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Hessenberg Varieties are not Pure Dimensional

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Dedicated to Robert MacPherson

Abstract: We study a family of subvarieties of the flag variety defined by certain linear conditions, called Hessenberg varieties. We compare them to Schubert varieties. We prove that some Schubert varieties can be realized as Hessenberg varieties and vice versa. Our proof explicitly identifies these Schubert varieties by their permutation and computes their dimension.

We use this to answer an open question by proving that Hessenberg varieties are not always pure dimensional. We give examples that neither semisimple nor nilpotent Hessenberg varieties need be pure; the latter are connected, non-pure-dimensional Hessenberg varieties. Our methods require us to generalize the definition of Hessenberg varieties.

1. Introduction: Background and notation

A flag is a nested collection of vector spaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$, where each V_i is i-dimensional. The full flag variety is the complex algebraic variety consisting of all flags; it is smooth and compact.

This paper studies two families of subvarieties of the full flag variety: Hessenberg varieties and Schubert varieties. The first family is defined using two parameters: a linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$ and a nondecreasing function $h : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$. We call h a Hessenberg function. The Hessenberg variety associated to X and h is denoted $\mathcal{H}(X, h)$ and defined by

$$\mathcal{H}(X, h) = \{\text{Flags } : XV_i \subseteq V_{h(i)} \text{ for all } i\}.$$

(This generalizes the original definition of [dMPS], as in Sections 1.3 and 4.)

For instance, if X is arbitrary and h has $h(i) = n$ for all i, then $\mathcal{H}(X, h)$ is the full flag variety. More interesting are the Springer fibers, namely the Hessenberg varieties such that X is nilpotent and $h(i) = i$ for each i. Springer fibers are

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used to construct geometric representations of the symmetric group ([CG] gives a survey). W. Borho and R. MacPherson generalize Springer representations to a class of Hessenberg varieties that blend these two examples: h is a *parabolic* function, defined in Section 4, and X is a nilpotent matrix whose Jordan blocks are subordinate to h (see [BM]). More Hessenberg varieties are in Section 1.3.

This paper answers an open question about Hessenberg varieties: are they all pure dimensional? The pure-dimensionality of Springer fibers is significant for Springer representations, which arise from permutation actions on top-dimensional cohomology. Until now, the answer was yes in all known cases.

We show two ways in which Hessenberg varieties can fail to be pure dimensional. In Section 3, we give an example in which X is a semisimple operator and $\mathcal{H}(X, h)$ is a disjoint union of smooth subvarieties of G/B of different dimensions. One case of this example came up in calculations that R. MacPherson and I performed while researching [MT]. In Section 2, we show that $\mathcal{H}(X, h)$ need not be pure dimensional even when X is nilpotent. Section 2 gives a family of examples that are connected but (in general) reducible Hessenberg varieties whose components have different dimensions.

To prove that nilpotent Hessenberg varieties are not always pure, we use Schubert varieties. Every invertible matrix g gives a flag $[g]$ whose i-dimensional subspace is spanned by the first i columns of g. For each permutation w, the Schubert variety Y_w is the closure of the set $\{[bw] : b \text{ is upper-triangular}\}$. Schubert varieties are important because they form a basis for the cohomology of the flag variety. Their geometry is a subject of intense scrutiny and is related to the combinatorics of the symmetric group. For instance, whether Y_w is singular is determined by substrings of w [BL, Chapters 5 and 8].

We show that certain Schubert varieties can be realized as Hessenberg varieties, and conversely that some Hessenberg varieties are unions of Schubert varieties. To construct these Schubert varieties, we take X to be the highest weight vector, namely $X = E_{1n}$. Section 2 describes these Hessenberg varieties in terms of their Schubert-variety components.

Most of this paper treats full flags in $GL_n(\mathbb{C})$. Section 4 discusses how to generalize these results to other Lie types. Section 5 contains open questions about Hessenberg varieties, including the question of whether every Schubert variety can be realized as a Hessenberg variety.

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1.1. **Descriptions of the flag variety.** This section is primarily an exposition of three classical ways to describe the flag variety, one geometric, one algebraic, and one combinatorial, all three of which will be used in this paper. This section also includes small lemmas needed elsewhere. Our motivation when selecting these proofs was diversity of approach.

1.1.1. *Geometric description of the flag variety.* Our initial definition was a geometric characterization of the variety of full flags in $\mathbb{C}^n$. We denote the flag $V_1 \subseteq \cdots \subseteq V_n$ by $V_\bullet$.

Throughout this paper, we use a fixed basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$. Each flag can be written explicitly in terms of this basis.

1.1.2. *Algebraic description of the flag variety.* The flag $V_\bullet$ can be realized (non-uniquely) as an invertible matrix g using the rule that the first i columns of g span V_i. In this case $V_\bullet$ is also denoted $[g]$. Let B denote the group of invertible upper-triangular matrices. The flag variety is the quotient GL_n/B.

The group GL_n acts on the flag variety by the rule that if h is in GL_n and $[g]$ is in GL_n/B then $h \cdot [g] = [hg]$. When this action is restricted to the upper-triangular matrices B, it partitions the flag variety into B-orbits whose closures are the Schubert varieties Y_w.

1.1.3. *Combinatorial description of the flag variety.* The permutation matrices index Schubert varieties and, as flags, are contained in Schubert varieties. We use w to refer both to the permutation matrix and to the permutation on the set $\{1, 2, \ldots, n\}$ defined by $we_i = e_{w(i)}$. We denote transpositions by s_{ij} and denote arbitrary permutations by w or v.

For each w, the *Schubert cell* $[Bw]$ is the interior of the Schubert variety Y_w. It can be described explicitly using the following subgroup of B.

Definition 1.1. *Fix a permutation w and let U_w be the subgroup of B defined by either one of the following equivalent conditions:*

(1) *U_w is the maximal subgroup of B such that $w^{-1}U_w w$ is lower-triangular with ones along the diagonal.*
(2) *U_w consists of all matrices in B with ones along the diagonal and whose (i,j) entry is zero for each pair $i < j$ with $w^{-1}(i) < w^{-1}(j)$.*

The next proposition follows from [H, Sections 28.3 and 28.4].

