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Combinatorial Identities in the Theory of SU(n) Casson Invariants of Knots

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*Dedicated to Professor Robert MacPherson
on the occasion of his 60th birthday.*

Abstract: We explicitly evaluate some combinatorial sums which occur in the theory of SU(n) Casson invariants of fibered knots, verifying a conjecture of Boden and Nicas in a special case.

1. Introduction

This paper is concerned with the explicit evaluation of finite sums of the form:

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{\ell=1}^{d}\left(\sum_{j=1}^{k}n_j^{2m_\ell+1}\right)}{\prod_{j=1}^{k}b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})}$$

where $n \geq 1$ and $m_1, \ldots, m_d \geq 0$ are integers, $b_q := 4^q q!(q-1)!$ and the interior sum is over of all compositions (i.e., ordered partitions) of n into k parts.

The motivation for studying these particular sums comes from topology. Given a fibered knot K in a closed oriented 3-manifold and $\alpha \in \mathrm{SU}(n)$ (the special unitary group), the SU(n) *Casson invariant of* K, denoted by $\lambda_{n,\alpha}(K)$, is an integer which can be viewed as an algebraic-topological count of the number of characters of SU(n) representations of the knot group which take a longitude into the conjugacy class of α (see [2, 3, 4]). For generic $\alpha \in \mathrm{SU}(n)$, including all generators

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of the center of $\mathrm{SU}(n)$, there exist universal polynomials $q_{n,\alpha}(y_0, y_2, \ldots, y_{2n-2})$ such that

$$\lambda_{n,\alpha}(K) = q_{n,\alpha}(C_0, C_2, \ldots, C_{2n-2})$$

for any fibered knot K with Conway polynomial $\nabla_K(z) = \sum_{i\geq 0} C_{2i} z^{2i}$.

The "wall-crossing" formulae of [2] imply that the weighted homogeneous part of the polynomial $\frac{1}{m_\alpha} q_{n,\alpha}(y_0, y_2, \ldots, y_{2n-2})$ (where $m_\alpha > 0$ is the Euler characteristic of the conjugacy class of $\alpha \in \mathrm{SU}(n)$ and y_{2i} has weighted degree $2i$) of highest weighted degree, denoted by ν_n, is independent of (generic) $\alpha \in \mathrm{SU}(n)$. Using Zagier's summation formula [6] for solving the Atiyah-Bott recursion [1], we showed in [3] how to express each coefficient of ν_n as an explicit linear combination of sums of the form

$$4 \sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{\prod_{\ell=1}^{d} \left[\sum_{j=1}^{k} \sum_{i=1}^{n_j} (2i-1)^{2\lambda_\ell} \right]}{\prod_{j=1}^{k} b_{n_j} \prod_{j=1}^{k-1} (n_j + n_{j+1})}$$

where λ_ℓ is a positive integer for $\ell = 1, \ldots, d$ such that $\lambda_1 + \cdots + \lambda_d < n$. Although these sums appear to be rather complicated, numerical evidence supports the following conjecture:

Conjecture 1.1. (Conjecture 1.16 of [3].) *For $n \geq 1$ and $\lambda_1, \ldots, \lambda_d > 0$,*

$$4 \sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{\prod_{\ell=1}^{d} \left[\sum_{j=1}^{k} \sum_{i=1}^{n_j} (2i-1)^{2\lambda_\ell} \right]}{\prod_{j=1}^{k} b_{n_j} \prod_{j=1}^{k-1} (n_j + n_{j+1})} = \begin{cases} 0 & \text{if } \lambda_1 + \cdots + \lambda_d < n-1 \\ n^{d-2} \prod_{\ell=1}^{d} \binom{2\lambda_\ell}{\lambda_\ell} & \text{if } \lambda_1 + \cdots + \lambda_d = n-1. \end{cases}$$

The case $d = 1$ of Conjecture 1.1 was proved in Theorem 2.18 of [3]. This was accomplished by first showing that Conjecture 1.1 is implied by Conjecture 1.2 below (in the case $d = 1$, but a straightforward generalization of argument used in the proof of Theorem 2.18 of [3] shows that Conjecture 1.2 implies Conjecture 1.1 for all $d \geq 1$):

Conjecture 1.2. (Conjecture 2.19 of [3].) *For $n \geq 1$ and $m_1, \ldots, m_d \geq 0$,*

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{\prod_{\ell=1}^{d} \left(\sum_{j=1}^{k} n_j^{2m_\ell+1} \right)}{\prod_{j=1}^{k} b_{n_j} \prod_{j=1}^{k-1} (n_j + n_{j+1})} = \begin{cases} 0 & \text{if } \sum_{\ell=1}^{d} m_\ell < n-1 \\ n^{d-2} 4^{-n} \prod_{\ell=1}^{d} \binom{2m_\ell}{m_\ell} (2m_\ell + 1) & \text{if } \sum_{\ell=1}^{d} m_\ell = n-1. \end{cases}$$

The case $d = 1$ of Conjecture 1.2 was proved in Proposition 2.17 of [3].

In this paper we prove Conjecture 1.2, and thus also Conjecture 1.1, in the case $d = 2$ (Theorem 5.10). The case $d = 2$ of Conjecture 1.2 is considerably more difficult to prove than the case $d = 1$. We expect that the techniques developed here to prove Theorem 5.10 should be effective to treat the case $d > 2$ (see Remark 5.12).

The paper is organized as follows. In §2, we summarize the integral equation technique introduced in [3] for analyzing sums of the form

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{a_{n_1}\cdots a_{n_k}}{\prod_{j=1}^{k-1}(n_j+n_{j+1})}$$

where the interior sum is over of all compositions of n into k parts and $\{a_n \mid n = 1,2,\ldots\}$ is a sequence of elements in an algebra defined over a field of characteristic 0. Of particular interest are integral equations of the form

$$(\star)\qquad \Phi(s,t)+\int_0^1\frac{\gamma(sx)}{x}\Phi(tx,t)\,dx = f(s,t)$$

where $\gamma(u) := \sum_{n=1}^{\infty} u^n/b_n$ with $b_n := 4^n n!(n-1)!$ and $f(s,t)$ is a formal power series such that $f(0,t) = 0$. We develop methods in §3 to explicitly solve $(\star)$ in terms of a particularly convenient "basis" of functions $\{\psi_j \mid j \geq 1\}$ given by (10); see Theorem 3.9. Some special solutions to $(\star)$ enjoy a remarkable "multiplicative" property (see Theorem 3.16) which used in the proof of Proposition 4.3, a key ingredient in the proof the main theorem (Theorem §5.10). In §4, we give an explicit formula for the series

$$(\star\star)\qquad \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{i=1}^{d}\left(\sum_{j=1}^{k} z_i^{n_j}\right)}{\prod_{j=1}^{k} b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})}s^{n_1}t^{n-n_1}\,,$$

where $z_1,\ldots,z_d$ are independent variables, in the cases $d = 0,1,2$ (in the case $d = 0$, interpret the product in the numerator to be 1); see (28), (32) and Proposition 4.1. The results of §3 and §4 together with exponential generating functions associated to $(\star\star)$ are used in §5 to prove the main theorem (Theorem §5.10) which verifies Conjecture 1.2 in the case $d = 2$.

2. Integral Equations

We summarize the integral equation technique introduced in §2 of [3] for analyzing sums of the form

$$(1)\qquad \sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{a_{n_1}\cdots a_{n_k}}{\prod_{j=1}^{k-1}(n_j+n_{j+1})}$$

where the interior sum is over of all compositions of n into k parts.

