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L^2-Cohomology of Locally Symmetric Spaces, I

Leslie Saper

In memory of Armand Borel

Abstract: Let X be a locally symmetric space associated to a reductive algebraic group G defined over $\mathbb{Q}$. $\mathcal{L}$-modules are a combinatorial analogue of constructible sheaves on the reductive Borel-Serre compactification $\widehat{X}$; they were introduced in [33]. That paper also introduced the micro-support of an $\mathcal{L}$-module, a combinatorial invariant that to a great extent characterizes the cohomology of the associated sheaf. The theory has been successfully applied to solve a number of problems concerning the intersection cohomology and weighted cohomology of $\widehat{X}$ [33], as well as the ordinary cohomology of X [36]. In this paper we extend the theory so that it covers L^2-cohomology. In particular we construct an $\mathcal{L}$-module $\Omega_{(2)}(X, E)$ whose cohomology is the L^2-cohomology $H_{(2)}(X; \mathbb{E})$ and we calculate its micro-support. As an application we obtain a new proof of the conjectures of Borel and Zucker.

Contents

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0. Introduction

The L^2-cohomology $H_{(2)}(X;\mathbb{E})$ of an arithmetic locally symmetric space X plays an important role in geometric analysis and number theory. In early work, such as [3] and [17], the application of L^2-growth conditions was to single out certain classes in ordinary cohomology, while later the focus shifted to an intrinsic notion of L^2-cohomology, as in for instance [4], [14], and [42]. Zucker conjectured [42] that the L^2-cohomology of a Hermitian locally symmetric space X is isomorphic to the middle-perversity intersection cohomology $I_pH(X^*;\mathbb{E})$ of the Baily-Borel-Satake compactification X^*. More precisely, the conjecture stated that there is a quasi-isomorphism $\mathit{\Omega}_{(2)}(X^*;\mathbb{E}) \cong \mathcal{I}_p\mathcal{C}(X^*;\mathbb{E})$ between complexes of sheaves which induces the above isomorphism on global cohomology. Since X^* is a projective algebraic variety defined over a number field, the conjecture is very relevant to Langlands's program and in particular the study of zeta functions. Zucker [42], [44] verified the conjecture in a number of examples. Borel [5] settled the conjecture in the case where X^* had only one singular stratum; the case of two singular strata was proved by Borel and Casselman [8]. The conjecture in general was resolved in the late 1980's by Stern and the author [37] and independently by Looijenga [26].

From the point of view of representation theory, it is natural to consider the situation where X is an equal-rank locally symmetric space, which is a more general condition than being Hermitian, and where X^* is a Satake compactification for which all real boundary components of the underlying symmetric space D are equal-rank. Borel proposed [6, §6.6], [44] that Zucker's conjecture be extended to this case. Soon after [37] appeared, Stern and the author (unpublished) verified that their arguments could be extended to settle Borel's conjecture; this relied partially on a case-by-case analysis.

However for the applications to Langlands's program, one wishes to compute the local contributions to a fixed-point formula for the action of a correspondence on $I_pH(X^*;\mathbb{E})$. This is complicated by the highly singular nature of X^*. Consequently it is desirable to work on a less singular compactification of X such as Zucker's reductive Borel-Serre compactification $\widehat{X}$ [42], which he showed [43] has a quotient map $\pi\colon \widehat{X} \to X^*$. Rapoport [30], [31] and independently Goresky and MacPherson [22] had conjectured that $I_pH(X^*;\mathbb{E}) \cong I_pH(\widehat{X};\mathbb{E})$; more precisely there should be a quasi-isomorphism $R\pi_*\mathcal{I}_p\mathcal{C}(\widehat{X};\mathbb{E}) \cong \mathcal{I}_p\mathcal{C}(X^*;\mathbb{E})$. We note also important related work involving weighted cohomology due to Goresky and MacPherson and their collaborators [19], [20], [23].

Rapoport's conjecture was proved in [33] for the equal-rank setting by using the theory of *$\mathcal{L}$-modules* and their *micro-support*. An $\mathcal{L}$-module $\mathcal{M}$ is a combinatorial model for a constructible complex of sheaves on $\widehat{X}$; the micro-support of an $\mathcal{L}$-module together with its associated *type* are combinatorial invariants that to a great extent characterize the cohomology of the associated sheaf $\mathcal{S}(\mathcal{M})$. The theory is quite general and can be applied to study many other types of cohomology groups associated to X, for example the weighted cohomology of $\widehat{X}$ [33] and the ordinary cohomology of X [36].

