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Height Bounds and Preperiodic Points for Families of Jointly Regular Affine Maps

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In honor of John Coates on the occasion of his 60th birthday

Introduction

The theory of height functions provides a powerful tool for studying the arithmetic of *morphisms* $\phi : \mathbb{P}^N \to \mathbb{P}^N$. For example, if ϕ is a morphism of degree $d \ge 2$ over a number field K, then the standard height estimate

$$h\big(\phi(P)\big) = d \cdot h(P) + O(1) \qquad \text{for all } P \in \mathbb{P}^N(\bar{K})$$

combined with the fact that there are only finitely many K-rational points of bounded height leads immediately to a proof of Northcott's Theorem [18] stating that ϕ has only finitely many K-rational preperiodic points.

The situation is more complicated if $\phi : \mathbb{P}^N \to \mathbb{P}^N$ is only required to be a rational map. An initial difficulty arises because there may be orbits $\mathcal{O}_\phi(P)$ that "terminate" because some iterate $\phi^n(P)$ arrives at a point where ϕ is not defined. In this paper we study morphisms $\phi : \mathbb{A}^M \to \mathbb{A}^N$ of affine space whose extension $\bar{\phi} : \mathbb{P}^M \to \mathbb{P}^N$ need not be a morphism. (For the application to dynamics, we take $M = N$.)

Example 1. The simplest automorphisms of $\mathbb{A}^2$ with interesting dynamics are the *Hénon maps*

$$\phi : \mathbb{A}^2 \to \mathbb{A}^2, \quad \phi(x,y) = \big(y, ax + f(y)\big), \text{ with } a \in \mathbb{C}^* \text{ and } f(y) \in \mathbb{C}[y].$$

The dynamics of these maps have been extensively studied ever since Hénon [4] introduced them as examples of systems $\mathbb{R}^2 \to \mathbb{R}^2$ having strange attractors.

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There are still many open questions regarding the geometric dynamics of Hénon maps, see for example [2, §2.9] or [6].

The extent to which a rational map $\bar{\phi} : \mathbb{P}^M \to \mathbb{P}^N$ fails to be a morphism is measured by its *locus of indeterminacy* $Z(\phi)$. Classically, a birational map $\bar{\phi} : \mathbb{P}^N \to \mathbb{P}^N$, i.e., a rational map with a rational inverse, is said to be *regular* if

$$Z(\phi) \cap Z(\phi^{-1}) = \emptyset.$$

A *regular affine automorphism* is an automorphism $\phi : \mathbb{A}^N \to \mathbb{A}^N$ whose natural extension $\bar{\phi} : \mathbb{P}^N \to \mathbb{P}^N$ to $\mathbb{P}^N$ is regular. The geometry and arithmetic of regular affine automorphisms has been the object of considerable study, see for example [8, 13, 16, 19]. More generally, we define a collection of morphisms (polynomial maps)

$$\phi_1, \ldots, \phi_r : \mathbb{A}^M \to \mathbb{A}^N$$

to be *jointly regular* if their extensions $\bar{\phi}_1, \ldots, \bar{\phi}_r : \mathbb{P}^M \to \mathbb{P}^N$ satisfy

$$Z(\phi_1) \cap \cdots \cap Z(\phi_r) = \emptyset.$$

Thus with this definition, a birational map $\phi : PP^N \to \mathbb{P}^N$ is regular if the pair $\{\phi, \phi^{-1}\}$ is jointly regular.

Before stating our first result, which gives a height estimate for jointly regular maps, we remind the reader of two definitions. The height of an affine point $P \in \mathbb{A}^N(\bar{\mathbb{Q}})$ is defined using the natural inclusion $\mathbb{A}^N(\bar{\mathbb{Q}}) \subset \mathbb{P}^N(\bar{\mathbb{Q}})$, thus

$$h(P) = h\big([1, x_1, \ldots, x_N]\big) \qquad \text{for } P = (x_1, \ldots, x_N) \in \mathbb{A}^N(\bar{\mathbb{Q}}).$$