Let $\mathcal{A}$ be a (not necessarily commutative) algebra over a field of characteristic 0. For a polynomial $p(y) = \sum_{i=0}^{N} a_i y^i$ over $\mathcal{A}$ in the commuting variable y, define the formal integral:

$$\int_0^1 p(y)dy := \sum_{i=0}^{N} \frac{a_i}{i+1} \ \in \mathcal{A}.$$

If $f(t,y) = \sum_{n=0}^{\infty} p_n(y)t^n$ is a formal power series in the commuting variable t with coefficients in the polynomial algebra $\mathcal{A}[y]$, define:

$$\int_0^1 f(t,y)dy := \sum_{n=0}^{\infty} \left(\int_0^1 p_n(y)dy\right) t^n.$$

Given a sequence $a_n \in \mathcal{A}$, $n \geq 1$, the associated *generating function* is the formal power series in the commuting variable s:

$$\rho(s) := \sum_{n=1}^{\infty} a_n s^n.$$

Define a formal power series over $\mathcal{A}$ in the commuting variables s, t by:

(2) $$\Phi(s,t) := \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{n=k}^{\infty} \sum_{n_1+\cdots+n_k=n} \frac{a_{n_1}\cdots a_{n_k}}{\prod_{j=1}^{k-1}(n_j+n_{j+1})} \ s^{n_1} t^{n-n_1}.$$

Observe that

$$\Phi(t,t) \ = \ \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{a_{n_1}\cdots a_{n_k}}{\prod_{j=1}^{k-1}(n_j+n_{j+1})} \right) t^n$$

is the generating function for the sequence of sums (1).

In [3], we showed that $\Phi(s,t)$ satisfies the *basic integral equation*:

(3) $$\Phi(s,t) + \int_0^1 \frac{\rho(sy)}{y} \Phi(ty,t)dy \ = \ \rho(s).$$

Let $f(s,t)$ be a formal power series over $\mathcal{A}$ with $f(0,t) = 0$. If a formal power series $\Theta(s,t) := \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} A_{ij} s^i t^j$ satisfies the integral equation:

(4) $$\Theta(s,t) + \int_0^1 \frac{\rho(sy)}{y} \Theta(ty,t)dy \ = \ f(s,t)$$

then a comparison of the coefficients on both sides of (4) yields the following recursion formula for the coefficients A_{ij}:

$$A_{i0} = f_{i0} \qquad i \geq 1, \tag{5}$$

$$A_{ij} = -a_i \sum_{q=0}^{j-1} \frac{A_{j-q,q}}{j-q+i} + f_{ij} \qquad j > 0 \tag{6}$$

where $f(s,t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} f_{ij} s^i t^j$. Conversely, if A_{ij} are elements in $\mathcal{A}$ defined by (5) and (6), then $\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} A_{ij} s^i t^j$ is a solution to (4). Since (5) and (6) uniquely define the A_{ij}'s, this formal power series solution to (4) is unique. In particular, $\Phi(s,t)$ given by (2) is the unique formal power series solution to the basic integral equation.

Definition 2.1. Let $\mathcal{A}[[s,t]]$ denote the algebra of formal power series over $\mathcal{A}$ in the commuting variables s, t and let $\mathcal{A}[[s,t]]'$ be the ideal of $\mathcal{A}[[s,t]]$ consisting of those $f(s,t) \in \mathcal{A}[[s,t]]$ for which $f(0,t) = 0$. Given a formal power series $\rho(s)$ over $\mathcal{A}$ with $\rho(0) = 0$, define the operator $I_\rho : \mathcal{A}[[s,t]] \to \mathcal{A}[[s,t]]$ by

$$(I_\rho \Theta)(s,t) := \Theta(s,t) + \int_0^1 \frac{\rho(sx)}{x} \Theta(tx,t)\; dx.$$

Note that I_ρ is linear over $\mathbb{F}[[t]]$, where $\mathbb{F}$ is the ground field, and $I_\rho(\mathcal{A}[[s,t]]') \subset \mathcal{A}[[s,t]]'$. If $\mathcal{A}$ is commutative or if ρ is defined over the ground field then I_ρ is $\mathcal{A}[[t]]$-linear. The equation (4) can be written as $I_\rho \Theta = f$.

Terminology. We refer to the unique formal power series solution to the integral equation $I_\rho \Theta = f$, where $f \in \mathcal{A}[[s,t]]'$, as "the solution to $I_\rho \Theta = f$".

3. Solutions to the integral equation $I_\gamma \Phi = f$

Let $b_n := 4^n n!(n-1)!$ and define the power series:

$$\gamma(u) := \sum_{n=1}^{\infty} \frac{u^n}{b_n}. \tag{7}$$

Let $\mathcal{A}$ be a commutative algebra over a field of characteristic 0. In this section, we are concerned with methods of explicitly solving the integral equation

$$\Phi(s,t) + \int_0^1 \frac{\gamma(sx)}{x} \Phi(tx,t)\; dx = f(s,t)$$

where $f(s,t)$ is a formal power series in the variables s, t over $\mathcal{A}$ such that $f(0,t) = 0$. We write this equation as $I_\gamma \Phi = f$ (see Definition 2.1).

Given a formal power series $g(t) = \sum_{n=0}^{\infty} a_n t^n$ and a non-negative integer p, the *p-th derivative of* $g(t)$, denoted by $g^{(p)}(t)$, is the formal power series:

$$g^{(p)}(t) := \sum_{\ell=0}^{\infty} \frac{(\ell+p)!}{\ell!} a_{\ell+p} t^{\ell}.$$

If $g(y_1, \ldots, y_m)$ is a formal power series in the variables $y_1, \ldots, y_m$ we also use the notations $\frac{\partial^p}{\partial y_j^p} g$ and $D_j^{(p)} g$ for the p-th partial derivative of g with respect to y_j.

Define the power series:

$$\mu(u) := \frac{\gamma(u)}{u} = \sum_{n=0}^{\infty} \frac{u^n}{b_{n+1}}. \tag{8}$$

It is straightforward to show that μ satisfies the differential equation:

$$u\mu^{(k)}(u) + k\mu^{(k-1)}(u) - \tfrac{1}{4}\mu^{(k-2)}(u) = 0 \tag{9}$$

where $k \geq 2$ is an integer. It will also be useful to define the $(k+1)$-th iterated anti-derivative of μ, denoted by $\mu^{(-1-k)}$, for $k \geq 0$, as follows:

$$\mu^{(-1-k)}(u) := \sum_{n=0}^{\infty} \frac{n!}{(n+k)!} \frac{u^{n+k}}{4^n (n!)^2} = \sum_{n=0}^{\infty} \frac{u^{n+k}}{4^n (n+k)! n!}.$$

With this definition, it is easy to show that the identity $\frac{d}{du}\, \mu^{(k)}(u) = \mu^{(k+1)}(u)$ and the differential equation (9) are valid for *all* integers k.

Leibnitz's rule and (9) (with "u" replaced by "ux") yield:

Lemma 3.1. *For all integers k,* $\dfrac{\partial}{\partial u}\, u^k \mu^{(k-1)}(ux) = \tfrac{1}{4} u^{k-1} \mu^{(k-2)}(ux)$. □

Repeated application of Lemma 3.1 yields:

Corollary 3.2. *For $j \geq 1$ and all k,* $\dfrac{\partial^j}{\partial u^j}\, u^k \mu^{(k-1)}(ux) = 4^{-j} u^{k-j} \mu^{(k-1-j)}(ux)$. □

A straightforward comparison of coefficients reveals the following relation between the derivatives and anti-derivatives of μ:

Lemma 3.3. *For all integers n,* $\mu^{(-n)}(u) = 4^{n-1} u^{n-1} \mu^{(n-2)}(u)$. □

The following formula for a product of $\mu^{(k-1)}$'s will be useful.

