D_Y$ acts on functions as another vector field which one denotes $\partial^{[p]}$. Then $\iota(\partial) := \partial^p - \partial^{[p]} \in D_Y$ commutes with functions. Since ι is p-linear we shall regard it as a linear map

$$\iota : TY^{(1)} \to \mathrm{Fr}_* D_Y.$$

In particular, there exists canonical quasi-coherent sheaf of algebras $\mathcal{D}_Y$ on $T^*Y^{(1)}$ together with an isomorphism $(\pi_Y^{(1)})_* \mathcal{D}_Y \simeq \mathrm{Fr}_* D_Y$.

The following result is proved in [3]:

Theorem 3.3. (1) *For every vector field ∂ defined on a Zariski open subset U of Y the element $\iota(\partial)$ is central in D_U.*

(2) *The map ι induces an isomorphism of sheaves between $\mathcal{O}_{(T^*Y)^{(1)}}$ and the center of $\mathcal{D}_Y$.*

(3) *$\mathcal{D}_Y$ is an Azumaya algebra on $(T^*Y)^{(1)}$ of rank p^{2d} where $d = \dim Y$.*

(4) *The Azumaya algebra $\mathcal{D}_Y$ on $T^*Y^{(1)}$ is non-trivial for every Y such that $\dim Y > 0$.*

Let us give a sketch of the proof of property (3) above since we are going to need it in the future. First of all, it is easy to check that $\mathcal{D}_Y$ is a locally free coherent sheaf of algebras on $(T^*Y)^{(1)}$ of rank p^{2d}. Moreover, there exists a natural coherent sheaf $(\mathcal{D}_Y)_{T^{*,1}Y}$ on $T^{*,1}Y$ such that $\mathcal{D}_Y = \eta_*(\mathcal{D}_Y)_{T^{*,1}Y}$ (recall that η denotes the natural morphism $T^{*,1}Y \to (T^*Y)^{(1)}$). Indeed, to construct $(\mathcal{D}_Y)_{T^{*,1}Y}$ is the same as to construct an action of the sheaf $\eta_* \mathcal{O}_{T^{*,1}Y}$ on $\mathcal{D}_Y$. Note that the sheaf of (commutative) algebras $\eta_* \mathcal{O}_{T^{*,1}Y}$ embeds naturally into $\mathcal{D}_Y$ since as a sheaf of algebras it is (by the definition) generated by $(\mathrm{Fr}_Y)_*(\mathcal{O}_Y)$ and $\mathcal{O}_{(T^*Y)^{(1)}}$. We now let it act on $\mathcal{D}_Y$ by *right* multiplication. It is clear that the sheaf $(\mathcal{D}_Y)_{T^{*,1}Y}$ is locally free of rank p^d on $T^{*,1}Y$.

To prove that $\mathcal{D}_Y$ is actually an Azumaya algebra it is enough to show (cf. [13]) that for some faithfully flat morphism $\rho : Z \to (T^*Y)^{(1)}$ the algebra $\rho^* \mathcal{D}_Y$ is isomorphic to the algebra of endomorphisms of a vector bundle E on Z. Let $Z = T^{*,1}Y$ and let ρ denote the natural morphism $T^{*,1}Y \to (T^*Y)^{(1)}$. Set also $E = (\mathcal{D}_Y)_{T^{*,1}Y}$. Then $\rho^* \mathcal{D}_Y$ acts on $(\mathcal{D}_Y)_{T^{*,1}Y}$ by left multiplication. This action commutes with the action of $\mathcal{O}_{T^{*,1}Y}$ since the latter came from right multiplication in $\mathcal{D}_Y$. Thus we get a homomorphism $\rho^* \mathcal{D}_Y \to \mathrm{End}_{\mathcal{O}_{T^{*,1}Y}}((\mathcal{D}_Y)_{T^{*,1}Y})$ of coherent sheaves of algebras on $T^{*,1}Y$. This homomorphism must be an embedding on the level of fibers (since $\mathcal{D}_Y$ has no zero divisors). Since both algebras have rank p^{2d} it follows that this map is an isomorphism generically.

3.4. **The "small" differential operators $\mathcal{D}_{X,0}$.** The restriction of $\mathcal{D}_Y$ to any closed subscheme Z of $T^*Y^{(1)}$ gives an Azumaya algebra on Z. In particular, we may take Z to be the zero section of $T^*Y^{(1)}$. In this way we get the algebra $\mathcal{D}_{Y,0}$ of *small differential operators*. This algebra is again generated by $\mathcal{O}_Y$ and TY and to get its relations we must add the relation

$$\partial^p = \partial^{[p]}, \ \partial \in T_Y$$

to (3.1).

It is easy to see that $\mathcal{D}_{Y,0}$ is the image of the canonical map $\mathcal{D}_Y \to \operatorname{End}_k \mathcal{O}_Y$. In fact, this gives an isomorpshism $\mathcal{D}_{Y,0} \simeq \operatorname{End}_{Y^{(1)}} \mathcal{O}_Y$ which shows that the Azumaya algebra $\mathcal{D}_{Y,0}$ on $Y^{(1)}$ is canonically split.

3.5. **p-curvature.** The construction of the algebra $\mathcal{D}_Y$ is closely related to the notion of p-curvature that we now recall. Let $\mathcal{F}$ be a D_Y-module which may regard as a quasi-coherent sheaf on Y endowed with a flat connection. Let also $\underline{\operatorname{End}}(\mathcal{F})$ denote the sheaf of endomorphisms of $\mathcal{F}$ over $\mathcal{O}_Y$. Then to ∇ we can canonically associate a section ψ_∇ of $\underline{\operatorname{End}}(\mathcal{F}) \otimes \operatorname{Fr}_Y^*(\Omega^1_{Y^{(1)}})$ which is called the *p-curvature* of ∇. To do that let us note that the space of global sections of $\underline{\operatorname{End}}(\mathcal{F}) \otimes \operatorname{Fr}_Y^*(\Omega^1_{Y^{(1)}})$ is the same as the space of global sections of $(\operatorname{Fr}_Y)_* \underline{\operatorname{End}}(\mathcal{F}) \otimes \Omega^1_{Y^{(1)}}$. To construct an element in the latter we need to construct an element $\psi_\nabla(\partial) \in \operatorname{End}(\mathcal{F})$ for each (locally defined) vector field ∂ so that the assignment $\partial \mapsto \psi_\nabla(\partial)$ is additive and satisfies

$$\psi_\nabla(f\partial) = f^p \psi_\nabla(\partial),$$

where f is any (locally defined) function on Y.

It is now clear that the assignment

$$\psi_\nabla(\partial) := \nabla(\partial)^p - \nabla(\partial^{[p]})$$

satisfies all the above requirements.

We shall denote by $\mathcal{M}(D_Y)$ the category of quasi-coherent sheaves of left D_Y-modules on Y; this category is equivalent to the category $\mathcal{M}(\mathcal{D}_Y)$ of quasi-coherent sheaves of left $\mathcal{D}_Y$-modules on $T^*Y^{(1)}$.

3.6. **Inverse image.** Let $\pi : Z \to W$ be a morphism of smooth varieties over k. We define the functor $\pi^! : \mathcal{M}(D_W) \to \mathcal{M}(D_Z)$ in the following way. [4] For any object $F \in \mathcal{M}(D_W)$ we set $\pi^! F$ to be equal to the pull-back of M in the sense of quasi-coherent sheaves. In other words,

$$\pi^! F = \mathcal{O}_Z \underset{\pi^\bullet \mathcal{O}_W}{\otimes} \pi^\bullet F$$

where $\pi^\bullet$ denotes the sheaf-theoretic pull-back. The sheaf TZ acts on $\pi^! F$ by means of the Leibniz formula.

Here is a standard reformulation of this definition. Set $D_{Z \to W} = \pi^! D_W$. This is a D_Z-module on Z which also admits a canonical right action of the sheaf

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[4]In fact, our definition differs from the standard definition (given usually in characteristic 0 case) by the shift by $\dim W - \dim Z$.

$\pi^{\cdot}D_W$ which commutes with the left D_Z-action. Then for every $F \in \mathcal{M}(\mathcal{D}_W)$ we have

$$\pi^! F = D_{Z \to W} \underset{\pi^{\bullet} D_W}{\otimes} \pi^{\bullet} F.$$

Let us reformulate this definition in terms of the algebras $\mathcal{D}_Z$ and $\mathcal{D}_W$.

