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On Borel's Regularization Theorem

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Introduction. Let G be a non compact semi simpel real Lie group with finite center and finitely many connected components. Assume that $\Gamma \subset G$ is an arithmetic subgroup such that $\Gamma\backslash G$ is non compact. If M is a finite dimensional representation of G, then Borel proves that there are natural isomorphisms

$$H^j(\Gamma, M) \longrightarrow H^j(\Omega^*(X, M)^{\Gamma}_{umg})$$

for all j, see [B 1, B 2, B 3, B 4]. Here $H^j(\Gamma, M)$ is the group cohomology of Γ acting on M and $\Omega^*(X, M)^{\Gamma}_{umg}$ is the complex of smooth M–valued Γ–invariant differential forms of uniform moderate growth on the symmetric space X attached to G. For the definition of growth conditions, see 1.1 and 2.4.

The main result of this paper, theorem 3.2, says that the above isomorphy holds if Γ is any discrete subgroup of a reductive Lie group G with finitely many connected components. Moreover M can be any smooth representation of G on a Frechet space.

Borel uses in his proof the Borel–Serre compactification of $\Gamma\backslash X$ and a tricky spectral–sequence argument to go from moderate growth conditions to uniform moderate growth conditions. In this paper we work essentially only on the symmetric space X. Here the uniform moderate growth conditions for the inverse map log of the exponential map exp $\mathfrak{p} \xrightarrow{\sim} X$ are most important, see 2.2 and 2.3. They are used to prove a global version of Poincaré's lemma with growth conditions on X, see 2.4.

There are other contributions to Borel's regularization theorem. J. Franke gives a different proof of Borel's result in the adelic context for reductive algebraic groups, see [F]. U. Bunke and M. Olbrich give an extension of the first version of Borel's result with moderate instead of uniform moderate growth conditions on $\Omega^*(X, M)$ to infinite dimensional coefficients M of moderate growth. Their result also holds for certain non arithmetic subgroups Γ of G, see [B–O].

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§ 1 Growth conditions

We fix our notation and define growth conditions on vectors $v \in V$ of a G–module V by growth conditions for the map $\varepsilon(v) \in C(G,V)$, where $\varepsilon(v)(g) = g^{-1}v, g \in G$.

1.1. Notations. We use standard notation whenever possible.

(i) By G we denote a reductive real Lie group with finitely many connected components. We assume that G is non compact. We fix a maximal compact subgroup K of G and a corresponding Cartan involution θ on G. Let $\mathfrak{g}$ resp. $\mathfrak{k}$ denote the Lie algebra of G resp. K. The Cartan involution on $\mathfrak{g}$ is also denoted by θ. We choose a connected central $\mathbb{R}$–split torus A_G in G. By $X := G/A_GK$ we denote the associated symmetric space. Since K meets all connected components of G the space X is connected. For most of the paper the choice of A_G is irrelevant. In connection with the theory of automorphic forms there is a natural choice of A_G, see § 3.4.

On the symmetric space $Y := G/K$ we fix a G–left–invariant Riemann metric, which comes from a non degenerate K–invariant bilinearform $B(\ ,\ )$ on $\mathfrak{g} \times \mathfrak{g}$. The corresponding distance of $y_1, y_2 \in Y$ is denoted by $d(y_1, y_2)$. Let $y_0 \in Y$ be the point determined by K. If $g \in G$ we define the *norm* $||g||$ of g by

$$||g|| = \exp(d(gy_0, y_0)) .$$

We write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{p}$, where $\mathfrak{a}$ is the Lie algebra of A_G and where $\mathfrak{p}$ is the orthogonal complement with respect to B of $\mathfrak{k} \oplus \mathfrak{a}$ in $\mathfrak{g}$. We identify $\mathfrak{g}/\mathfrak{k} = \mathfrak{a} \oplus \mathfrak{p}$ with the tangent space at y_0 of Y. Let $x_0 \in X$ be the point of X given by A_GK. Then $\mathfrak{p}$ is identified with the tangent space of x_0 of X. Let $\exp : \mathfrak{g} \longrightarrow G$ be the exponential map. If $Z \in \mathfrak{a} \oplus \mathfrak{p}$ then $t \longmapsto \exp(tZ)y_0$ is the geodesic in Y starting at y_0 with velocity Z. If $g = \exp(Z)y_0$ we get $||g|| = \exp||Z||$, where $||Z||$ is the norm of Z with respect to B.

(ii) In this paper a topological vector space V is a vector space over $\mathbb{R}$ or $\mathbb{C}$. The topology of V is Hausdorff, locally convex and it is defined by a family $S(V)$ of semi norms.

If V is a topological vector space and if G is a Lie group we denote by $C^\infty(G,V)$ the vector space of all maps $\varphi : G \longrightarrow V$ such that with respect to local coordinates on G all partial derivatives exist and are continuous. If $D \in \mathfrak{g}$ and $\varphi \in C^\infty(G,V)$ then $D\varphi(g) := \frac{d}{ds}|_{s=0}\varphi(\exp(-sD)g)$ exists for all $g \in G$ and $D\varphi \in C^\infty(G,V)$. If D is an element of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$

then $D\varphi$ is defined and $D\varphi \in C^\infty(G,V)$. Let $| \ |_\alpha \in S(V)$ be a semi norm and let $C \subset G$ be a compact subset. We define the C^∞–topology on $C^\infty(G,V)$ by the family of semi norms q depending on C, p and D where

$$q(\varphi) = \sup_{a\in C} |(D\varphi))(a)|_\alpha, D \in U(\mathfrak{g}) \ .$$

It is well known that in the C^∞–topology $C^\infty(G,V)$ is complete if V is complete.

(iii) Let V be a topological vectorspace with a continuous left G–action i.e. a continuous representation written as $(g,v) \longmapsto gv$. Then we have an injection of G–modules

$$\varepsilon : V \longrightarrow C(G,V)$$

given by $\varepsilon(v)(g) = g^{-1}v, g \in G, v \in V$. Here $C(G,V)$ is the space of continuous maps from G to V and $a \in G$ acts on $\varphi \in C(G,V)$ by $(a\varphi)(g) = \varphi(a^{-1}g), g \in G$. We say that V is a smooth G–module if $\varepsilon(V) \subset C^\infty(G,V)$ and if the C^∞–topology induces the topology on V. We put $V^\infty = \varepsilon^{-1}C^\infty(G,V)$ and give V^∞ the topology induced by the C^∞–topology on $C^\infty(G,V)$. Then V^∞ is a smooth G–module. An evaluation map $\mu : C^\infty(G,V) \longrightarrow V$ is given by $\varphi \longmapsto \varphi(e)$. The map μ is continuous and as topological vector space (without G–action) V^∞ is a direct summand of $C^\infty(G,V)$.