Lemma 3.4. *For all integers a, b,*

$$\mu^{(a-1)}(u)\, \mu^{(b-1)}(u) = \sum_{k=M}^{\infty} \frac{1}{4^{a+b+k} k! (k+a+b)!} \binom{2k+a+b}{k+a} u^k$$

where $M = \max(0,\, -a,\, -b,\, -a-b)$.

Proof. By making use of Lemma 3.3, it suffices to consider the case $a, b \geq 0$.

$$
\begin{aligned}
\mu^{(a-1)}(u)\,\mu^{(b-1)}(u) &= \left(\sum_{n=0}^{\infty} \frac{u^n}{4^{n+a}(n+a)!n!}\right)\left(\sum_{m=0}^{\infty} \frac{u^m}{4^{m+b}(m+b)!m!}\right) \\
&= \sum_{k=0}^{\infty} \sum_{m+n=k} \frac{u^{m+n}}{4^{m+n+a+b}(n+a)!(m+b)!n!m!} \\
&= \sum_{k=0}^{\infty} \frac{1}{4^{a+b+k}k!(k+a+b)!} \left(\sum_{j=0}^{k} \binom{k+a+b}{j+a}\binom{k}{j}\right) u^k \\
&= \sum_{k=0}^{\infty} \frac{1}{4^{a+b+k}k!(k+a+b)!} \binom{2k+a+b}{k+a} u^k \,.
\end{aligned}
$$

The identity

$$\sum_{j=0}^{k} \binom{k+a+b}{j+a}\binom{k}{j} = \binom{2k+a+b}{k+a}$$

is a consequence of the Vandermonde convolution formula, see §1.3 of [5]. □

For $j \geq 1$, define power series:

$$\psi_j(s,t) := \left((s-t)^j - (-1)^j t^j\right) \mu^{(j-1)}(s-t)\ . \tag{10}$$

Proposition 3.5. *For* $q \geq 1$*, the solution to the equation* $I_\gamma P_q\,(s,t) = s^q$ *is given by*

$$P_q(s,t) = q!\ (-1)^q \sum_{n=1}^{\infty} (-1)^n 4^n \psi_n(s,t) \mu^{(n-q-1)}(-t).$$

Proof. By Proposition 2.9 of [3],

$$P_q(s,t) = s^q + (-1)^q 4^q q! \sum_{n=q+1}^{\infty} \sum_{k=0}^{n-q-1} \frac{(-1)^{n+k}(2n-k-q-2)!}{b_n\ (n-q)!(n-k-1-q)!} \binom{n-1}{k} s^{k+1} t^{n-k-1}$$

and so the coefficient of $s^i t^j$ in $P_q(s,t)$ is

$$(-1)^q 4^q q! (-1)^{j-1} \frac{(i+2j-q-1)!}{b_{i+j}(i+j-q)!(j-q)!} \binom{i+j-1}{i-1}$$

for $j \geq q$, $i \geq 1$ and $\delta_{iq}\delta_{0j}$ otherwise (where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise). The Taylor expansion of $\psi_n(s,t)$ in the variable s, computed with the aid of

Corollary 3.2, is

$$\psi_n(s,t) \;=\; \sum_{i=1}^{\infty}\left(4^{-i}(-t)^{n-i}\mu^{(n-i-1)}(-t) - (-t)^n\mu^{(n+i-1)}(-t)\right)\frac{s^i}{i!}$$

and so the coefficient of s^it^j in $q!\ (-1)^q\sum_{n=1}^{\infty}(-1)^n4^n\psi_n(s,t)\mu^{(n-q-1)}(-t)$ is the same as the coefficient of t^j in

$$\frac{q!\ (-1)^q}{i!}\sum_{n=1}^{\infty}(-1)^n4^n\left(4^{-i}(-t)^{n-i}\mu^{(n-i-1)}(-t) - (-t)^n\mu^{(n+i-1)}(-t)\right)\mu^{(n-q-1)}(-t).$$

This coefficient, when computed with the assistance of Lemma 3.4, is seen to coincide with the coefficient of s^it^j in $P_q(s,t)$ given above. ☐

The following version of Proposition 2.11 of [3] will be useful.

Proposition 3.6. *The solution to $I_\gamma\Phi = f$, where $f \in \mathcal{A}[[s,t]]'$, is given by:*

$$\Phi(s,t) \;=\; \sum_{i=1}^{\infty}\frac{P_i(s,t)}{i!}D_1^{(i)}f\,(0,t)$$

where, for $i \geq 1$, $P_i(s,t)$ is the solution to $I_\gamma P\,(s,t) = s^i$.

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{A}[[s,t]]$ consisting of those formal power series in s,t with vanishing constant term. By Proposition 3.5, $P_i(s,t) \in \mathfrak{m}^i$ for $i \geq 1$. Hence the expression $\Phi(s,t) := \sum_{i=1}^{\infty}(1/i!)P_i(s,t)D_1^{(i)}f\,(0,t)$ is valid as a formal power series; furthermore,

$$\begin{aligned} I_\gamma\Phi(s,t) &= I_\gamma\left(\sum_{i=1}^{\infty}\frac{P_i(s,t)}{i!}D_1^{(i)}f\,(0,t)\right)\\ &= \sum_{i=1}^{\infty}\frac{I_\gamma P_i(s,t)}{i!}D_1^{(i)}f\,(0,t) = \sum_{i=1}^{\infty}\frac{s^i}{i!}D_1^{(i)}f\,(0,t) = f(s,t). \end{aligned}$$

The last equality is the Taylor expansion of f in the first variable (the term corresponding to $i = 0$ vanishes because $f(0,t) = 0$, i.e., $f \in \mathcal{A}[[s,t]]'$). ☐

Definition 3.7. Define the $\mathcal{A}[[t]]$-linear operator $\mathcal{L} : \mathcal{A}[[u,t]] \to \mathcal{A}[[u,t]]$ by

$$\mathcal{L}f\,(u,t) := \sum_{i=0}^{\infty}4^iu^i\mu^{(i-1)}(-u)D_1^{(i)}f\,(0,t).$$

Remark 3.8. The operator $\mathcal{L}$ can be expressed using integrals as follows:

$$\mathcal{L}f\,(u,t) = f(u,t) - u\int_0^1\mu(u(x-1))f(ux,t)dx.$$

Furthermore, $\mathcal{L}$ is invertible and its inverse is given by

$$\mathcal{L}^{-1}f\,(u,t) = f(u,t)+u\int_0^1 \mu(-u(x-1))f(ux,t)dx = \sum_{i=0}^{\infty} 4^i u^i \mu^{(i-1)}(u)D_1^{(i)}f\,(0,t).$$

Theorem 3.9. *Suppose* $f(s,t) \in \mathcal{A}[[s,t]]'$. *Then the solution to* $I_\gamma \Phi = f$ *is given by:*

$$\Phi(s,t) = \sum_{n=1}^{\infty} 4^n \psi_n(s,t) D_1^{(n)} \mathcal{L}f\,(t,t).$$

Proof. By Definition 3.7, $\mathcal{L}f\,(u,t) = \sum_{i=1}^{\infty} 4^i u^i \mu^{(i-1)}(-u)D_1^{(i)}f\,(0,t)$ (the sum starts at $i = 1$ because $f(0,t) = 0$, i.e., $f(s,t) \in \mathcal{A}[[s,t]]'$). Differentiating this expression n times with respect to u using Corollary 3.2 and evaluating at $u = t$ yields:

$$D_1^{(n)}\mathcal{L}f\,(t,t) = \sum_{i=1}^{\infty} 4^{i-n} t^{i-n} \mu^{(i-n-1)}(-t) D_1^{(i)} f\,(0,t). \tag{11}$$