Let $d\pi : Z^{(1)} \times_{W^{(1)}} T^* W^{(1)} \to T^* Z^{(1)}$ be (the Frobenius twist of) the differential of π. On $Z^{(1)} \times_{W^{(1)}} T^* W^{(1)}$ we get two Azumaya algebras: $d\pi^*(\mathcal{D}_Z)$ and $\mathrm{pr}^*(\mathcal{D}_W)$ (where pr is the projection to the second factor).

Proposition 3.7. *The Azumaya algebras $d\pi^*(\mathcal{D}_Z)$ and $pr^*(\mathcal{D}_W)$ are canonically equivalent.*

Proof. To prove the Proposition it suffices to construct a splitting for the Azumaya algebra $\mathcal{A} = d\pi^*(\mathcal{D}_W)^{op} \otimes \mathrm{pr}^*(\mathcal{D}_Z)$, i.e. to provide a vector bundle of rank $\sqrt{rank(\mathcal{A})} = p^{\dim Z + \dim W}$ equipped with an action of $\mathcal{A}$ (a splitting module for $\mathcal{A}$).

Recall that we have the left D_Z-module $D_{Z \to W}$ endowed with a natural right action of $\pi^{\bullet} D_W$. Thus there exists a natural $\mathcal{D}_Z \boxtimes \mathcal{D}_W^{op}$-module $\mathcal{D}_{Z \to W}$ whose direct image to $T^* Z^{(1)}$ is identified with $(\mathrm{Fr}_Z)_* D_{Z \to W}$. In fact it is clear that $\mathcal{D}_{Z \to W}$ is supported on $T^* Z^{(1)} \times_{W^{(1)}} T^* W^{(1)} \subset T^* Z^{(1)} \times T^* W^{(1)}$. This follows from the fact that the right action of the central subalgebra $\pi^{\bullet}(\mathcal{O}_{W^{(1)}}) \subset \pi^{\bullet}(\mathcal{O}_{T^* W^{(1)}}) \subset \pi^{\bullet}(\mathcal{D}_W)$ coincides with the one factoring through the left action of $\mathcal{O}_{Z^{(1)}} \subset D_Z$. Thus $\pi^*(\mathcal{D}_W)$ can be viewed as a quasi-coherent sheaf on $T^* Z^{(1)} \times_{W^{(1)}} T^* W^{(1)}$ equipped with an action of the Azumaya algebra $\mathrm{pr}_1^*(\mathcal{D}_Z) \otimes \mathrm{pr}_2^*(\mathcal{D}_W)^{op}$.

Lemma 3.8. *The sheaf $\pi^*(\mathcal{D}_W)$ is supported on the closed subscheme*
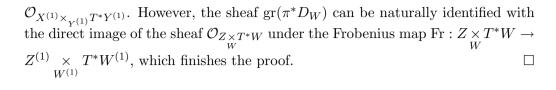
$$Z^{(1)} \times_{W^{(1)}} T^* W^{(1)} \subset T^* Z^{(1)} \times_{W^{(1)}} T^* W^{(1)}$$

(the graph of $d\pi$). It is locally free of rank $p^{\dim(Z) + \dim(W)}$ on this subscheme.

Note that it follows from Lemma 3.8 that $\mathcal{D}_{Z \to W}$ is a splitting module for $\mathcal{A}$. Thus Lemma 3.8 implies Proposition 3.7.

Proof. To check the first statement it suffices to see that if v is a vector field on an open $U \subset Z$ with constant horizontal component, i.e. $d\pi(v) = \pi^*(w) \in \Gamma(U, \pi^* T_W)$ for some vector field w on an open neighborhood of $\pi(U)$ then the left action of $v^p - v^{[p]}$ on $\pi^*(D_W)$ coincides with the right action of $\pi^{\bullet}(w^p - w^{[p]})$. This follows from the fact that both operators commute with the action of $\mathcal{O}_Z \subset D_Z$, and obviously coincide on the image of $\pi^{\bullet}(D_W) \to \pi^*(D_W)$.

To check the second assertion it is enough to see that the associated graded of $\mathrm{gr}(\pi^* D_W)$ of $\pi^* D_W$ with respect to the standard filtration by the order of a differential operator is locally free of rank $p^{\dim(X) + \dim(Y)}$ over $\mathrm{gr}(\mathcal{O}_{X^{(1)} \times_{Y^{(1)}} T^* Y^{(1)}}) =$

$\mathcal{O}_{X^{(1)} \times_{Y^{(1)}} T^*Y^{(1)}}$. However, the sheaf $\mathrm{gr}(\pi^* D_W)$ can be naturally identified with the direct image of the sheaf $\mathcal{O}_{Z \underset{W}{\times} T^*W}$ under the Frobenius map $\mathrm{Fr} : Z \underset{W}{\times} T^*W \to Z^{(1)} \underset{W^{(1)}}{\times} T^*W^{(1)}$, which finishes the proof. $\square$

$\square$

Let us now reformulate the definition of the inverse image functor using Proposition 3.7. Namely, it follows from Proposition 3.7 that we have a natural equivalence of categories $\mathcal{M}(d\pi^* \mathcal{D}_Z) \simeq \mathcal{M}(\mathrm{pr}^* \mathcal{D}_W)$. It is now easy to see that the functor $\pi^!$ defined above is equal to the composition of the pullback functor $\mathcal{M}(\mathcal{D}_W) \to \mathcal{M}(\mathrm{pr}^* \mathcal{D}_W)$, the above equivalence, and the push-forward functor $\mathcal{M}(d\pi^* \mathcal{D}_Z) \to \mathcal{M}(\mathcal{D}_Z)$.

3.9. **Direct image.** Let $\pi : Z \to W$ be again a morphism of smooth varieties over k. The usual definition of the direct image functor works also in our case. However, we would like to use another definition in terms of the algebras $\mathcal{D}_Z$, $\mathcal{D}_W$. Namely, Proposition 3.7 yields as before a canonical equivalence between the categories $\mathcal{M}(d\pi^* \mathcal{D}_Z)$ and $\mathcal{M}(\mathrm{pr}^* \mathcal{D}_W)$. Composing this equivalence with the pull-back functor $\mathcal{M}(\mathcal{D}_Z) \to \mathcal{M}(d\pi^* \mathcal{D}_Z)$ on the left, and the push-forward functor $\mathcal{M}(\mathrm{pr}^* \mathcal{D}_W) \to \mathcal{M}(\mathcal{D}_W)$ on the right we get the functor of direct image from $\mathcal{M}(\mathcal{D}_Z)$ to $\mathcal{M}(\mathcal{D}_W)$.

3.10. **The algebra $\mathcal{D}_{Y,\theta}$.** A 1-form θ on $Y^{(1)}$ defines a section $\theta : Y^{(1)} \to T^*Y^{(1)}$. We let $\mathcal{D}_{Y,\theta} = \theta^*(\mathcal{D}_Y)$ denote the pull-back of the Azumaya algebra $\mathcal{D}_Y$ under θ.

For example, for $\theta = 0$ we recover the algebra $\mathcal{D}_{Y,0}$ of small differential operators which, as we have seen before, is canonically split. More generally, the category (more precisely, $\mathcal{O}_{Y^{(1)}}^\times$-gerbe) of splittings of $\mathcal{D}_{Y,\theta}$ is canonically equivalent to the category of line bundles on Y equipped with a flat connection whose p-curvature equals $\mathrm{Fr}^*\theta$.

Let us denote the above gerbe by $\mathcal{G}_{Y,\theta}$. It follows from the above description that $\mathcal{G}_{Y,\theta}$ is functorial in (Y,θ).

Suppose that θ equals $\omega - C(\omega)$ for a 1-form ω on Y (where $C : \mathrm{Fr}_*(\Omega^1_{cl}) \to \Omega^1_{Y^{(1)}}$ is the Cartier operator, and where we omit from the notation the tautological Frobenius-linear isomorphism between $\Gamma(Y, \Omega^1) = \Gamma(Y^{(1)}, \Omega^1)$), then θ is the p-curvature of the connection $d + \omega$ on the trivial line bundle, so in this case the choice of such a 1-form ω defines a splitting of the Azumaya algebra $\mathcal{D}_{Y,\theta}$.

Recall now that the cotangent bundle to any smooth variety is endowed with a canonical one form (whose differential equals the canonical symplectic form).