Proposition 1.2. *For each permutation w, the following hold:*

(1) *The set $U_w w$ consists of the matrices*

$$\left\{ w + u : \begin{array}{c} u \text{ is nonzero only in entries that are} \\ \textbf{both} \text{ above } \textbf{and} \text{ to the left of a nonzero entry in } w \end{array} \right\}.$$

(2) *The matrices $U_w w$ are a set of distinct coset representatives for the flags in the Schubert cell $[Bw]$. (See Figure 1.)*

$$\begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} a & b & 1 \\ c & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

FIGURE 1. Examples of $U_w w$ when $n = 3$ (a, b, and c are free)

For each permutation w, let $w = s_{i_1,i_1+1}s_{i_2,i_2+1}\cdots s_{i_k,i_k+1}$ be a factorization with k as small as possible. We call k the *length* of w, denoted $\ell(w)$. The length of w relates the geometric, algebraic, and combinatorial descriptions of GL_n/B.

Proposition 1.3. *For each permutation w, the following hold:*

$$\begin{aligned} \ell(w) &= \dim(\overline{[Bw]}) \\ &= \text{the number of nonzero entries (strictly) above the diagonal in } U_w \\ &= \text{the number of pairs } i < j \text{ such that } w^{-1}(i) > w^{-1}(j). \end{aligned}$$

Each pair $i < j$ that satisfies $w^{-1}(i) > w^{-1}(j)$ is called an *inversion* for w.

The symmetric group is partially ordered by the *Bruhat order*. The geometric definition is that $v \leq w$ if and only if $[Bv] \subseteq \overline{[Bw]}$. Combinatorially, we say $v \leq w$ if and only if there is a factorization $w = s_{i_1,i_1+1}s_{i_2,i_2+1}\cdots s_{i_k,i_k+1}$ so that v can be written as the product of a substring of the s_{i_j,i_j+1}.

1.2. **Properties of permutations and Schubert cells.** Several lemmas that follow from these properties will be used later in this paper. The difficulty of the proofs depends on which characterization of the flag variety is used. (Each of them is a nice exercise for the reader!)

Lemma 1.4. *Fix $j < k \leq n$. For each permutation w, the following hold:*

1. *The permutation $ws_{j,j+1}$ satisfies $\ell(ws_{j,j+1}) = \ell(w) - 1$ if and only if $w(j) > w(j+1)$. Otherwise $\ell(ws_{j,j+1}) = \ell(w) + 1$.*
2. *If $w(j) > w(k)$ then $w > ws_{jk}$ in the Bruhat order.*

Proof. The first part is classical, proven by noting that the sets of inversions of w and of $ws_{j,j+1}$ differ exactly by $(w(j), w(j+1))$.

To prove the next part, we show the closure of $[Bw]$ contains ws_{jk}. Let $u_{w(k),w(j)}(a)$ be the upper-triangular matrix with a in position $(w(k), w(j))$, ones on the diagonal, and zeroes elsewhere. Figure 2 is a schematic of $u_{w(k),w(j)}(a)w$.

Denote the flag $[u_{w(k),w(j)}(a)w]$ by $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$. The first $j-1$ subspaces and last $n-k+1$ subspaces of this flag agree with those of the flag $[ws_{jk}]$. The

$$\left(\begin{array}{c|c|c|c|c}
 & \vdots & & \vdots & \\
 & 0 & & 0 & \\
\hline
0\cdots 0 & a & 0\cdots 0 & 1 & 0\cdots 0 \\
\hline
 & 0 & & 0 & \\
 & \vdots & & \vdots & \\
 & 0 & & 0 & \\
\hline
0\cdots 0 & 1 & 0\cdots 0 & 0 & 0\cdots 0 \\
\hline
 & 0 & & 0 & \\
 & \vdots & & \vdots & \\
 & 0 & & 0 &
\end{array}\right)$$

FIGURE 2. Schematic of $u_{w(k),w(j)}(a)w$

other subspaces are

$$\begin{aligned}\langle V_{j-1}, e_{w(j)} + ae_{w(k)}\rangle &\subseteq \langle V_{j-1}, e_{w(j)} + ae_{w(k)}, e_{w(j+1)}\rangle \subseteq \cdots \\ &\subseteq \langle V_{j-1}, e_{w(j)} + ae_{w(k)}, e_{w(j+1)}, e_{w(j+2)}, \ldots, e_{w(k-1)}\rangle \\ &\subseteq \langle V_{j-1}, e_{w(j)}, e_{w(j+1)}, e_{w(j+2)}, \ldots, e_{w(k-1)}, e_{w(k)}\rangle = V_k.\end{aligned}$$

As a approaches ∞, these subspaces approach the subspaces

$$\begin{aligned}\langle V_{j-1}, e_{w(k)}\rangle &\subseteq \langle V_{j-1}, e_{w(k)}, e_{w(j+1)}\rangle \subseteq \cdots \\ &\subseteq \langle V_{j-1}, e_{w(k)}, e_{w(j+1)}, e_{w(j+2)}, \ldots, e_{w(k-1)}\rangle \\ &\subseteq \langle V_{j-1}, e_{w(k)}, e_{w(j+1)}, e_{w(j+2)}, \ldots, e_{w(k-1)}, e_{w(j)}\rangle = V_k,\end{aligned}$$

which are the corresponding parts of $[ws_{jk}]$. Thus $\lim_{a\mapsto\infty}[u_{w(k),w(j)}(a)w] = [ws_{jk}]$. It follows that $[Bws_{jk}] \subseteq \overline{[Bw]}$, and so $ws_{jk} < w$. $\square$

1.3. **Hessenberg varieties.** In this section, we define Hessenberg varieties algebraically. We also discuss some technical issues that arise.