We have:

$$\begin{aligned}
\Phi(s,t) &= \sum_{i=1}^{\infty} \frac{P_i(s,t)}{i!} D_1^{(i)} f\,(0,t) \qquad \text{(by Proposition 3.6)}\\
&= \sum_{i=1}^{\infty} (-1)^i \left(\sum_{n=1}^{\infty} (-1)^n 4^n \psi_n(s,t) \mu^{(n-i-1)}(-t) \right) D_1^{(i)} f\,(0,t)\\
&= \sum_{n=1}^{\infty} 4^n \psi_n(s,t)(-1)^n \left(\sum_{i=1}^{\infty} (-1)^i \mu^{(n-i-1)}(-t) D_1^{(i)} f\,(0,t) \right)\\
&= \sum_{n=1}^{\infty} 4^n \psi_n(s,t) \left(\sum_{i=1}^{\infty} 4^{i-n} t^{i-n} \mu^{(i-n-1)}(-t) D_1^{(i)} f\,(0,t) \right)\\
&= \sum_{n=1}^{\infty} 4^n \psi_n(s,t) D_1^{(n)} \mathcal{L}f\,(t,t) \qquad \text{(by (11))}.
\end{aligned}$$

The second line follows from Proposition 3.5 and the fourth line from Lemma 3.3. □

Corollary 3.10. *For* Φ *and* f *as in Theorem 3.9,*

$$\Phi(t,t) = \mathcal{L}f\,(t,t) = \sum_{i=1}^{\infty} 4^i t^i \mu^{(i-1)}(-t) D_1^{(i)} f\,(0,t).$$

Proof. Note that $\psi_n(t,t) = -(-t)^n \mu^{(n-1)}(0) = -(-t)^n/(4^n n!)$. Hence

$$\Phi(t,t) = \sum_{n=1}^{\infty} 4^n \psi_n(t,t) D_1^{(n)} \mathcal{L}f\,(t,t) = \sum_{n=1}^{\infty} \frac{-(-t)^n}{n!} D_1^{(n)} \mathcal{L}f\,(t,t) = \mathcal{L}f\,(t,t).$$

The last equality is deduced by taking the Taylor expansion of $\mathcal{L}f\,(u,t)$ with respect to u centered at t. □

Remark 3.11. Corollary 3.10 can also be deduced from Proposition 3.6 and the observation that $P_i(t,t) = 4^i i!\, t^i \mu^{(i-1)}(-t)$.

We will be interested in equations of the form $I_\gamma \Phi\,(s,t) = f(s,t)$ where f depends on a set of parameters $\{y_1, \ldots, y_n\}$. This is interpreted by taking the algebra $\mathcal{A}$ to be the algebra of formal power series in the variables $y_1, \ldots, y_n$ over a field of characteristic 0. We write $f(s,t;y_1,\ldots,y_n)$ to indicate the dependence of f on the parameters.

Theorem 3.12. *For $k \geq 1$, let $G_k(s,t;z)$ be the solution to $I_\gamma G_k\,(s,t;z) = \psi_k(sz,t)$. Then $G_k(s,t;z) = \sum_{j=1}^{\infty} 4^j \psi_j(s,t) R_{k,j}(t;z)$ where*

$R_{k,j}(t;z) :=$
$\sum_{\ell=1}^{\infty} \left(4^{\ell-k-j} t^{\ell-j} \mu^{(\ell-k-1)}(-t) \;-\; 4^{\ell-j}(-1)^k t^{k+\ell-j} \mu^{(\ell+k-1)}(-t)\right) \mu^{(\ell-j-1)}(-t) z^\ell.$

Proof. Let $g(s,t;z) := \psi_k(sz,t)$. Differentiating g with the assistance of Corollary 3.2 and applying Lemma 3.3 yields:

$$\begin{aligned} D_1^{(\ell)} g\,(0,t;z) &= \left(4^{-\ell}(-t)^{k-\ell}\mu^{(k-\ell-1)}(-t) - (-t)^k \mu^{(k+\ell-1)}(-t)\right) z^\ell \\ &= \left(4^{-k}\mu^{(\ell-k-1)}(-t) - (-t)^k \mu^{(k+\ell-1)}(-t)\right) z^\ell \end{aligned}$$

and so by (11)

$$D_1^{(j)} \mathcal{L}g\,(t,t;z) = \sum_{\ell=1}^{\infty} 4^{\ell-j} t^{\ell-j} \mu^{(\ell-j-1)}(-t) D_1^{(\ell)} g\,(0,t;z).$$

The conclusion of the theorem now follows from Theorem 3.9. □

Corollary 3.10 and the proof Theorem 3.12 yield:

Corollary 3.13. *For $k \geq 1$, $G_k(t,t;z) = R_{k,0}(t;z)$.* □

The following proposition and its corollary will be used to derive several identities involving the $R_{k,j}(t;z)$'s.

Proposition 3.14. *For $n, j \geq 1$, define*

$$Q_{n,j}(t) := 4^{n-j} n!\, t^{n-j} \mu^{(n-j-1)}(-t) - 4^n n! (-1)^j t^n \mu^{(n+j-1)}(-t).$$

Then $\sum_{j=1}^{\infty} 4^j \psi_j(s,t) Q_{n,j}(t) = s^n$.

Proof. For $n \geq 1$, define $h_n(s,t) := I_\gamma s^n$. Then

$$h_n(s,t) = s^n + \int_0^1 \frac{\gamma(sx)}{x}(tx)^n dx$$
$$= s^n + t^n s \int_0^1 \mu(sx) x^n dx = s^n + \sum_{k=1}^{\infty} \frac{s^k t^n}{4^k k!(k-1)!(k+n)}.$$

Calculating $\mathcal{L}h_n$ yields $\mathcal{L}h_n\,(u,t) = 4^n n!\,\mu^{(n-1)}(-u)(u^n - t^n) + t^n$. Hence, for $j \geq 1$,

$$D_1^{(j)} \mathcal{L}h_n\,(t,t) = 4^{n-j} n!\, t^{n-j} \mu^{(n-j-1)}(-t) - 4^n n! (-1)^j t^n \mu^{(n+j-1)}(-t) = Q_{n,j}(t).$$

The conclusion of the theorem now follows from Theorem 3.9. □

Corollary 3.15. *Suppose $g(s,t) \in \mathcal{A}[[s,t]]'$. Then*

$$\sum_{j=1}^{\infty} 4^j \psi_j(s,t) \left(\sum_{n=1}^{\infty} \frac{Q_{n,j}(t)}{n!} D_1^{(n)} g\,(0,t) \right) = g(s,t).$$

Proof. Using Theorem 3.14, we have:

$$\sum_{j=1}^{\infty} 4^j \psi_j(s,t) \left(\sum_{n=1}^{\infty} \frac{Q_{n,j}(t)}{n!} D_1^{(n)} g\,(0,t) \right) = \sum_{n=1}^{\infty} \frac{1}{n!} D_1^{(n)} g\,(0,t) \left(\sum_{j=1}^{\infty} 4^j \psi_j(s,t) Q_{n,j}(t) \right)$$
$$= \sum_{n=1}^{\infty} \frac{s^n}{n!} D_1^{(n)} g\,(0,t) = g(s,t)$$

where the last equality is the Taylor expansion of $g(s,t)$ in s (the term corresponding to $n = 0$ vanishes because $g(0,t) = 0$). □

For $k, j \geq 1$, define:

$$X_{k,j}(t;z) := \sum_{\ell=1}^{\infty} \left(4^{\ell-k} t^{\ell-k} \mu^{(\ell-k-1)}(-t) \;-\; 4^{\ell}(-1)^k t^{\ell} \mu^{(\ell+k-1)}(-t) \right) \mu^{(\ell+j-1)}(-t) z^{\ell}.$$