Proposition 3.11. *Let θ be the canonical 1-form on $T^*Y^{(1)}$. Then the Azumaya algebra $\mathcal{D}_{T^*Y,\theta}$ (of rank p^{4d}) is canonically equivalent to the Azumaya algebra $\mathcal{D}_Y$ (of rank $2d$).*

Proof. We apply Proposition 3.7 to the projection $\pi : T^*Y \to Y$. We get an equivalence $\eta^* \mathcal{D}_{T^*Y} \sim \mathrm{pr}_2^*(\mathcal{D}_Y)$ between the two Azumaya algebras on $T^*Y^{(1)} \times_{Y^{(1)}} T^*Y^{(1)}$; here η stands for the closed imbedding $T^*Y^{(1)} \times_{Y^{(1)}} T^*Y^{(1)} \to T^*(T^*Y)^{(1)}$, and pr_2 is the second projection. The section $\theta : T^*Y^{(1)} \to T^*(T^*Y)^{(1)}$ lands in the image of η; moreover, we have $\theta = \delta \circ \eta$, where $\delta : T^*Y^{(1)} \to T^*Y^{(1)} \times_{Y^{(1)}} T^*Y^{(1)}$ is the diagonal embedding.

Thus

$$D_{T^*Y,\theta} = \delta^* \iota^* \mathcal{D}_{T^*Y} \sim \delta^* \mathrm{pr}_2^*(\mathcal{D}_Y) = \mathcal{D}_Y,$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 3.12. *Let $f : Y_1 \to Y_2$ be a morphism of smooth algebraic varieties over k. Let $\theta_i \in \Gamma(Y_i^{(1)}, \Omega^1_{Y_i^{(1)}})$ (where $i = 1, 2$). Assume that $(f^{(1)})^* \theta_2 = \theta_1$. Then the algebras $\mathcal{D}_{Y_1,\theta_1}$ and $(f^{(1)})^* \mathcal{D}_{Y_2,\theta_2}$ are canonically equivalent.*

3.13. **Stack version.** Let now Y is a smooth irreducible algebraic stack. Assume that Y is good in the sense [2]; in other words we assume that $\dim T^*Y = 2 \dim Y$ (in this case the stack T^*Y is automatically equidimensional and irreducible). We assume in addition that T^*Y has an open-substack T^*Y^0 which is a smooth Deligne-Mumford stack.

Recall that when k has characteristic 0 and Y is as above one can define canonical quasi-coherent sheaf of algebras D_Y (cf. [2], Chapter 1). More precisely, we have the following data. Let $f : S \to Y$ be a smooth map from a scheme S to Y. Then we can define a sheaf $(D_Y)_S$ of algebras on S (this sheaf is the sheaf-theoretic pull-back of D_Y to S; note that this is not a sheaf of $\mathcal{O}_S$-modules). In addition we can also define a D_S-module $(D_Y)_S^\sharp$ on S endowed with a right action of $(D_Y)_S$ and with a morphism $(D_Y)_S \to (D_Y)_S^\sharp$ of $(D_Y)_S$-modules (this is the pull-back of D_Y in the sense of $\mathcal{O}$-modules). Both $(D_Y)_S$ and $(D_Y)_S^\sharp$ must be sheaves (which we denote by D_Y and $D_Y^\sharp$) on the smooth site of Y and the morphism $D_Y \to D_Y^\sharp$ must induce an isomorphism of global sections on any open subset $U \subset Y$.

The definition of the above sheaves (when $\mathrm{char}(k) = 0$) is as follows. Let $\widetilde{D}_S \subset D_S$ be the normalizer of the left ideal $I_S = D_S \cdot T(S/Y)$ (here $T(S/Y)$ stands for the sheaf of vector fields on S which are vertical with respect to the morphism $f : S \to Y$). Then (after [2]) we set $(D_Y)_S = \widetilde{D}_S / I_S$ and $(D_Y)_S^\sharp = D_S / I_S$. Note that that if Y is a scheme then $(D_Y)_S^\sharp$ is nothing else but $D_{S \to Y}$. Also, it is clear

that we have the natural identification

$$(3.2) \qquad\qquad (D_Y)_S = \mathcal{E}nd_{D_S}((D_Y)_S^\sharp).$$

Let us now turn to the case $\mathrm{char}(k) = p > 0$. In this case we leave the definition of $(D_Y)_S^\sharp$ unchanged. However, it is easy to see that the definition of $(D_Y)_S$ has to be modified (if we define $(D_Y)_S$ as in (3.2) then this algebra will contain the center of D_S which we don't want to be there).

In fact we don't know a good definition of $(D_Y)_S$ in this case. In other words we don't know how to define an algebra structure on the sheaf defined by the collection of all the $(D_Y)_S^\sharp$. Instead we are going to proceed as follows.

Let as before let $\pi : T^*Y \to Y$ denote the natural projection; let also $\pi^{(1)} : T^*Y^{(1)} \to Y^{(1)}$ denote its Frobenius twist.

Lemma 3.14.   (1) *There exists a natural coherent sheaf of algebras $\mathcal{D}_Y$ on $T^*Y^{(1)}$ together with the natural isomorphism*

$$\pi_*^{(1)}\mathcal{D}_Y \simeq \mathrm{Fr}_*D_Y.$$

(2) *The restriction of $\mathcal{D}_Y$ to $(T^*Y^0)^{(1)}$ is an Azumaya algebra on $(T^*Y^0)^{(1)}$.*

In particular, under the above conditions it makes sense to speak about the category of $\mathcal{D}_Y$-modules.

Proof. First, let us construct a coherent sheaf of algebras $\mathcal{D}_Y$ on $T^*Y^{(1)}$ whose direct image to $Y^{(1)}$ coincides with Fr_*D_Y. To do that let us note the following. Let $f : S \to Y$ be as above. Let us denote by $(T^*Y)_S$ the orthogonal complement of $T(S/Y)$ in T^*S; this is a closed subscheme of T^*S. We have a Cartesian square

$$
\begin{array}{ccc}
(T^*Y)_S & \xrightarrow{\;\widetilde{f}\;} & T^*Y \\
{\scriptstyle (\pi_Y)_S}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\
S & \xrightarrow{\;f\;} & Y
\end{array}
$$

In particular, the map $\widetilde{f} : (T^*Y)_S \to T^*Y$ is a smooth covering. Also, given two smooth maps $f : S \to Y$ and $f' : S' \to Y$ together with a morphism $\beta : S' \to S$ (of schemes over Y) we have a natural morphism $\widetilde{\beta} : (T^*Y)_{S'} \to (T^*Y)_S$ of schemes over T^*Y. Thus in order to define the sheaf $\mathcal{D}_Y$ we need to define the following data:

- A coherent sheaf of algebras $(\mathcal{D}_Y)_S$ on $(T^*Y)_S^{(1)}$ for each S as above;

- An isomorphism $(\widetilde{\beta}^{(1)})^*(\mathcal{D}_Y)_S \simeq (\mathcal{D}_Y)_{S'}$ for every β as above.

This data must satisfy the standard "cocycle" condition.

Consider the D_S-module $(D_Y)_S^\sharp$. We denote by $(\mathcal{D}_Y)_S^\sharp$ the corresponding $\mathcal{D}_S$-module. We claim that it is supported on $(T^*Y^{(1)})_S$. Indeed, the module $(D_Y)_S^\sharp$ is generated by one section 1 which is annihilated by any local section of $T(S/Y)$. Hence $(\mathcal{D}_Y)_S^\sharp$ is also generated by the section 1 which is annihilated by any local section of $T(S/Y)$; hence 1 is also annihilated by all their p-th powers. This means that 1 is annihilated by any local section of $T^*(S/Y)^{(1)}$. Since 1 is a generator and since the sections of $T(Y/S)^{(1)}$ lie in the center of $\mathcal{D}_S$ it follows that any local section of $T(Y/S)^{(1)}$ acts by zero on $(\mathcal{D}_Y)_S^\sharp$ which means that it is supported on $(T^*Y)_S^{(1)}$.

Define now

$$(\mathcal{D}_Y)_S = \mathcal{E}nd_{\mathcal{D}_S}((\mathcal{D}_Y)_S^\sharp)^{op}.$$

We have the natural isomorphism

$$((\pi_Y)_S^{(1)})_*(\mathcal{D}_Y)_S = (\mathrm{Fr}_S)_*(D_Y)_S$$

which follows immediately from (3.2) (in particular, this gives another definition of $(\mathcal{D}_Y)_S$). The construction of the above data is straightforward.

