Pure and Applied Mathematics Quarterly
Volume 2, Number 4
(*Special Issue: In honor of
Robert MacPherson, Part 2 of 3*)
1217—1236, 2006

Discrete Torsion, Orbifold Elliptic Genera and the Chiral de Rham Complex

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To Robert MacPherson on his 60th birthday

Abstract: Given a compact complex algebraic variety with an effective action of a finite group G, and a class $\alpha \in H^2(G, U(1))$, we introduce an orbifold elliptic genus with discrete torsion α, denoted $Ell^{\alpha}_{orb}(X, G, q, y)$. We give an interpretation of this genus in terms of the chiral de Rham complex attached to the orbifold $[X/G]$. If X is Calabi-Yau and G preserves the volume form, $Ell^{\alpha}_{orb}(X, G, q, y)$ is a weak Jacobi form. We also obtain a formula for the generating function of the elliptic genera of symmetric products with discrete torsion.

1. Introduction

The two-variable elliptic genus (see for example [Kri]) of a compact complex manifold X is a generating function

$$Ell(X, y, q) = \sum_{m,l} c(m, l) q^m y^l \tag{1.1}$$

which captures important topological information about X. For appropriate values of y and q, $Ell(X, y, q)$ specializes to the L, $\widehat{A}$, and χ_y genera respectively. Mathematically, the elliptic genus can be defined as follows. For a holomorphic vector bundle V on X and a formal variable t, let

$$\mathrm{Sym}_t V = 1 + tV + t^2 \mathrm{Sym}^2 V + t^3 \mathrm{Sym}^3 V + \cdot \ \in K_0(X)[[t]]$$

and

$$\Lambda_t V = 1 + tV + t^2 \Lambda^2 V + t^3 \Lambda^3 V + \cdot \ \in K_0(X)[[t]]$$

Received October 13, 2005.

Let T_X, T_X^* denote the holomorphic tangent and cotangent bundles respectively, and

$$\mathcal{E}ll(X,q,y) = y^{-\frac{\dim X}{2}} \otimes_{n\geq 1} (\Lambda_{-yq^{n-1}} T_X^* \otimes \Lambda_{-y^{-1}q^n} T_X \otimes S_{q^n} T_X^* \otimes S_{q^n} T_X) \tag{1.2}$$

viewed as an element of $K_0(X)[[q]][[y^{\pm\frac{1}{2}}]]$. Then

$$Ell(X,q,y) = \chi(\mathcal{E}ll(X,q,y)).$$

In physics, $Ell(X,q,y)$ is part of the partition function of a two-dimensional conformal field theory with target X (cf. [EOTY]).

In this paper, we will say that X is Calabi-Yau if K_X is trivial - this is of course weaker than the usual mathematical Calabi-Yau condition, but agrees with the physics terminology. When X is Calabi-Yau, $Ell(X,q,y)$ has nice modular properties. Let $\mathbb{H}$ denote the upper half plane. A *weak Jacobi form* of weight $k \in \mathbb{Z}$ and index $r \in \frac{1}{2}\mathbb{Z}$ is a holomorphic function on $\mathbb{H} \times \mathbb{C}$ satisfying the transformation property

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi i \frac{rcz^2}{c\tau+d}} \phi(\tau,z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\phi(\tau, z+m\tau+n) = (-1)^{2r(m+n)} e^{-2\pi i r(m^2\tau+2mz)} \phi(\tau,z), \quad (m,n) \in \mathbb{Z}^2$$

that in addition has a Fourier expansion $\sum_{l,m} c_{m,l} y^l q^m$ with nonnegative m (see [EZ]), where $q = e^{2\pi i\tau}, y = e^{2\pi iz}$. It is shown in [BL, Gri] that if X is Calabi-Yau then $Ell(X,q,y)$ is a weak Jacobi form of weight 0 and index $\dim(X)/2$.

In [BL1], the authors introduced a notion of orbifold elliptic genus $Ell(X,G,q,y)$ attached to the global quotient orbifold $[X/G]$, where X is a smooth compact complex manifold, and $G \in Aut(X)$ is a finite group (see also [DLiM] for general reduced orbifolds). A mathematical definition of this genus is given in section 2.1. Roughly, $Ell(X,G,q,y)$ is obtained by adding the contributions of Euler characteristics of bundles analogous to 1.2 over the various fixed-point sets $X^g, g \in G$ of the G–action on X. Furthermore, for each $g \in G$, the contribution takes into account the eigenvalues of g on $TX|_{X^g}$. It is also shown in [BL1] that if X is Calabi-Yau, and G preserves the volume form, then $Ell(X,G,q,y)$ is a weak Jacobi form of weight 0 and index $\dim(X)/2$.

It was observed in [BL] that the ordinary elliptic genus $Ell(X,y,q)$ can be interpreted in terms of an object called the *chiral de Rham complex*, denoted Ω_X^{ch}. The latter is a sheaf of vertex superalgebras attached to any smooth complex manifold X, introduced in [MSV] (for a brief discussion of vertex algebras, see section 3.1). The sheaf Ω_X^{ch} has an increasing filtration $F^\bullet\Omega_X^{ch}$, and also a compatible bigrading by two operators L_0, J_0. One can therefore describe the associated graded sheaf $gr_F\Omega_X^{ch}$ in terms of the bigraded components. One finds

that Supertrace$(q^{L_0}y^{J_0}, \Omega_X^{ch})$ is the sheaf 1.2 above. It follows from this observation that

$$Ell(X, y, q) = \text{Supertrace}(q^{L_0}y^{J_0-\dim(X)/2}, H^*(X, \Omega_X^{ch}))$$

It is believed that $H^*(X, \Omega_X^{ch})$ captures some of the information of the two-dimensional quantum field theory on X mentioned above. Ω_X^{ch} also carries a differential $Q_{BRST} \in \text{End}(\Omega_X^{ch})$, $Q_{BRST}^2 = 0$ (which is why it is called a complex). The "de Rham" part of the name comes from the fact that the complex $(\Omega_X^{ch}, Q_{BRST})$ is quasiisomorphic to the holomorphic de Rham complex (Ω_{dR}, ∂).