Also, the degree of an affine morphism $\phi : \mathbb{A}^M \to \mathbb{A}^N$ is the largest total degree of the monomials that appear in the coordinate functions of ϕ. Equivalently, the degree of ϕ is d if

$$\bar{\phi}^* O_{\mathbb{P}^N}(1) = O_{\mathbb{P}^M}(d).$$

Theorem 1. *Let $\phi_1, \ldots, \phi_r : \mathbb{A}^M \to \mathbb{A}^N$ be a collection of jointly regular morphisms defined over $\bar{\mathbb{Q}}$ and let $d_i = \deg(\phi_i)$. There is a constant $C = C(\phi_1, \ldots, \phi_r)$ so that for all $P \in \mathbb{A}^N(\bar{\mathbb{Q}})$,*

$$\frac{1}{d_1} h\big(\phi_1(P)\big) + \frac{1}{d_2} h\big(\phi_2(P)\big) + \cdots + \frac{1}{d_r} h\big(\phi_r(P)\big) \ge h(P) - C. \tag{1}$$

An immediate corollary is an analogous estimate for regular affine automorphisms.

Corollary 2. *Let $\phi : \mathbb{A}^N \to \mathbb{A}^N$ be a regular affine automorphism defined over $\bar{\mathbb{Q}}$, i.e., ϕ is a polynomial map with a polynomial inverse such that ϕ and ϕ^{-1} are jointly regular. Let $d_1 = \deg(\phi)$ and $d_2 = \deg(\phi^{-1})$. Then for all $P \in \mathbb{A}^N(\bar{\mathbb{Q}})$,*

$$\frac{1}{d_1} h\big(\phi(P)\big) + \frac{1}{d_2} h\big(\phi^{-1}(P)\big) \ge h(P) - C(\phi).$$

Kawaguchi [8] has suggested an improvement in the lower bound in Corollary 2 and has given a proof in dimension 2. Although the improvement may appear to be minor, it has important consequences for the construction of canonical height functions. (See also [9, 10].)

Conjecture 3. *Let $\phi : \mathbb{A}^N \to \mathbb{A}^N$ be a regular affine automorphism defined over $\bar{\mathbb{Q}}$. Let $d_1 = \deg(\phi)$ and $d_2 = \deg(\phi^{-1})$. Then for all $P \in \mathbb{A}^N(\bar{\mathbb{Q}})$,*

$$\frac{1}{d_1}h\bigl(\phi(P)\bigr) + \frac{1}{d_2}h\bigl(\phi^{-1}(P)\bigr) \ge \left(1 + \frac{1}{d_1 d_2}\right) h(P) - C(\phi).$$

Under suitable conditions on the degrees of the maps, the height estimate given in Theorem 1 implies that the set of points with finite orbits form a set of bounded height. This application is our second main result.

Theorem 4. *Let $\phi_1, \ldots, \phi_r : \mathbb{A}^N \to \mathbb{A}^N$ be a collection of jointly regular morphisms defined over $\bar{\mathbb{Q}}$, and let Φ be the monoid of maps generated by $\phi_1, \ldots, \phi_r$ under composition. For any point $P \in \mathbb{A}^N$, let $\Phi(P) = \bigl\{\psi(P) : \psi \in \Phi\bigr\}$ be the full orbit of P under the maps $\phi_1, \ldots, \phi_r$, and let*

$$\operatorname{PrePer}\bigl(\Phi, \mathbb{A}^N(\bar{\mathbb{Q}})\bigr) = \bigl\{P \in \mathbb{A}^N(\bar{\mathbb{Q}}) : \Phi(P) \text{ is finite}\bigr\}$$

be the set of (strongly) preperiodic points for Φ.

Assume that the degrees $d_i = \deg(\phi_i)$ of the maps satisfy

$$\delta := \frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_r} < 1. \tag{2}$$

Then

$$\operatorname{PrePer}\bigl(\Phi, \mathbb{A}^N(\bar{\mathbb{Q}})\bigr) \text{ is a set of bounded height.}$$

We briefly describe the history of estimates of the sort given in Corollary 2 and Conjecture 3, which in turn imply boundedness as in Theorem 4. The first result of this type was a proof of Conjecture 3 by the author [20] for Hénon maps of degree 2 in dimension 2. The proof involves an explicit construction of a blowup of $\mathbb{P}^2$ so that ϕ and ϕ^{-1} extend to morphisms. Denis [1] found an ingenious way to prove a weaker height estimate without using blowups that still suffices to prove Theorem 4. Denis's argument applies in particular to all regular affine automorphisms in dimension 2. Marcello [12, 13, 16] (see also [14, 15, 17]) extended Denis's work and proved a height bound of the form

$$h\bigl(\phi(P)\bigr) + h\bigl(\phi^{-1}(P)\bigr) \ge \min\{d_1, d_2\}h(P) - C(\phi)$$

for regular affine automorphisms of arbitrary dimension. Although again not quite as strong as Corollary 2, it is strong enough to prove Theorem 4 for such maps.