Now we must show that the sheaf $\mathcal{D}_Y|_{(T^*Y^0)^{(1)}}$ is an Azumaya algebra of rank $p^{2\dim Y}$. In other words, we have to show that for every S as above the algebra $((\mathcal{D}_Y)_S)|_{(T^*Y^0)_S^{(1)}}$ is an Azumaya algebra of rank $p^{2\dim Y}$. Here $(T^*Y^0)_S$ denotes the preimage of T^*Y^0 in $(T^*Y)_S$. In fact, we are going to show that this Azumaya algebra is equivalent to $\mathcal{D}_S|_{(T^*Y^0)^{(1)}}$. Indeed, consider again the sheaf $(\mathcal{D}_Y)_S^\sharp$. By the definition, it is endowed with a left action of $\mathcal{D}_S|_{(T^*Y)_S^{(1)}}$ and with a right action of $(\mathcal{D}_Y)_S$. Since $\mathcal{D}_S|_{(T^*Y)_S^{(1)}}$ is an Azumaya algebra of rank $p^{2\dim S}$ the required statement follows from the following

Lemma 3.15. *The restriction of $(\mathcal{D}_Y)_S^\sharp$ to $(T^*Y^0)_S^{(1)}$ is locally free of rank $p^{\dim Y + \dim S}$.*

Proof. It is enough to prove that $\mathrm{gr}((\mathcal{D}_Y)_S^\sharp)$ restricted to $(T^*Y^0)_S^{(1)}$ is locally free of rank $p^{\dim Y + \dim S}$. However, we have the natural isomorphism

$$\mathrm{gr}((\mathcal{D}_Y)_S^\sharp) \simeq \mathrm{Fr}_*(\mathcal{O}_{(T^*Y)_S})$$

which immediately implies what we need (since $(T^*Y^0)_S$ is smooth).   □

     □

In the sequel we are going to need the following lemma whose proof is explained in [14].

Lemma 3.16. *Let θ be a one-form on a group stack $\mathcal{Y}$ (over a base $\mathcal{W}$). Assume that*

(3.3) $$m^*\theta = p_1^*\theta + p_2^*\theta.$$

Then the algebra $\mathcal{D}_{\mathcal{Y},\theta}$ has a natural group structure.

4. D-modules on Bun_n and the Hitchin fibration

In this section k is an arbitrary algebraically closed field and X is a smooth projective irreducible curve over k of genus $g > 1$. For $n > 0$ we let Bun_n denote the moduli stack of rank n vector bundles on X. We denote by Ω_X the canonical sheaf of X; we shall also use the notation T^*X for the corresponding geometric object (i.e. the total space of the corresponding line bundle). We also denote by $^iT^*X$ the total space of $\Omega_X^{\otimes i}$.

4.1. **The Hitchin map.** The stack $T^*\mathrm{Bun}_n$ parametrises pairs $(\mathcal{F}, A)$ where $\mathcal{F} \in \mathrm{Bun}_n$ and $A : \mathcal{F} \to \mathcal{F} \otimes \Omega_X$ is an arbitrary map. Let

$$\mathrm{Hitch}_n = \bigoplus_{i=1}^{n} H^0(X, \Omega_X^{\otimes i}).$$

Define the map $h : T^*\mathrm{Bun}_n \to \mathrm{Hitch}_n$ in the following way:

$$h : (\mathcal{F}, A) \mapsto (\tau_1(A), ..., \tau_n(A)) := (\mathrm{tr}(A), \mathrm{tr}(\Lambda^2 A), ..., \mathrm{tr}(\Lambda^n A) = \det A).$$

4.2. **Spectral curves.** Let $\chi : \mathrm{Hitch}_n \times T^*X \to {}^nT^*X$ be the map sending the point $(\tau_1, ..., \tau_n), \xi$ to

$$\sum (-1)^i \tau_i \otimes \xi^{n-i}.$$

For $(\mathcal{F}, A) \in T^*\mathrm{Bun}_n$ one can think of $\chi(h((\mathcal{F}, \mathcal{A})))$ as the characteristic polynomial of A.

We let $\widetilde{X}$ be the (scheme-theoretic) pre-image of the zero section in $^nT^*X$. This is a closed subscheme of $\mathrm{Hitch}_n \times T^*X$ which we shall call *the total spectral curve*.

Let $\pi : \widetilde{X} \to \mathrm{Hitch}_n \times X$ be the natural morphism obtained by composing the embedding $\widetilde{X} \hookrightarrow \mathrm{Hitch}_n \times T^*X$ with the natural projection $\mathrm{Hitch}_n \times T^*X \to X$. Then π is a finite flat morphism of degree n. We also denote by $\mathrm{pr}_1 : \widetilde{X} \to \mathrm{Hitch}_n$ and $\mathrm{pr}_2 : \widetilde{X} \to T^*X$ the corresponding projections.

Given a scheme S over k and an S-point τ of Hitch_n we let $\widetilde{X}_\tau$ denote the corresponding closed subscheme of $S \times T^*X$ (obtained by base change from $\widetilde{X}$). In particular, if $S = \mathrm{Spec}\, k$ then $\widetilde{X}_\tau$ is a closed susbcheme of T^*X which if flat and finite of degree n over X.

Proposition 4.3. *There exists a non-empty open subset Hitch_n^0 of Hitch_n over which pr_1 is smooth.*

Let $\widetilde{X}^0 = \mathrm{pr}_1^{-1}(\mathrm{Hitch}_n^0)$. Let also $\theta = \mathrm{pr}_2^* \theta_X$.

4.4. Fibers of the Hitchin map via line bundles on $\widetilde{X}$.
Let S a k-scheme and let τ be an S-point of Hitch_n^0. It is well-known that the fiber of h over τ can be canonically identified with with the stack $\mathrm{Pic}(\widetilde{X}_\tau)$. Let us recall this identification. Let $\mathcal{L}$ be a line bundle on $\widetilde{X}_\tau$. The embedding $\widetilde{X}_\tau \subset S \times T^* X$ gives rise to a map

$$a : \mathcal{L} \to \mathcal{L} \otimes \pi^*(\mathcal{O}_S \boxtimes \Omega_X).$$

Then $\mathcal{F} = \pi_* \mathcal{L}$ is a vector bundle on $S \times X$ of rank n and the push-forward of a gives rise to a Higgs field

$$A : \mathcal{F} \to \mathcal{F} \otimes \Omega_X.$$

In particular when τ is a k-point of Hitch_n then $h^{-1}(\tau)$ can be identified with the stack $\mathrm{Pic}(\widetilde{X}_\tau)$.

Corollary 4.5. *The stack $T^* \mathrm{Bun}_n^0$ can be naturally identified with $\mathrm{Pic}(\widetilde{X}^0 / \mathrm{Hitch}_n^0)$.*

Corollary 4.5 shows in particular that the automorphism group of every k-point of $T^* \mathrm{Bun}_n^0$ is equal to $\mathbb{G}_m$.

From now on we assume that k has characteristic $p > 0$.

4.6. The algebra $\mathcal{D}_{\mathbf{Bun}_n}$.
The stack Bun_n is not good in the terminology of Section 3.13. Therefore, we must explain what we mean by $\mathcal{D}_{\mathrm{Bun}_n}$. We claim that there exists a stack $\underline{\mathrm{Bun}}_n$ and a canonical morphism

$$\kappa_n : \mathrm{Bun}_n \to \underline{\mathrm{Bun}}_n$$

such that

1) Bun_n is a $\mathbb{G}_m$-gerbe over $\underline{\mathrm{Bun}}_n$

2) Every connected component of $\underline{\mathrm{Bun}}_n$ is very good in the sense of Section 3.13.

It follows from the above that the stack $T^* \mathrm{Bun}_n$ is a $\mathbb{G}_m$-gerbe over $T^* \underline{\mathrm{Bun}}_n$. We define the algebra $\mathcal{D}_{\mathrm{Bun}_n}$ to be the pullback of $\mathcal{D}_{\underline{\mathrm{Bun}}_n}$ from $T^* \underline{\mathrm{Bun}}_n$.

Let us explain the construction of the stack $\underline{\mathrm{Bun}}_n$. Let us define a functor

$$F : \text{Schemes over } k \to \text{Groupoids}$$

in the following way. For any scheme S over k let us define the category $F(S)$ in the following way:

- Objects of $F(S)$ are vector bundles on $S \times X$ of rank n

• Morphisms between two vector bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ on $S \times X$ in the category $F(S)$ consist of isomorphism classes of pairs $(\mathcal{L}, \iota)$ where $\mathcal{L}$ is a line bundle on S and ι is an isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_2 \otimes p_S^* \mathcal{L}$ where $p_S : S \times X \to S$ denotes the natural projection.

It is easy to see that a pair $(\mathcal{L}, \iota)$ as above does not have any non-trivial automorphisms, hence looking at the isomorphism classes of such pairs is really a harmless operation (any such isomorphism is unique).