In [FS], the construction of Ω_X^{ch} was extended to orbifolds (another construction of the chiral de Rham complex for orbifolds was obtained independently by A. Vaintrob). For each $g \in G$, one introduces sheaves $\Omega_X^{ch,g}$ supported on X^g, which are modules over Ω_X^{ch}. Each $\Omega_X^{ch,g}$ carries a canonical $C(g)$–equivariant structure, where $C(g)$ denotes the centralizer of g in G. The sheaves $\Omega_X^{ch,g}$ allow one to interpret some of the "stringy" geometric invariants of the orbifold $[X/G]$. In particular, it is shown in [FS] that

$$Ell(X, G, q, y) = \text{Supertrace}(q^{L_0}y^{J_0-\dim(X)/2}, \bigoplus_{[g]} H^*(X, \Omega_X^{ch,g})^{C(g)}) \tag{1.3}$$

and

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch,g}, Q_{BRST})^{C(g)} \cong \bigoplus_{[g]} H_{dR}^*(X^g/C(g), \mathbb{C}) \tag{1.4}$$

where $[g]$ denotes a set of representatives for the conjugacy classes in G. The object on the right in 1.4, with an appropriate grading and ring structure is called the Chen-Ruan cohomology of $[X/G]$ (see [CR]). The isomorphism 1.4 is an isomorphism of graded vector-spaces.

We now come to *discrete torsion*. This term was introduced in the physics literature to refer to the discovery (see [Va1, Va2]) that an orbifold quantum field theory on $[X/G]$ could be "twisted" by a cocycle $\alpha \in H^2(G, U(1))$. In terms of physics, the path integral can be written as a sum of contributions from sectors parametrized by commuting pairs of elements (g, h) in $G \times G$, and the contribution from the (g, h)–sector is multiplied by the phase $\delta(g, h) = \alpha(g, h)/\alpha(h, g)$. This modification produces a consistent physical theory and leads to α–twisted invariants of the orbifold $[X/G]$. For a mathematical discussion of various aspects of discrete torsion see [AF, R, Ka]. In this paper, we build on the results in [BL1, FS] to give a mathematical treatment of orbifold elliptic genera with discrete torsion $Ell^\alpha(X, G, q, y)$. One way to define this object along the lines of [BL1] is to multiply each contribution in the sum 2.3 by the appropriate phase $\delta(g, h)$. From the point of view of the chiral de Rham complex, this definition can be cast in a manner somewhat closer to the original physics approach as follows. Recall that in 1.3 one uses the $C(g)$–equivariant structure on $\Omega_X^{ch,g}$ to project

on the $C(g)$–invariant part of $H^*(X, \Omega_X^{ch,g})$. A cocycle $\alpha \in H^2(G, U(1))$ induces characters $\alpha_g : C(g) \mapsto U(1)$ by $\alpha_g(h) = \delta(g,h)$. The character α_g allows us to twist the $C(g)$ equivariant structure, and taking invariants with respect to this twisted structure projects on a different subspace. We can therefore define

$$Ell^{\alpha}(X,G,q,y) = \text{Supertrace}(q^{L_0}y^{J_0-\dim(X)/2}, \bigoplus_{[g]} H^*(X,\Omega_X^{ch,g})^{C(g)_\alpha}) \tag{1.5}$$

where $C(g)_\alpha$ indicates that the twisted action is being used. We show that if X is Calabi-Yau, $Ell^{\alpha}(X,G,q,y)$ is a Jacobi form of weight 0 and index $\dim(X)/2$. We also show that there is an isomorphism of graded vector spaces

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch,g}, Q_{BRST})^{C(g)_\alpha} \cong \bigoplus_{[g]} H^*_{dR}(X^g/C(g), \mathcal{L}_g^{\alpha})$$

where the object on the right denotes the α–twisted Chen-Ruan cohomology of $[X/G]$ valued in the collection of local systems $\mathcal{L}_g^{\alpha}$, introduced in [R].

An important example of discrete torsion arises in the case of symmetric products (see [Di]). S_N, the symmetric group on N letters, acts on the hyperplane in $\mathbb{R}^N$ given by the equation $x_1 + \cdots + x_N = 0$. This yields an embedding $S_N \hookrightarrow O(N-1)$. Pulling back the double cover $Pin(N-1) \mapsto O(N-1)$ yields a central extension of S_N by $\mathbb{Z}/2\mathbb{Z}$, which we denote $\widehat{S}_N$ - i.e.

$$1 \mapsto \mathbb{Z}/2\mathbb{Z} \mapsto \widehat{S}_N \mapsto S_N \mapsto 1 \tag{1.6}$$

The extension 1.6 is non-split for $N \geq 4$, and therefore yields a non-zero class $\alpha \in H^2(S_N, \mathbb{Z}/2\mathbb{Z})$, which via the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow U(1)$ can be pushed into $H^2(S_N, U(1))$.

The orbifold elliptic genera of symmetric products can be arranged into remarkable generating functions. It was proved in [BL1] following a physics derivation in [DMVV] that

$$\sum_{N \in \mathbb{Z}_+} p^N Ell_{orb}(X^N, S_N, y, q) = \prod_{n,m,\ell \geq 0} (1 - p^n q^m y^{\ell})^{-c(nm,\ell)}.$$

where the $c(m,l)$'s are as in 1.1. In section 4.2 we obtain a generalization of this formula with discrete torsion given by α above, which was originally obtained by Dijkgraaf ([Di]) in the physics literature.

Acknowledgements: This project originally began with Lev Borisov. We would like to thank him for many valuable conversations. During the course of this work the second author was supported by NSF grant DMS −0401619.

2. Orbifold elliptic genera

2.1. **The orbifold elliptic genus.** Let X be a complex manifold on which a finite group G acts effectively via holomorphic transformations. Let X^h will be the fixed point set of $h \in G$ and $X^{g,h} = X^g \cap X^h (g, h \in G)$. Let

$$TX|_{X^h} = \oplus_{\lambda(h)\in \mathbf{Q}\cap[0,1)} V_\lambda. \tag{2.1}$$

where the bundle V_λ on X^h is determined by the requirement that h acts on V_λ via multiplication by $e^{2\pi i\lambda(h)}$. For a connected component of X^h (which by abuse of notation we also will denote X^h), the fermionic shift is defined as $F(h, X^h \subseteq X) = \sum_\lambda \lambda(h)$ (cf. [Z], [BD]). Let us consider the bundle:

$$V_{h,X^h\subseteq X} := \otimes_{k\geq 1}\Big[(\Lambda^\bullet_{yq^{k-1}} V_0^*) \otimes (\Lambda^\bullet_{y^{-1}q^k} V_0) \otimes (Sym^\bullet_{q^k} V_0^*) \otimes (Sym^\bullet_{q^k} V_0)\otimes$$

$$\otimes\Big[\otimes_{\lambda\neq 0}(\Lambda^\bullet_{yq^{k-1+\lambda(h)}} V_\lambda^*) \otimes (\Lambda^\bullet_{y^{-1}q^{k-\lambda(h)}} V_\lambda) \otimes (Sym^\bullet_{q^{k-1+\lambda(h)}} V_\lambda^*) \otimes (Sym^\bullet_{q^{k-\lambda(h)}} V_\lambda)\Big]\Big] \tag{2.2}$$

Definition 2.1. The orbifold elliptic genus of a G-manifold X is the function on $H \times \mathbf{C}$ given by:

$$Ell_{orb}(X, G, q, y) := y^{-\dim/X/2} \sum_{[g],X^g} y^{F(g,X^g\subseteq X)} \frac{1}{|C(g)|} \sum_{h\in C(h)} L(h, V_{g,X^g\subseteq X})$$

where the summation in the first sum is over all conjugacy classes in G and connected components X^g of an element $g \in [g]$, $C(g)$ is the centralizer of $g \in G$ and

$$L(h, V_{g,X^g\subseteq X}) = \sum_i (-1)^i \mathrm{tr}(h, H^i(V_{g,X^g\subseteq X}))$$

is the holomorphic Lefschetz number.