Despite the utility of $\mathcal{L}$-modules, they have not yet been used to study L^2-cohomology itself. (Although L^2-cohomology was used as a tool in [33] to prove the vanishing theorem recalled in §6 below, it was not itself the focus of study.) Of course, L^2-cohomology is by now fairly well-understood; besides the above references, we note for example other work of Borel and Casselman [7] and Franke [16]. Still it would be valuable to treat L^2-cohomology and intersection cohomology within the same combinatorial framework. One difficulty that arises is that the original definition of an $\mathcal{L}$-module does not allow for the infinite dimensional local cohomology groups which can arise with L^2-cohomology. More seriously, technical analytic problems arise in trying to represent L^2-cohomology as an $\mathcal{L}$-module.

In this paper we overcome these issues and construct a generalized $\mathcal{L}$-module $\Omega_{(2)}(E)$ whose cohomology is the L^2-cohomology $H_{(2)}(X;\mathbb{E})$. We also calculate the micro-support of $\Omega_{(2)}(E)$. These results apply to any locally symmetric space, without the equal-rank or Hermitian hypothesis. In a sequel to this paper, we will modify $\Omega_{(2)}(E)$ to obtain an $\mathcal{L}$-module whose cohomology is the "reduced" L^2-cohomology isomorphic to the space of L^2-harmonic differential forms and compute its micro-support.

As an application of our micro-support calculation and the techniques of [33] we obtain here a new proof of the conjectures of Borel and Zucker. Elsewhere we will show that a morphism between $\mathcal{L}$-modules which induces an isomorphism on micro-support and its type also induces an isomorphism on global cohomology. Consequently when the micro-support of $\Omega_{(2)}(E)$ is finite-dimensional (which occurs precisely under the condition given by Borel and Casselman [7]) we recover Nair's identification of L^2-cohomology and weighted cohomology [27]. More generally if $(E|_{^0G})^* \cong \overline{E|_{^0G}}$ then we will establish a relation between the reduced L^2-cohomology, the weighted cohomology, and the intersection cohomology of $\widehat{X}$, even beyond the equal-rank situation. (The condition $(E|_{^0G})^* \cong \overline{E|_{^0G}}$ is standard in this context; without it both the L^2-cohomology and the weighted cohomology vanish.) Unlike the situation of the Borel and Zucker conjectures, this will not in general be induced from a local isomorphism on a Satake compactification X^*. We note that the relation between reduced L^2-cohomology and weighted cohomology can likely also be proven using results of Borel and Garland [10], Franke [16], Langlands [25], and Nair [27].

The paper begins in §1 by reviewing the notation that we will use; in particular D will be the symmetric space associated to a reductive algebraic group G defined over $\mathbb{Q}$, and X will be the quotient $\Gamma\backslash D$ for an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$. In §2 we briefly recall the definition of L^2-cohomology and the L^2-cohomology sheaf. We give special attention to the case of a locally symmetric space X with coefficients $\mathbb{E}$ determined by a regular G-module E (that is, where $G \to \mathrm{GL}(E)$ is a morphism of varieties). In §3 we outline the construction of the reductive Borel-Serre compactification $\widehat{X}$ of X; it is a stratified space whose strata are indexed by $\mathcal{P}$, the partially ordered set of Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G. The stratum X_P associated to $P \in \mathcal{P}$ is a locally symmetric space associated to a certain reductive group, namely the Levi quotient $L_P = P/N_P$, where N_P is the unipotent radical of P.

In §4 we recall the notion of special differential forms on X [19]; these are needed in order to associate a sheaf $\mathcal{S}(\mathcal{M})$ to an $\mathcal{L}$-module $\mathcal{M}$. The important fact for us will be that a special differential form on X has a well-defined restriction to a special differential form on any boundary stratum X_P of $\widehat{X}$. The definition of an $\mathcal{L}$-module is recalled in §5. Briefly an $\mathcal{L}$-module $\mathcal{M}$ consists of a collection of graded regular L_P-modules E_P, one for each $P \in \mathcal{P}$, together with connecting morphisms $f_{PQ}\colon H(\mathfrak{n}_P^Q; E_Q) \to E_P[1]$ whenever $P \leq Q$; here $\mathfrak{n}_P^Q$ is the Lie algebra of N_P/N_Q. These data must satisfy a "differential" type condition (33). We also recall the associated sheaf $\mathcal{S}(\mathcal{M})$ on $\widehat{X}$ as well as pullback and pushforward functors for $\mathcal{L}$-modules which are analogues of those for sheaves. In §6 we recall the micro-support of an $\mathcal{L}$-module and state a vanishing theorem proved in [33]. This theorem asserts the vanishing of $H(\widehat{X}; \mathcal{S}(\mathcal{M}))$ in degrees outside a range determined by the micro-support of $\mathcal{M}$ and its type.