To obtain an algebraic characterization of Hessenberg varieties, we use subspaces of $n \times n$ matrices rather than the Hessenberg function h. The matrix basis unit that is zero except in entry (i, j), where it is one, is denoted E_{ij}. Each Hessenberg function defines a subspace of $n \times n$ matrices by $H_h = \langle E_{ij} : i \leq h(j)\rangle$. We call H_h a Hessenberg space. The Hessenberg variety of X and h is

$$\mathcal{H}(X, h) = \{\text{Flags } [g] : g^{-1}Xg \in H_h\}.$$

Many examples of Hessenberg spaces come from classical Lie theory. If h is the Hessenberg function with $h(i) = i$ for each i then H_h is the set of upper-triangular matrices. If h is the Hessenberg function given by $h(i) = n$ for each i then H_h consists of all $n \times n$ matrices. In fact, if H_h is any parabolic subalgebra, then H_h

is a Hessenberg space and the corresponding h is one of the parabolic Hessenberg functions from the Introduction.

Most Hessenberg spaces are not parabolic. For instance, the Hessenberg function given by $h(i) = i+1$ when $i \neq n$ and $h(n) = n$ corresponds to the subspace H_h which is zero below the subdiagonal. Figure 3 shows this for $n = 4$. Hessenberg

$$\begin{array}{l} h(1)=2 \\ h(2)=3 \\ h(3)=4 \\ h(4)=4 \end{array} \longleftrightarrow \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

FIGURE 3. One Hessenberg function and space when $n = 4$

varieties with this Hessenberg function are important in various applications, including numerical analysis [dMPS] and computing quantum cohomology of the flag variety (see [K] and [R]).

Our definition of Hessenberg functions omits one condition from the original definition in [dMPS], which also requires $h(i) \geq i$ for each i. This paper studies a strictly larger collection of varieties than in [dMPS]. Our generalization is particularly useful when X is nilpotent. (When X is regular semisimple, the variety $\mathcal{H}(X,h)$ will be empty if $h(i) < i$ for each i.) Nilpotent Hessenberg varieties arise naturally when studying representations of the symmetric group on Hessenberg varieties that generalize Springer's correspondance [MT].

Section 4 generalizes this definition (and other results) to all Lie types.

Our first proposition establishes that nilpotent Hessenberg varieties depend only on the i for which the Hessenberg function does not satisfy $h(i) = i$.

Proposition 1.5. *Fix n and fix i such that $1 \leq i \leq n$. Suppose h is a Hessenberg function with $h(i) = i$ and that the function h' defined by*

$$h'(j) = \begin{cases} h(j) & \text{if } j \neq i, \text{ and} \\ i-1 & \text{for } i = j \end{cases}$$

is also a Hessenberg function. If X is nilpotent then $\mathcal{H}(X,h) = \mathcal{H}(X,h')$.

Proof. If $g^{-1}Xg \in H_{h'}$ then $g^{-1}Xg \in H_h$ since $H_{h'} \subseteq H_h$. Now assume $g^{-1}Xg \in H_h$. We have

$$(g^{-1}Xg)e_i \in ce_i + \langle e_1, \ldots, e_{i-1} \rangle$$

where e_j are the standard basis vectors for $\mathbb{C}^n$. Also

$$(g^{-1}Xg)\langle e_1, \ldots, e_{i-1} \rangle \subseteq \langle e_1, \ldots, e_{h(i-1)} \rangle \subseteq \langle e_1, \ldots, e_{i-1} \rangle,$$

since $h(i-1) < h(i) = i$. Since X is nilpotent, applying $g^{-1}Xg$ to e_i sufficiently many (e.g. n) times should give zero. On the other hand we have

$$(g^{-1}Xg)^n e_i \in c^n e_i + \langle e_1, \ldots, e_{i-1} \rangle.$$

Therefore $c = 0$, and as a consequence $g^{-1}Xg$ lies in $H_{h'}$. □

Comments from K. Rietsch greatly improved this proof. This lemma motivates the following definition, also suggested by K. Rietsch.

Definition 1.6. *For each linear operator X, the Hessenberg spaces H and H' are X-equivalent if $\mathcal{H}(X,H) = \mathcal{H}(X,H')$. In this case, we write $H \sim_X H'$ and say that H and H' are in the same X-equivalence class.*

X-equivalence of Hessenberg functions is defined the same way.

The X-equivalence class of Hessenberg spaces (or functions) depends only on the conjugacy class of X since $\mathcal{H}(X,H) \cong \mathcal{H}(g^{-1}Xg, H)$ (see [T, Proposition 2.7]).

For instance, if $X = 0$ then there is only one X-equivalence class of Hessenberg spaces. If X is nilpotent, then the Hessenberg function defined by $h(i) = i$ for all i is X-equivalent to the function defined by $h'(i) = i-1$ for all i. Alternatively, the Hessenberg space consisting of all upper-triangular matrices is X-equivalent to the space of all strictly upper-triangular matrices. (This fact is used frequently in Springer theory.) We generalize this in the next corollary, whose proof is immediate from Proposition 1.5.

Corollary 1.7. *For each nilpotent linear operator X, there is a unique minimal element of each X-equivalence class of Hessenberg functions (respectively Hessenberg spaces). This minimal element satisfies*

- *if there exists i such that $h(i) = i$, then $h(i-1) = i$ as well;*
- *if there exists a matrix $\sum c_{jk}E_{jk}$ in H_h and i such that the coefficient $c_{ii} \neq 0$, then E_{ii} and $E_{i-1,i}$ are both in H_h.*

Typically, we assume H and h are minimal in their X-equivalence classes.

2. Geometry and topology of $\mathcal{X}_h$

In this section, we fix X to be the matrix E_{1n} and study the Hessenberg varieties

$$\mathcal{X}_h = \{\text{Flags } [g] : g^{-1}E_{1n}g \in H_h\} = \{\text{Flags } V_1 \subseteq \cdots \subseteq V_n : E_{1n}V_i \subseteq V_{h(i)}\}.$$

We will show that these Hessenberg varieties are unions of Schubert varieties. Loosely speaking, each Schubert variety comes from one "corner" of the Hessenberg space. We will identify explicitly these Hessenberg varieties, including which

Schubert varieties arise and their dimensions. We will also show that many of these Hessenberg varieties are not pure-dimensional.