(iv) Let V be a smooth G–module and $\varphi \in C^\infty(G,V)$. We say that φ is of *moderate growth* if for all $D \in U(\mathfrak{g})$ and all $| \ |_\alpha \in S(V)$ there are constants $0 < N = N(\varphi, D, | \ |_\alpha)$ and $C = C(D, \varphi, | \ |_\alpha)$ such that

$$|(D\varphi)(a)|_\alpha \leq ||a||^N C$$

for all $a \in G$. If in this estimate $N(D, \varphi, | \ |_\alpha) = N(\varphi, | \ |_\alpha)$ does not depend on D, then φ is said to be of *uniform moderate growth.*

We denote by $C^{mg}(G,V)$ the subspace of $C^\infty(G,V)$ consisting of elements of moderate growth. On $C^{mg}(G,V)$ we take the subspace topology. Then $C^{mg}(G,V)$ is a smooth G–module. We say that a smooth G–module V is of moderate growth if $\varepsilon(V) \subset C^{mg}(G,V)$. The G–module $C^{mg}(G,V)$ is of moderate growth. In the obvious way we define $C^{umg}(G,V)$ to consist of maps of uniform moderate growth and call V of uniform moderate growth if $\varepsilon(V) \subset C^{umg}(G,V)$.

We recall that a continuous representation V of G is called a *Banach representation* if V is a Banach space. The following is well known, see [C: § 1].

1.2. Lemma. *Let V be a Banach representation. Then V^∞ is of uniform moderate growth.*

Proof. We have to show that $\varepsilon : V^\infty \longrightarrow C^\infty(G,V)$ maps V^∞ to $C^\infty(G,V)^{umg}$. If $v \in V^\infty, g \in G$ then $\varepsilon(v)(g) = g^{-1}v := \pi(g^{-1})v$ where $\pi : G \longrightarrow GL(V)$ is the given representation. It is well known that there is a $c > 0$ and an $r > 0$ such that $||\pi(g)|| \leq c||g||^r$ for all $g \in G$, see [W: example p. 282]. Here $||\pi(g)||$ is the operator norm of $\pi(g)$ on V.

If $D \in \mathfrak{g}, v \in V^\infty$ then $D(\varepsilon(v))(g) = \frac{d}{ds}|_{s=0}\varepsilon(v)(\exp(-sD)g) = \\ = \pi(g^{-1})\frac{d}{ds}|_{s=0}\varepsilon(v)\exp(-sD) = \pi(g^{-1})(D\varepsilon(v))(e) = \pi(g^{-1})\frac{d}{dt}|_{t=0}\pi(\exp(tD))v = \pi(g^{-1})(Dv)$. If $D = D_nD_{n-1}\cdot D_1, D_i \in \mathfrak{g}$ we use induction n and apply the estimate of $||\pi(g^{-1})||$. q.e.d.

1.3. The growth conditions of 1.1 depend on the action of $g \in G$ on $\varphi \in C^\infty(G,V)$ by $(g\varphi)(a) = \varphi(g^{-1}a)$ for all $a \in G$. We will consider also the actions of g on φ given by $(g_r\varphi)(a) = \varphi(ag)$, called *action by right translation*, and the *diagonal action* given by $(g_d\varphi)(a) = g\varphi(ag)$. The subspaces of moderate resp. uniform moderate growth of $C^\infty(G,V)$ with respect to these actions are denoted by

$$C^{mg}(G,V)_r, C^{mg}(G,V)_d \text{ resp. } C^{umg}(G,V)_r \text{ resp. } C^{umg}(G,V)_d\,.$$

1.4. Proposition. *We use the Notation of* 1.3. *Then:*

(i) $C^{mg}(G,V)_r, C^{umg}(G,V)_r, C^{mg}(G,V)_d$ and $C^{umg}(G,V)_d$ *are G–submodules of* $C^\infty(G,V)$.
(ii) $C^{mg}(G,V) = C^{mg}(G,V)_r$.
(iii) *If V is of moderate growth, then* $C^{mg}(G,V)_r = C^{mg}(G,V)_d$.
(iv) *If V is of uniform moderate growth, then* $C^{umg}(G,V)_r = C^{umg}(G,V)_d$.

Proof. (i) We consider $\varphi \in C^{mg}(G,V)_r, a, g \in G$ and $D \in \mathfrak{g}$. We have to show that $g\varphi \in C^{mg}(G,V)_r$. Since $(D_r\varphi)(a) := \frac{d}{dt}|_{t=0}\varphi(a\exp tD)$ and since the actions on G by right and left translation commute we get $(D_r(g\varphi))a = (D_r\varphi)(g^{-1}a)$. If $|\ |_\alpha \in S(V)$ we get

$$|D_r(g\varphi)(a)| \leq ||g^{-1}a||^{N(\varphi,D,|\ |_\alpha)}C(\varphi, D, |\ |_\alpha).$$

By induction on n for $D_nD_{n-1}D_1 \in U(\mathfrak{g})$ we see that $C^{mg}(G,V)_r$ and $C^{umg}(G,V)_r$ are G–submodules of $C^\infty(G,V)$.

Assume that $\varphi \in C^{mg}(G,V)_d$. We use the above notation and abbreviate

$$(D_d\varphi)(a) := \frac{d}{dt}|_{t=0}\exp(tD)\varphi(a\exp tD) = \frac{d}{dt}|_{t=0}\exp(tD)\varphi(a) + (D_r\varphi)(a)\ .$$

If $\pi(g) : V \longrightarrow V$ denotes the action of $g \in G$ on V, then $\frac{d}{dt}|_{t=0}\exp(tD)v =: \pi(D)v, v \in V$, where $\pi(D) : V \longrightarrow V$ is linear. Let $|\ |_\alpha \in S(V)$. Then $|\pi(D)\varphi(g^{-1}a)|_\alpha \leq |\varphi(g^{-1}a)|_\beta$ for some $|\ |_\beta \in S(V)$ and all $g, a \in G$. Since φ is of moderate growth we get $|\pi(D)(g\varphi)(a)|_\alpha \leq ||g^{-1}a||^{N(\varphi,D,|\ |_\beta)}C(\varphi, D, |\ |_\beta)$. Since we have estimated $|(D_r(g\varphi))(a)|_\alpha$ at the beginning of the proof, we get by induction $g\varphi \in C^{mg}(G,V)_d$. If $\varphi \in C^{umg}(G,V)_d$ the above proof shows that $g\varphi \in C^{umg}(G,V)_d$. Hence (i) is proved.