The new material of the paper begins in §7. The component E_P of an $\mathcal{L}$-module is actually a complex under the differential f_{PP}; its cohomology represents the local cohomology $H(i_P^! \mathcal{S}(\mathcal{M}))$ with supports along a stratum $i_P \colon X_P \hookrightarrow \widehat{X}$. Since these groups are often infinite dimensional for L^2-cohomology, we need to generalize the notion of an $\mathcal{L}$-module to allow E_P to be a locally regular L_P-module, that is, the tensor product of a regular module and a possibly infinite dimensional vector space on which L_P acts trivially. We introduce such $\mathcal{L}$-modules and their associated sheaves in §7 and verify that the vanishing theorem continues to hold in this context.

The definition of the $\mathcal{L}$-module $\Omega_{(2)}(E)$ is presented in §8. Here is the idea underlying the definition. We may assume by induction that $j_P^* \Omega_{(2)}(E)$ has already been defined, where $j_P \colon U \setminus X_P \hookrightarrow U$ and U is a neighborhood of some stratum X_P. In order to extend the definition to all of U, one must define a complex (E_P, f_{PP}) of locally regular L_P-modules which represents the local L^2-cohomology with supports along X_P, together with a map $\bigoplus_{Q>P} f_{PQ}$ from the link complex $i_P^* j_{P*} j_P^* \Omega_{(2)}(E)$ to (E_P, f_{PP}). Zucker's work [42] provides us with a complex of locally regular L_P-modules whose cohomology is the local L_2-cohomology along X_P (without supports), namely $(\Omega_{(2)}(\bar{A}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty, d_{A_P^G})$; here $\bar{A}_P^G$ is the compactified split component transverse to X_P, h_P is a certain weight function, and we are taking germs of forms at infinity. It is natural to define (E_P, f_{PP}) as the mapping cone (with a degree shift of -1) of an attaching map $\Omega_{(2)}(\bar{A}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty \to i_P^* j_{P*} j_P^* \Omega_{(2)}(X, E)$. However the existence of this attaching map, from forms on A_P^G to forms on smaller split components A_Q^G, is not apparent. To resolve the problem, we replace $i_P^* j_{P*} j_P^* \Omega_{(2)}(E)$ by a quasi-isomorphic complex of forms on A_P^G, with no additional growth conditions in the new directions, before forming the mapping cone.

Having defined the $\mathcal{L}$-module $\Omega_{(2)}(E)$, we calculate in §9 that the associated sheaf $\mathcal{S}(\Omega_{(2)}(E))$ and the L^2-cohomology sheaf $\Omega_{(2)}(\widehat{X}; \mathbb{E})$ have the same local cohomology. However this is not sufficient to establish that they are quasi-isomorphic since we don't yet know the local quasi-isomorphisms are induced by a global map of sheaves. To construct such a global map requires a complex of forms on X for which both (i) there is a subcomplex whose cohomology is L^2-cohomology, and (ii) there is a restriction map to a similar complex on any boundary stratum X_P. Special differential forms have the second property but not the first; smooth forms satisfy the first property but not the second. In §10 we introduce the complex of quasi-special forms and prove it has both desired properties; this is the technical heart of the paper. A form is quasi-special if it is decomposable near any point on the boundary and if the restriction to a boundary stratum (viewed as a form with coefficients in the sheaf of germs of forms in the transverse direction) is (recursively) a quasi-special form. In §11 we use quasi-special forms to prove that $\mathcal{S}(\Omega_{(2)}(E))$ and $\Omega_{(2)}(\widehat{X};\mathbb{E})$ are quasi-isomorphic.

Finally the micro-support of $\Omega_{(2)}(E)$ is calculated in §12 following the analogous calculation for weighted cohomology in [33]. We deduce the conjectures of Borel and Zucker in §13.