Proposition 2.1. *$\mathcal{X}_h$ is a union of Schubert varieties $\bigcup Y_w$.*

Proof. Each flag can be written in row echelon form as $[uw]$ for some invertible upper-triangular u and permutation matrix w. The flag $[uw]$ is in $\mathcal{X}_h$ if and only if $w^{-1}u^{-1}E_{1n}uw$ is in H_h. Direct calculation shows that $u^{-1}E_{1n}u$ is a nonzero scalar multiple of E_{1n} for each upper-triangular u. Thus, the flag $[uw]$ is in $\mathcal{X}_h$ if and only if $[w]$ is in $\mathcal{X}_h$.

This means $\mathcal{X}_h$ is a union of Schubert cells, say $\mathcal{X}_h = \bigcup [Bw]$, and so $\mathcal{X}_h \subseteq \bigcup Y_w$. Since $\mathcal{X}_h$ is closed, it also contains the closures $\bigcup \overline{[Bw]} = \bigcup Y_w$. □

In general the variety $\mathcal{H}(X, h)$ is not a union of Schubert cells $[Bw]$. In fact, if $g^{-1}Xg$ is another element of the conjugacy class of X, then typically at most one of $\mathcal{H}(X, h)$ and $\mathcal{H}(g^{-1}Xg, h)$ is a union of cells $[Bw]$, even though the two varieties are homeomorphic [T, Proposition 2.7]. For instance, suppose $n = 3$ and the Hessenberg function satisfies $h(i) = i$ for each i. Each of $\mathcal{H}(E_{12}, h)$ and $\mathcal{H}(E_{13}, h)$ is homeomorphic to two copies of $\mathbb{P}^1$ glued together at a point. However, the variety $\mathcal{H}(E_{13}, h)$ is the union of the Schubert varieties $Y_{s_1} \cup Y_{s_2}$, while $\mathcal{H}(E_{12}, h)$ is a one-dimensional closed subvariety of $Y_{s_2 s_1}$.

For each $i \neq j$, let h_{ij} be the Hessenberg function defined by

$$h_{ij}(k) = \begin{cases} 0 & \text{if } k < j \text{ and} \\ i & \text{if } k \geq j. \end{cases}$$

The corresponding Hessenberg space H_{ij} is spanned by the matrix basis units E_{kl} with $k \leq i$ and $l \geq j$. In other words, H_{ij} is the subspace of matrices which are zero outside of the upper-right $i \times (n - j + 1)$ rectangle, as in Figure 4. For

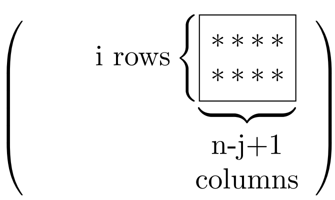


Figure 4. Schematic diagram of H_{ij}

example, H_{n1} consists of all $n \times n$ matrices and H_{1n} is just the span of E_{1n}. If the sun rises at the far left of the i^{th} row, travels around the bottom left corner of the matrix, and sets at the bottom of the j^{th} column, then H_{ij} is the shadow cast by the matrix basis unit E_{ij} during the course of the 'day'. (A. Ottazzi created this image in [O].)

Lemma 2.2. *For each pair $i \neq j$, let w be the permutation that has e_n in column j, e_1 in column i, and the other vectors inserted in decreasing order (e_{n-1}, e_{n-2}, …, e_2) in the remaining columns. Then $\mathcal{X}_{H_{ij}} = Y_w$.*

Proof. The proof has three parts. First, we show that if s is a permutation, then $[s] \in X_{H_{ij}}$ if and only if s has e_1 somewhere in its first i columns and e_n in its last $n-j+1$ columns. For each such s, we form the permutation s' by moving e_1 to the i^{th} column, moving e_n to the j^{th} column, and keeping the other columns in the same order as in s. We then show that $s' \geq s$. Finally, we show that $w \geq s'$.

Suppose the matrix s has e_1 in its k^{th} column and e_n in its l^{th} column. Since $s^{-1} = s^t$, the matrix s^{-1} has e_1 in its k^{th} row. So

$$s^{-1}E_{1n}s = E_{kl}. \tag{1}$$

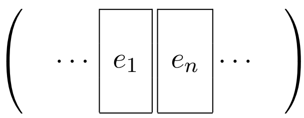
This is in H_{ij} if and only if $k \leq i$ and $l \geq j$. We conclude that the flag $[s] \in X_{H_{ij}}$ if and only if $s^{-1}e_1 = e_k$ for $k \leq i$ and $s^{-1}e_n = e_l$ for $l \geq j$.

The permutation w satisfies this condition so $\mathcal{X}_{H_{ij}} \supseteq Y_w$. We now show that for any permutation s of this form, the flag $[s]$ is in the variety Y_w.

We begin by moving the column with e_n to the left or the column with e_1 to the right, as long as one of those moves is possible. Suppose $l > j$ and either the $(l-1)^{th}$ column is not e_1 or it is e_1 and $l-1 \neq i$. The flag $[ss_{l-1,l}]$ is also in $\mathcal{X}_{H_{ij}}$. Lemma 1.4 Part 2 showed $ss_{l-1,l} > s$, so the corresponding Schubert varieties satisfy $Y_s \subseteq Y_{ss_{l-1,l}}$. (When $k < i$ and either the $(k+1)^{th}$ column is not e_n or it is e_n but $k+1 \neq j$, use the flag $[ss_{k,k+1}]$ in a symmetric argument.)

A move of this sort will be impossible exactly when $j < i$ and either

- $i = k$ and $l = k+1 = i+1$ or
- $l = j$ and $k = j-1 = l-1$.



The diagram is a schematic for these cases: the vectors e_1 and e_n are adjacent, and the i^{th} column is in place (respectively j^{th}) while e_n is moving to the left (respectively e_1 to the right). Lemma 1.4 Part 2 shows that the permutation obtained from s by exchanging its $(i+1)^{th}$ and $(i-1)^{th}$ columns is greater than s in the Bruhat order (respectively $j-1$ and $j+1$).

Once e_1 is to the right of e_n, successively multiply s on the right by $s_{k,k+1}$ or $s_{l-1,l}$ to obtain a permutation s' with $s' \geq s$, so that $s'(e_i) = e_1$ and $s'(e_j) = e_n$.