To prove (ii) let $\varphi \in C^\infty(G,V), D \in \mathfrak{g}, a \in G$. Then

$$(D\varphi)(a) = \frac{d}{dt}|_{t=0}\varphi(\exp(-Dt)a) = \cdots$$

$$\cdots \tfrac{d}{dt}|_{t=0}\varphi(a\exp(-tAd(a^{-1})D)) = ((-Ad(a^{-1})(D))_r\varphi)(a).$$

Let $Y^1, \ldots, Y^t$ be an $\mathbb{R}$–basis of $\mathfrak{g}$. Then there are $\varphi_i \in C^{umg}(G,\mathbb{R})$ such that $-Ad(a^{-1})(D) = \sum_{i=1}^t \varphi_i(a)Y^i$. Here we use 1.2. Therefore

$$(D\varphi)(a) = \sum_{i=1}^t \varphi_i(a)(Y^i_r\varphi)(a).$$

Let $D = D_nD_{n-1}\ldots D_1 \in U(\mathfrak{g})$. By induction n we see that $\varphi \in C^{mg}(G,V)$ if $\varphi \in C^{mg}(G,V)_r$. The other inclusion follows in the same way.

To prove (iii) and (iv) we use the notation as in the proof of (i). If $\varphi \in C^\infty(G,V)$ then $(D_d\varphi)(a) = \pi(D)\varphi(a) + (D_r\varphi)(a)$. Since V is of uniform moderate growth or of moderate growth we can argue as in (i) and see that $\varphi \in C^{umg}(G,V)_d$ iff $\varphi \in C^{umg}(G,V)_r$ and $\varphi \in C^{mg}(G,V)_d$ iff $\varphi \in C^{mg}(G,V)_r$. q.e.d.

1.5. Remark. Let M be a Frechet space defined by continuous seminorms $S(V)$. Assume that G acts continuously on M. Then Casselman calls M a G–module of *moderate growth* if for all $|\,|_\alpha \in S(V)$ there is an $N > 0$ and a semi norm $|\,|_\beta \in S(V)$ such that

$$|gv|_\alpha \leq ||g||^N |v|_\beta \quad \textit{for all} \quad v \in V \quad \textit{and all} \quad g \in G$$

see [C: § 1]. For $M = C^\infty(G, V)$ the growth condition of this paper of course differs from Casselman's. He shows that a given finitely generated $(\mathfrak{g}, K)$–module occurs as module of K–finite vectors of a representation of moderate growth (in his sense), see [C § 1].

§ 2 Standard complexes with growth condition

The relative Lie–Algebra cohomology $H^j(\mathfrak{g}, A_G K, V)$ of a smooth G–module V is computed as j–th homology of the G–invariants of the complex $(\Omega^*(X, V), d)$ of smooth V–valued differential forms with exterior differentials d. Here

$$0 \longrightarrow V \xrightarrow{\varepsilon} \Omega^0(X, V) \longrightarrow \ldots \xrightarrow{d} \Omega^i(X, V) \xrightarrow{d}$$

is exact. We define G–modules $\Omega^i(X, V)_d^{umg}$ consisting of V–valued i–forms with uniform moderate growth conditions and show that the subcomplex $0 \longrightarrow V \xrightarrow{\varepsilon} \Omega^*(X, V)_d^{umg}$ is exact.

2.1. Preliminaries

We use the standard notation of differential geometry on $X = G/A_G K$ and on G as in [He]. In particular if $a \in G$ then $L_a(x) = ax$, for $x \in X$ with differential $dL_a : T_x X \longrightarrow T_{ax} X$, where $T_z X$ is the tangent space of X at z. Sometimes we write $L_{a*} = dL_a$. We use the same notation for the corresponding left translations on G. The natural projection $G \longrightarrow X$ is denoted by π. The tangent space at e of G is identified with the Lie algebra $\mathfrak{g}$ and the tangent space $T_{x_0}(X)$ at X at the point x_0 given by $A_G K$ is identified with $\mathfrak{p}$.

(i) Let $v \in \mathfrak{p}$ and $u \in A_G K$. Then u acts on $\mathfrak{p}$ by the adjoint action and we write $uv = Ad(u)(v), v \in \mathfrak{p}$. We have a right $A_G K$–action on $G \times \mathfrak{p} \ni (g, v)$ by $(g, v)u = (gu, u^{-1}v)$. The quotient $G \times \mathfrak{p}/A_G K$ is a vector bundle over X and the bundle is denoted by $G \prod_{A_G K} \mathfrak{p}$. A smooth section Y of $G \prod_{A_G K} \mathfrak{p}$ over X is a smooth map $Y : G \longrightarrow \mathfrak{p}$ such that $Y(gu) = u^{-1} Y(g)$. We put $C^\infty_{A_G K}(G, \mathfrak{p}) = \{Y \in C^\infty(G, \mathfrak{p}) / Y(gu) = u^{-1} Y(g)\}$ and observe that $C^\infty_{A_G K}(G, \mathfrak{p})$ is a sub G–module of $C^\infty(G, \mathfrak{p})$. In particular $C^{mg}_{A_G K}(G, \mathfrak{p}) := C^\infty_{A_G K}(G, \mathfrak{p}) \cap C^{mg}(G, \mathfrak{p})$ makes sense. We use the corresponding notation if mg is replaced by umg .

(ii) The map $\pi : G \longrightarrow X$ induces an identification of G–modules

$$C^\infty(X,\mathbb{R}) \xrightarrow{\sim} C^\infty_{A_G K}(G,\mathbb{R})\,.$$

We write the Cartan decomposition of $g \in G$ as $g = \exp(D(g))u(g)$ for $u(g) \in A_G K$ and $D(g) \in \mathfrak{p}$. If $Y \in \mathfrak{p}$ we define $Y(g) := u(g)^{-1}Y \in \mathfrak{p}$, i.e. we consider Y as element of $C^\infty_{A_G K}(G,\mathfrak{p})$.