We now define $\underline{\mathrm{Bun}}_n$ to be the sheafification of the functor F in the smooth topology. It is easy to see that it satisfies all the above properties.

4.7. **The stack Loc_n.** Let Loc_n denote the stack parametrising "de Rham local systems of rank n" on X. In other words, for a test scheme S we define $\mathrm{Hom}(S, \mathrm{Loc}_n)$ to be the groupoid of all vector bundles $\mathcal{E}$ on $S \times X$ of rank n endowed with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathrm{pr}_X^* \Omega_X$ where $\mathrm{pr}_X : S \times X \to X$ is the natural projection.

We claim that there exists a natural map $c : \mathrm{Loc}_n \to \mathrm{Hitch}_n^{(1)}$. To construct it let $(\mathcal{F}, \nabla)$ be a point in Loc_n. Recall that to ∇ there corresponds the p-curvature operator

$$\psi_\nabla : \mathcal{F} \to \mathcal{F} \otimes \mathrm{Fr}_X^* \Omega_{X^{(1)}}$$

which can also be regarded as a section of $(\mathrm{Fr}_X)_* (\underline{\mathrm{End}}(\mathcal{F})) \otimes \Omega_{X^{(1)}}^1$. Applying the standard invariant polynomials to the first multiple as in Section 4.1 we obtain a point of $\mathrm{Hitch}_n^{(1)}$ which we set to be $c((\mathcal{F}, \nabla))$.

Let us note that the identification $T^* \mathrm{Bun}_n^0 = \mathrm{Pic}(\widetilde{X}^0 / \mathrm{Hitch}_n^0)$ induces a group structure on the former as a stack over Hitch_n^0. Set now $\mathrm{Loc}_n^0 = c^{-1}((\mathrm{Hitch}_n^0)^{(1)})$.

Lemma 4.8. *Loc_n^0 has a natural structure of a $(T^* Bun_n^0)^{(1)} = (\mathrm{Pic}(\widetilde{X}^0 / Hitch_n^0))^{(1)}$-torsor (as a stack over $(Hitch_n^0)^{(1)}$).*

Proof. To prove Lemma 4.8 we are going to rephrase the definition of the stack Loc_n^0. Namely, we claim that an S-point of Loc_n^0 is the same as the following data:

1) A morphism $S \to (\mathrm{Hitch}_n^0)^{(1)}$;

2) A splitting of the pull-back of the algebra $\mathcal{D}_X$ from $T^* X^{(1)}$ to $S \underset{\mathrm{Hitch}_n^{(1)}}{\times} \widetilde{X}^{(1)}$

(note that we have a natural map $S \underset{\mathrm{Hitch}_n^{(1)}}{\times} \widetilde{X}^{(1)} \to T^* X^{(1)}$ coming from $\mathrm{pr}_2^{(1)} : \widetilde{X}^{(1)} \to T^* X^{(1)}$).

We leave the verification of the fact that the above functor is indeed represented by Loc_n^0 to the reader (note however, that we do not have such a simple description of the whole stack Loc_n).

Now since the category of splittings of an Azumaya algebra on $S \underset{\mathrm{Hitch}_n^{(1)}}{\times} \widetilde{X}^{(1)}$ is a Picard category over the category of line bundles on $S \underset{\mathrm{Hitch}_n^{(1)}}{\times} \widetilde{X}^{(1)}$ the statement of Lemma 4.8 follows. $\qquad\square$

4.9. **The main result.** The next theorem is the main result of this section. Let us denote by $\mathcal{D}_{\mathrm{Bun}_n}^0$ the restriction of $\mathcal{D}_{\mathrm{Bun}_n}$ to $(T^*\mathrm{Bun}_n^0)^{(1)}$.

Theorem 4.10.   (1) *The algebra $\mathcal{D}_{\mathrm{Bun}_n}^0$ has a natural group structure (with respect to the above group structure on $(T^*\mathrm{Bun}_n^0)^{(1)}$).*
   (2) *The $\mathrm{Pic}((\widetilde{X}^0)^{(1)}/(\mathit{Hitch}_n^0)^{(1)})$-torsor Loc_n^0 is canonically equivalent to the dual torsor of $\mathcal{Y}_{D_{\mathrm{Bun}_n}^0}$. In particular, we have a canonical equivalence of derived categories $\Phi_n : D^b(\mathcal{M}(\mathcal{D}_{\mathrm{Bun}_n}^0)) \simeq D^b(\mathcal{M}(\mathcal{O}_{\mathit{Loc}_n^0}))$.*

Proof. We are going to give two different proofs (though they are based on the same idea). The first one makes use of Corollary 3.12. The second proof is more direct; it will be used in the next section.

4.11. **First proof.** Consider the addition map
(4.1)
$$a : \widetilde{X}^0 \underset{\mathrm{Hitch}_n}{\times} \mathrm{Pic}(\widetilde{X}^0/\mathrm{Hitch}_n^0) = \widetilde{X}^0 \times T^*\mathrm{Bun}_n^0 \to \mathrm{Pic}(\widetilde{X}^0/\mathrm{Hitch}_n^0) = T^*\mathrm{Bun}_n^0$$

We now claim the following (see section 4.16 for a proof):

Theorem 4.12. *$a^*\theta_{\mathrm{Bun}_n} = \mathrm{pr}_2^*\,\theta_X \boxtimes \theta_{\mathrm{Bun}_n}$. In particular, the Azumaya algebras $(a^{(1)})^*\mathcal{D}_{\mathrm{Bun}_n}^0$ and $(\mathrm{pr}_2^{(1)})^*\mathcal{D}_X \boxtimes \mathcal{D}_{\mathrm{Bun}_n}$ are equivalent.*

Restricting the above equality to the product of $\widetilde{X}^0$ with the unit section in $\mathrm{Pic}(\widetilde{X}^0/\mathrm{Hitch}_n^0)$ we get the following corollary:

Corollary 4.13. *Consider the natural map $\kappa : \widetilde{X}^0 \to \mathrm{Pic}(\widetilde{X}^0/\mathit{Hitch}_n^0) = T^*\mathrm{Bun}_n^0$ sending a point $\widetilde{x} \in \widetilde{X}^0$ to the bundle $\mathcal{O}_{\widetilde{x}}$. Then we have*
$$\mathrm{pr}_2^*\,\theta_X = \kappa^*\theta_{\mathrm{Bun}_n}.$$
In particular, the Azumaya algebras $(\kappa^{(1)})^*\mathcal{D}_{\mathrm{Bun}_n}^0$ and $(\mathrm{pr}_2^{(1)})^*\mathcal{D}_X$ are canonically equivalent. Thus also there is a canonical equivalence of Azumaya algebras
(4.2)
$$(\mathrm{pr}_2^{(1)})^*\mathcal{D}_X \simeq (\kappa^{(1)})^*\mathcal{D}_{\mathrm{Bun}_n}^0.$$

Let us first explain how Theorem 4.12 implies Theorem 4.10.

4.14. **The group structure on $\mathcal{D}^0_{\mathbf{Bun}_n}$: first construction.** It is enough to check that the form $\theta^0_{\mathrm{Bun}_n} := \theta_{\mathrm{Bun}_n}|_{T^*\mathrm{Bun}_n^0}$ satisfies the condition of Lemma 3.16. For any $d, d' \in \mathbb{Z}$ let us denote by $m_{d,d'}$ the addition map

$$\mathrm{Pic}^d(\widetilde{X}^0/\mathrm{Hitch}_n^0) \times \mathrm{Pic}^{d'}(\widetilde{X}^0/\mathrm{Hitch}_n^0) \to \mathrm{Pic}^{d+d'}(\widetilde{X}^0/\mathrm{Hitch}_n^0).$$

It is enough to show that $m^*_{d,d'}\theta^0_{\mathrm{Bun}_n} = \theta^0_{\mathrm{Bun}_n} \boxtimes \theta^0_{\mathrm{Bun}_n}$ for d large enough. So let us assume that $d > 2n^2(g-1)$. Let Y_d denote the d-th Cartesian power of $\widetilde{X}^0$ over Hitch_n^0. Let us also denote by $\kappa_d : Y_d \to \mathrm{Pic}^d(\widetilde{X}^0/\mathrm{Hitch}_n^0)$ the natural map sending a point $(x_1, ..., x_d)$ to $\mathcal{O}(x_1 + ... + x_d)$. Then Y_d has an open subset on which the map κ_d is dominant and smooth. Hence it is enough to show that

$$(\kappa_d \times \mathrm{Id})^* m^*_{d,d'}\theta^0_{\mathrm{Bun}_n} = \kappa_d^*\theta^0_{\mathrm{Bun}_n} \boxtimes \theta_{\mathrm{Bun}_n^0}.$$

However, iterating the assertion of Theorem 4.12 d-times we see that the left hand side is equal to

$$\underbrace{\mathrm{pr}_2^* \theta_X \boxtimes ... \boxtimes \mathrm{pr}_2^* \theta_X}_{d \text{ times}}.$$

However, iterating the assertion of Corollary 4.13 we see that the latter form is equal to $\kappa_d^*\theta_{\mathrm{Bun}_n}$ which finishes the proof.