We now prove by induction that $s' \leq w$. Assume that the first t columns of s' and w agree and the $(t+1)^{th}$ does not. The $(t+1)^{th}$ column of w is filled with $e_{w(t+1)}$. Neither $s'(t+1)$ nor $w(t+1)$ is in $\{1, n\}$ because $s'(t+1) \neq w(t+1)$. Since w and s' agree in the first t columns, the column vector $w(e_{t+1})$ is none of $e_{s'(1)}$,

$e_{s'(2)}, \ldots, e_{s'(t)}$, so there is a positive integer t_1 such that $s'(e_{t+1+t_1}) = e_{w(t+1)}$. The permutation $s'' = s's_{t+1,t+1+t_1}$ satisfies $s'' \geq s'$ by Lemma 1.4 Part 2. Since neither e_1 nor e_n moved, s'' has $s''(e_i) = e_1$ and $s''(e_j) = e_n$, and also agrees with w in its first $t+1$ columns. By induction, the claim follows. □

The following corollary restates the condition on w.

Corollary 2.3. *For each pair $i \neq j$, let w be the largest permutation in the Bruhat order that satisfies $w^{-1}E_{1n}w = E_{ij}$. Then $X_{H_{ij}} = Y_w$.*

We can factor w explicitly in terms of simple transpositions.

Corollary 2.4. *Let w_0 be the permutation with $w_0 e_k = e_{n-k+1}$ for each $k = 1, \ldots, n$. For each pair $i \neq j$, the Hessenberg variety $X_{H_{ij}} = Y_w$, where*

$$w = \begin{cases} w_0 s_{12}s_{23}\cdots s_{j-1,j}s_{n,n-1}\cdots s_{i+1,i} & \text{if } j < i \text{ and} \\ w_0 s_{12}s_{23}\cdots s_{j-2,j-1}s_{n,n-1}\cdots s_{i+1,i} & \text{if } j > i. \end{cases}$$

Proof. For each matrix M, the product $Ms_{12}s_{23}\cdots s_{k,k+1}$ cyclically permutes the first $k+1$ columns of M, sending the first column to the $(k+1)^{th}$ position and moving each of the other columns one position to the left. Similarly, the product $Ms_{n,n-1}s_{n-1,n-2}\cdots s_{k+1,k}$ cyclically permutes the last $n-k+1$ columns, moving the last column to the k^{th} and moving the others one column to the right. Cyclically permuting the first j (respectively $j-1$) columns and the last $n-i+1$ columns of w_0 gives the permutation w of Lemma 2.2. □

This gives a closed formula for the dimension of $X_{H_{ij}}$.

Corollary 2.5. *For each $i \neq j$, the dimension of $X_{H_{ij}}$ is*

$$\begin{cases} \binom{n}{2} - (j-1+n-i) \text{ if } j < i \text{ and} \\ \binom{n}{2} - (j-2+n-i) \text{ if } j > i. \end{cases}$$

Proof. The length of the permutation w_0 is $\binom{n}{2}$. Let $w = w_0 \prod s_{k,k+1}$ be the factorization from Corollary 2.4. Each simple transposition in this factorization reduces the length of w_0 by one, from Lemma 1.4 Part 1. □

2.1. **The components of $\mathcal{X}_H$.** It is usually difficult to identify the irreducible components of Hessenberg varieties. However, when $X = E_{1n}$, it can be done.

Proposition 2.6. *For all H and H', we have $\mathcal{X}_{H \cup H'} = \mathcal{X}_H \cup \mathcal{X}_{H'}$.*

Proof. The flag $[w]$ is in $\mathcal{X}_{H \cup H'}$ if and only if $w^{-1}E_{1n}w$ is in $H \cup H'$. Since $w^{-1}E_{1n}w$ is a matrix basis unit, it is in $H \cup H'$ if and only if either $w^{-1}E_{1n}w$ is in H or $w^{-1}E_{1n}w$ is in H'. This holds if and only if the flag $[w]$ is in $\mathcal{X}_H \cup \mathcal{X}_{H'}$. □

Lemma 2.7. *Let H and H' be Hessenberg spaces that are minimal in their E_{1n}-equivalence classes. Then $\mathcal{X}_H \subseteq \mathcal{X}_{H'}$ if and only if $H \subseteq H'$.*

Proof. We reduce to the case when $\mathcal{X}_H$ and $\mathcal{X}_{H'}$ are Schubert varieties. Write $H = \bigcup H_{ij}$ and $H' = \bigcup H'_{i'j'}$. Each H_{ij} satisfies $i \neq j$ since H is minimal in its E_{1n}-equivalence class (respectively $i' \neq j'$). For each pair $i \neq j$, the matrix E_{ij} is in H' if and only if $E_{ij} \in H'_{i'j'}$ for some i', j'. This holds if and only if $H_{ij} \subseteq H'_{i'j'}$. Consequently $H \subseteq H'$ if and only if for each i, j there exist i', j' such that $H_{ij} \subseteq H'_{i'j'}$. We know $\mathcal{X}_H = \bigcup \mathcal{X}_{H_{ij}}$ and $\mathcal{X}_{H'} = \bigcup \mathcal{X}_{H'_{i'j'}}$ from Proposition 2.6. It suffices to show that $\mathcal{X}_{H_{ij}} \subseteq \mathcal{X}_{H'_{i'j'}}$ if and only if $H_{ij} \subseteq H_{i'j'}$.

Both $\mathcal{X}_{H_{ij}}$ and $\mathcal{X}_{H_{i'j'}}$ are a disjoint union of Schubert cells by Proposition 2.1. This means the inclusion $\mathcal{X}_{H_{ij}} \subseteq \mathcal{X}_{H_{i'j'}}$ holds if and only if each permutation flag $[s]$ in $\mathcal{X}_{H_{ij}}$ is also contained in $\mathcal{X}_{H_{i'j'}}$. Equation 1 shows that $[s]$ is in $\mathcal{X}_{H_{ij}}$ if and only if $s^{-1}E_{1n}s = E_{kl}$, where k and l satisfy the conditions $k \leq i$ and $l \geq j$. It follows that each permutation flag $[s]$ in $\mathcal{X}_{H_{ij}}$ is also in $\mathcal{X}_{H_{i'j'}}$ if and only if $i \leq i'$ and $j \geq j'$, which is true if and only if $H_{ij} \subseteq H_{i'j'}$. □

Definition 2.8. *A maximal decomposition of the Hessenberg space H is a union $H = \bigcup H_{ij}$ so that no pair H_{ij}, $H_{i'j'}$ satisfies $H_{ij} \subseteq H_{i'j'}$.*

If H is minimal in its E_{1n}-equivalence class, then a maximal decomposition $H = \bigcup H_{ij}$ further satisfies $i \neq j$ for each H_{ij}.