Let $\tilde{Y} \in \mathcal{X}(X)$ be a smooth vectorfield on X. If $x = gx_0 \in X, g \in G$ we put $Y(g) := d(L_{g^{-1}})(\tilde{Y}|_x) \in T_e(X) = \mathfrak{p}$. We see that $Y \in C^\infty_{A_G K}(G,\mathfrak{p})$ and get a natural isomorphism of G–modules $\mathcal{X}(X) \xrightarrow{\sim} C^\infty_{A_G K}(G,\mathfrak{p})$. If $Y \in C^\infty_{A_G K}(G,\mathfrak{p})$ and $gx_0 \in X$ we define $\theta_Y(x,t) := g\exp(tY(g))x_0$. Then $\theta_Y(x,t)$ is well defined and θ_Y is a global flow on X with corresponding vectorfield $\tilde{Y}$. In particular $(Y\varphi)(g)$ is defined by $\frac{d}{dt}|_{t=0}\varphi(g\exp(tY(g))x_0)$ for $\varphi \in C^\infty_{A_G K}(G,\mathbb{R})$ and $(Y\varphi)(g) = (\tilde{Y}\varphi)(gx_0)$.

(iii) Let $\exp : \mathfrak{p} \longrightarrow G$ be the exponential map. Then $\exp$ induces a diffeomorphism $\mathfrak{p} \xrightarrow{\sim} X$, where $v \in \mathfrak{p}$ is mapped to $\exp(v)x_0$. The inverse of this diffeomorphism is denoted by $\log : X \xrightarrow{\sim} \mathfrak{p}$. If $u \in A_G K$ then $u\exp(v)x_0 = \exp(Ad(u)v)ux_0 = \exp(Ad(u)(v))x_0$. Hence $\log(ux) = Ad(u)\log(x) = u\log(x)$. Let $g \in G$. We put $\log(g) = -\log(g^{-1}x_0)$. To motivate the signs we observe that for $g = \exp(v), v \in \mathfrak{p}$ we get $g^{-1}x_0 = \exp(-v)x_0$ and $-\log(g^{-1}x_0) = v$. We see $\log(gu) = u^{-1}\log(g)$. Hence $\log \in C^\infty_{A_G K}(G,\mathfrak{p})$ determines a vectorfield $\widetilde{\log} \in \mathcal{X}(X)$ by $\widetilde{\log}|_x = dL_g(\log(g))$, for $x = gx_0$. We notice that $\widetilde{\log}|_x = dL_{\exp(\log(x))}(\log(x))$, where $\log(x) \in \mathfrak{p}$.

(iv) Let V be a smooth G–module. The i–th exterior power of $Ad(u), u \in A_G K$ acts on $Y \in \wedge^i\mathfrak{p}$ by $uY := \wedge^i Ad(u)$. Then u acts on $\varphi \in \mathrm{Hom}_{\mathbb{R}}(\wedge^i\mathfrak{p},V)$ by $(u\varphi)(Y) = u\varphi(u^{-1}Y)$. We get right action of u an $G \times \mathrm{Hom}(\wedge^i\mathfrak{p},V) \ni (g,\varphi)$ by $(g,\varphi)u = (gu,u^{-1}\varphi)$. Hence the vector bundle $(G \times \mathrm{Hom}(\wedge^i\mathfrak{p},V))/A_G K = G\prod_{A_G K}\mathrm{Hom}_{\mathbb{R}}(\wedge^i\mathfrak{p},V)$ over X is defined with smooth sections $C^\infty_{A_G K}(G,\mathrm{Hom}(\wedge^i\mathfrak{p},V)) =: \Omega^i(X,V)$.

(v) Let $s \in C^\infty_{A_G K}(G,V)$ For $a \in G$ define $\phi(s)(a) = a(s(a))$. Then ϕ induces an isomorphism $\phi : C^\infty_{A_G K}(G,V) \xrightarrow{\sim} C^\infty(X,V)$. If $Y \in C^\infty_{A_G K}(G,\mathfrak{p})$ we view Y as vectorfield $\tilde{Y}$ on X and define a connection $\bigtriangledown$ by $\bigtriangledown_Y s := \tilde{Y}(\phi(s))$. Then

$$(\bigtriangledown_Y s)(g) = \frac{d}{dt}|_{t=0}g\exp(t(Y(g)))s(g\exp(tY(g)))$$

and $\bigtriangledown$ is a flat connection on the bundle $G\prod_{A_GK}V$. We write $\tilde{Y}s$ instead of $\bigtriangledown_Y s$. For the exterior differentials $d := d_{\bigtriangledown}$ we get $d \circ d = 0$. Hence $(\Omega^i(X,V), d)$ is a complex. We consider the exterior differentials as maps $d : C^\infty_{A_GK}(G, \mathrm{Hom}(\wedge^i \mathfrak{p}, V)) \longrightarrow C^\infty_{A_GK}(G, \mathrm{Hom}(\wedge^{i+1}\mathfrak{p}, V))$ where for $Y_0, Y_1, \ldots, Y_i \in \mathfrak{p}$ we have

$$(d\omega)(Y_0, \ldots, Y_i)(g) = \sum_{j=0}^{i} (-1)^j \tilde{Y}_j(\omega(Y_1, \ldots, \hat{Y}_j, \ldots, Y_i))(g) \, .$$

Here we use (ii) and that $[\tilde{Y}_i, \tilde{Y}_j] = 0$, since $(\tilde{Y}_i(\tilde{Y}_j\varphi))(x) = \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}\varphi(exp(D(g))\exp(tY_i)\exp(sY_j)x_0)$ and $[Y_i, Y_j] \in \mathfrak{k}, \varphi \in C^\infty_{A_GK}(G, \mathbb{R})$, $x = gx_0, g \in G$. Then $C^\infty_{A_GK}(G, \mathrm{Hom}(\wedge^i\mathfrak{p}, V))$ is a $(\mathfrak{g}, A_GK)$–module, where the $\mathfrak{g}$–action comes from the diagonal G–action on $C^\infty(G, \mathrm{Hom}(\wedge^i\mathfrak{p}, V))$. Moreover d commutes with the action of $g \in G$ given by left translation on G.