Calculation using Corollary 3.15 yields the following identities:

$$\sum_{k=1}^{\infty} 4^k \psi_k(s,t) X_{k,j}(t;z) \;=\; \mu^{(j-1)}(sz - t) - \mu^{(j-1)}(-t), \tag{12}$$

$$\sum_{k=1}^{\infty} 4^k \psi_k(s,t)\, t^{j-k} R_{k,j}(t;z) \;=\; (sz - t)^j \mu^{(j-1)}(sz - t) - (-t)^j \mu^{(j-1)}(-t). \tag{13}$$

Combining (12) and (13) yields the identity:

$$\sum_{k=1}^{\infty} 4^k \psi_k(s,t) \left(t^{j-k} R_{k,j}(t;z) - (-t)^j X_{k,j}(t;z)\right) \;=\; \psi_j(sz,t)\ . \tag{14}$$

Defining

$$C_{k,j}(t;z) := t^{j-k} R_{k,j}(t;z) - (-t)^j X_{k,j}(t;z)\ , \tag{15}$$

we write the identity (14) as:

$$\sum_{k=1}^{\infty} 4^k \psi_k(s,t) C_{k,j}(t;z) \;=\; \psi_j(sz,t)\ . \tag{16}$$

The identity (16) implies the following remarkable "multiplicative" property of the series $G_k(s,t;z)$ of Theorem 3.12.

Theorem 3.16. *For $j \geq 1$, $\sum_{k=1}^{\infty} 4^k G_k(s,t;w) C_{k,j}(t;z) = G_j(s,t;wz)$.*

Proof. Replacing s with sw in (16) yields:

$$\sum_{k=1}^{\infty} 4^k \psi_k(sw,t) C_{k,j}(t;z) \;=\; \psi_j(swz,t)\ . \tag{17}$$

Let $I_\gamma^{-1} f$ denote the solution to the equation $I_\gamma \Phi = f$. Recall that $G_k(s,t;z)$ is, by definition, $I_\gamma^{-1}\psi_k(sz,t)$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} 4^k G_k(s,t;w) C_{k,j}(t;z) &= \sum_{k=1}^{\infty} 4^k \left(I_\gamma^{-1}\psi_k(sw,t)\right) C_{k,j}(t;z) \\
&= I_\gamma^{-1} \left(\sum_{k=1}^{\infty} 4^k \psi_k(sw,t) C_{k,j}(t;z) \right) \\
&= I_\gamma^{-1} \psi_j(swz,t) \qquad \text{by (17)} \\
&= G_j(s,t;wz)\ . \quad \square
\end{aligned}$$

With the assistance of Lemma 3.4, the series $R_{k,j}(t,z)$ and $C_{k,j}(t,z)$ can be written explicitly as follows:

$$\begin{aligned}
R_{k,j}(t;z) = & \sum_{n=\max(0,k-j)}^{\infty} \left(\sum_{\ell=1}^{n+j} (-1)^\ell z^\ell \binom{2n-k+j}{n-\ell+j} \right) \frac{(-1)^{n+j} t^n}{4^{n+j} n!(n-k+j)!} \\
& - \sum_{n=k}^{\infty} \left(\sum_{\ell=1}^{n-k+j} (-1)^\ell z^\ell \binom{2n-k+j}{n+\ell} \right) \frac{(-1)^{n+j} t^n}{4^{n+j} (n+j)!(n-k)!}\ .
\end{aligned} \tag{18}$$

$$(19)\; C_{k,j}(t;z) = \sum_{n=\max(0,j-k)}^{\infty}\left(\sum_{\ell=1}^{n+k}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n+\ell-j}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}n!(n-j+k)!}$$
$$-\sum_{n=j}^{\infty}\left(\sum_{\ell=1}^{n}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n-\ell}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}(n+k)!(n-j)!}$$
$$-\sum_{n=j}^{\infty}\left(\sum_{\ell=1}^{n+k-j}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n+\ell}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}(n+k)!(n-j)!}$$
$$+\sum_{n=j+1}^{\infty}\left(\sum_{\ell=1}^{n-j}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n-\ell-j}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}n!(n-j+k)!}\;.$$

We will also make use (see Proposition 4.3) of the series

$$(20)\qquad E_{k,j}(t;z) := R_{j,k}(t;z) - \tfrac{1}{2}C_{k,j}(t;z)\;.$$

Combining (18) and (19), we obtain:

$$(21)\; -2E_{k,j}(t;z) = C_{k,j}(t;z) - 2R_{j,k}(t;z)$$
$$= -\sum_{n=\max(0,j-k)}^{\infty}\left(\sum_{\ell=1}^{n+k}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n+\ell-j}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}n!(n-j+k)!}$$
$$-\sum_{n=j}^{\infty}\left(\sum_{\ell=1}^{n}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n-\ell}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}(n+k)!(n-j)!}$$
$$+\sum_{n=j}^{\infty}\left(\sum_{\ell=1}^{n+k-j}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n+\ell}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}(n+k)!(n-j)!}$$
$$+\sum_{n=j+1}^{\infty}\left(\sum_{\ell=1}^{n-j}(-1)^{\ell}z^{\ell}\binom{2n+k-j}{n-\ell-j}\right)\frac{(-1)^{n+k}t^{n}}{4^{n+k}n!(n-j+k)!}\;.$$

4. Polynomial product power sums

In this section we give an explicit formula for the series

(22)

$$\Phi_d(s,t;z_1,\ldots,z_d) := \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{i=1}^{d}\left(\sum_{j=1}^{k}z_i^{n_j}\right)}{\prod_{j=1}^{k}b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})}s^{n_1}t^{n-n_1},$$

where $z_1, \dots, z_d$ are parameters, in the cases $d = 0, 1, 2$ (in the case $d = 0$, interpret the product in the numerator to be 1); see (28), (32) and Proposition 4.1.

Let $u_1, \dots, u_d$ and $z_1, \dots, z_d$ be parameters. Define

$$\text{(23)}\qquad \widehat{\Phi}_d(s, t; z_1, \dots, z_d; u_1, \dots, u_d) := \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{j=1}^{k}\prod_{i=1}^{d}(1+u_i z_i^{n_j})}{\prod_{j=1}^{k} b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})} s^{n_1}t^{n-n_1} ,$$

and

$$\text{(24)}\qquad \rho_d(s; z_1, \dots, z_d; u_1, \dots, u_d) := \sum_{n=1}^{\infty}\left(\prod_{i=1}^{d}(1+u_i z_i^n)\right)\frac{s^n}{b_n}$$
$$= \gamma(s) + \sum_i \gamma(z_i s)u_i + \sum_{i<j}\gamma(z_i z_j s)u_i u_j + \cdots + \gamma(z_1 \cdots z_d s)u_1 \cdots u_d .$$

where γ is the series (7). The series $\widehat{\Phi}_d$ is of the type (2) considered in §2 and thus satisfies the basic integral equation (3), i.e.,

$$\text{(25)}\qquad I_{\rho_d}\widehat{\Phi}_d\,(s, t; \mathbf{p}) \;=\; \widehat{\Phi}_d(s, t; \mathbf{p}) + \int_0^1 \frac{\rho_d(sx; \mathbf{p})}{x}\widehat{\Phi}_d(tx, t; \mathbf{p})dx \;=\; \rho_d(s, t; \mathbf{p})$$

where $\mathbf{p} := z_1, \dots, z_d; u_1, \dots, u_d$ is the list of parameters. Using the expansion

$$\prod_{j=1}^{k}(1+u_i z_i^{n_j}) = 1 + \left(\sum_{j=1}^{k} z_i^{n_j}\right)u_i + O(u_i^2) ,$$

where the notation "$O(u_i^2)$" indicates a polynomial (or a formal power series) in the principal ideal generated by u_i^2, we obtain:

$$\begin{aligned}\widehat{\Phi}_d(s, t; z_1, \dots, z_d; u_1, \dots, u_d) = \Phi_0(s, t) &+ \sum_i \Phi_1(s, t; z_i)u_i \\ &+ \sum_{i<j}\Phi_2(s, t; z_i, z_j)u_i u_j + \cdots \\ &+ \Phi_d(s, t; z_1, \dots, z_d)u_1 \cdots u_d + \sum_i O(u_i^2) .\end{aligned}$$