I would like to thank Steve Zucker and Rafe Mazzeo for urging me to write up this work. I would also like to thank an anonymous referee for many thoughtful and insightful comments and suggestions. I spoke about these results in July 2004 at the International Conference in Memory of Armand Borel in Hangzhou. The L^2-cohomology of arithmetic locally symmetric spaces was a subject that greatly interested Borel, as evidenced by the many papers he wrote on this subject, particularly during the 1980's. Thus it seems fitting to dedicate this paper to his memory.

1. Notation

1.1. **Algebraic Groups.** For any algebraic group P defined over $\mathbb{Q}$, let $X(P)$ denote the regular or rationally defined characters of P and let $X(P)_{\mathbb{Q}}$ denote the subgroup of characters defined over $\mathbb{Q}$. Set

$$^0P = \bigcap_{\chi \in X(P)_{\mathbb{Q}}} \operatorname{Ker} \chi^2 .$$

The Lie algebra of $P(\mathbb{R})$ will be denoted by $\mathfrak{p}$. Let N_P denote the unipotent radical of P and let $L_P = P/N_P$ be its Levi quotient. The center of P is denoted by $Z(P)$ and the derived group by $\mathscr{D}P$. Let S_P be the maximal $\mathbb{Q}$-split torus in the $Z(L_P)$ and set $A_P = S_P(\mathbb{R})^0$. We will identify $X(S_P)\otimes\mathbb{R}$ with $\mathfrak{a}_P^*$, the dual of the Lie algebra of A_P.

Throughout the paper G will be a connected, reductive algebraic group G defined over $\mathbb{Q}$ and the notation of the previous paragraph will primarily be applied when P is a parabolic $\mathbb{Q}$-subgroup of G, as we now assume. If $P \subseteq Q$ are parabolic $\mathbb{Q}$-subgroups of G, there are natural inclusions $N_P \subseteq N_Q$ and $A_Q \subseteq A_P$. We let $N_P^Q = N_P/N_Q$ denote the unipotent radical of P/N_Q viewed as a parabolic subgroup of L_Q. There is a natural complement A_P^Q to $A_Q \subseteq A_P$ which will be recalled in (7) and hence a decomposition $A_P = A_Q \times A_P^Q$. For $a \in A_P$ we write $a = a_Q a^Q$ according to this decomposition. The same notation will be used for elements of $\mathfrak{a}_P = \mathfrak{a}_Q \oplus \mathfrak{a}_P^Q$ and $\mathfrak{a}_P^* = \mathfrak{a}_Q^* \oplus \mathfrak{a}_P^{Q*}$.

Let $\Delta_P \subseteq X(S_P)$ denote the simple weights of the adjoint action of S_P on the Lie algebra $\mathfrak{n}_{P\mathbb{C}}$ of N_P. (Although this action depends on the choice of a lift $\widetilde{S}_P \subseteq P$, its weights do not.) By abuse of notation we will call these *roots*. If P is minimal, Δ_P are the simple roots for some ordering of the $\mathbb{Q}$-root system of G and we have the *coroots* $\{\alpha^\vee\}_{\alpha\in\Delta_P}$ in $\mathfrak{a}_P$. In general to define the coroot $\alpha^\vee \in \mathfrak{a}_P$ for $\alpha \in \Delta_P$, let $P_0 \subset P$ be a minimal parabolic $\mathbb{Q}$-subgroup and let γ be the unique element of $\Delta_{P_0} \setminus \Delta_{P_0}^P$ such that $\gamma|_{\mathfrak{a}_P} = \alpha$. Following [1] we define $\alpha^\vee$ as the projection of $\gamma^\vee \in \mathfrak{a}_{P_0} = \mathfrak{a}_P \oplus \mathfrak{a}_{P_0}^P$ to $\mathfrak{a}_P$.

For parabolic $\mathbb{Q}$-subgroups $P \subseteq Q$, let $\Delta_P^Q \subseteq \Delta_P$ denote those roots which restrict trivially to A_Q; they form a basis of $\mathfrak{a}_P^{Q*}$. The coroots $\{\alpha^\vee\}_{\alpha\in\Delta_P^Q}$ are a basis of $\mathfrak{a}_P^Q$ and we let $\{\beta_\alpha^Q\}_{\alpha\in\Delta_P^Q}$ denote the corresponding dual basis of $\mathfrak{a}_P^{Q*}$. Likewise let $\{\beta_\alpha^{Q\vee}\}_{\alpha\in\Delta_P^Q}$ denote the basis of $\mathfrak{a}_P^Q$ dual to Δ_P^Q. Let