(vi) Let $d : \Omega^i(X,V) \longrightarrow \Omega^{i+1}(X,V)$ be the exterior differential. Then Poincaré's Lemma follows from the existence of linear maps $e^{j+1} : \Omega^{j+1}(X,V) \longrightarrow \Omega^j(X,V)$ such that

$$de^j + e^{j+1}d = Id|\Omega^j(X,V) \, .$$

We need a coordinate free description of the e^j considered as maps

$$e^j : C^\infty_{A_GK}(G, \mathrm{Hom}(\wedge^j\mathfrak{p}, V)) \longrightarrow C^\infty_{A_GK}(G, \mathrm{Hom}(\wedge^{j-1}\mathfrak{p}, V)).$$

For this we recall the construction of e^j, see [G – H – V].

Put $I = [0,1]$ and define a vectorfield ∂ on the manifold $X \times I$ by $\partial|_{(x,s)}f = \frac{d}{dt}_{t=s}\varphi(x,t), f \in C^\infty(X \times I, \mathbb{R})$. Let $h : X \times I \longrightarrow X$ be the retraction to x_0 given by $h(x,s) := \exp(s\log(x))x_0$ Define $j_s : X \longrightarrow X \times I$ by $j_s(x) = (x,s), x \in X, s \in I$. If $\omega \in \Omega^i(X,V)$ then $j_s^*h^*\omega \in \Omega^i(X,V)$ is for $s \in [0,1]$ a smooth family of differential forms. If $Z_1, \ldots, Z_i$ are vector fields on $X \times I$ and if $\omega \in \Omega^{i+1}(X \times I, V)$ then $i_\partial\eta \in \Omega^i(X \times I, V)$ is defined by $(i_\partial\omega)(Z_1, \ldots, Z_i) = \omega(\partial, Z_1, \ldots, Z_i)$. According to [G–H–V]

$$e^{i+1} : \Omega^{i+1}(X,V) \longrightarrow \Omega^i(X,V)$$

is defined by

$$(e^{i+1}\omega)(Z_1, \ldots, Z_i) := \int_0^1 j_s^*(i_\partial(h^*\omega))\,(Z_1, \ldots, Z_i)ds$$

where Z_k are vector fields on X for $k = 1, 2, \ldots, i$.

We have $j_s^*(i_\partial h^*\eta)(Z_1, \ldots, Z_j)(x) = \eta(h_*\partial, h_* j_{s*} Z_1, \ldots, h_* j_{s*} Z_j)(h(x,s))$. If $\varphi \in C^\infty(X, \mathbb{R})$ then $h_*(\partial|_{x,s})\varphi = \partial_{(x,s)}\varphi \circ h = \frac{d}{dt}|_{t=s}\varphi(\exp(s\log(x))) = \frac{d}{dt}|_{t=0}\varphi(\exp\ (s\log(x)) \cdot \exp(t\log(x)))$. Hence

$$h_*(\partial|(x,s)) = (L_{\exp(s\log(x))})_* \log(x) = \widetilde{\log}|_{h(x,s)}.$$

If Z is a vectorfield on X and $\varphi \in C^\infty(X, \mathbb{R})$ we get $(h_* j_{s^*} Z|_x)\varphi = \frac{d}{dt}|_{t=0}\varphi(\exp(s\log(g\exp(tZ(g))x_0))x_0)$. Put $c(t) := s\log(g\exp(tZ(g))x_0)$. Then $\frac{d}{dt}|_{t=0}c(t) = \frac{d}{dt}|_{t=0}\log(g\exp(stZ(g))x_0)$. Here we identify as usual all tangent spaces to $\mathfrak{p}$ with $\mathfrak{p}$. Hence $h_* j_{s*}(Z|_x) = sZ|_{h(x,s)}$.

We put $h(g,t) := \exp(t\log(g))$ and get: If $\omega \in C^\infty_{A_G K}(G, \mathrm{Hom}(\wedge^{i+1}\mathfrak{p}, V))$ and if

$$(e^{i+1}\omega)(g) := \int_0^1 t^i (i_{\log}\omega)(h(g,t))dt$$

then $de^i + e^{i+1}d = Id$ on $C^\infty_{A_G K}(G, \mathrm{Hom}(\wedge^i\mathfrak{p}, V))$. Here $e^0 : C^\infty_{A_G K}(G, V) \longrightarrow V$ is defined by $e^\circ(f) = f(e)$ and $\log \in C^\infty_{A_G K}(G, V)$.

2.2. Proposition. *The vector field* $\log \in C^\infty_{A_G K}(G, \mathfrak{p})$ *is of uniform moderate growth.*

Proof. We will need some standard results on the structure of G. In order to fix the notation we recall them briefly. A reference is for example [He: VI 3.6, X 1.16].

Let θ be the Cartan involution corresponding to K on G. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{p}$ and $\theta|\mathfrak{p} + \mathfrak{a} = -Id$, $\theta|\mathfrak{k} = Id$. We choose a maximal abelian subalgebra $\mathfrak{y} \subset \mathfrak{p}$ such that all adH, $H \in \mathfrak{y}$ are semi simple. Let $\mathfrak{t}$ be the centralizer of $\mathfrak{y}$ in $\mathfrak{k}$ with respect to the adjoint action. Write $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ and $\mathfrak{g}_\mathbb{C} = k_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$ in obvious notation. Then $\mathfrak{h} := \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C} \oplus \mathfrak{y}_\mathbb{C}$ is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\mathbb{C}$. Denote by $X_\alpha \in \mathfrak{g}_\mathbb{C}$ a root vector corresponding to a root α of $\mathfrak{h}$ and choose a system R_+ of positive roots. We have $\alpha(\mathfrak{y}) \subset \mathbb{R}$. We can write $\mathfrak{p}_\mathbb{C} = \mathfrak{y}_\mathbb{C} \oplus_{\alpha \in R_+} \mathbb{C}(X_\alpha - \theta(X_\alpha))$, $\mathfrak{k}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in R_+} \mathbb{C}(X_\alpha + \theta X_\alpha)$

We put $A = \exp\mathfrak{y}$ and $A' = \exp\mathfrak{y}'$, $\mathfrak{y}' = \{H \in \mathfrak{y} | \alpha(H) \neq 0,\ \alpha \in R_+\}$. Then A' is open in A. Let T be the centralizer of $\mathfrak{y}$ in K with respect to the adjoint action. Then $LieT = \mathfrak{t}$. We have an injection $m : K/T \times \mathfrak{y}' \longrightarrow X$ with open image $X' \subset X = G/A_GK$ given by $m(k_0T, H_0) = k_0 \exp(H_0)x_0$.