Equating the coefficients of the term $u_1 \cdots u_n$ on both sides of (25) yields the integral equation:

$$\begin{aligned}
(26)\ \Phi_d(s,t;z_1,\ldots,z_d) &+ \int_0^1 \frac{\gamma(sx)}{x}\Phi_d(tx,t;z_1,\ldots,z_d)dx \\
&+ \sum_i \int_0^1 \frac{\gamma(sz_ix)}{x}\Phi_{d-1}(tx,t;z_1,\ldots,\hat{z}_i,\ldots,z_d)dx \\
&+ \sum_{i<j} \int_0^1 \frac{\gamma(sz_iz_jx)}{x}\Phi_{d-2}(tx,t;z_1,\ldots,\hat{z}_i,\ldots,\hat{z}_j,\ldots,z_d)dx \\
&+ \cdots + \int_0^1 \frac{\gamma(sz_1\cdots z_dx)}{x}\Phi_0(tx,t)dx \\
&= \gamma(sz_1\cdots z_d)
\end{aligned}$$

where "$\hat{z}_i$" indicates that z_i has been omitted from the parameter list.

We solve the integral equation (26) explicitly in the cases $d = 0, 1, 2$.

In the case $d = 0$, (26) is the equation:

$$\Phi_0(s,t) + \int_0^1 \frac{\gamma(sx)}{x}\Phi_0(tx,t)dx = \gamma(s)\ . \tag{27}$$

The solution to (27) is

$$\Phi_0(s,t) = s\mu(s-t) = \psi_1(s,t) \tag{28}$$

by Proposition 2.8 of [3] or, alternatively, deduce (28) from Theorem 3.9. (Recall that μ is defined by (8) and ψ_j by (10)).

In the case $d = 1$, (26) is the equation (writing z for z_1):

$$\Phi_1(s,t;z) + \int_0^1 \frac{\gamma(sx)}{x}\Phi_1(tx,t;z)dx + \int_0^1 \frac{\gamma(szx)}{x}\Phi_0(tx,t) = \gamma(sz)\ . \tag{29}$$

Replacing s with sz in (27) yields

$$\Phi_0(sz,t) + \int_0^1 \frac{\gamma(szx)}{x}\Phi_0(tx,t)dx = \gamma(sz)$$

and so (29) is equivalent to

$$I_\gamma\Phi_1\,(s,t;z) = \Phi_1(s,t;z) + \int_0^1 \frac{\gamma(sx)}{x}\Phi_1(tx,t;z)dx = \Phi_0(sz,t) \tag{30}$$

which, by (28), is the same as

$$I_\gamma\Phi_1(s,t;z) = \psi_1(sz,t)\ . \tag{31}$$

By Theorem 3.12, the solution to (31) is

$$\Phi_1(s,t;z) = G_1(s,t;z) = \sum_{j=1}^{\infty} 4^j \psi_j(s,t) R_{1,j}(t;z) \tag{32}$$

where $R_{1,j}(t;z)$ is given explicitly by (18).

In the case $d=2$, (26) is the equation (writing w for z_1 and z for z_2):

$$\begin{aligned} &\Phi_2(s,t;w,z) + \int_0^1 \frac{\gamma(sx)}{x}\Phi_2(tx,t;w,z)dx \\ &+ \int_0^1 \frac{\gamma(swx)}{x}\Phi_1(tx,t;z)dx + \int_0^1 \frac{\gamma(szx)}{x}\Phi_1(tx,t;w)dx \\ &+ \int_0^1 \frac{\gamma(swzx)}{x}\Phi_0(tx,t)dx \;=\; \gamma(swz)\ . \end{aligned} \tag{33}$$

Equations (27) and (30) yield:

$$\begin{aligned} \int_0^1 \frac{\gamma(swzx)}{x}\Phi_0(tx,t)dx &= -\Phi_0(swz,t) + \gamma(swz)\ , \\ \int_0^1 \frac{\gamma(swx)}{x}\Phi_1(tx,t;z)dx &= -\Phi_1(sw,t;z) + \Phi_0(swz,t)\ , \\ \int_0^1 \frac{\gamma(szx)}{x}\Phi_1(tx,t;w)dx &= -\Phi_1(sz,t;w) + \Phi_0(swz,t)\ . \end{aligned}$$

Substituting these identities into (30) yields the equation:

$$I_\gamma\Phi_2\,(s,t;w,z) = \Phi_1(sw,t;z) + \Phi_1(sz,t;w) - \Phi_0(swz,t). \tag{34}$$

Proposition 4.1. *The series $\Phi_2(s,t;w,z)$ is given by the formula:*

$$\Phi_2(s,t;w,z) = -G_1(s,t;wz) + \sum_{i=1}^{\infty} 4^i \left(G_i(s,t;w)R_{1,i}(t;z) + G_i(s,t;z)R_{1,i}(t;w)\right)$$

where $G_i(s,t;u) = \sum_{j=1}^{\infty} 4^j \psi_j(s,t) R_{i,j}(t;u)$.

Proof. Equation (34) and (32), (28) yield:

$$\begin{aligned} I_\gamma\Phi_2\,(s,t;w,z) &= -\Phi_0(swz,t) + \Phi_1(sw,t;z) + \Phi_1(sz,t;w) \\ &= -\psi_1(swz,t) + \sum_{i=1}^{\infty} 4^i \left(\psi_i(sw,t)R_{1,i}(t;z) + \psi_i(sz,t)R_{1,i}(t;w)\right) \end{aligned}$$

Writing $I_\gamma^{-1} f$ for the solution to the equation $I_\gamma \Theta = f$, we have:

$$\begin{aligned}\Phi_2(s,t;w,z) &= I_\gamma^{-1}\left(-\psi_1(swz,t) + \sum_{i=1}^{\infty} 4^i\left(\psi_i(sw,t)R_{1,i}(t;z) + \psi_i(sz,t)R_{1,i}(t;w)\right)\right)\\ &= -I_\gamma^{-1}\psi_1(swz,t) + \sum_{i=1}^{\infty} 4^i\left(I_\gamma^{-1}\psi_i(sw,t)R_{1,i}(t;z) + I_\gamma^{-1}\psi_i(sz,t)R_{1,i}(t;w)\right)\\ &= -G_1(s,t;wz) + \sum_{i=1}^{\infty} 4^i\left(G_i(s,t;w)R_{1,i}(t;z) + G_i(s,t;w)R_{1,i}(t;z)\right)\end{aligned}$$

where $I_\gamma^{-1}\psi_i(su,t) = G_i(s,t;u)$ by Theorem 3.12. □

Remark 4.2. For general d, further analysis of (26) shows that the series $\Phi_d(s,t;z_1,\ldots,z_d)$ can be expressed explicitly in terms of the $G_i(s,t;z_j)$'s and products of the $R_{k,i}(t;z_j)$'s.

The following proposition will be used in the proof of Theorem 5.10.

Proposition 4.3.

$$\Phi_2(s,t;w,z) = \sum_{i=1}^{\infty} 4^i\left(G_i(s,t;w)E_{i,1}(t;z) + G_i(s,t;z)E_{i,1}(t;w)\right)$$

where $E_{i,1}(t;u)$ *is given by* (20).