Kawaguchi [8] returns to the idea of resolving rational maps via blowups and gives two proofs of Conjecture 3 for all regular affine automorphisms in dimension 2. The first uses an ingeneious intersection theory argument and the second uses an explicit resolution of Hénon maps by Hubbard, Papadopol and Veselov [7].

The present paper was motivated by an (unsuccessful) attempt to prove Conjecture 3 for regular affine automorphisms of arbitrary dimension. Thus aside from its intrinsic interest, the author hopes that Theorem 1 may help to clarify the extent to which any proof of the conjecture must make use of both the invertibility of ϕ and the joint regularity of ϕ and ϕ^{-1}.

1. A height bound for jointly regular morphisms

In this section we recall some basic definitons, set some notation, and then prove Theorem 1.

For a rational map of projective varieties $\bar{\phi} : V \to W$, let

$$Z(\bar{\phi}) = \text{the } \textit{locus of indeterminacy of } \phi.$$

See [3, Example 7.17.3] for the precise definition. Later we will give an explicit description for the maps that we are studying.

Definition 1. Let $\bar{\phi}_i : V \to W_i$ for $1 \le i \le r$ be a collection of rational maps between projective varieties. We say that the set $\{\phi_i\}_{1\le i\le r}$ is *jointly regular* if

$$Z(\bar{\phi}_1) \cap Z(\bar{\phi}_2) \cap \cdots \cap Z(\bar{\phi}_r) = \emptyset.$$

Let $\phi : \mathbb{A}^M \to \mathbb{A}^N$ be a morphism, say given by polynomials

$$\phi = [F_1, F_2, \ldots, F_N] \qquad \text{with } F_1, \ldots, F_N \in K[X_1, \ldots, X_M].$$

The *degree of* ϕ is the maximal degree of the monomials appearing in polynomials $F_1, \ldots, F_N$. If $\deg(\phi) = d$, we let

$$\bar{F}_i(X_0, X_1, \ldots, X_M) = X_0^d F_i\left(\frac{X_1}{X_0}, \frac{X_2}{X_0}, \ldots, \frac{X_M}{X_0}\right) \quad \text{for } i = 1, 2, \ldots, N$$

and define the *homogeniziation of* ϕ to be the map

$$\bar{\phi} = [\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_N].$$

Then $\bar{\phi}$ is a rational map $\bar{\phi} : \mathbb{P}^M \to \mathbb{P}^N$ extending the affine morphism $\phi : \mathbb{A}^M \to \mathbb{A}^N$.

With this notation, we define

$$\bar{F}_i^*(X_1, \ldots, X_M) = \bar{F}_i(0, X_1, \ldots, X_M).$$

Thus $\bar{F}_i^*$ is the degree d part of $\bar{F}_i$. (The value of d should be clear from the context. Note that d is the degree of ϕ, not necessarily the degree of F_i.) Then the locus of indeterminacy of ϕ is easily seen to be

$$Z(\phi) = \bigl\{P \in \mathbb{P}^M : \bar{F}_1^*(P) = \bar{F}_2^*(P) = \cdots = \bar{F}_N^*(P) = 0\bigr\}.$$

Proof of Theorem 1. Write

$$\phi_i = (F_{i1}, F_{i2}, \ldots, F_{iN}) \qquad \text{for } i = 1, 2, \ldots, r.$$

Let

$$d_i = \deg(\phi_i) \qquad \text{and} \qquad D = d_1 d_2 \ldots d_r.$$

We combine the homogenizations of $\phi_1, \ldots, \phi_N$ into a single map

$$\psi : \mathbb{P}^M \to \mathbb{P}^{rN},$$
$$\psi = \Bigl[X_0^D, \bar{F}_{11}^{D/d_1}, \ldots, \bar{F}_{1N}^{D/d_1}, \bar{F}_{21}^{D/d_2}, \ldots, \bar{F}_{2N}^{D/d_2}, \ldots, \bar{F}_{r1}^{D/d_r}, \ldots, \bar{F}_{rN}^{D/d_r}\Bigr].$$