4.15. **Proof of Theorem 4.10(2).** It is enough to construct a map $(\mathcal{Y}_{D^0_{\mathrm{Bun}_n}})_1^\vee \to \mathrm{Loc}_n^0$ of $\mathrm{Pic}((\widetilde{X}^0)^{(1)}/(\mathrm{Hitch}_n^0)^{(1)})$-torsors; here $(\mathcal{Y}_{D^0_{\mathrm{Bun}_n}})_1^\vee$ denotes the preimage of 1 under the natural map $(\mathcal{Y}_{D^0_{\mathrm{Bun}_n}})^\vee \to \mathbb{Z}$ (cf. Section 2.8). Let us do that on the level of k-points (the construction on the level of S-points is basically a word-by-word repetition). A k-point of $(\mathcal{Y}_{D^0_{\mathrm{Bun}_n}})^\vee$ is a splitting of $\mathcal{D}^0_{\mathrm{Bun}_n}|_{(h^{(1)})^{-1}(\tau)}$ compatible with the group structure for some $\tau \in (\mathrm{Hitch}_n^{(1)})^0$. Restricting this splitting to the image of $\widetilde{X}^{(1)}_\tau$ in $(h^{(1)})^{-1}(\tau) = \mathrm{Pic}(\widetilde{X}^{(1)}_\tau)$ and applying (4.2) we get a splitting of $\mathcal{D}_X|_{\widetilde{X}^{(1)}_\tau}$, i.e. a point of Loc_n^0 which lies in $c^{-1}(\tau)$. This clearly defines a morphism $(\mathcal{Y}_{D^0_{\mathrm{Bun}_n}})^\vee \to \mathrm{Loc}_n^0$. The fact that this is a map of $\mathrm{Pic}((\widetilde{X}^0)^{(1)}/(\mathrm{Hitch}_n^0)^{(1)})$-torsors is obvious from the definitions.

4.16. **Proof of Theorem 4.12.** Denote by $\mathcal{H}$ [5] the stack parametrising the following data:

1) $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Bun}_n$ and $x \in X$;

2) An embedding $\mathcal{F}_1 \subset \mathcal{F}_2$ such that the quotient is a coherent sheaf of length one concentrated at the point x.

---

[5]In the next section we are going to define stacks $\mathcal{H}^r$ for $r = 1, ..., n$; the stack $\mathcal{H}$ discussed here will be denoted by $\mathcal{H}^1$ in the next section.

Let us denote by $q_1 : \mathcal{H} \to X \times \mathrm{Bun}_n$ sending the above data to the pair $(x, \mathcal{F}_1)$; also we denote by $q_2 : \mathcal{H} \to \mathrm{Bun}_n$ the map sending the above data to $\mathcal{F}_2$.

It is easy to see that both q_1 and q_2 are smooth maps. Therefore, we have the closed embeddings $q_1^* T^*(X \times \mathrm{Bun}_n) \to T^*\mathcal{H}$ and $q_2^* T^*\mathrm{Bun}_n \to T^*\mathcal{H}$. Set

$$Z = q_1^* T^*(X \times \mathrm{Bun}_n) \cap q_2^* T^*\mathrm{Bun}_n; \qquad Z^0 = q_1^*(T^*X \times T^*\mathrm{Bun}_n^0) \cap q_2^* T^*\mathrm{Bun}_n^0.$$

Clearly, we have the natural maps $\alpha_1 : Z^0 \to T^*X \times T^*\mathrm{Bun}_n^0$ and $\alpha_2 : Z^0 \to T^*\mathrm{Bun}_n^0$. We claim now that there is a natural isomorphism $Z^0 \simeq \widetilde{X}^0 \underset{\mathrm{Hitch}_n}{\times} T^*\mathrm{Bun}_n^0 = \widetilde{X}^0 \underset{\mathrm{Hitch}_n}{\times} \mathrm{Pic}(\widetilde{X}^0 \times \mathrm{Hitch}_n^0)$ such that:

a) The map α_1 is the Cartesian product of the map $\mathrm{pr}_2 : \widetilde{X}^0 \to T^*X$ and the identity map $T^*\mathrm{Bun}_n^0 \to T^*\mathrm{Bun}_n^0$;

b) The map α_2 is equal to the addition map (4.1).

In order to construct such an isomorphism let us note that if $((x, \xi), (\mathcal{F}_1, A_1), (\mathcal{F}_2, A_2))$ lies in Z^0 then $(\mathcal{F}_1, A_1)$ and $(\mathcal{F}_2, A_2)$ lie over the same point τ of Hitch_n since we have $A_1 = A_2$ outside of $x \in X$ (this makes sense since $\mathcal{F}_1$ and $\mathcal{F}_2$ are identified outside of x). This also implies that the embedding $\mathcal{F}_1 \subset \mathcal{F}_2$ comes from the embedding $\mathcal{L}_1 \subset \mathcal{L}_2$ where $\mathcal{L}_i$ $(i = 1, 2)$ is the line bundle on $\widetilde{X}_\tau$ corresponding to the pair $(\mathcal{F}_i, A_i)$. Since the quotient of $\mathcal{L}_2$ by $\mathcal{L}_1$ must be one-dimensional it is actually equal to the skyscraper sheaf of a point $\widetilde{x} \in \widetilde{X}_\tau$. It is easy to see that we must have $\widetilde{x} = (x, \xi)$; in particular, $(x, \xi) \in \widetilde{X}_\tau$. Now it is clear that sending the triple $((x, \xi), (\mathcal{F}_1, A_1), (\mathcal{F}_2, A_2))$ to $(\widetilde{x}, \mathcal{L}_1)$ identifies Z^0 with $\widetilde{X}^0 \underset{\mathrm{Hitch}_n}{\times} \mathrm{Pic}(\widetilde{X}^0 \times \mathrm{Hitch}_n^0)$ and this identification satisfies a) and b) formulated above.

Now we want to prove the statement of Theorem 4.12. From the above we see that it is enough to show that $\alpha_1^*(\theta_X \boxtimes \theta_{\mathrm{Bun}_n}) = \alpha_2^*(\theta_{\mathrm{Bun}_n})$ on Z^0. However, both these forms coincide with the restriction of $\theta_{\mathcal{H}}$ to Z^0 which finishes the proof.

4.17. **Second proof of Theorem 4.10.** We want to give another proof of Theorem 4.10 which does not use 1-forms. In fact we are going to give a different proof of the second assertion of Theorem 4.12 which does not appeal to one-forms and leave the details of the other parts of the proof to the reader. It is also not difficult to check that the equivalence as in Theorem 4.10 constructed below coincides with the one constructed above using 1-forms.

We need to establish an equivalence

$$(4.3) \qquad (a^{(1)})^* \mathcal{D}_{\mathrm{Bun}_n}^0 \simeq (\mathrm{pr}_2^{(1)})^* \mathcal{D}_X \boxtimes \mathcal{D}_{\mathrm{Bun}_n}$$

of Azumaya algebras on $(\widetilde{X}^0)^{(1)} \underset{(\mathrm{Hitch}_n^0)tw}{\times} \mathrm{Pic}((\widetilde{X}^0)^{(1)}/(\mathrm{Hitch}_n^0)^{(1)})$. Recall that the latter stack is identified with $(T^*Z^0)^{(1)} \subset T^*\mathcal{H}^{(1)}$. It is clear that the LHS of (4.3) is identified $(dq_2^{(1)})^*\mathcal{D}_{\mathrm{Bun}_n}^0$ and the RHS of (4.3) is identified with $(dq_1^{(1)})^*(\mathcal{D}_X \boxtimes \mathcal{D}_{\mathrm{Bun}_n})$. We claim now that both these algebras can be identified with the restriction of $\mathcal{D}_\mathcal{H}$ to $(Z^0)^{(1)}$. This is an immediate corollary of Proposition 3.7. □

Remark. Let S be any smooth k-variety. Then it is easy to see that a slight generalization of the above construction gives an equivalence of categories

$$\Phi_{n,S} : D^b(\mathcal{D}_{\mathrm{Bun}_n \times S}^0 - mod) \widetilde{\rightarrow} D^b(\mathcal{O}_{\mathrm{Loc}_n^0} \boxtimes \mathcal{D}_S) - mod.$$

4.18. **The modules** $\mathrm{Aut}_\mathcal{E}$. Given $(\mathcal{E}, \nabla) \in \mathrm{Loc}_n^0$ we shall denote by $\mathrm{Aut}_\mathcal{E}$ the corresponding $\mathcal{D}_{\mathrm{Bun}_n}^0$-module. This module defines a splitting of $\mathcal{D}_{\mathrm{Bun}_n}^0$ on the corresponding Hitchin fiber. Since this fiber is closed in $T^*\mathrm{Bun}_n$ it follows that we may regard $\mathrm{Aut}_\mathcal{E}$ as a $\mathcal{D}_{\mathrm{Bun}_n}$-module (rather than $\mathcal{D}_{\mathrm{Bun}_n}^0$-module). In the next section we are going to show that $\mathrm{Aut}_\mathcal{E}$ is a Hecke eigen-module (in the sense explained in the next section).