Proof. By Theorem 3.16,

$$G_1(s,t;wz) = \tfrac{1}{2}\sum_{k=1}^{\infty} 4^k\left(G_k(s,t;w)C_{k,1}(t;z) + G_k(s,t;z)C_{k,1}(t;w)\right)$$

and hence by Proposition 4.1,

$$\begin{aligned}&\Phi_2(s,t;w,z)\\ &= -G_1(s,t;wz) + \sum_{i=1}^{\infty} 4^i\left(G_i(s,t;w)R_{1,i}(t;z) + G_i(s,t;z)R_{1,i}(t;w)\right)\\ &= \sum_{i=1}^{\infty} 4^i\left(G_i(s,t;w)(R_{1,i}(t;z) - \tfrac{1}{2}C_{i,1}(t;z)) + G_i(s,t;z)(R_{1,i}(t;w) - \tfrac{1}{2}C_{i,1}(t;w))\right)\\ &= \sum_{i=1}^{\infty} 4^i\left(G_i(s,t;w)E_{i,1}(t;z) + G_i(s,t;z)E_{i,1}(t;w)\right)\ .\quad \square\end{aligned}$$

5. Integral product power sums

For non-negative integers $q_1, \ldots, q_d$, define
(35)
$$\bar{\Phi}_d(s,t;q_1,\ldots,q_d) := \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{i=1}^{d}\left(\sum_{j=1}^{k} n_j^{q_i}\right)}{\prod_{j=1}^{k} b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})}s^{n_1}t^{n-n_1}\ .$$
Let $e^w := \sum_{j=0}^{\infty} w^j/j!$. The series $\bar{\Phi}_d$ is related to the series Φ_d (see (22)) by
$$\bar{\Phi}_d(s,t;q_1,\ldots,q_d) = \frac{\partial^{q_1+\cdots+q_p}}{\partial w_1^{q_1}\cdots\partial w_d^{q_d}}\,\Phi_d(s,t;e^{w_1},\ldots,e^{w_1})\bigg|_{w_1=\cdots=w_d=0}\ ,$$
i.e., the series $\Phi_d(s,t;e^{w_1},\ldots,e^{w_1})$ is the exponential generating function of the $\bar{\Phi}_d(s,t;q_1,\ldots,q_d)$'s. Also define, for $q \geq 0$,
$$\begin{aligned}\bar{G}_k(s,t;q) &:= \partial^q/\partial w^q\ G_k(s,t;e^w)|_{w=0}\ ,\\ \bar{R}_{k,j}(t;q) &:= \partial^q/\partial w^q\ R_{k,j}(t;e^w)|_{w=0}\ ,\\ \bar{E}_{k,j}(t;q) &:= \partial^q/\partial w^q\ E_{k,j}(t;e^w)|_{w=0}\ .\end{aligned}$$
(See Theorem 3.12 for the definition of $G_k(s,t;z)$ and $R_{kj}(t,z)$ and see (20) for the definition of $E_{k,j}(t;z)$.) Note that the effect of the operation $\partial^q/\partial w^q\, f(t,e^w)|_{w=0}$ on a series $f(t,z)$ is to replace occurrences of z^n with n^q.

Lemma 5.1. *For $2n+m \geq 0$,*
$$(-1)^n e^{-nz}(1-e^z)^{2n+m} = \sum_{\ell=1}^{n}(-1)^{\ell}e^{-\ell z}\binom{2n+m}{n-\ell}+\binom{2n+m}{n}+\sum_{\ell=1}^{n+m}(-1)^{\ell}e^{\ell z}\binom{2n+m}{n+\ell}\ .$$

Proof. Binomial expansion of $(1-e^z)^{2n+m}$ gives:
$$(-1)^n e^{-nz}(1-e^z)^{2n+m} = \sum_{j=0}^{2n+m}(-1)^{j+n}e^{(j-n)z}\binom{2n+m}{j}$$
$$= \sum_{j=0}^{n-1}(-1)^{j+n}e^{(j-n)z}\binom{2n+m}{j}+\binom{2n+m}{n}+\sum_{j=n+1}^{2n+m}(-1)^{j+n}e^{(j-n)z}\binom{2n+m}{j}\ .$$
Reindexing the last two sums yields the conclusion. □

Lemma 5.2. *For $2n+m \geq 0$,*
$$(-1)^n e^{-nz}(1-e^z)^{2n+m} = (-1)^{n+m}\left(z^{2n+m}+\tfrac{m}{2}z^{2n+m+1}\right)+O(z^{2n+m+2})\ .$$

Proof. The series for e^z yields:
$$\begin{aligned}(e^z-1)^{2n+m} &= (z+\tfrac{1}{2}z^2+O(z^3))^{2n+m} = z^{2n+m}+\tfrac{1}{2}(2n+m)z^{2n+m+1}+O(z^{2n+m+2})\ ,\\ e^{-nz} &= 1-nz+O(z^2)\ .\end{aligned}$$

Hence,

$$\begin{aligned}&(-1)^n e^{-nz}(1-e^z)^{2n+m} = (-1)^{n+m}e^{-nz}(e^z-1)^{2n+m}\\ &= (-1)^{n+m}\left(1-nz+O(z^2)\right)\left(z^{2n+m}+\tfrac{1}{2}(2n+m)z^{2n+m+1}+O(z^{2n+m+2})\right)\\ &= (-1)^{n+m}\left(z^{2n+m}+\tfrac{m}{2}z^{2n+m+1}\right)+O(z^{2n+m+2})\ . \quad \square\end{aligned}$$

Proposition 5.3. *For $q \geq 1$ and $2n+k-j \geq 0$,*

$$(-1)^q \sum_{\ell=1}^{n}(-1)^\ell \ell^q \binom{2n+k-j}{n-\ell} + \sum_{\ell=1}^{n+k-j}(-1)^\ell \ell^q \binom{2n+k-j}{n+\ell} = \begin{cases} 0 & \text{if } q < 2n+k-j,\\ (-1)^{n+k-j}q! & \text{if } q = 2n+k-j,\\ (-1)^{n+k-j}q!\,\frac{k-j}{2} & \text{if } q = 2n+k-j+1.\end{cases}$$

Proof. A comparison of two computations of the q-th derivative of $(-1)^n e^{-nz}(1-e^z)^{2n+k-j}$ at $z=0$, first using Lemma 5.1 and then using Lemma 5.2 yields the conclusion. □

Similarly:

Proposition 5.4. *For $q \geq 1$ and $2n+k-j \geq 0$,*

$$(-1)^q \sum_{\ell=1}^{n-j}(-1)^\ell \ell^q \binom{2n+k-j}{n-\ell-j} + \sum_{\ell=1}^{n+k}(-1)^\ell \ell^q \binom{2n+k-j}{n+\ell-j} = \begin{cases} 0 & \text{if } q < 2n+k-j,\\ (-1)^{n+k}q! & \text{if } q = 2n+k-j,\\ (-1)^{n+k}q!\,\frac{k+j}{2} & \text{if } q = 2n+k-j+1. \quad \square\end{cases}$$

Notation. Given a polynomial $p(x) = \sum_{i=0}^n a_i x^i$ of degree n, we write $p(x) = a_n x^n +$ l.d.t. or $p(x) =$ l.d.t. $+ a_n x^n$ ("l.d.t." = lower degree terms).