$$\mathfrak{a}_P^{Q+} = \{\, H \in \mathfrak{a}_P^Q \mid \langle \alpha, H\rangle > 0 \text{ for all } \alpha \in \Delta_P^Q \,\},$$
$$^+\mathfrak{a}_P^{Q} = \{\, H \in \mathfrak{a}_P^Q \mid \langle \beta_\alpha, H\rangle > 0 \text{ for all } \alpha \in \Delta_P^Q \,\}$$

denote the strictly dominant cone and its open dual cone; similarly define $\mathfrak{a}_P^{Q*+}$ and $^+\mathfrak{a}_P^{Q*}$. Set $\mathfrak{a}_P^+ = \mathfrak{a}_G \oplus \mathfrak{a}_P^{G+}$, etc. If P is minimal we may omit it from the notation.

Let $\operatorname{cl}(Y)$ denote the closure of a subspace Y of a topological space. We will often use the standard facts that $\alpha|_{\mathfrak{a}_P^Q} \in -\operatorname{cl}(\mathfrak{a}_P^{Q*+})$ for $\alpha \in \Delta_P \setminus \Delta_P^Q$ and that $\mathfrak{a}_P^{Q*+} \subseteq {}^+\mathfrak{a}_P^{Q*}$.

Let $\rho_P \in X(L_P)_{\mathbb{Q}} \otimes \mathbb{Q}$ denote one-half the character by which L_P acts on $\bigwedge^{\dim \mathfrak{n}_P} \mathfrak{n}_P$; we have $\rho_P \in \mathfrak{a}_P^{*+}$. If $P \subseteq Q$, then $\rho_P|_{\mathfrak{a}_Q} = \rho_Q$. Also define

$$\tau_P^Q = \sum_{\alpha\in\Delta_P^Q} \beta_\alpha^Q \in \mathfrak{a}_P^{Q*+} \qquad \text{and} \qquad \tau_P^{Q\vee} = \sum_{\alpha\in\Delta_P^Q} \beta_\alpha^{Q\vee} \in \mathfrak{a}_P^{Q+}. \tag{1}$$

1.2. **Regular Representations.** By a *regular representation of* G (or a *regular* G*-module*) we mean a finite dimensional complex vector space E together with a morphism $\sigma\colon G \to \mathrm{GL}(E)$ of algebraic varieties. In other words, the representation is rationally defined. Let $\mathfrak{Mod}(G)$ denote the category of regular G-modules.

If E is a regular G-module, let $E|_{^0G}$ denote the corresponding regular ^0G-module; if E is irreducible or more generally isotypical, let $\xi_E \in X(S_G)$ denote the character by which S_G acts on E.

If V is an irreducible regular G-module, let E_V denote the V-isotypical component, that is, $E_V \cong V \times \operatorname{Hom}_G(V, E)$.

1.3. **Homological Algebra.** For an additive category $\mathcal{C}$ we let $\operatorname{Gr}(\mathcal{C})$ denote the category of graded objects of $\mathcal{C}$ and we let $\mathbf{C}(\mathcal{C})$ denote the category of (cochain) complexes of objects of $\mathcal{C}$. If C is an object of $\operatorname{Gr}(\mathcal{C})$ and $k \in \mathbb{Z}$, the *shifted object* $C[k]$ is defined by $C[k]^i = C^{k+i}$. For a complex (C, d_C) in $\mathbf{C}(\mathcal{C})$, define the *shifted* *complex* $(C[k], d_{C[k]})$ by $d_{C[k]} = (-1)^k d_C$. The *mapping cone* $M(f)$ of a morphism $f\colon (C, d_C) \to (D, d_D)$ of complexes is the complex $(C[1] \oplus D, -d_C + d_D + f)$.

Consider a functor F from $\mathcal{C}$ to $\mathbf{C}(\mathcal{C}')$, where $\mathcal{C}'$ is another additive category. For example, F may be the functor $\mathbb{E} \mapsto A(X;\mathbb{E})$ sending a local system $\mathbb{E}$ on a manifold X to the complex of differential forms with coefficients in $\mathbb{E}$. In this case we extend F to a functor $\mathrm{Gr}(\mathcal{C}) \to \mathbf{C}(\mathcal{C}')$ by defining

$$F(E) = \bigoplus_k F(E^k)[-k]. \tag{2}$$

Occasionally we further extend F to a functor $\mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C}')$ by means of the associated total complex.