Let $g \in G$. There are $k_1, k_2 \in K, a \in A_G$ and an $H \in \mathfrak{y}$ such that $g = k_1 \exp(H)k_2a$. Then $\log(g) = -\log(g^{-1}x_0) = -\log(k_2^{-1}\exp(-H)k_1^{-1}x_0) = -\log(k_2^{-1}\exp(-H)x_0) = -\log(\exp(Ad(k_2^{-1})(-H))x_0) = ad(k_2^{-1})H$. Since $||g|| \geq \exp||(H)||$ we see $||\log(g)|| \leq \log(||g||)$.

Let $D \in \mathfrak{g}, g \in G$. We compute $(D\log)(g) = \frac{d}{ds}|_{s=0}\log(\exp(-sD)g) = \frac{d}{ds}|_{s=0} - \log(g^{-1}\exp(sD)x_0)$. This depends only on the image of D in $\mathfrak{g}/\mathfrak{a}\oplus\mathfrak{k} = \mathfrak{p}$. Hence we assume $D \in \mathfrak{p}$. We define $\gamma : K/T \times \mathfrak{y}' \longrightarrow \mathfrak{p}$ by $\gamma(k_0T, H_0) = Ad(k_0)(H_0)$. Then $\log \circ m = \gamma$ and on X' we have $(\log)_* = \gamma_* \circ (m_*)^{-1}$. We compute γ_* and m_*^{-1} on X'. The explicite formula for $\log_*|X'$ then will extended by continuity to a formula for $\log_*$ on X.

Let $(U, H) \in \mathfrak{k} \oplus \mathfrak{y}$. Then (U, H) determines by left translation a tangent vector at $(a_0T_0, H) \in K/T \times \mathfrak{y}'$. For $\varphi \in C^\infty((K/T) \times \mathfrak{y}', \mathbb{R})$ it acts by $(U, H)|_{(k_0T,H_0)}\varphi = \frac{d}{ds}|_{s=0}\varphi(k_0\exp(sU)T, H_0 + sH)$. Put $a_0 = \exp H_0$. If $f \in C^\infty(X', \mathbb{R})$ we get $m_*(U, H)|_{(k_oT,H_0)}f = \frac{d}{ds}|_{s=0}f(k_0\exp(sU)\exp(H_0 + sH)x_0) = \frac{d}{ds}|_{s=0}f(k_0a_0\exp(sAd(a_0^{-1})U)\exp(sH)x_0) = \frac{d}{ds}|_{s=0}f(k_0a_0\exp(sAd(a_0^{-1})U)x_0) + \frac{d}{ds}|_{s=0}f(k_0a_0\exp(sH)x_0) = Ad(a_0^{-1})U|_{a_0k_0x_0}f + H|_{k_0a_0x_0}f$. Since $\gamma_*|_{(k_0T,H_0)}(X_\alpha + \theta(X_\alpha)) = -Ad(k_0)(\alpha(H_0)(X_\alpha - \theta(X_\alpha))$, we get $d\log_x|_{k_0a_0x_0}(X_\alpha - \theta(x_\alpha) + H) = \gamma_* \circ m_*^{-1}((X_\alpha - \theta(X_\alpha)) + H)|_{k_0a_0x_0} = Ad(k_0)\left(\frac{-\alpha(H_0)}{e^{-\alpha(H_0)} - e^{+\alpha(H_0)}}(X_\alpha - \theta(X_\alpha)) + H\right) \in \mathfrak{p}$,

where $H_0 = \log(a_0) \in \mathfrak{y}'$. Now $\varphi(x) = \frac{x}{e^x - e^{-x}}$ is defined for all $x \in \mathbb{R}$ and it is together with all its derivatives bounded. By continuity the above formula is valid for all $H_0 \in \mathfrak{y}$. We get

$$d\log|_{k_0a_0x_0}(D) = Ad(k_0)(\psi(a_0)D),$$

where $\psi(a_0) : \mathfrak{p} \longrightarrow \mathfrak{p}$ is linear and can be diagonalized with diagonal coefficients which are together with all their derivatives bounded as functions in a_0. Recall that $\log(k_0a_0x_0) = Ad(k_0)(H_0), \exp(H_0) = a_0$. Hence the above argument can be applied to $Ad(k_0)(\psi(a_0)D)$. If $D_1 \dots D_n \in \mathfrak{g}$ we get inductively

$$||(D_n \cdot \ldots \cdot D_1 \log)(k_0 a_0 x_0)|| \le C_n ||D_n|| \ldots ||D_1||$$

where C_n depends on n and not on $k_0 a_0 x_0$. qed.

We have $\log \in C^\infty_{A_G K}(G, \mathfrak{p}) \subset C^\infty(G, \mathfrak{p})$. On $\varphi \in C^\infty(G, \mathfrak{p})$ we let $a \in G$ act by right–translation i.e. $(a_r \varphi)(g) = \varphi(ga)$ for all $g \in G$.

2.3. Corollary. *Consider* $\log \in C^\infty(G, \mathfrak{p})$. *Then* $\log$ *is of uniform moderate growth with respect to right translation.*

Proof. Let $D \in \mathfrak{g}$ and $g \in G$. Then $(D_r \log)(g) := \frac{d}{ds}|_{s=0} \log(g \exp sD) = \frac{d}{ds}|_{s=0}(-\log(\exp(-sD)g^{-1}x_0))$. Since we can write $g^{-1} = \exp(D_1)x_0, D_1 \in \mathfrak{p}$, it suffices to estimate

$\frac{d}{ds}|_{s=0} \log(\exp(-sD)\exp(D_1)x_0) = \frac{d}{ds}|_{s=0} \log(\exp(e^{ad(-sD)}(D_1))\exp(-sD)x_0) = -[D, D_1]_0 + \frac{d}{ds}|_{s=0} \log(\exp(D_1)\exp(-sD)x_0) =$

$$-[D, D_1]_0 + (D \log)(g^{-1}x_0),$$

where $[D, D_1]_0$ is the $\mathfrak{p}$–component of $[D, D_1] \in \mathfrak{g}$. We use 2.2 and induction to estimate $((D_n \cdot \ldots \cdot D_1)_r \log)(g)$ and get the desired result. qed.