We claim that ψ is a morphism. Clearly ψ is well defined at any point with $X_0 \neq 0$. And for $X_0 = 0$ we have

$$\bar{F}_{ij}(0, X_1, \ldots, X_M) = \Bigl[0, \bar{F}_{ij}^*(X_1, \ldots, X_M)^{D/d_i}\Bigr]_{\substack{1 \le i \le r \\ 1 \le j \le N}},$$

from which we see that

$$Z(\psi) = Z(\phi_1) \cap Z(\phi_2) \cap \cdots \cap Z(\phi_r) = \emptyset.$$

This proves that ψ is a morphism, and its degree is equal to D, so standard properties of height functions (see [11, Chapter 4] or [5, Theorem B.2.5(b)]) give the estimate

$$h\bigl(\psi(P)\bigr) \ge Dh(P) + O(1) \qquad \text{for all } P \in \mathbb{P}^M(\bar{\mathbb{Q}}). \tag{3}$$

In the other direction, we use that fact that for any point

$$Q = [Z, Y_{11}, \ldots, Y_{1N}, Y_{21}, \ldots, Y_{2N}, \ldots, Y_{r1}, \ldots, Y_{rN}] \in \mathbb{P}^{rN}(\bar{\mathbb{Q}})$$

with $Z \neq 0$ there is an elementary inequality

$$h(Q) \le \sum_{i=1}^{r} h\bigl([Z, Y_{i1}, Y_{i2}, \ldots, Y_{iN}]\bigr). \tag{4}$$

To see this, first divide all of the coordinates by Z, let $U_{ij} = Y_{ij}/Z$, then use the fact that for any absolute value $|\cdot|$,

$$\max\bigl\{1, |U_{11}|, \ldots, |U_{1N}|, |U_{21}|, \ldots, |U_{2N}|, \ldots, |U_{r1}|, \ldots, |U_{rN}|\bigr\}$$
$$\le \prod_{i=1}^{r} \max\bigl\{1, |U_{i1}|, \ldots, |U_{iN}|\bigr\}.$$

Applying (4) with $Q = \psi(P)$ for a point $P \in \mathbb{A}^M(\bar{\mathbb{Q}})$ and using (3) yields

$$\sum_{i=1}^{r} \frac{D}{d_i} h\bigl(\phi_i(P)\bigr) \geq Dh(P) + O(1),$$

which completes the proof of Theorem 1. $\square$

Example 2. Let $\phi_1(x,y) = (x^2, xy)$ and $\phi_2(x,y) = (x, y^2)$. The homogenizations of ϕ_1 and ϕ_2 are

$$\bar{\phi}_1([X_0, X_1, X_2]) = [X_0^2, X_1^2, X_1X_2],$$
$$\bar{\phi}_2([X_0, X_1, X_2]) = [X_0^2, X_0X_1, X_2^2].$$

Notice that $Z(\bar{\phi}_1) = \bigl\{[0,0,1]\bigr\}$ and $Z(\bar{\phi}_2) = \bigl\{[0,1,0]\bigr\}$ are disjoint, so $\{\phi_1, \phi_2\}$ is a jointly regular set of maps. We consider points of the form $P = (0, b) \in \mathbb{A}^2$ with $b \in \mathbb{Z}$. In homogeneous coordinates $P = [1, 0, b]$, so

$$\bar{\phi}_1(P) = [1,0,0] \qquad \text{and} \qquad \bar{\phi}_2(P) = [1, 0, b^2].$$

Hence

$$\frac{1}{2} h\bigl(\phi_1(P)\bigr) + \frac{1}{2} h\bigl(\phi_2(P)\bigr) = \frac{1}{2} h\bigl((0,0)\bigr) + \frac{1}{2} h\bigl((0,b^2)\bigr) = 0 + \log|b| = h(P).$$

This shows that the estimate (1) in Theorem 1 cannot be improved, in the sense that the lower bound cannot be replaced by $(1+\epsilon)h(P) - C$ with $\epsilon > 0$ unless some further restriction is placed on the maps $\phi_1, \ldots, \phi_r$. In Section 3 we continue the discussion of the extent to which Theorem 1 can be strengthened.