5. The Hecke eigenvalue property

5.1. **The Hecke correspondences.** Let r be an integer such that $1 \leq r \leq n$. Denote by $\mathcal{H}_r$ the stack which classifies the following data:

1) A triple $(\mathcal{F}_1, \mathcal{F}_2, x) \in \mathrm{Bun}_n \times \mathrm{Bun}_n \times X$

2) An embedding $\mathcal{F}_1 \subset \mathcal{F}_2$ such that the quotient is scheme-theoretically concentrated at x and has length r.

We denote $\overleftarrow{q}_r : \mathcal{H}_r \rightarrow \mathrm{Bun}_n \times X$ the map sending the above data to the pair $(\mathcal{F}_1, x)$. Similarly we let $\overrightarrow{q}_r : \mathcal{H}_r :\rightarrow \mathrm{Bun}_n$ be the map sending the above data to $\mathcal{F}_2$. It is well-known that the maps $\overrightarrow{q}_r, \overleftarrow{q}_r$ (and thus the stack $\mathcal{H}_r$) are smooth.

5.2. **The Hecke functors.** We denote by $\mathbf{H}_r$ the functor from the category $D^b(\mathcal{M}(\mathcal{D}_{\mathrm{Bun}_n}))$ to the category $D^b(\mathcal{M}(\mathcal{D}_{\mathrm{Bun}_n \times X}))$ defined by

$$\mathbf{H}_r(M) = (\overleftarrow{q}_r)_* \overrightarrow{q}_r^!(M).$$

It is easy to see that the functor $\mathbf{H}_r$ is in fact a functor between $D^b(\mathcal{M}(\mathcal{D}_{\mathrm{Bun}_n}))$ and $D^b(\mathcal{M}(\mathcal{D}_{\mathrm{Bun}_n \times X}))$ as categories over $\mathrm{Hitch}_n^{(1)}$; in particular, we may restrict it to $(\mathrm{Hitch}_n^0)^{(1)}$. We shall denote the corresponding functor by $\mathbf{H}_r^0$.

On the other hand, for any number r as above we can define the functor $\mathbf{T}_r : D^b(\mathrm{Loc}_n) \rightarrow D^b(\mathcal{M}(\mathcal{O}_{\mathrm{Loc}_n} \boxtimes \mathcal{D}_X))$ as follows. Let $\mathcal{E} \in \mathcal{M}(\mathcal{O}_{\mathrm{Loc}_n} \boxtimes \mathcal{D}_X)$

denote the "universal local system". Let $\mathrm{pr}_1 : \mathrm{Loc}_n \times X \to \mathrm{Loc}_n$ denote the projection to the first multiple. Then (for every $\mathcal{F} \in D^b(\mathrm{Loc}_n)$) we set[6]

$$\mathbf{T}_r(\mathcal{F}) = \mathrm{pr}_1^* \, \mathcal{F} \otimes \wedge^r \mathcal{E}.$$

We denote by $\mathbf{T}_r^0$ the corresponding functor from $D^b(\mathrm{Loc}_n^0)$ to $D^b(\mathcal{M}(\mathcal{O}_{\mathrm{Loc}_n} \boxtimes \mathcal{D}_X))$.

5.3. The Hecke eigenvalue property. Here is the main result of this section.

Theorem 5.4. *For $n < p$ there is canonical isomorphism of functors*

$$\Phi_{n,X} \circ \mathbf{H}_r^0 \simeq \mathbf{T}_r^0.$$

Remark. One can define the analog of the functors $\mathbf{T}_r$ for every finite-dimensional representation of $GL(n)$ (the functors $\mathbf{T}_r$ correspond to the wedge powers of the standard representation). However we do not know how to define the analogs of the functors $\mathbf{H}_r$ in this case (unlike the case when k has characteristic 0 where such functors are well-known).

5.5. Proof of Theorem 5.4 for $r = 1$. Let $\mathcal{P}$ denote the universal $\mathcal{D}_{\mathrm{Bun}_n}^0 \boxtimes \mathcal{O}_{\mathrm{Loc}_n^0}$-module. The statement of Theorem 5.4 is equivalent to the existence of an isomorphism

$$(5.1) \qquad\qquad {}^1\mathbf{H}_r^0(\mathcal{P}) \simeq {}^2\mathbf{T}_r^0(\mathcal{P}),$$

where the superscript on the left means that we apply the corresponding functor either along the first or the second factor.

Let us now concentrate on the case $r = 1$ (we shall see later that the proof in the general case is almost a word-by-word repetition of the proof for $r = 1$ but notationally it is a bit more complicated). Let Z, Z^0 be as in the previous section. Recall that $Z^0 \simeq T^*\mathrm{Bun}_n^0 \underset{\mathrm{Hitch}_n^0}{\times} \widetilde{X}^0$. We have the natural maps $\alpha_1 : Z^0 \to T^*X \times T^*\mathrm{Bun}_n^0$ and $\alpha_2 : Z^0 \to T^*\mathrm{Bun}_n^0$, where α_1 is a closed embedding and α_2 is smooth. Also we have the natural equivalence of Azumaya algebras

$$(5.2) \qquad\qquad (\alpha_1^{(1)})^*(\mathcal{D}_X \boxtimes \mathcal{D}_{\mathrm{Bun}_n}^0) \simeq (\alpha_2^{(1)})^*(\mathcal{D}_{\mathrm{Bun}_n}^0).$$

By the definition, in order to compute $\mathbf{H}_1(M)$ for any $\mathcal{D}_{\mathrm{Bun}_n}^0$-module M we just need to look at $(\alpha_2^{(1)})^*(M)$ and regard it as a $\mathcal{D}_X \boxtimes \mathcal{D}_{\mathrm{Bun}_n}^0$-module using (5.2). Let us apply it to the module $\mathcal{P}$ and recall the following:

1) The stack Loc_n^0 parametrises splittings of $\mathcal{D}_{\mathrm{Bun}_n^0}$ compatible with the group structure.

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[6]Note that $\mathbf{T}_r$ lands indeed in $D^b(\mathcal{M}(\mathcal{O}_{\mathrm{Loc}_n} \boxtimes \mathcal{D}_X))$ since $\mathcal{E}$ is flat.

2) The map α_2 is the composition of the natural embedding $\kappa : \widetilde{X}^0 \to \mathrm{Pic}(\widetilde{X}^0/\mathrm{Hitch}_n^0) = T^*\mathrm{Bun}_n^0$ and the addition map

$$\mathrm{Pic}(\widetilde{X}^0/\mathrm{Hitch}_n^0) \underset{\mathrm{Hitch}_n^0}{\times} \mathrm{Pic}(\widetilde{X}^0/\mathrm{Hitch}_n^0) \to \mathrm{Pic}(\widetilde{X}^0/\mathrm{Hitch}_n^0).$$

3) The Azumaya algebra $(\kappa^{(1)})^*\mathcal{D}_{\mathrm{Bun}_n}^0$ is naturally equivalent to $(\mathrm{pr}_2^{(1)})^*\mathcal{D}_X$ (recall that pr_2 denotes the natural map $\widetilde{X} \to T^*X$). Moreover, under this equivalence the splitting $(\kappa^{(1)} \times \mathrm{id})^*\mathcal{P}$ of $(\kappa^{(1)})^*\mathcal{D}_{\mathrm{Bun}_n}^0 \boxtimes \mathcal{O}_{\mathrm{Loc}_n^0}$ goes over to the splitting $(\mathrm{pr}_2^{(1)} \times \mathrm{id})^*\mathcal{E}$ of $(\mathrm{pr}_2^{(1)})^*\mathcal{D}_X \boxtimes \mathcal{O}_{\mathrm{Loc}_n^0}$.