Corollary 2.9. *Let H be minimal in its E_{1n}-equivalence class. If $H = \bigcup H_{ij}$ is a maximal decomposition, the components of $\mathcal{X}_H$ are the Schubert varieties $\mathcal{X}_{H_{ij}}$.*

Proof. Write $\mathcal{X}_H = \bigcup \mathcal{X}_{H_{ij}}$ as in Proposition 2.6. For each H_{ij}, there is a unique permutation w_{ij} such that $[Bw_{ij}]$ is dense in $\mathcal{X}_{H_{ij}}$ by Lemma 2.2. For every $(i', j') \neq (i, j)$, Lemma 2.7 shows that $[w_{i'j'}]$ is not in $\mathcal{X}_{H_{ij}}$, and so $[Bw_{i'j'}] \cap \mathcal{X}_{H_{ij}}$ is empty. This means $\mathcal{X}_{H_{ij}}$ is an irreducible component of $\mathcal{X}_H$. □

Corollary 2.10. *Fix H, a minimal Hessenberg space in its E_{1n}-equivalence class. The Hessenberg variety $\mathcal{X}_H$ is pure dimensional if and only if there exists an integer $k \in \{1, 2, \ldots, n-1\}$ and a subset $I \subseteq \{1, 2, \ldots, n-k\}$ such that either $H = \bigcup_{i \in I} H_{i,i+k}$ or $H = \bigcup_{i \in I} H_{i,i-k}$.*

Proof. Let $H = \bigcup H_{ij}$ be a maximal decomposition and write $\mathcal{X}_H$ as a union of its irreducible components $X_{H_{ij}}$. Each $X_{H_{ij}}$ is a Schubert variety that has dimension $\binom{n}{2} - (j - 1 + n - i)$ if $j < i$ and $\binom{n}{2} - (j - 2 + n - i)$ if $j > i$ by

Corollary 2.5. Given pairs (i,j) and (i',j'), the varieties $X_{H_{ij}}$ and $X_{H_{i'j'}}$ have the same dimension if and only if $j-i=j'-i'$. $\square$

This gives a collection of examples of nilpotent Hessenberg varieties that are connected and not pure dimensional. For instance, the Hessenberg space $H = H_{41} \cup H_{54}$ of 5×5 matrices gives a variety $\mathcal{X}_H$ in GL_5/B that is not pure.

3. A semisimple Hessenberg variety that is not pure dimensional

In this section, we describe another way that Hessenberg varieties can fail to be pure dimensional. The next proposition generalizes an example that R. MacPherson and I discovered.

Proposition 3.1. *Fix $X = \sum_{i=1}^{n-1} E_{ii}$. Let h be the Hessenberg function with $h(i) = n-1$ for all $i \leq n-1$, and $h(n) = n$. The variety $\mathcal{H}(X,h)$ is the disjoint union of two components, one of which is homeomorphic to GL_{n-1}/B and the other of which is homeomorphic to a fiber bundle over $\mathbb{P}^{n-2}$ with fiber GL_{n-1}/B. In particular, the Hessenberg variety $\mathcal{H}(X,h)$ is not pure dimensional.*

Proof. By definition, each flag $V_\bullet$ in $\mathcal{H}(X,h)$ satisfies $XV_{n-1} \subseteq V_{n-1}$. Since $X(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^{n-1} a_i e_i$, either

(1) $e_n \in V_{n-1}$ or
(2) $V_{n-1} = \langle e_1, e_2, \ldots, e_{n-1} \rangle$.

These conditions are closed and so define two closed subvarieties $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, in GL_n/B. The two conditions cannot be simultaneously satisfied so $\mathcal{H}(X,h)$ is the disjoint union $\mathcal{Y}_1 \cup \mathcal{Y}_2$. We now describe these subvarieties.

First we show that $\mathcal{Y}_2 \cong GL_{n-1}/B$. The flag $[g]$ satisfies Condition 2 if and only if the matrix g is in

$$P = \left(\begin{array}{c|c} & * \\ & * \\ GL_{n-1} & \vdots \\ & * \\ \hline 0\,0 \cdots 0 & \mathbb{C}^* \end{array} \right).$$

In other words, the component $\mathcal{Y}_2$ is isomorphic to GL_{n-1}/B via the isomorphism that sends $V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n$ to the flag $V_1 \subseteq \cdots \subseteq V_{n-1}$ inside $\langle e_1, \ldots, e_{n-1} \rangle$.

Now we study $\mathcal{Y}_1$. Denote the Grassmannian of $n-1$-planes in $\mathbb{C}^n$ by $G(n-1,n)$. Write $\pi_{n-1} : GL_n/B \longrightarrow G(n-1,n)$ for the projection that sends the flag $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$ to the subspace V_{n-1}. This is a continuous map; in fact, it is the quotient map $\pi_{n-1} : GL_n/B \longrightarrow GL_n/P$.

Restrict the map to $\pi_{n-1}|_{\mathcal{Y}_1} : \mathcal{Y}_1 \longrightarrow G(n-1,n)$. The image $\pi_{n-1}(\mathcal{Y}_1)$ is

$$\pi_{n-1}(\mathcal{Y}_1) = \{\text{Subspaces } V_{n-1} \text{ such that } e_n \in V_{n-1}\}.$$

This is isomorphic to the set of $n-2$-dimensional subspaces in $\langle e_1, \ldots, e_{n-1}\rangle$, so $\pi_{n-1}(\mathcal{Y}_1) \cong G(n-2, n-1)$. Since $G(n-2,n-1) \cong \mathbb{P}^{n-2}$, we conclude that the image $\pi_{n-1}(\mathcal{Y}_1) \cong \mathbb{P}^{n-2}$.