2. An application to arithmetic dynamics

As in the previous section, we begin with some definitions and notation, after which we prove Theorem 4.

Fix an integer $r \geq 1$ and let $\phi_1, \ldots, \phi_r : S \to S$ be functions from some set S to itself. For each $k \geq 0$, let W_k be the collection of ordered k-tuples chosen from $\{1, 2, \ldots, r\}$,

$$W_k = \bigl\{(i_1, i_2, \ldots, i_k) : i_j \in \{1, 2, \ldots, r\}\bigr\} = \{1, 2, \ldots, r\}^k,$$

and let

$$W_* = \bigcup_{k \geq 0} W_k.$$

Thus W_* is the collection of words on r symbols.

For any $I = (i_1, i_2, \ldots, i_k) \in W_k$, let ϕ_I denote the corresponding composition of the functions $\phi_1, \ldots, \phi_r$,

$$\phi_I = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_k}.$$

Definition 2. We denote the monoid of maps $S \to S$ generated by $\phi_1, \ldots, \phi_r$ under composition by

$$\Phi = \{\phi_I : I \in W_*\}.$$

Let $P \in S$. The Φ-*orbit of* P is

$$\Phi(P) = \{\psi(P) : \psi \in \Phi\}.$$

The set of (*strongly*) Φ-*preperiodic points of* S is the set

$$\mathrm{PrePer}(\Phi, S) = \{P \in S : \Phi(P) \text{ is finite}\}.$$

We are now ready to prove Theorem 4, which is our principal application of Theorem 1. We recall that Theorem 4 says that if $\phi_1, \ldots, \phi_r$ form a jointly regular collection of affine morphisms whose degrees d_i satisfy

$$\delta := \frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_r} < 1,$$

then the set of strongly preperiodic points of $\phi_1, \ldots, \phi_r$ is a set of bounded height.

Proof of Theorem 4. Theorem 1 tells us that there is a constant C so that

$$h(Q) - C \le \sum_{i=1}^{r} \frac{1}{d_i} h(\phi_i(Q)) \qquad \text{for all } Q \in \mathbb{A}^N(\bar{\mathbb{Q}}). \tag{5}$$

We fix a point $P \in \mathbb{A}^N(\bar{\mathbb{Q}})$. For each map $\psi \in \Phi$, we apply (5) to the point $\psi(P)$ to get

$$0 \le \sum_{i=1}^{r} \frac{1}{d_i} h(\phi_i \psi(P)) - h(\psi(P)) + C. \tag{6}$$

We define a map $\mu : W_* \to \mathbb{R}$ by the following rule:

$$\mu_I = \mu_{(i_1, i_2, \ldots, i_k)} = \frac{1}{d_1^{\#\{j : i_j = 1\}} \cdot d_2^{\#\{j : i_j = 2\}} \cdots d_r^{\#\{j : i_j = r\}}}.$$

Notice that these are the numbers that appear in the multonomial expansion

$$\delta^k = \left(\frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_r}\right)^k = \sum_{I \in W_k} \frac{1}{d_{i_1} d_{i_2} \cdots d_{i_k}} = \sum_{I \in W_k} \mu_I. \tag{7}$$

We now apply (6) with $\psi = \phi_I$, multiply both sides by μ_I, and sum over all k-tuples I with $k \le K$. This yields

$$0 \le \sum_{k=0}^{K} \sum_{I \in W_k} \mu_I \left(\sum_{i=1}^{r} \frac{1}{d_i} h(\phi_i \phi_I(P)) - h(\phi_I(P)) + C \right). \tag{8}$$

We first evaluate the sum over C. Using the multinomial formula (7) for the inner sum and the assumption (2) that $\delta < 1$, we find that

$$\sum_{k=0}^{K} \sum_{I \in W_k} \mu_I = \sum_{k=0}^{K} \left(\frac{1}{d_1} + \frac{1}{d_2} + \dots + \frac{1}{d_r} \right)^k = \sum_{k=0}^{K} \delta^k \le \frac{1}{1-\delta}.$$