We put $\Omega^i(X, V)^{umg}_d := C^{umg}_{A_G K}(G, \mathrm{Hom}(\wedge^i \mathfrak{p}, V))_d := C^\infty_{A_G K}(G, \mathrm{Hom}(\wedge^i \mathfrak{p}, V)) \cap C^{umg}(G, \mathrm{Hom}(\wedge^i \mathfrak{p}, V))_d$. These are G–modules by 1.4 (iv). Since $\log \in C^\infty_{A_G K}(G, \mathfrak{p})$ is of uniform moderate growth with respect to the action of G by right translation we see that e^{i+1} maps $C^{umg}_{A_G K}(G, \mathrm{Hom}(\wedge^{i+1}, V))_d$ to $C^{umg}_{A_G K}(G, \mathrm{Hom}(\wedge^i \mathfrak{p}, V))_d$. Here we use that integration over a compact path in the Frechet space $C^\infty_{A_G K}(G, \mathrm{Hom}(\wedge^i p, V))$ preserves uniform moderate growth. We observe that $\varepsilon : V \longrightarrow C^\infty_{A_G K}(G, V)$ has image in $C^{umg}_{A_G K}(G, V))_d = \Omega^0(X, V)^{umg}_d$. Therefore we get

2.4. Proposition. *Let* $n = \dim X$.

The sequence of G*–modules*

$$0 \longrightarrow V \xrightarrow{\varepsilon} \Omega^0(X, V)^{umg}_d \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(X, V)^{umg}_d \longrightarrow 0$$

is exact.

§ 3 The main result

We apply Prop. 2.4 to get the main result. It implies Borel's regularization theorem for arithmetic groups and gives an extension of his result to reductive algebraic group, discrete subgroups instead of arithmetic subgroups and infinite dimensional coefficients. We also represent an adelic version of the main result.

3.1. If $\omega \in C^\infty_{A_GK}(G, \mathrm{Hom}(\wedge^i\mathfrak{p}, V)), a \in G, Y \in \wedge^i\mathfrak{p}$ we consider ω as element of $\mathrm{Hom}_{A_GK}(\wedge^i\mathfrak{p}, C^\infty(G,V))$ where $\omega(Y)(a) := \omega(a)(Y)$. If $\Gamma \subset G$ is a discrete subgroup, we see

$$C^\infty_{A_GK}(G, \mathrm{Hom}(\wedge^i\mathfrak{p}, V))^\Gamma = \mathrm{Hom}_{A_GK}(\wedge^i\mathfrak{p}, C^\infty(\Gamma\backslash G, V))\ .$$

If $\varphi \in C^\infty(\Gamma\backslash G, V), a, g \in G$ we put $(g_d\varphi)(a) = g\varphi(ag)$. Then $C^\infty(\Gamma\backslash G, V)$ is a G–module and hence a $(\mathfrak{g}, A_GK)$–module.

We know that $(\mathrm{Hom}_{A_GK}(\wedge^*\mathfrak{p}, C^\infty(\Gamma\backslash G, V)), d)$ is a complex. By definition of the Lie–algebra cohomology the i–th Homology of the complex is denoted by $H^i(\mathfrak{g}, A_GK, C^\infty(\Gamma\backslash G, V))$ and it is well know that $H^i(\mathfrak{g}, A_GK, C^\infty(\Gamma\backslash G, V)) = H^i(\Gamma, V)$. Here $H^i(\Gamma, V)$ is the group cohomology of the action of $\Gamma \subset G$ on V and $\Gamma \cap A_G = \{e\}$. For all this see [B–W].

3.2. Theorem. *Let M be a Frechet space with continuous G–action. Let V be the G–module of smooth vectors of M Assume that $\Gamma \cap A_G = \{e\}$. Then*

$$H^i(\Gamma, V) = H^i(\mathfrak{g}, A_GK, C^{umg}(\Gamma\backslash G, V)_d) \ \text{ for all } \ i \in \mathbb{N}$$

Proof. The cohomology $H^j(\Gamma, V)$ is computed as j–th homology of the Γ–invariants of the complex

$$C^\infty_{A_GK}(G\ \mathrm{Hom}(\wedge^*\mathfrak{p}, V)) = \Omega^*(X, V)$$

By 2.4 the complex $0 \longrightarrow V \longrightarrow \Omega^*(X,V)^{umg}_d$ is exact. If we show that $H^j(\Gamma, \Omega^i(X,V)^{umg}_d) = 0$ for all $j \geq 1$ and all $i \geq 0$ then $H^j(\Gamma, V)$ can be computed from the Γ–invariants of the complex $\Omega^*(X,V)^{umg}_d$, i.e. the theorem holds.

For short we write $M := \mathrm{Hom}(\wedge^i\mathfrak{p}, V)$ and recall that M is a $(\mathfrak{g}, A_GK)$–module. By a switch of variables we have an isomorphism of $(\mathfrak{g}, A_GK)$–modules

$$s : C^\infty(\Gamma\backslash G, C^{umg}_{A_GK}(G, M)_d) \xrightarrow{\sim} C^{umg}_{A_GK}(G, C^\infty(\Gamma\backslash G, M))_d$$

where

$$C^{umg}_{A_GK}(G, C^\infty(\Gamma\backslash G, M))_d = C^{umg}(G, C^\infty(G, M))_d \cap C^\infty_{A_GK}(G, C^\infty(\Gamma\backslash G, M))$$

and where the growth condition is with respect to the diagonal G–action which is given on G by right translation and on $M = \mathrm{Hom}(\wedge^i \mathfrak{p}, V)$ by the G–action on V. We use 2.1 (iv) and see maps:

$$\mathrm{Hom}_{A_GK}(\wedge^{j+1}\mathfrak{p}, C^{umg}_{A_GK}(G, C^\infty(\Gamma\backslash G, M))_d) \cdots$$

$$\cdots \stackrel{e^{j+1}}{\longrightarrow} \mathrm{Hom}_{A_GK}(\wedge^{j}\mathfrak{p}, C^{umg}_{A_GK}(G, C^\infty(\Gamma\backslash G, M))_d) \ .$$

As in the proof of 2.4 we get $H^j(\mathfrak{g}, A_GK, C^{umg}_{A_GK}(G, C^\infty(\Gamma\backslash G, M))_d) = 0$ if $j \geq 1$. Hence the theorem holds. q.e.d.