We now identify the fiber $\left(\pi_{n-1}|_{\mathcal{Y}_1}\right)^{-1}(V_{n-1})$ of each $V_{n-1} \in \pi_{n-1}(\mathcal{Y}_1)$. The flag $W_1 \subseteq \cdots \subseteq W_n$ is in $\left(\pi_{n-1}|_{\mathcal{Y}_1}\right)^{-1}(V_{n-1})$ if and only if $W_{n-1} = V_{n-1}$. Every flag in GL_n/B satisfies $W_n = \mathbb{C}^n$, so the fiber is characterized by

$$\left(\pi_{n-1}|_{\mathcal{Y}_1}\right)^{-1}(V_{n-1}) = \{\text{Flags such that } W_1 \subseteq W_2 \subseteq \cdots W_{n-2} \subseteq V_{n-1}\}.$$

This is the set of complete flags in V_{n-1} and is homeomorphic to GL_{n-1}/B.

Consequently, the map $\pi_{n-1} : \mathcal{Y}_1 \longrightarrow \pi_{n-1}(\mathcal{Y}_1)$ is a fiber bundle whose base space is homeomorphic to $\mathbb{P}^{n-2}$ and whose fiber is homeomorphic to GL_{n-1}/B. □

For example, when $n = 3$ the Hessenberg variety $\mathcal{H}(X, h)$ is a disjoint union of $\mathbb{P}^1$ and a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

4. Generalizing to all Lie types

In this section, we discuss generalizations of these results to arbitrary Lie type. Our exposition is brief; we assume our reader is familiar with the general theory.

Let G be a complex reductive linear algebraic group, $\mathfrak{g}$ its Lie algebra, B a fixed Borel subgroup, and $\mathfrak{b}$ its Lie algebra. The full flag variety is G/B and its elements are written $[g]$. Let T be a maximal torus contained in B and $\mathfrak{t}$ be the Cartan subalgebra associated to T. We will also use $\mathfrak{n}^-$, the maximal nilpotent subalgebra in the opposite Borel subalgebra $\mathfrak{b}^-$. Let W be the Weyl group.

The positive roots in the root system corresponding to $\mathfrak{g}$ are denoted Φ^+ and the negative roots are Φ^-. The inner product on Φ is written $\langle \cdot, \cdot \rangle$. We refer to the length of roots, which can be either short or long. If α and β are two roots, then $\alpha \succ \beta$ means $\alpha - \beta$ is a *sum* of positive roots. (Note that this is not the partial ordering where $\alpha > \beta$ means $\alpha - \beta$ is a positive root.) If $\alpha = \sum c_i \alpha_i$ is a (reduced) sum of simple roots, then the support of α is the set $\text{supp}(\alpha) = \{\alpha_i : c_i \neq 0\}$. Given α, we write E_α for a root vector corresponding to α.

A Hessenberg space H is a linear subspace of matrices such that $[H, \mathfrak{b}] \subseteq H$. (This definition omits one condition from that found in [dMPS].) Suppose X is in $\mathfrak{g}$ and H is a Hessenberg space. The Hessenberg variety of (X, H) is given by

$$\mathcal{H}(X, H) = \{[g] \in G/B : g^{-1}Xg \in H\}.$$

Proposition 4.1. *Let E_θ be a weight vector for the highest weight θ. For each Hessenberg space H, the variety $\mathcal{H}(E_\theta, H)$ is a union of Schubert varieties.*

This generalizes Proposition 2.1. The proof is the same as in Proposition 2.1: the flag $[bw]$ is in $\mathcal{H}(E_\theta, H)$ if and only if $w^{-1}b^{-1}E_\theta bw$ is in H, and the adjoint action of B multiplies E_θ by a nonzero constant factor.

Definition 4.2. *For each root α, define H_α to be minimal with respect to inclusion among all Hessenberg spaces that contain the root vector E_α.*

If α is positive then H_α is the span of the root vectors E_β with $\beta \succeq \alpha$. However, this is not true when α is negative. In that case, every positive root β satisfies $\beta \succeq \alpha$, but H_α need not contain $\mathfrak{b}$.

Let $N(\mathrm{supp}(\alpha)) = \{\alpha_j : \exists \alpha_i \in \mathrm{supp}(\alpha) \text{ with } \langle \alpha_j, \alpha_i \rangle \neq 0\}$. In other words, $N(\mathrm{supp}(\alpha))$ consists of $\mathrm{supp}(\alpha)$ as well as the simple roots that are joined to a root in $\mathrm{supp}(\alpha)$ by an edge in the Dynkin diagram for $\mathfrak{g}$.

Lemma 4.3. *Let $\alpha \in \Phi^-$. If $H_\alpha^+ = \langle E_\beta : \beta \in \Phi^+, \mathrm{supp}(\beta) \cap N(\mathrm{supp}(\alpha)) \neq \emptyset \rangle$, $H_\alpha^- = \langle E_\beta : \beta \in \Phi^- \text{ has } \beta \succeq \alpha \rangle$, and $T_\alpha = \langle [E_{\alpha_i}, E_{-\alpha_i}] : \alpha_i \in \mathrm{supp}(\alpha) \rangle$, then*

$$H_\alpha = H_\alpha^- \oplus H_\alpha^+ \oplus T_\alpha.$$

Proof. Recall that $[E_\beta, E_\gamma]$ is a nonzero multiple of $E_{\beta+\gamma}$ if $\beta + \gamma$ is a root, an element T_β of the Cartan subalgebra if $\gamma = -\beta$, and zero otherwise.

This identity implies that

$$H_\alpha^- = \bigcap_{\substack{\text{Hess. spaces } H \\ \text{s.t. } E_\alpha \in H}} H \cap \mathfrak{n}^-$$

and that the Cartan subalgebra $\mathfrak{t}$ intersects $[H_\alpha^-, \mathfrak{b}]$ exactly in T_α. The $[\mathfrak{b}, \cdot]$-closure of T_α is H_α^+.