If $I = (i_1, \dots, i_k) \in W_k$ is a k-tuple, it is convenient to adopt the notation iI for the $k+1$-tuple $(i, i_1, \dots, i_k)$. Notice that with this notation, the definition of μ tells us that

$$\mu_{iI} = \frac{\mu_I}{d_i} \qquad \text{for all } I \in W_* \text{ and all } 1 \le i \le r. \tag{9}$$

The following calculation shows that most of the terms in (8) telescope to zero:

$$\begin{aligned}
&\left(\sum_{k=0}^{K-1} \sum_{I \in W_k} \mu_I \sum_{i=1}^{r} \frac{1}{d_i} h\bigl(\phi_i \phi_I(P)\bigr) \right) - \left(\sum_{k=1}^{K} \sum_{I \in W_k} \mu_I h\bigl(\phi_I(P)\bigr) \right) \\
&= \left(\sum_{k=0}^{K-1} \sum_{I \in W_k} \sum_{i=1}^{r} \frac{\mu_I}{d_i} h\bigl(\phi_i \phi_I(P)\bigr) \right) - \left(\sum_{k=0}^{K-1} \sum_{I \in W_k} \sum_{i=1}^{r} \mu_{iI} h\bigl(\phi_i \phi_I(P)\bigr) \right) \\
&= \sum_{k=0}^{K-1} \sum_{I \in W_k} \sum_{i=1}^{r} \left(\frac{\mu_I}{d_i} - \mu_{iI} \right) h\bigl(\phi_i \phi_I(P)\bigr) \\
&= 0 \qquad \text{from (9).}
\end{aligned}$$

The remaining terms in (8) are

$$0 \le \left(\sum_{I \in W_K} \mu_I \sum_{i=1}^{r} \frac{1}{d_i} h\bigl(\phi_i \phi_I(P)\bigr) \right) - h(P) + \frac{C}{1-\delta}.$$

Now suppose that $P \in \operatorname{PrePer}\bigl(\Phi, \mathbb{A}^N(\bar{\mathbb{Q}})\bigr)$ is preperiodic. Then the points $\phi_i \phi_I(P)$ take on only finitely many values as i and I vary, so their height is bounded independently of i and I by

$$h\bigl(\Phi(P)\bigr) := \sup_{Q \in \Phi(P)} h(Q).$$

Hence

$$h(P) \le \left(\sum_{I \in W_K} \mu_I \sum_{i=1}^{r} \frac{1}{d_i} \right) h\bigl(\Phi(P)\bigr) + \frac{C}{1-\delta}.$$

As above, we can use the multinomial expansion (7) to calculate the sum,

$$\sum_{I \in W_K} \mu_I \sum_{i=1}^{r} \frac{1}{d_i} = \sum_{I \in W_K} \sum_{i=1}^{r} \mu_{iI} = \sum_{I \in W_{K+1}} \mu_I = \delta^{K+1}.$$

This gives the inequality

$$h(P) \le \delta^{K+1} \cdot h\bigl(\Phi(P)\bigr) + \frac{C}{1-\delta}.$$

By assumption, $h\bigl(\Phi(P)\bigr)$ is finite and $\delta < 1$, so letting $K \to \infty$ shows that $h(P)$ is bounded by a constant that depends only on the maps $\phi_1, \ldots, \phi_r$. □

Applying the theorem to a pair $\{\phi, \phi^{-1}\}$ consisting of a regular affine automorphism and its inverse, we recover Marcello's theorem.

Corollary 5. (Marcello [13, 16]) *Let $\phi : \mathbb{A}^N \to \mathbb{A}^N$ be a regular affine automorphism of degree at least* 2 *defined over $\bar{\mathbb{Q}}$, i.e., ϕ has a polynomial inverse and ϕ and ϕ^{-1} are jointly regular. Then* $\operatorname{Per}\bigl(\phi, \mathbb{A}^N(\bar{\mathbb{Q}})\bigr)$ *is a set of bounded height.*