Remark. In most cases we will make a distinction between a graded object C and a complex (C, d_C) created using C and a morphism $d_C\colon C \to C[1]$, particularly when working with $\mathcal{L}$-modules. This is because often a particular graded object or morphism will enter into the definition of several complexes. However for the complex of differential forms we will simply write $A(X;\mathbb{E})$ instead of $(A(X;\mathbb{E}), d_X)$ and similarly for the corresponding complex of sheaves.

2. L^2-cohomology

2.1. **Definition of L^2-cohomology.** Let $\mathbb{E}$ be a locally constant sheaf on a manifold X, that is, $\mathbb{E}$ is the sheaf of locally flat sections of a flat vector bundle on X which we will also denote $\mathbb{E}$. Let $A(X;\mathbb{E})$ denote the complex of smooth differential forms with coefficients in $\mathbb{E}$; the differential is the exterior derivative $d = d_X$. By de Rham's theorem, the cohomology of $A(X;\mathbb{E})$ represents the topological or sheaf cohomology $H(X;\mathbb{E})$. Assume X has a Riemannian metric and $\mathbb{E}$ has a fiber metric (which may not be locally constant) and for $\omega \in A(X;\mathbb{E})$ define the L^2-norm (which may be infinite) by

$$\|\omega\| = \left(\int_X |\omega|^2 \, dV \right)^{\frac{1}{2}}.$$

Let $A_{(2)}(X;\mathbb{E}) \subseteq A(X;\mathbb{E})$ denote the subcomplex consisting of forms ω such that ω and $d\omega$ are L^2, that is, such that $\|\omega\|, \|d\omega\| < \infty$. The cohomology $H_{(2)}(X;\mathbb{E})$ of $A_{(2)}(X;\mathbb{E})$ is called the *L^2-cohomology of X with coefficients in* $\mathbb{E}$. We also consider the weighted L^2-norm[1] $\|\omega\|_h = \|h\omega\|$ obtained by multiplying the norm on $\mathbb{E}$ by a weight function $h\colon X \to (0,\infty)$. The cohomology of the corresponding complex $A_{(2)}(X;\mathbb{E},h)$ is the *weighted L^2-cohomology* $H_{(2)}(X;\mathbb{E},h)$. If X is noncompact (our case of interest) then $H_{(2)}(X;\mathbb{E})$ and $H_{(2)}(X;\mathbb{E},h)$ are no longer topological invariants of X, but depend on the quasi-isometry class of h and the metrics.

[1]The notation is consistent with [16] whereas in [42] our norm would be associated to the weight function h^2.

All of the above extends to the case of a Riemannian orbifold X and a metrized orbifold locally constant sheaf $\mathbb{E}$. The notion of an orbifold (originally a *V-manifold*) was introduced by Satake [38]; for more details see [15]. We also may allow $\mathbb{E}$ to be graded (by applying (2)).

2.2. **Localization of L^2-cohomology.** Let $\Omega(X;\mathbb{E})$ be the complex of sheaves associated to the presheaf $U \mapsto A(U;\mathbb{E})$. From this point of view, the de Rham isomorphism follows from the facts that $\Omega(X;\mathbb{E})$ is a fine sheaf and the inclusion $\mathbb{E} \to \Omega(X;\mathbb{E})$ is a *quasi-isomorphism* (a morphism which induces an isomorphism on local cohomology sheaves). If we apply the analogous localization to $A_{(2)}(X;\mathbb{E})$, the L^2 growth conditions disappear and we obtain the same sheaf $\Omega(X;\mathbb{E})$. Instead, consider a *partial compactification* $\widehat{X}$ of X; by this we mean a topological space $\widehat{X}$ (not necessarily a manifold) which contains X as a dense subspace. Define the *L^2-cohomology sheaf* $\Omega_{(2)}(\widehat{X};\mathbb{E})$ to be the complex of sheaves associated to the presheaf $U \mapsto A_{(2)}(U \cap X;\mathbb{E})$. If $\widehat{X}$ is compact and $\Omega_{(2)}(\widehat{X};\mathbb{E})$ is fine, then the L^2-cohomology is isomorphic to the hypercohomology of $\Omega_{(2)}(\widehat{X};\mathbb{E})$.