It follows now easily from 1,2,3 above that the $\mathcal{D}_X \boxtimes \mathcal{D}_{\mathrm{Bun}_n}^0 \boxtimes \mathcal{O}_{\mathrm{Loc}_n^0}$-module corresponding to $(\alpha_2^{(1)})^*(\mathcal{P})$ via (5.2) is naturally isomorphic to $\mathcal{E}^{13} \otimes \mathcal{P}^{23}$ (where the double superscript means that the sheaf in question is lifted from the corresponding couple of multiples of $T^*X^{(1)} \times (T^*\mathrm{Bun}_n^0)^{(1)} \times \mathrm{Loc}_n^0$). This finishes the proof.

5.6. **Proof of Theorem 5.4 in the general case.** Let us explain how to generalize this proof to arbitrary r. In fact we are only going to give a sketch of the proof here, breaking it into several (simple) steps whose proofs we are going to leave to the reader.

Let $\mathrm{Sym}^r(\widetilde{X}^0/\mathrm{Hitch}_n^0)$ denote the relative symmetric power of $\widetilde{X}$ over Hitch_n^0. Alternatively, we can say that an S-point of $\mathrm{Sym}^r(\widetilde{X}^0/\mathrm{Hitch}_n^0)$ is the same as a morphism $S \to \mathrm{Hitch}_n^0$ and a zero-dimensional subscheme $\widetilde{X}^0 \underset{\mathrm{Hitch}_n^0}{\times} S$ which is flat over S and has length r over any closed point of S. Also we define $\mathrm{Hilb}^r(\widetilde{X}^0/X \times \mathrm{Hitch}_n^0)$ to be the closed subscheme of $\mathrm{Sym}^r(\widetilde{X}^0/\mathrm{Hitch}_n^0)$ whose S-points consist of the following data:

1) A morphism $S \to X \times \mathrm{Hitch}_n^0$

2) A zero-dimensional subscheme $\widetilde{X}^0 \underset{X \times \mathrm{Hitch}_n^0}{\times} S$ which is flat over S and has length r over any closed point of S.

The proof of the following lemma is left to the reader.

Lemma 5.7.   (1) *The scheme* $\mathrm{Hilb}^r(\widetilde{X}^0/X \times \mathrm{Hitch}_n^0)$ *is flat and finite of degree* $\binom{n}{r}$ *over* $X \times \mathrm{Hitch}_n^0$.

   (2) *There exists a natural map* $\eta : \mathrm{Hilb}^r(\widetilde{X}^0/X \times \mathrm{Hitch}_n^0) \to T^*X$ *satisfying the following property: let* τ *be a k-point of* Hitch_n^0 *and let* $x \in X(k)$ *be such that* $\widetilde{X}_\tau$ *is unramified over x. Let also* $\mathcal{T} \subset \widetilde{X}_\tau$ *be any collection of r points of* $\widetilde{X}_\tau$ *lying over x (which naturally defines a point in* $\mathrm{Hilb}^r(\widetilde{X}^0/X \times$

Hitch_n^0). *Then*

$$\eta(\mathcal{T}) = \sum_{\widetilde{x} \in \mathcal{T}} \widetilde{x}$$

where the summation on the right is taken inside $T_x^* X$.

Let now τ be an S-point of Hitch_n^0 and let $\mathcal{L}$ be a line bundle on $\widetilde{X}_\tau$. Denote by $\mathcal{L}^{(r)}$ its r-th symmetric power restricted to $\mathrm{Hilb}^r(\widetilde{X}_\tau / X \times S)$ (the latter is defined as the base change of $\mathrm{Hilb}^r(\widetilde{X}^0 / X \times \mathrm{Hitch}_n^0)$ to S). Let $\eta_S : \mathrm{Hilb}^r(\widetilde{X}_\tau / X \times S) \to T^*X \times S$ be the corresponding base change of η multiplied by id_S.

Set now $\mathcal{E}_\mathcal{L}$ to be the direct image of $\mathcal{L}$ under the natural map $\widetilde{X}_\tau \to T^*X \times S$. Let us think of $\mathcal{E}_\mathcal{L}$ as an S-point of $T^*\mathrm{Bun}_n^0$, i.e. we want to think of it as a vector bundle of rank n on $X \times S$ together with a Higgs field

$$A : \mathcal{E}_\mathcal{L} \to \mathcal{E}_\mathcal{L} \otimes (\Omega_X \boxtimes \mathcal{O}_S).$$

We denote by $\Lambda^r(\mathcal{E}_\mathcal{L})$ the r-th exterior power of $\mathcal{E}_\mathcal{L}$ endowed with a Higgs field $\Lambda^r(A)$ defined by

$$\Lambda^r(A)(e_1 \wedge ... \wedge e_r) = A(e_1) \wedge e_2 \wedge ... \wedge e_r + ... + e_1 \wedge e_2 \wedge ... \wedge e_{r-1} \wedge A(e_r).$$

Note that $\Lambda^r(\mathcal{E}_\mathcal{L})$ can again be considered as a sheaf on $T^*X \times S$.

Lemma 5.8. *We have the natural isomorphism*

$$(\eta_S)_* \mathcal{L}^{(r)} = \Lambda^r(\mathcal{E}_\mathcal{L}).$$

Define now

$$Z_r = (\overleftarrow{q}_r)^*(T^*\mathrm{Bun}_n \times T^*X) \cap (\overrightarrow{q}_r)^*(T^*\mathrm{Bun}_n) \subset T^*\mathbf{H}_r;$$

$$Z_r^0 = (\overleftarrow{q}_r)^*(T^*\mathrm{Bun}_n^0 \times T^*X) \cap (\overrightarrow{q}_r)^*(T^*\mathrm{Bun}_n^0).$$

We claim that Z_r^0 can be canonically identified with $\mathrm{Pic}(\widetilde{X}^0 / \mathrm{Hitch}_n^0) \times \mathrm{Hilb}^r$ $(\widetilde{X}^0 / X \times \mathrm{Hitch}_n^0)$. Note that the latter scheme can be identified with the scheme classifying 5-tuples $(\tau, \mathcal{L}_1, \mathcal{L}_2, x, \iota)$ where τ is an (S)-point of Hitch_n^0, $\mathcal{L}_1, \mathcal{L}_2$ are two line bundles on $\widetilde{X}_\tau$, x is a point of X and ι is an embedding $\mathcal{L}_1 \to \mathcal{L}_2$ such that the corresponding ideal sheaf in $\mathcal{O}_{\widetilde{X}_\tau}$ defines a subscheme of length r lying in the preimage of x. Under this identification the isomorphism $Z_r^0 \simeq \mathrm{Pic}(\widetilde{X}^0 / \mathrm{Hitch}_n^0) \times \mathrm{Hilb}^r(\widetilde{X}^0 / X) \times \mathrm{Hitch}_n^0$ satisfies the following properties:

1) The natural map $\overleftarrow{\alpha} : Z_r^0 \to T^*\mathrm{Bun}_n \times T^*X$ sends $(\tau, \mathcal{L}_1, \mathcal{L}_2, x, \iota)$ to $(\mathcal{L}_1, \eta(\mathcal{T}))$ where $\mathcal{T}$ is the corresponding point of $\mathrm{Hilb}^r(\widetilde{X}^0 / X \times \mathrm{Hitch}_n^0)$.

2) The natural map $\overrightarrow{\alpha} : Z_r^0 \to T^*\mathrm{Bun}_n$ sends the above 5-tuple to $\mathcal{L}_2$.

The proof of this claim is identical to the proof of the corresponding statement for $r = 1$ discussed in Section 4.16.

Now the rest of the proof of Theorem 5.4 is the same as in Section 5.5

5.9. **Hecke eigen-modules.** Let now $\mathcal{E}$ be any k-point of Loc_n^0 and let $\mathrm{Aut}_{\mathcal{E}}$ be the corresponding D-module on Bun_n considered in Section 4.18. We now claim that $\mathrm{Aut}_{\mathcal{E}}$ is a "Hecke eigen-module" in the following sense:

Theorem 5.10. *For any $r = 1, ..., n$ there is a canonical isomorphism*

$$\mathbf{H}_r(\mathrm{Aut}_{\mathcal{E}}) \simeq \mathrm{Aut}_{\mathcal{E}} \boxtimes \Lambda^r(\mathcal{E}).$$

The proof is immediate from Theorem 5.4

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