Using the holomorphic Lefschetz fixed-point formula ([AS]) one can rewrite this definition as follows.

Theorem 2.1. [BL1] Let $TX|_{X^{g,h}} = \oplus W_\lambda$ and let x_λ be the collection of Chern roots of W_λ. Let

$$\theta(z,\tau) = q^{\frac{1}{8}}(2\sin\pi z) \prod_{l=1}^{l=\infty}(1-q^l) \prod_{l=1}^{l=\infty}(1-q^l \mathrm{e}^{2\pi \mathrm{i} z})(1-q^l \mathrm{e}^{-2\pi \mathrm{i} z})$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ be the Jacobi's theta function and let

$$\Phi(g,h,\lambda,z,\tau,x) = \frac{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x}{2\pi i} + \lambda(g) - \tau\lambda(h))} e^{2\pi i z\lambda(h)}.$$

Then:

$$Ell_{orb}(X, G, z, \tau) = \frac{1}{|G|} \sum_{gh=hg} \prod_{\lambda(g)=\lambda(h)=0} x_\lambda \prod_\lambda \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}]. \tag{2.3}$$

The orbifold elliptic genus so defined specializes for $q = 0$ to

$$Ell_{orb}(X, G, 0, y) = y^{-\dim X}\chi_{-y}(X, G)$$

where

$$\chi_y(X, G) = \sum_{[g],X^g} y^{F(g,X^g\subset X)} \sum_{p,q}(-1)^q \dim H^{p,q}(X^g)^{C(g)}$$

On the other hand $\chi_y(X, G)$ is the value of the orbifold E-function

$$E(u, v, G) = \sum_{[g],X^g} (uv)^{F(g,X^g\subset X)} \sum_{p,q} \dim H^{p,q}(X^g)^{C(g)} u^p v^q$$

for $u = y, v = -1$. In paticular $Ell_{orb}(X, G, 0, 1)$ coincides with the orbifold Euler characteristic: $e_{orb}(X, G) = \frac{1}{|G|}\sum_{fg=gf} e(X^{f,g})$.

2.2. **Discrete torsion.**

Definition 2.2. Let $\alpha \in H^2(G, U(1))$, and define

$$\delta(g, h) = \frac{\alpha(g, h)}{\alpha(h, g)}$$

The orbifold elliptic genus with discrete torsion α, written $Ell^{\alpha}_{orb}(X, G, q, y)$, is defined as

(2.4)

$$Ell^{\alpha}_{orb}(X, G, q, y) := y^{-\dim X/2} \sum_{[g],X^g} y^{F(g,X^g\subseteq X)} \frac{1}{|C(g)|} \sum_{h\in C(h)} \delta(g, h)L(h, V_{g,X^g\subseteq X}).$$

As above, using the holomorphic Lefschetz fixed-point formula this can be rewritten as

$$Ell^{\alpha}_{orb}(X, G; y, q) = \frac{1}{|G|} \sum_{gh=hg} \delta(g, h) \prod_{\lambda(g)=\lambda(h)=0} x_\lambda \prod_{\lambda} \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}].$$

Such twisted elliptic genus has specialization properties similar to the case $\alpha = 0$. Using Dolbeault cohomology corresponding to the inner local systems L_α defined by α (cf. [R]) one can define twisted E-function:

$$E^{\alpha}(u, v, G) = \sum_{[g],X^g} (uv)^{F(g,X^g\subset X)} \sum_{p,q} \dim H^{p,q}(X^g, L_\alpha)^{C(g)} u^p v^q$$

which for $u = 1, v = -1$ yields:

$$e^{\alpha}(X, G) = \frac{1}{|G|} \sum_{fg=gf} \delta(f.g)e(X^{f,g})$$

The elliptic genus 2.4 satisfies:

$$Ell^{\alpha}(0, y, G) = y^{\frac{\dim X}{2}} E^{\alpha}(y, -1, G)$$

We proceed to investigate the modularity properties of this twisted orbifold elliptic genus.

2.2.1. *Jacobi forms.* Let $\mathbb{H}$ denote the upper half plane. A *weak Jacobi form* of weight $k \in \mathbb{Z}$ and index $r \in \frac{1}{2}\mathbb{Z}$ is a function on $\mathbb{H}\times\mathbb{C}$ satisfying the transformation property

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi i \frac{rcz^2}{c\tau+d}}\phi(\tau,z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\phi(\tau, z+m\tau+n) = (-1)^{2r(m+n)}e^{-2\pi i r(m^2\tau+2mz)}\phi(\tau,z), \quad (m,n)\in\mathbb{Z}^2$$

and has a Fourier expansion $\sum_{l,m} c_{m,l}y^l q^m$ with nonnegative m. Equivalently, we can say that a Jacobi form is an automorphic form for the Jacobi group $\Gamma^J = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ generated by the four transformations:

$$(z,\tau)\to(z+1,\tau),\ (z,\tau)\to(z+\tau,\tau),\ (z,\tau)\to(z,\tau+1),\ (z,\tau)\to(\frac{z}{\tau},-\frac{1}{\tau}).$$

Theorem 2.2. *Let X be a compact complex manifold of dimension d such that K_X is trivial, G a finite group acting effectively on X, and $\alpha \in H^2(G,U(1))$. Let n denote the order of G in $\mathrm{Aut}(H^0(X,K_X))$. Then $Ell^\alpha_{orb}(X,G)$ is a weak Jacobi form of weight 0 and index $d/2$ with respect to subgroup of the Jacobi group Γ^J generated by transformations*

$$(z,\tau)\to(z+n,\tau),\ (z,\tau)\to(z+n\tau,\tau),\ (z,\tau)\to(z,\tau+1),\ (z,\tau)\to(\frac{z}{\tau},-\frac{1}{\tau}).$$

In particular, if the action preserves holomorphic volume then $Ell_{orb}(X,G)$ is a weak Jacobi form of weight 0 and index $d/2$ for the full Jacobi group.