2.3. **L^2-cohomology of Locally Symmetric Spaces.** Let G be a connected reductive algebraic group defined over $\mathbb{Q}$; we will use the notation established in §1.1. Given a maximal compact subgroup K of $G(\mathbb{R})$ we obtain a symmetric space $G(\mathbb{R})/KA_G$. If K and K' are two maximal compact subgroups then $K' = hKh^{-1}$ for some $h \in \mathscr{D}G(\mathbb{R})$ which is unique modulo $K \cap \mathscr{D}G(\mathbb{R})$. We identify $G(\mathbb{R})/K'A_G \xrightarrow{\sim} G(\mathbb{R})/KA_G$ by mapping $gK'A_G \mapsto ghKA_G$; the resulting $G(\mathbb{R})$-homogeneous space is the *symmetric space associated to* G and we denote it D. If $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup we let $X = \Gamma\backslash D$ denote the corresponding *locally symmetric space associated to* G *and* Γ.

Note that the symmetric space D above may have Euclidean factors since the maximal $\mathbb{R}$-split torus ${}_{\mathbb{R}}S_G$ in $Z(G)$ may be strictly larger than S_G. Set ${}_{\mathbb{R}}A_G = {}_{\mathbb{R}}S_G(\mathbb{R})^0$. The choice of a basepoint $x_0 \in D$ is equivalent to the choice of a maximal compact subgroup K and a point $a \in {}_{\mathbb{R}}A_G/A_G$ so that $x_0 = aKA_G$. For simplicity we will only consider basepoints with $a = e$. The choice of a maximal compact subgroup K in turn determines a unique involutive automorphism θ of G (the *Cartan involution*) whose fixed point set in $G(\mathbb{R})$ is K [11, §1.6]. Unless otherwise specified we will not assume that a specific basepoint has been chosen.

A regular representation E of G determines a locally constant sheaf $\mathbb{E} = D \times_\Gamma E$. In general X is an orbifold and $\mathbb{E}$ is an orbifold locally constant sheaf, but we will not mention this explicitly from now on. Note that there always exists neat (in particular, torsion-free) subgroups $\Gamma' \subseteq \Gamma$ with finite index; for such Γ', $\Gamma'\backslash D$ is smooth and $D \times_{\Gamma'} E$ is an honest flat vector bundle.

Let $x_0 \in D$ be a basepoint and let KA_G and θ be the associated stabilizer and Cartan involution. Choose a Hermitian inner product on E such that $\sigma(g)^* = \sigma(\theta g)^{-1}$ for all $g \in G(\mathbb{R})$; such an inner product always exists and is called *admissible for* x_0. If E is irreducible an admissible inner product is uniquely determined up to a positive scalar multiple. The admissible inner product on E determines a fiber metric on $\mathbb{E}$; in the case that E is isotypical this is given explicitly as

$$|(gKA_G, v)|_{\mathbb{E}} = |\xi_E(g)| \cdot |g^{-1}v|_E. \tag{3}$$

(Properly speaking one should write $|\xi_E^k(g)|^{\frac{1}{k}}$ instead of $|\xi_E(g)|$, where $k \in \mathbb{N}$ is such that $\xi_E^k \in X(S_G)$ extends to a character on G, but we make this abuse of notation.) If $x_0' = hx_0$ (where $h \in \mathscr{D}G(\mathbb{R})$) is another basepoint then $v \mapsto |h^{-1}v|_E$ is admissible for x_0'; it induces the same fiber metric on $\mathbb{E}$.

There exists an invariant nondegenerate bilinear form B on the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ such that the Hermitian inner product $\langle X, Y\rangle = B(\overline{X}, \theta Y)$ is positive definite on $\mathfrak{g}_{\mathbb{C}}$. This inner product on $\mathfrak{g}_{\mathbb{C}}$ is admissible for x_0 under the adjoint representation. In addition it induces an inner product on $T_{x_0}D$ and hence a $G(\mathbb{R})$-invariant Riemannian metric on D. We give X the induced Riemannian metric.

We now apply §2.1 to define $A_{(2)}(X;\mathbb{E})$ and $H_{(2)}(X;\mathbb{E})$ in this context. These are well-defined since the choices above yield quasi-isometric metrics.

3. Compactifications

We outline the construction of the Borel-Serre compactification following [11] however we use the principal homogeneous spaces $\mathcal{A}_P^G$ and $\mathcal{N}_P(\mathbb{R})$ introduced in [33] in order to write decompositions independent of a choice of basepoint.

We also recall the reductive Borel-Serre compactification and use it to represent L^2-cohomology as the hypercohomology of a complex of sheaves.