Proposition 5.5. *Let q be a positive odd integer and $k \geq 1$. Then $\bar{G}_k(t,t;q) = 0$ if $q < k$ and for $q \geq k$,*

$$\bar{G}_k(t,t;q) = \begin{cases} 4^{-(q+k)/2}\dbinom{q}{(q-k)/2} t^{(q+k)/2} + \text{ l.d.t.} & \text{if } k \text{ is odd},\\[2ex] 4^{-(q+k-1)/2}\,\dfrac{qk}{2}\dbinom{q-1}{(q-k-1)/2} t^{(q+k-1)/2} + \text{ l.d.t.} & \text{if } k \text{ is even}.\end{cases}$$

Proof. By Corollary 3.13, $G_k(t,t;z) = R_{k,0}(t;z)$ and hence $\bar{G}_k(t,t;q) = \bar{R}_{k,0}(t;q)$. The conclusion follows from Proposition 5.3 (or Proposition 5.4) applied to (18). □

Combining Proposition 5.5 and (32) yields:

Corollary 5.6. *Let q be a positive odd integer. Then*

$$\bar{\Phi}_1(t,t;q) = \bar{G}_1(t,t;q) = 4^{-(q+1)/2}\binom{q}{(q-1)/2} t^{(q+1)/2} + \text{ l.d.t.} \quad \square$$

Corollary 5.6 gives another proof of Conjecture 1.2 in the case $d = 1$ (this case of Conjecture 1.2 was previously verified in Proposition 2.17 of [3]).

Proposition 5.7. *Let q be a positive odd integer.*

(1) *If $q < k - j$ then $\bar{E}_{k,j}(t;q) = 0$.*

(2) *If $k - j$ is odd and $q \geq k - j$ then $-2\bar{E}_{k,j}(t;q) =$ l.d.t. +*

$$4^{-(q+k+j)/2}\left((-1)^j\binom{q}{(q+k+j)/2} - \binom{q}{(q-k+j)/2}\right) t^{(q-k+j)/2} .$$

(3) *If $k - j$ is even and $q \geq k - j$ then $-2\bar{E}_{k,j}(t;q) =$ l.d.t. +*

$$4^{-(q+k+j-1)/2} q\left(\tfrac{(k-j)}{2}(-1)^j\binom{q-1}{(q+k+j-1)/2} - \tfrac{(k+j)}{2}\binom{q-1}{(q-k+j-1)/2}\right) t^{(q-k+j-1)/2} .$$

Proof. Apply Propositions 5.3 and 5.4 to (21). $\square$

Specializing to the case $j = 1$ in Proposition 5.7 gives:

Corollary 5.8. *Let q be a positive odd integer.*

(1) *If $q < k - 1$ then $\bar{E}_{k,1}(t;q) = 0$.*

(2) *If k is even and $q \geq k - 1$ then $-2\bar{E}_{k,1}(t;q) =$ l.d.t. +*

$$4^{-(q+k+1)/2}\left(-\binom{q}{(q+k+1)/2} - \binom{q}{(q-k+1)/2}\right) t^{(q-k+1)/2} .$$

(3) *If k is odd and $q \geq k - 1$ then $-2\bar{E}_{k,1}(t;q) =$ l.d.t. +*

$$4^{-(q+k)/2} q\left(-\tfrac{(k-1)}{2}\binom{q-1}{(q+k)/2} - \tfrac{(k+1)}{2}\binom{q-1}{(q-k)/2}\right) t^{(q-k)/2} .$$

Lemma 5.9. *Let p, q be positive odd integers. Then*

$$\begin{aligned} S_{\text{even}}(p,q) &:= \sum_{k \,\in\, \text{even}^+} \frac{k}{2}\binom{p-1}{(p-k-1)/2}\left(\binom{q}{(q+k+1)/2} + \binom{q}{(q-k+1)/2}\right) \\ &= \binom{p-1}{(p-1)/2}\binom{q-1}{(q-1)/2}\frac{q(p-1)}{(p+q)} , \end{aligned}$$

where even$^+$ *is the set of positive even integers, and*

$$\begin{aligned} S_{\text{odd}}(p,q) &:= \sum_{k\,\in\,\text{odd}^+} \binom{p}{(p-k)/2}\left(\frac{(k-1)}{2}\binom{q-1}{(q+k)/2} + \frac{(k+1)}{2}\binom{q-1}{(q-k)/2}\right) \\ &= \binom{p-1}{(p-1)/2}\binom{q-1}{(q-1)/2}\frac{p(q+1)}{(p+q)} \end{aligned}$$

where odd$^+$ *is the set of positive odd integers.*

Proof. For $m, n \geq 0$,

$$\begin{aligned} S_{\text{even}}(2m+1,2n+1) &:= \sum_{i=1}^{m} i\binom{2m}{m-i}\left(\binom{2n+1}{n+i+1} + \binom{2n+1}{n-i+1}\right) \\ &= \sum_{i=1}^{m} i\binom{2m}{m-i}\binom{2n+2}{n+i+1}. \end{aligned}$$

Multiplying this identity by $((m+n+1)!)^2/((2n+2)!(2m)!)$ and letting $A = m+n+1$, it is easy to see that the conclusion of the Lemma for S_{even} is equivalent to the identity

$$\sum_{i=1}^{m} i\binom{A}{m-i}\binom{A}{m+i} = \frac{m}{2}\binom{A}{m}\binom{A-1}{m}$$

which can be established by induction. The proof for S_{odd} is similar. □

Theorem 5.10. *Let p, q be positive odd integers. Then*

$$\bar{\Phi}_2(t,t;p,q) \;=\; 4^{-(p+q)/2}\,pq\binom{p-1}{(p-1)/2}\binom{q-1}{(q-1)/2}t^{(p+q)/2} \;+\; \text{l.d.t.}$$

Hence Conjecture 1.2 is true for $d = 2$.

Proof. Proposition 4.3 implies:

$$\bar{\Phi}_2(t,t;p,q) = \sum_{k=1}^{\infty} 4^k\left(\bar{G}_k(t,t;p)\bar{E}_{k,1}(t;q) + \bar{G}_k(t,t;q)\bar{E}_{k,1}(t;p)\right). \tag{36}$$

By Propositions 5.5 and 5.8, the sum on the right of (36) is equal to

$$4^{-(p+q)/2}\left(\tfrac{p}{2}S_{\text{even}}(p,q) + \tfrac{q}{2}S_{\text{even}}(q,p) + \tfrac{q}{2}S_{\text{odd}}(p,q) + \tfrac{p}{2}S_{\text{odd}}(q,p)\right)t^{(p+q)/2} + \text{ l.d.t.}$$

(See Lemma 5.9 for the notation S_{even} and S_{odd}.) By Lemma 5.9,

$$\tfrac{p}{2}S_{\text{even}}(p,q) + \tfrac{q}{2}S_{\text{even}}(q,p) + \tfrac{q}{2}S_{\text{odd}}(p,q) + \tfrac{p}{2}S_{\text{odd}}(q,p) \ =$$
$$\binom{p-1}{(p-1)/2}\binom{q-1}{(q-1)/2}\left(\frac{pq(p-1)}{2(p+q)} + \frac{pq(q-1)}{2(p+q)} + \frac{pq(q+1)}{2(p+q)} + \frac{pq(p+1)}{2(p+q)}\right) \ =$$
$$\binom{p-1}{(p-1)/2}\binom{q-1}{(q-1)/2}pq\ . \quad \square$$

Corollary 5.11. *Conjecture 1.1 is true for $d = 2$.* □

Remark 5.12. The techniques of this paper can be used to show that for $d \geq 3$ and non-negative integers $m_1, \dots, m_d$ the series $\bar{\Phi}_d(t, t; 2m_1 + 1, \dots, 2m_d + 1)$ is a polynomial of degree less than or equal to $d + \sum_{i=1}^{d} m_i$ and so the sums in Conjecture 1.2 vanish for $\sum_{i=1}^{d} m_i < n - d$ (the conjectured vanishing range is $\sum_{i=1}^{d} m_i < n - 1$).

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