Proof. We use the notation $Ell^\alpha_{orb}(X,G,z,\tau)$ rather than $Ell^\alpha(X,G,q,y)$ to emphasize the dependence on τ and z. It is shown in [BL1] that

$$\Phi(g,h,\lambda,z+1,\tau,x) = -\mathrm{e}^{2\pi\mathrm{i}\lambda(h)}\cdot\Phi(g,h,\lambda,z,\tau,x)$$

which implies that

$$\prod_\lambda \Phi(g,h,\lambda,z+n,\tau,x_\lambda)[X^{g,h}] = (-1)^{dn}\mathrm{e}^{2n\pi\mathrm{i}\sum\lambda(h)}\prod_\lambda\Phi(g,h,\lambda,z,\tau,x_\lambda)[X^{g,h}]$$

Now, $n\sum\lambda(h)\in\mathbb{Z}$ by the assumption that h^n acts trivially on $H^0(X,K_X)$. Thus

$$Ell^\alpha_{orb}(X,G,z+n,\tau) = (-1)^{dn}Ell^\alpha_{orb}(X,G,z,\tau).$$

The following formulas are also obtained in [BL1]:

$$\Phi(g,h,\lambda,z,\tau+1,x) = \Phi(gh^{-1},h,\lambda,z,\tau,x) \tag{2.5}$$

$$\Phi(g,h,\lambda,z+n\tau,\tau,x) = (-1)^n\mathrm{e}^{-2\pi\mathrm{i}nz-\pi\mathrm{i}n^2\tau}\mathrm{e}^{nx+2\pi\mathrm{i}n\lambda(g)}\cdot\Phi(g,h,\lambda,z,\tau,x) \tag{2.6}$$

$$\Phi(g,h,\lambda,\frac{z}{\tau},\ -\frac{1}{\tau},\frac{x}{\tau}) = e^{\frac{\pi\mathrm{i}z^2}{\tau}-\frac{zx}{\tau}}\cdot\Phi(h,g^{-1},\lambda,z,\tau,x). \tag{2.7}$$

Equation 2.5 and $\delta(g,h)=\delta(gh^{-1},h)$ imply that

$$Ell^{\alpha}_{orb}(X,G,z,\tau+1)=Ell^{\alpha}_{orb}(X,G,z,\tau).$$

Equation 2.6 implies that

$$Ell^{\alpha}_{orb}(X,G,z+n\tau,\tau)=(-1)^{dn}e^{-2\pi \mathrm{i} dnz-\pi \mathrm{i} dn^2\tau}Ell^{\alpha}_{orb}(X,G,z,\tau)$$

In order to see how

$$Ell^{\alpha}_{orb}(X,G,z/\tau,-1/\tau)=e^{\frac{\pi i dz^2}{\tau}}Ell^{\alpha}_{orb}(X,G,z,\tau) \tag{2.8}$$

follows, we write

$$\prod_{\lambda(g)=\lambda(h)=0}x_\lambda\prod_{\lambda}\Phi(g,h,\lambda,z,\tau,x_\lambda)=\sum_{\mathbf{k}_\lambda}Q(g,h,z,\tau)\mathbf{x}_\lambda{}^{\mathbf{k}_\lambda}$$

where $\mathbf{k}_\lambda$ are multiindices and $\mathbf{x}_\lambda$ are the corresponding monomials. We thus obtain,

$$\prod_{\lambda(g)=\lambda(h)=0}\frac{x_\lambda}{\tau}\prod_{\lambda}\Phi(g,h,\lambda,\frac{z}{\tau},\frac{-1}{\tau},\frac{x_\lambda}{\tau})=\sum_{\mathbf{k}_\lambda}(\frac{1}{\tau})^{deg(\mathbf{k}_\lambda)}Q(g,h,\frac{z}{\tau},\frac{-1}{\tau})\mathbf{x}_\lambda{}^{\mathbf{k}_\lambda}$$

whereas 2.7 implies that

$$\begin{aligned}\prod_{\lambda(g)=\lambda(h)=0}\frac{x_\lambda}{\tau}\prod_{\lambda}\Phi(g,h,\lambda,\frac{z}{\tau},\frac{-1}{\tau},\frac{x_\lambda}{\tau})&=e^{\frac{\pi i dz^2}{\tau}}\prod_{\lambda(g)=\lambda(h)=0}\frac{x_\lambda}{\tau}\prod_{\lambda}\Phi(h,g^{-1},\lambda,z,\tau,x)\\&=\tau^{-dim(X^{g,h})}\sum_{\mathbf{k}_\lambda}Q(h,g^{-1},z,\tau)\mathbf{x}_\lambda{}^{\mathbf{k}_\lambda}\end{aligned}$$

Thus for multiindices $\mathbf{k}_\lambda$ such that $deg(\mathbf{k}_\lambda)=dim(X^{g,h})$, we find

$$Q_k(g,h,\frac{z}{\tau},\frac{-1}{\tau})=Q_k(h,g^{-1},z,\tau)$$

Finally, $\delta(g,h)=\delta(h,g^{-1})$ ensures that 2.8 holds. □

3. Discrete torsion and the chiral de Rham complex

Let X be a smooth complex algebraic variety, and G a finite group acting effectively on X. In this section, we briefly review the construction of the chiral de Rham complex of an orbifold introduced in [FS]. Another construction of this object was obtained independently by A. Vaintrob.

3.1. **Vertex algebras and twisted modules.** In this section we will use the language of vertex superalgebras, their modules, and twisted modules. For an introduction to vertex algebras and their modules [FLM, K, FB], and for background on twisted modules, see [FFR, D, DLM, FS].

We recall that a conformal vertex superalgebra is a $\mathbb{Z}_+$–graded super vector space

$$V = \bigoplus_{n=0}^{\infty} V_n,$$

$$V_n = V_n^{\bar{0}} \oplus V_n^{\bar{1}}$$

together with a vacuum vector $|0\rangle \in V_0^{\bar{0}}$, an even translation operator T of degree 1, a conformal vector $\omega \in V_2^{\bar{0}}$ and an even linear map

$$\begin{aligned} Y : V &\to \operatorname{End} V[[z^{\pm 1}]], \\ A &\mapsto Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}. \end{aligned}$$

These data must satisfy certain axioms (see [FLM, K, FB]). In what follows we will denote the collection of such data simply by V, and the parity of an element $A \in V$ homogeneous with respect to the $\mathbb{Z}/2\mathbb{Z}$ grading by $p(A)$.