Proof. This follows immediately from Theorem 4 applied to the jointly regular maps $\{\phi, \phi^{-1}\}$ unless both ϕ and ϕ^{-1} have degree 2. If they do both have degree 2, then we use the fact [19] that ϕ^2 is regular, $\deg(\phi^2) = \deg(\phi)^2$, and $\operatorname{Per}(\phi) = \operatorname{Per}(\phi^2)$ and apply the theorem to ϕ^2. (Although not needed for the proof, we remark that $\deg \phi = \deg \phi^{-1}$ is only possible for even values of N, in which case one must also have $\dim Z(\phi) = \dim Z(\phi^{-1}) = (N-2)/2$. See [19].) □

3. Further questions

Let $\phi_1, \ldots, \phi_r : \mathbb{A}^N \to \mathbb{A}^N$ be morphisms defined over $\bar{\mathbb{Q}}$ with $d_i = \deg(\phi_i)$ as usual. Define the *height expansion factor* of $\phi_1, \ldots, \phi_r$ to be the quantity

$$\kappa(\phi_1, \ldots, \phi_r) = \liminf_{\substack{P \in \mathbb{A}^N(\bar{\mathbb{Q}}) \\ h(P) \to \infty}} \frac{1}{h(P)} \sum_{i=1}^{r} \frac{1}{d_i} h\bigl(\phi_i(P)\bigr).$$

It has the following properties:

- If $\phi_1, \ldots, \phi_r$ are jointly regular, then $\kappa(\phi_1, \ldots, \phi_r) \ge 1$. This is a weak form of Theorem 1.
- There exist jointly regular maps ϕ_1, ϕ_2 so that $\kappa(\phi_1, \phi_2) = 1$. This follows from Example 2.
- Suppose that ϕ_1 is an automorphism (not necessarily regular) and let $\phi_2 = \phi_1^{-1}$. Then Kawaguchi [8, Proposition 4.2] proves that
$$\kappa(\phi_1, \phi_2) \le 1 + (d_1 d_2)^{-1}.$$
- Suppose that ϕ_1 is a regular automorphism and let $\phi_2 = \phi_1^{-1}$. Then Conjecture 3 and Kawaguchi's result imply that
$$\kappa(\phi_1, \phi_2) = 1 + (d_1 d_2)^{-1}.$$

  For $N = 2$, Kawaguchi [8, Theorem 3.6] proves this unconditionally.

Question 1. Are there natural conditions on $\phi_1, \dots, \phi_r$ that imply $\kappa(\phi_1, \dots, \phi_r) > 1$? In particular, is $\kappa > 1$ if

$$\phi_1, \dots, \phi_r : \mathbb{A}^N \longrightarrow \mathbb{A}^N$$

are jointly regular and finite (i.e., everywhere finite-to-one)?

It is also interesting to ask whether there are natural conditions under which the monoid of maps generated by a collection of affine morphisms $\phi_1, \dots, \phi_r$ has only finitely many *complex* preperiodic points. Clearly we need to require that $r \geq 2$, but in addition, the maps must be independent in some sense. For simplicity, we consider the case of maximal independence, as in the following definition.

Definition 3. Let $\phi_1, \dots, \phi_r : S \to S$ be maps from a set to itself and let Φ be the monoid generated by $\phi_1, \dots, \phi_r$ under composition. We say that $\phi_1, \dots, \phi_r$ are *totally independent* if the map

$$W_* \longrightarrow \mathrm{Map}(S, S), \qquad I \longmapsto \phi_I,$$

is injective. In other words, $\phi_1, \dots, \phi_r$ are totally independent if they satisfy no nontrivial relations under composition.

Now let $\phi_1, \dots, \phi_r : \mathbb{A}^N \to \mathbb{A}^N$ be morphisms defined over $\mathbb{C}$ of degree $d_i = \deg(\phi_i)$, and consider the following four properties:

(1) $\phi_1, \dots, \phi_r$ are totally independent.

(2) $\phi_1, \dots, \phi_r$ are jointly regular.

(3) $d_1^{-1} + d_2^{-1} + \cdots + d_r^{-1} < 1$.

(4) $r \geq 2$.

Question 2. Let $\phi_1, \dots, \phi_r : \mathbb{A}^N \to \mathbb{A}^N$ be affine morphisms satisfying properties (1)–(4). Can Φ have nonisolated preperiodic points? Does $\mathrm{PrePer}\big(\Phi, \mathbb{A}^N(\mathbb{C})\big)$ contain only finitely many (isolated) points?

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