3.3. Remark. If V is of uniform moderate growth we apply 1.4 (iv) and get $H^i(\Gamma, V) = H^i(\mathfrak{g}, A_GK, C^{umg}(\Gamma\backslash G, V)_r)$. The growth conditions with respect to the right translation on G are the ones Borel uses.

Let $G_0/\mathbb{Q}$ be a connected reductive algebraic group which is defined over $\mathbb{Q}$. Denote by $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ the ring of adeles over $\mathbb{Q}$. This is a locally compact ring. We have $G_0(\mathbb{A}) = G_0(\mathbb{R}) \times G_0(\mathbb{A}_f)$ with the topology induced by the topology on $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. If V is a smooth G–module we put $C^\infty(G_0(\mathbb{A}_f), V) = \{\varphi : G_0(\mathbb{A}_f) \longrightarrow V | \varphi$ is continuous and there is an open and compact subgroup $K_f \subset G_0(\mathbb{A}_f)$ such that $\varphi(ak) = \varphi(a)$ for all $a \in G_0(\mathbb{A}_f)$ and all $k \in K_f\}$. Put $C^\infty(G_0(\mathbb{A}), V) = C^\infty(G_0(\mathbb{R}), C^\infty(G_0(\mathbb{A}_f), V)$. If V is quasi complete, then $C^\infty(G_0(\mathbb{A}), V)$ is quasi complete, see [B–W: Chap. X, 1.3]. In particular a continuous map $c : [0,1] \longrightarrow C^\infty(G_0(\mathbb{A}), V)$ can be integrated. Let $C^\infty(G(\mathbb{A}), V)^{umg}_d = C^{umg}(G_0(\mathbb{R}), C^\infty(G(\mathbb{A}_f, V))_d$. Now $G_0(\mathbb{Q}) \subset G_0(\mathbb{A})$ acts by left translation on $C^\infty(G_0(\mathbb{A}), V)^{umg}_d$ and the fixpoint set of this action is $C^\infty(G_0(\mathbb{Q})\backslash G_0(\mathbb{A}), V)^{umg}_d$. We can view $G_0(\mathbb{Q})$ also as subgroup of $G_0(\mathbb{A}_f)$. Hence $G_0(\mathbb{Q})$ acts on $C^\infty(G_0(\mathbb{A}_f), V)$ by left translation and the j-th group cohomology $H^j(G_0(\mathbb{Q}), C^\infty(G_0(\mathbb{A}_f), V))$ is defined. Let A_G be the connected component of 1 of the real points of the maximal $\mathbb{Q}$–split central torus of G_0. Then $A_G \cap G_0(\mathbb{Q}) = \{e\}$. We get

3.4. Corollary. *With the above notation one has natural isomorphisms*

$$H^j(G_0(\mathbb{Q}), C^\infty(G_0(\mathbb{A}_f), V)) \stackrel{\sim}{\longrightarrow} H^j(\mathfrak{g}, A_GK, C^\infty(G_0(\mathbb{Q})\backslash G_0(\mathbb{A}), V))$$
$$\stackrel{\sim}{\longrightarrow} H^j(\mathfrak{g}, A_GK, C^\infty(G_0(\mathbb{Q})\backslash G_0(\mathbb{A}), V)^{umg}_d).$$

Proof. Let K_f be an open and compact subgroup of $G_0(\mathbb{A}_f)$ Since cohomology commutes with direct limits it and since taking K_f–invariants is an exact functor suffices to prove a corresponding claim, where $G_0(\mathbb{A}_f)$ is replaced by $G_0(\mathbb{A}_f)/K_f$ and $G_0(\mathbb{A})$ is replaced by $G_0(\mathbb{A})/K_f$. By the finiteness of the class number of $G_0/\mathbb{Q}$ there are $a_i \in G_0(\mathbb{A}_f)$ such that $G_0(\mathbb{A}_f) = \bigcup_{i=1}^h G_0(\mathbb{Q})a_iK_f$ as disjoint union. Put $\Gamma_i = G_0(\mathbb{Q}) \cap a_iK_fa_i^{-1}$. Then $C^\infty(G_0(\mathbb{A}_f)/K_f, M) = \oplus_{i=1}^h \mathrm{Ind}_{\Gamma_i}^G(V)$ and by Shapiro's lemma we have $H^j(G_0(\mathbb{Q}), C^\infty(G_0(\mathbb{A}_f)/K_f, M)) = \bigoplus_{i=1}^h H^j(\Gamma_i, V)$. In the same way $C^\infty(G_0(\mathbb{Q})\backslash G_0(\mathbb{A})/K_f, V) = \bigoplus_{i=1}^h C^\infty(\Gamma_i\backslash G_0(\mathbb{R}), V)$. Hence

$$H^j(\mathfrak{g}, A_GK, C^\infty(G_0(\mathbb{Q})\backslash G_0(\mathbb{A})/K_f, V) = \bigoplus_{i=1}^h H^j(\mathfrak{g}, A_GK, C^\infty(\Gamma_i\backslash G_0(\mathbb{R}), V))$$

and the corresponding formula holds if the index umg is attached. We now apply 3.2. and our claim holds. qed.

3.5. Remark. Let M be a finite dimensional representation of G. Borel uses his result to show that the inclusion $\Omega^*_{cusp}(\Gamma\backslash X) \otimes M \longrightarrow \Omega^*(\Gamma\backslash X)^{mg} \otimes M$ induces an inclusion of the cuspidal cohomology $H^*_{cusp}(\Gamma, M)$ to $H^*(\Gamma, M)$. A. Deitmar and J. Hilgert observe that the generalisation of this to an infinite dimensional M of moderate growth is false in general, see [D–H]. Hence it is desirable, to find a cohomologically useful version of Langland's decomposition theorem with infinite dimensional coefficients and growth conditions, see [B–L–S: 2.4].

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