A vector superspace M is called a V–module if it is equipped with an even linear map

$$\begin{aligned} Y^M : V &\to \operatorname{End} M[[z^{\pm 1}]], \\ A &\mapsto Y^M(A,z) = \sum_{n\in\mathbb{Z}} A^M_{(n)} z^{-n-1} \end{aligned}$$

such that for any $v \in M$ we have $A^M_{(n)} v = 0$ for large enough n. This operation must satisfy the following axioms:

- $Y^M(|0\rangle, z) = \mathrm{Id}_M$;
- For any $v \in M$ and homogeneous $A, B \in V$ there exists an element
$$f_v \in M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$
such that the formal power series
$$Y^M(A,z)Y^M(B,w)v, \quad (-1)^{p(A)p(B)} Y^M(B,w)Y^M(A,z)v, \qquad \text{and}$$
$$Y_M(Y(A,z-w)B,w)v$$
are expansions of f_v in $M((z))((w))$, $M((w))((z))$ and $M((w))((z-w))$, respectively.

The power series $Y^M(A,z)$ are called vertex operators. We write the vertex operator corresponding to ω as

$$Y^M(\omega,z)=\sum_{n\in\mathbb{Z}} L_n^M z^{-n-2},$$

where L_n^M are linear operators on V generating the Virasoro algebra. Following [D], we call M *admissible* if L_0^M acts semi-simply with integral eigenvalues.

Now let σ_V be a conformal automorphism of V, i.e., an even automorphism of the underlying vector superspace preserving all of the above structures (so in particular $\sigma_V(\omega)=\omega$). We will assume that σ_V has finite order $m>1$. A vector space M^σ is called a *σ_V–twisted V–*module (or simply twisted module) if it is equipped with an even linear map

$$\begin{aligned} Y^{M^\sigma} &: V \to \operatorname{End} M^\sigma[[z^{\pm\frac{1}{m}}]],\\ &A \mapsto Y^{M^\sigma}(A,z^{\frac{1}{m}})=\sum_{n\in\frac{1}{m}\mathbb{Z}} A_{(n)}^{M^\sigma} z^{-n-1} \end{aligned}$$

such that for any $v\in M^\sigma$ we have $A_{(n)}^{M^\sigma}v=0$ for large enough n. Please note that we use the notation $Y^{M^\sigma}(A,z^{\frac{1}{m}})$ rather than $Y^{M^\sigma}(A,z)$ in the twisted setting. This operation must satisfy the following axioms (see [FFR, D, DLM, Li, FS]):

- $Y^{M^\sigma}(|0\rangle,z^{\frac{1}{m}})=\mathrm{Id}_{M^\sigma}$;
- For any $v\in M^\sigma$ and homogeneous $A,B\in V$, there exists an element

  $$f_v\in M^\sigma[[z^{\frac{1}{m}},w^{\frac{1}{m}}]][z^{-\frac{1}{m}},w^{-\frac{1}{m}},(z-w)^{-1}]$$

  such that the formal power series

  $$Y^{M^\sigma}(A,z^{\frac{1}{m}})Y^{M^\sigma}(B,w^{\frac{1}{m}})v,\quad (-1)^{p(A)p(B)}Y^{M^\sigma}(B,w^{\frac{1}{m}})Y^{M^\sigma}(A,z^{\frac{1}{m}})v,\qquad \text{and}$$

  $$Y^{M^\sigma}(Y(A,z-w)B,w^{\frac{1}{m}})v$$

  are expansions of f_v in $M^\sigma((z^{\frac{1}{m}}))((w^{\frac{1}{m}}))$, $M^\sigma((w^{\frac{1}{m}}))((z^{\frac{1}{m}}))$ and $M^\sigma((w^{\frac{1}{m}}))((z-w))$, respectively.
- If $A\in V$ is such that $\sigma_V(A)=e^{\frac{2\pi ik}{m}}A$, then $A_{(n)}^{M^\sigma}=0$ unless $n\in\frac{k}{m}+\mathbb{Z}$.

The series $Y^{M^\sigma}(A,z)$ are called twisted vertex operators. In particular, the Fourier coefficients of the twisted vertex operator

$$Y^{M^\sigma}(\omega,z^{\frac{1}{m}})=\sum_{n\in\mathbb{Z}} L_n^{M^\sigma} z^{-n-2},$$

generate an action of the Virasoro algebra on M^σ.

3.2. **The chiral de Rham complex of an orbifold.** For $g \in G$, let X^g denote the fixed-point set of g, and denote by

$$i^g : X^g \hookrightarrow X$$

the inclusion map of X^g into X. The following results were obtained in [FS].

- For each $g \in G$, there exists a sheaf $\Omega_X^{ch,g}$ supported on X^g. When $g = 1$, this is a sheaf of vertex superalgebras, originally introduced in [MSV], and called the chiral de Rham complex. We denote it simply by Ω_X^{ch}. Being a sheaf of vertex superalgebras means that for each open $U \in X$, $\Omega_X^{ch}(U)$ is a vertex superalgebra.
- Let $g, h \in G$, and let $g' = hgh^{-1}$. There exist isomorphisms of sheaves

$$R^h_{g,hgh^{-1}} : \Omega_X^{ch,g} \mapsto h^*\Omega_X^{ch,hgh^{-1}} \tag{3.1}$$

satisfying

$$R^k_{g',g''} \circ R^h_{g,g'} = R^{kh}_{g,g''}$$

where $k \in G$ and $g'' = khgh^{-1}k^{-1}$.

- When $g \neq 1$, $\Omega_X^{ch,g}$ is a sheaf of g–twisted modules, meaning that for each g–invariant U, $\Omega_X^{ch,g}(U)$ is a g–twisted $\Omega_X^{ch}(U)$–module. This structure induces a corresponding twisted module structure on $H^*(X, \Omega_X^{ch,g})$.
- $\Omega_X^{ch,g}$ carries a bigrading by two operators L_0^g, J_0^g. This bigrading induces a bigrading on $H^*(X, \Omega_X^{ch,g})$.
- $\Omega_X^{ch,g}$ carries a differential Q^g, such that $(Q^g)^2 = 0$. Furthermore, there exists an inclusion of the de Rham complex of X^g

$$i^g_*(\Omega_{dR}(X^g, d)) \hookrightarrow (\Omega_X^{ch,g}, Q^g)$$

which is a quasiisomorphism. This implies in particular that

$$\mathbb{H}^*(\Omega_X^{ch,g}, Q^g) \cong H^*_{dR}(X^g, \mathbb{C})$$

3.1 implies that $C(g)$, the centralizer of g, acts on $(\Omega_X^{ch,g}, Q^g)$, and therefore on its hypercohomology. This gives an isomorphism

$$\mathbb{H}^*(\Omega_X^{ch,g}, Q^g)^{C(g)} \cong H^*_{dR}(X, \mathbb{C})^{C(g)} \cong H^*_{dR}(X^g/C(g), \mathbb{C})$$

We therefore have

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch,g}, Q^g)^{C(g)} \cong \bigoplus_{[g]} H^*_{dR}(X^g/C(g), \mathbb{C}) \tag{3.2}$$

The right-hand side is isomorphic as a vector space to the Chen-Ruan orbifold cohomology of $[X/G]$. Furthermore, the operators J_0^g acting on $\Omega_X^{ch,g}$, induce a gradation on the left which coincides with the Chen-Ruan gradation shifted by the fermionic shift (see [CR, Z]). 3.2 is therefore an isomorphism of graded vector spaces.