We must show that $[H_\alpha^-, \mathfrak{b}] \cap \mathfrak{b} \subseteq T_\alpha \oplus H_\alpha^+$. Suppose $E_\gamma \in \mathfrak{b}$ and $E_\beta \in H_\alpha^-$ satisfy $\gamma + \beta \in \Phi^+$. We will find a simple root $\alpha_i \in \mathrm{supp}(\gamma+\beta) \cap N(\mathrm{supp}(\alpha))$. If the support of β is contained in the support of $\gamma + \beta$, then any $\alpha_i \in \mathrm{supp}(\beta)$ is as desired, since $\mathrm{supp}(\beta) \subseteq \mathrm{supp}(\alpha)$ by definition of H_α^-. If $\mathrm{supp}(\beta) \not\subseteq \mathrm{supp}(\gamma + \beta)$ then we may write the support of γ as the (not necessarily disjoint) union $\mathrm{supp}(\gamma) = \mathrm{supp}(\beta) \cup \mathrm{supp}(\gamma + \beta)$. Suppose for every $\alpha_i \in \mathrm{supp}(\gamma + \beta)$ and every $\alpha_j \in \mathrm{supp}(\alpha)$ we have $\langle \alpha_i, \alpha_j \rangle = 0$. Then the support of $\gamma + \beta$ is not connected to the support of α in the Dynkin diagram, and consequently the support of $\gamma + \beta$ is not connected to the support of β. In other words, the support of γ is a disconnected subset of the Dynkin diagram. This contradicts the fact that the support of each root is a connected subset of the Dynkin diagram (see [B, page 169]). So $E_{\gamma+\beta} \in H_\alpha^+$. □

The next two results generalize Lemma 2.2.

Lemma 4.4. *If α is a root of the same length as θ, there is a unique maximal Weyl group element w that satisfies $w^{-1}\theta = \alpha$.*

Proof. Denote the stabilizer of θ in W by $\mathrm{Stab}(\theta)$. Consider the left cosets $\mathrm{Stab}(\theta)\backslash W$. Each coset has a unique maximal element since $\mathrm{Stab}(\theta)$ is parabolic. Also, each coset $\mathrm{Stab}(\theta)u$ is determined by the root $u^{-1}\theta$. Since α and θ are in the same W-orbit, there exists w with $w^{-1}\theta = \alpha$. □

Proposition 4.5. *Let α be a root of the same length as θ and let w be the maximal Weyl group element with $w^{-1}\theta = \alpha$. Then $\mathcal{H}(E_\theta, H_\alpha) = Y_w$.*

Proof. For each element u in W, the flag $[u]$ is in $\mathcal{H}(E_\theta, H_\alpha)$ if and only if $u^{-1}E_\theta u \in H_\alpha$. Since $u^{-1}E_\theta u = E_{u^{-1}\theta}$, the Hessenberg variety $\mathcal{H}(E_\theta, H_\alpha)$ is a union of Schubert cells indexed by the elements in cosets of $\mathrm{Stab}(\theta)\backslash W$. We must show that if $E_{u^{-1}\theta} \in H_\alpha$ then $w \geq u$ in the Bruhat order.

The roots $u^{-1}\theta$ and $w^{-1}\theta$ have the same length. If $u^{-1}\theta$ and $w^{-1}\theta$ have the same sign, then $u^{-1}\theta \succeq w^{-1}\theta$ if and only if $w \geq u$ by [St, Proposition 3.2]. Now suppose $u^{-1}\theta$ is positive and $w^{-1}\theta$ is negative. Without loss of generality, let $u^{-1}\theta = \alpha_i$ be simple. If α_i is in the support of $w^{-1}\theta$ then $s_i u^{-1}\theta \succeq w^{-1}\theta$ and so $w \geq us_i$. Moreover, we know $us_i > u$ since $us_i\alpha_i \in \Phi^-$. This gives $w \geq u$.

If α_i is not in the support of $w^{-1}\theta$ then there exists an $\alpha_j \in \mathrm{supp}(w^{-1}\theta)$ such that $\langle \alpha_i, \alpha_j \rangle \neq 0$ by Lemma 4.3. At least one simple root in $\mathrm{supp}(w^{-1}\theta)$ has the same length as $w^{-1}\theta$ and hence as α_i. If the Dynkin diagram for $\mathfrak{g}$ has a multiedge, then the simple roots are long on one side of the multiedge and short on the other. So the edge from α_i to α_j cannot be a multiedge. This means that $s_j\alpha_i = s_i\alpha_j = \alpha_i + \alpha_j$. The root $s_j u^{-1}\theta = \alpha_i + \alpha_j$ and so $u > us_j$ by [St, Proposition 3.2]. Let $u = vs_j$ be a reduced factorization. Then $s_j s_i v^{-1}\theta = -\alpha_j$ and $w \geq vs_is_j$, again by [St, Proposition 3.2]. The factorization vs_is_j is reduced because $vs_i > v$ ([St, Proposition 3.2]) and $vs_is_j > vs_i$ (because $vs_is_j\alpha_j \in \Phi^-$). We conclude that $w \geq vs_is_j > u$. □

5. Questions

We first ask about the relation between Schubert and Hessenberg varieties.

Question 5.1. *Are all Schubert varieties Hessenberg varieties? If not, describe explicitly the Schubert varieties that are also Hessenberg varieties.*

The matrices in the Hessenberg spaces of Corollary 2.10 are said to be in *banded Hessenberg form*, a form used in numerical analysis (see [dMPS]). We ask if this algebraic property is related to the geometric condition of purity.

Question 5.2. *Let X be any linear operator. If H is in banded Hessenberg form, is the Hessenberg variety $\mathcal{H}(X, H)$ necessarily pure-dimensional?*

The next question arises because of the representations on the cohomology of Springer fibers. We wonder whether the highest-weight Hessenberg varieties of Sections 2 and 4 also carry interesting geometric actions.

Question 5.3. *Does the cohomology of the highest-weight Hessenberg varieties carry interesting group actions? Is there an interesting group action on the highest-weight Hessenberg variety that permutes its irreducible components?*

[dMPS] proved that regular semisimple Hessenberg varieties are smooth.

Question 5.4. *Are all semisimple Hessenberg varieties smooth?*

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