- There exists an increasing exhaustive filtration on $\Omega_X^{ch,g}$

$$F^0\Omega_X^{ch,g} \subset F^1\Omega_X^{ch,g} \subset F^2\Omega_X^{ch,g} \subset \cdots . \tag{3.3}$$

Let $\overline{\Omega}_X^{ch,g}$ denote the restriction of $\Omega_X^{ch,g}$ to X^g, which inhertis a filtration from 3.3. The bigrading operators J_0^g, L_0^g are compatible with 3.3, and so the associated graded $gr_F(\overline{\Omega}_X^{ch,g})$ can be described in terms of its decomposition into eigenbundles for J_0^g, L_0^g. We have:

$$gr_F(\overline{\Omega}_X^{ch,g}) = \bigotimes_{k\geq 1}\Big(\Lambda^\bullet_{yq^{k-1}}V_0^* \otimes \Lambda^\bullet_{y^{-1}q^k}V_0 \otimes \mathrm{Sym}^\bullet_{q^k}V_0^* \otimes \mathrm{Sym}^\bullet_{q^k}V_0 \otimes \tag{3.4}$$

$$\bigotimes_{\lambda\neq 0}(\Lambda^\bullet_{yq^{k-1+\lambda(g)}}V_\lambda^* \otimes \Lambda^\bullet_{y^{-1}q^{k-\lambda(g)}}V_\lambda \otimes \mathrm{Sym}^\bullet_{q^{k-1+\lambda(g)}}V_\lambda^* \otimes \mathrm{Sym}^\bullet_{q^{k-\lambda(g)}}V_\lambda)\Big) .$$

where

$$TX|_{X^g} = \bigoplus V_\lambda.$$

If we now form

$$\mathcal{H}_{orb}(X,G) = \bigoplus_{[g]} H^*(X, \Omega_X^{ch,g})^{C(g)}$$

then as shown in [FS]

$$Ell_{orb}(X,G,q,y) = \mathrm{Supertrace}(q^{L_0}y^{J_0-\dim(X)/2}, \mathcal{H}_{orb}(X,G)).$$

3.2.1. *Adding discrete torsion to the chiral de Rham complex.* In this section we show how to incorporate discrete torsion in the above setup. Suppose that Y is a G–manifold, and W a G–equivariant sheaf on Y. This means that for each $g \in G$, we are given an isomorphism

$$T_g : W \mapsto g^*W$$

such that

$$T_gT_h = T_{gh}$$

Suppose now that $\chi : G \mapsto \mathbb{C}^\times$ is a character of G. Then

$$T'_g = \chi(g)T_g$$

is a new G equivariant structure on W.

We apply this observation to the sheaves $\Omega_X^{ch,g}$ and $C(g)$ rather than G. A class $\alpha \in H^2(G, U(1))$ yields characters

$$\alpha_g : C(g) \mapsto U(1)$$

defined by $\alpha_g(h) = \delta(g,h)$. We can now twist the $C(g)$–equivariant structure on $\Omega_X^{ch,g}$ described above, given by the $R^h_{g,g'}$, to obtain a new $C(g)$–equivariant

structure, denoted $C(g)_\alpha$.The following theorem is an immediate consequence of the above discussion.

Theorem 3.1. *i)*

$$\mathbb{H}^*(\Omega_X^{ch}, Q_{BRST})^{C(g)_\alpha} \cong H^*_{dR}(X^g/C(g), \mathcal{L}_g^\alpha)$$

where the right-hand side is the de Rham cohomology of $[X^g/C(g)]$ *with values in the orbifold local system* $\mathcal{L}_g^\alpha$ *described in* [R]. *Thus we have an isomorphism of graded vector spaces*

$$\bigoplus_{[g]} \mathbb{H}^*(\Omega_X^{ch}, Q_{BRST})^{C(g)_\alpha} \cong H^*_{orb,\alpha}([X/G], \mathbb{C})$$

where the right-hand side is the α*–twisted Chen-Ruan cohomology of* $[X/G]$ *(see* [R]*).*
ii) Let

$$\mathcal{H}^\alpha_{orb}(X,G) = \bigoplus_{[g]} H^*(X, \Omega_X^{ch,g})^{C(g)}_\alpha$$

Then

$$Ell^\alpha_{orb}(X,G,q,y) = \text{Supertrace}(q^{L_0} y^{J_0 - \dim(X)/2}, \mathcal{H}^\alpha_{orb}(X,G))$$

4. Symmetric products and discrete torsion

4.1. **The spin double cover of S_N.** We begin by reviewing discrete torsion for the symmetric group following [Di].

Let S_N denote the symmetric group on N letters. It is well-known (see eg. [Kar]) that for $N \geq 4$

$$H^2(S_N, U(1)) \cong \mathbb{Z}_2. \tag{4.1}$$

which implies that for $N \geq 4$, there is a unique non-trivial central extension of the permutation group

$$1 \to \mathbb{Z}_2 \to \widehat{S}_N \to S_N \to 1. \tag{4.2}$$

The extension $\widehat{S}_N$ can be constructed as follows. S_N acts on the hyperplane in $\mathbb{R}^N$ given by

$$x_1 + \cdots x_N = 0$$

preserving the standard inner product. This yields an embedding

$$S_N \hookrightarrow O(N-1).$$

Now, $O(N-1)$ has a double cover $\text{Pin}(N-1)$. Pulling back this central extension to S_N yields $\widehat{S}_N$. We call the latter the *spin double cover* of S_N.

In terms of generators and relations, $\widehat{S}_N$ can be described as follows. It is generated by elements $1, z, \hat{t}_1, \cdots, \hat{t}_{N-1}$, where z is central, subject to the relations:

$$\begin{aligned} z^2&=1,\\ \hat{t}_i^2&=z,\\ \hat{t}_i\hat{t}_{i+1}\hat{t}_i&=\hat{t}_{i+1}\hat{t}_i\hat{t}_{i+1},\\ \hat{t}_i\hat{t}_j&=z\,\hat{t}_j\hat{t}_i, \qquad j>i+1. \end{aligned} \tag{4.3}$$

The map $\widehat{S}_N \mapsto S_N$ amounts to sending $\hat{t}_i$ to the transposition t_i interchanging the ith and $i+1$st letters, and sending z to 1. We can think of z as being -1.

4.2. **Generating functions.** Suppose that the elliptic genus of X is given by

$$Ell(X;q,y)=\sum_{m,\ell} c(m,\ell)q^m y^\ell \tag{4.4}$$

As shown in [BL1, DMVV], the generating function of the orbifold elliptic genera of the symmetric products is

$$Z(p,q,y)=\sum_{N\geq 0} p^N Ell_{orb}(X^N,S_N,q,y)=\prod_{n>0,\,m,\ell}(1-p^nq^my^\ell)^{-c(nm,\ell)} \tag{4.5}$$

In this section, we obtain a formula for the generating function of elliptic genera of symmetric products with discrete torsion. Let

$$Z^\alpha(p,q,y)=\sum_{N\geq 0} p^N Ell^\alpha_{orb}(X^N,S_N,q,y) \tag{4.6}$$

and let

$$Z_{++}(p,q,y) = \prod_{n>0,\, m,l\geq 0} \frac{\left(1+p^{2n}q^{m-\frac{1}{2}}y^{\ell}\right)^{c(n(2m-1),\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}}$$

$$Z_{+-}(p,q,y) = \prod_{n>0,\, m,l\geq 0} \frac{\left(1-p^{2n}q^{m-\frac{1}{2}}y^{\ell}\right)^{c(n(2m-1),\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}}$$

$$Z_{-+}(p,q,y) = \prod_{n>0,\, m,l\geq 0} \frac{\left(1+p^{2n}q^{m}y^{\ell}\right)^{c(2nm,\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}}$$

$$Z_{--}(p,q,y) = -\prod_{n>0,\, m,l\geq 0} \frac{\left(1-p^{2n}q^{m}y^{\ell}\right)^{c(2nm,\ell)}}{\left(1-p^{2n-1}q^{m}y^{\ell}\right)^{c((2n-1)m,\ell)}} \tag{4.7}$$

Theorem 4.1.

$$Z^{\alpha}(p,q,y) = \frac{1}{2}\left(Z_{++}+Z_{+-}+Z_{-+}+Z_{--}\right). \tag{4.8}$$

We begin by recalling a variation on Lemma 4.5 from [BL1]

Lemma 4.1. *Let $V = V_0 \oplus V_1$ be a super vector space, and A and B two commuting operators acting semisimply on V and preserving the parity decomposition of V. Assume furthermore that B only has non-negative eigenvalues in $\frac{1}{2}\mathbb{Z}$, and that the bigraded pieces $V_{m,l} = \{v \in V | Av = lv, Bv = mv\}$ are finite-dimensional. Let $d(m,l) = sdim(V_{m,l})$, where sdim denotes superdimension. Define the superdimension of V with respect to A, B to be the series*

$$\chi(V)(y,q) = \text{Supertrace}(V, y^A q^B) = \text{tr}(V_0, y^A q^B) - \text{tr}(V_1, y^A q^B) = \sum_{m,l} d(m,l) q^m y^l$$

Let $Sym^N V$ denote the Nth supersymmetric product of V. The operators A and B act on $Sym^N V$, and

$$\sum_N p^N \,\text{Supertrace}(Sym^N V, y^A q^B) = \prod_{m,l} \frac{1}{(1-pq^m y^l)^{d(m,l)}}$$

where the right hand side is expanded in a power series in q and p.

Let $\Lambda^N V$ denote the Nth supersymmetric wedge product of V. Since $\Lambda^N V$ is isomorphic to $Sym^N \overline{V}$, where $\overline{V}$ denotes V with its parity reversed (or directly from the argument in the proof of lemma 4.5 in [BL1]), we obtain the following:

Corollary 4.1. *Let V be as in Lemma 4.1. Then*

$$\sum_N p^N \,\mathrm{Supertrace}(\Lambda^N V, y^A q^B) = \prod_{m,l}(1-pq^m y^l)^{d(m,l)}$$

Proof. (Of Theorem 4.1)
Let S_N denote the symmetric group on N letters. We recall that conjugacy classes in S_N are parametrized by partitions of N. The conjugacy class of an element $g \in S_N$ is therefore uniquely determined by the numbers a_j of j–cycles in the cycle decomposition of g. Recall moreover that the centralizer of an element with cycle type $[g] = (1)^{a_1}(2)^{a_2}\cdots(k)^{a_k}$ is

$$\prod_{i=1}^k S_{a_i} \ltimes (\mathbb{Z}/i\mathbb{Z})^{a_i}$$

where the $\mathbb{Z}/i\mathbb{Z}$ act by powers of the i–cycles and S_{a_i} permutes the i–cycles among themselves.

Let $c_j \in S_j$ be a j–cycle, and denote $\Omega^{ch,c_j}_{X^j}$ simply by $\Omega^{ch,j}_{X^j}$. Recall that this is a sheaf on X^j supported on X diagonally embedded, whose fibers are twisted modules over the chiral de Rham vertex algebra. Let $\mathcal{H}_{[g]} = H^*(X^N, \Omega^{ch,g}_{X^N})$ and $\mathcal{H}_j = H^*(X^j, \Omega^{ch,j}_{X^j})$, viewed as a super vector space where the parity is given by the sum of the cohomology index and the fermionic charge grading. Furthermore, introduce the operator D which acts on $\mathcal{H}_j$ by multiplication by $-j\dim(X)/2$. We have

$$\Omega^{ch,g}_{X^N} = \boxtimes_{j=1}^k (\Omega^{ch,j}_{X^j})^{\boxtimes a_j}$$

and so by the Kunneth formula

$$\mathcal{H}_{[g]} = \bigotimes_{j=1}^k \mathcal{H}_j^{\otimes a_j}$$

We have

$$\sum_N p^N Ell^\alpha_{orb}(X^N, S_N, q, y) = \sum_N p^N \sum_{[g]\in S_N} \mathrm{Supertrace}(q^{L_0} y^{J_0+D}, \mathcal{H}_{[g]}^{C(g)_\alpha})$$

where the subscript on $C(g)_\alpha$ indicates that invariants are being taken with respect to the α-twisted action of $C(g)$. If $h \in C(g)$, and T_h denotes the operator of h acting on $\mathcal{H}_{[g]}$ untwisted by α, then the α–twisted action is given by $\delta(g,h)T_h$.

As explained in for instance [Di], the value $\delta(g,h)$ depends on the parity of g and h, where the latter is given by

$$\begin{aligned} p(g) &= \sum_{j=1}^{k}(j-1)a_j \mod 2 \\ &= \sum_{j=1,\, j \text{ even}}^{k} a_j \mod 2 \end{aligned}$$

$C(g)$ is generated by transpositions $\tau(j)_{ab}$ interchanging two cycles of length j, as well as the j–cycles c_j in the cycle decomposition of g (we use the short-hand notation $c_j \in g$). The result is as follows:

$$\delta(g,\tau(j)_{ab}) = (-1)^{j-1} \tag{4.9}$$

and if $c_j \in g$, then

$$\delta(g,c_j) = \begin{cases} 1, & \text{if } j \text{ is odd,} \\ (-1)^{p(g)-1} & \text{if } j \text{ is even} \end{cases} \tag{4.10}$$

It follows from 4.9 and 4.10 that

$$\mathcal{H}_{[g]}^{C(g)_\alpha} = \bigotimes_{j=1 \text{ even}}^{k} \mathrm{Sym}^{a_j}(\mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha}) \otimes \bigotimes_{j=1 \text{ odd}}^{k} \Lambda^{a_j}(\mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha})$$

The space $\mathcal{H}^{\mathbb{Z}/j\mathbb{Z}_\alpha}$ will depend on how α twists the $\mathbb{Z}/j\mathbb{Z}$–action. For j even, there is only one possibility, and

$$\mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha} = \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}}$$

When j is odd, let

$$\begin{aligned} \mathcal{H}_j^{+} &= \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha}, \quad \text{when } [g] \text{ is odd} \\ \mathcal{H}_j^{-} &= \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}_\alpha}, \quad \text{when } [g] \text{ is even} \end{aligned}$$

It was shown in [BL1] that with $\mathcal{H} = \mathcal{H}_j^{\mathbb{Z}/j\mathbb{Z}}$ or $\mathcal{H}_j^{+}$

$$\begin{aligned} \mathrm{Supertrace}(q^{L_0}y^{J_0+D}, \mathcal{H}) &= \frac{1}{j}\sum_{r=0}^{j-1} Ell(X, q^{\frac{1}{j}}\xi^r, y) \\ &= \frac{1}{j}\sum_{m,l}(\sum_{r=0}^{j-1}\xi^{mr})q^{\frac{m}{j}}y^l \\ &= \sum_{m,l} c(mj,l)y^l q^m. \end{aligned} \tag{4.11}$$

where $\xi = \exp(2\pi i/j)$. Similarly, using the holomorphic Lefschetz fixed-point formula, one finds

$$\begin{aligned}\text{Supertrace}(q^{L_0}y^{J_0+D}, \mathcal{H}_j^-) &= \frac{1}{j}\sum_{r=0}^{j-1}(-1)^r Ell(X, q^{\frac{1}{j}}\xi^r, y)\\ &= \frac{1}{j}\sum_{m,l}(\sum_{r=0}^{j-1}(-1)^m\xi^{mr})q^{\frac{m}{j}}y^l\\ &= \sum_{m,l} c((m-\frac{1}{2})j, l)y^l q^{m-\frac{1}{2}}. \qquad (4.12)\end{aligned}$$

Let

$$\mathbb{S} = \bigotimes_{j\,\text{odd}} \text{Sym}_{p^j}\,\mathcal{H}^{\mathbb{Z}/j\mathbb{Z}}$$

and let

$$\Lambda^+ \in \bigotimes_{j\,\text{even}} \Lambda_{p^j}\mathcal{H}_j^+ \qquad (4.13)$$

$$\Lambda^- \in \bigotimes_{j\,\text{even}} \Lambda_{p^j}\mathcal{H}_j^- \qquad (4.14)$$

denote the subspaces corresponding to permutations of odd (resp. even) parity. We have that

$$\begin{aligned}\sum_N p^N Ell^\alpha_{orb}(X^N, S_n, q, y) = &\,\text{Supertrace}(q^{L_0}y^{J_0+D}, \mathbb{S}\bigotimes\Lambda^+)\\ &+ \text{Supertrace}(q^{L_0}y^{J_0+D}, \mathbb{S}\bigotimes\Lambda^-)\end{aligned}$$

The result now follows from Lemma 4.1, Corollary 4.1, and the observation that

$$\begin{aligned}\text{Supertrace}(q^{L_0}y^{J_0+D}, \Lambda^+) = &\frac{1}{2}\,\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}}\Lambda_{p^j}\mathcal{H}_j^+)\\ &-\frac{1}{2}\,\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}}\Lambda_{-p^j}\mathcal{H}_j^+)\end{aligned}$$

and

$$\begin{aligned}\text{Supertrace}(q^{L_0}y^{J_0+D}, \Lambda^-) = &\frac{1}{2}\,\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}}\Lambda_{p^j}\mathcal{H}_j^-)\\ &+\frac{1}{2}\,\text{Supertrace}(q^{L_0}y^{J_0+D}, \bigotimes_{j\,\text{even}}\Lambda_{-p^j}\mathcal{H}_j^-).\end{aligned}$$

□

Remark. There is an equivariant version of the theorem 4.1 which is also a twisted form of the product formula for the generating functions for the wreath products (conjectured in [WZ] and proven in [BL1] Remark 4.6. p.341). Let X and G be as above and let

$$Ell(X,G;q,y)=\sum_{m,\ell} c_G(m,\ell)q^m y^\ell \tag{4.15}$$

The wreath product $G\wr S_N$ (consisting of pairs $((g_1,...,g_N);\sigma), g_i\in G, \sigma\in S_N$ with multiplication: $((g_1,...,g_N);\sigma_1)\cdot((h_1,...,h_N);\sigma_2) = ((g_1\cdot h_{\sigma_1^{-1}(1)},...,g_N\cdot h_{\sigma_1^{-1}(N)});\sigma_1\sigma_2)$ acts on the symmetric products X^N. The nontrivial class in $H^2(S_N,U(1))$ can be pulled back to the class in $H^2(G\wr S_N,U(1))$ which we denote again as α. Then the generating function $Z^\alpha(X,G,p,q,y)=\sum_{N\geq 0}p^N Ell^\alpha_{orb}(X^N, G\wr S_N,q,y)$ is given by the theorem 4.1 with the coefficients $c(m,l)$ in the formulas 4.6 being replaced by $c_G(m,l)$ from 4.15

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