3.1. **Geodesic Action.** Let $x_0 \in D$ be a basepoint with corresponding stabilizer KA_G and Cartan involution θ. For Q a parabolic $\mathbb{Q}$-subgroup of G, there is a unique lift of $L_Q(\mathbb{R})$ to $\widetilde{L}_Q(\mathbb{R}) \subseteq Q(\mathbb{R})$ which is θ-stable; for $z \in L_Q(\mathbb{R})$ let $\tilde{z} \in \widetilde{L}_Q(\mathbb{R})$ denote the corresponding lift. Since $G(\mathbb{R}) = Q(\mathbb{R})K$, any $x \in D$ may be written as qKA_G for some $q = nr \in Q(\mathbb{R}) = N_Q(\mathbb{R})\widetilde{L}_Q(\mathbb{R})$. The *geodesic action* of $z \in L_Q(\mathbb{R})$ on $x \in D$ is defined by

$$z \,\mathbf{o}\, x = n\tilde{z}rKA_G. \tag{4}$$

For $z = a \in A_Q$ this agrees with the definition given in [11, §3.2]; in general see [33, §1.1]. The geodesic action of $L_Q(\mathbb{R})$ is independent of the choice of x_0

and commutes with the action of $N_Q(\mathbb{R})$; the geodesic action of A_Q furthermore commutes with the action of $Q(\mathbb{R})$.

Suppose $P \subseteq Q$ are parabolic $\mathbb{Q}$-subgroups of G. Since P/N_Q is a parabolic subgroup of L_Q, the maximal $\mathbb{Q}$-split torus in $Z(P/N_Q)$ is simply S_Q. Then since P/N_Q projects onto L_P, we may identify S_Q with a subtorus of S_P and A_Q with a subgroup of A_P. The geodesic action of $a \in A_Q$ is the same whether a is viewed in A_Q or in A_P.

3.2. **Geodesic Decompositions.** We may view A_G as a subgroup of A_Q; since A_G acts trivially, the geodesic action of A_Q descends to $A_Q^G = A_Q/A_G$. The quotient $\mathcal{A}_Q^G = {}^0Q(\mathbb{R})\backslash D$ is a principal A_Q^G-homogeneous space under the geodesic action and the geodesic quotient $e_Q = A_Q^G\backslash D$ is a ${}^0Q(\mathbb{R})$-homogeneous space. (A choice of a basepoint in D determines a basepoint in $\mathcal{A}_Q^G$ and hence a unique isomorphism of A_Q^G-spaces $\mathcal{A}_Q^G \cong A_Q^G$ sending the basepoint to the identity.) The projections yield

$$D \cong \mathcal{A}_Q^G \times e_Q, \tag{5}$$

an isomorphism of $(A_Q^G \times {}^0Q(\mathbb{R}))$-homogeneous spaces [11, §3.8]. (This follows from the identity $Q(\mathbb{R}) = \widetilde{A}_Q \times {}^0Q(\mathbb{R})$ for any lift $\widetilde{A}_Q$ of A_Q.) We denote by

$$\operatorname{pr}_Q\colon D \longrightarrow \mathcal{A}_Q^G \qquad \text{and} \qquad \operatorname{pr}^Q\colon D \longrightarrow e_Q \tag{6}$$

the corresponding projections; the latter is called *geodesic retraction*. We will propagate this notation and terminology to the induced decompositions of various quotients and compactifications of D to be considered below, for example (8), (11), and (17).

For $P \subseteq Q$ note that the geodesic action of A_P on D descends to an action on e_Q. We now define a subgroup $A_P^Q \subseteq A_P$ which is complementary to $A_Q \subseteq A_P$ and acts freely on e_Q. Note there is an injection $X(Q)_\mathbb{Q} = X(L_Q)_\mathbb{Q} \hookrightarrow X(P/N_Q)_\mathbb{Q} = X(L_P)_\mathbb{Q} \hookrightarrow X(S_P)$, $\chi \mapsto \chi_P$. Then set

$$S_P^Q = \Big(\bigcap_{\chi \in X(Q)_\mathbb{Q}} \operatorname{Ker} \chi_P \Big)^0 \subseteq S_P$$

and define $A_P^Q = S_P^Q(\mathbb{R})^0$. There is a direct product decomposition [44, 1.3(15)][2]

$$A_P = A_Q \times A_P^Q \tag{7}$$

and (5) is an isomorphism of $(A_Q^G \times A_P^Q)$-homogeneous spaces.

[2]Note A_P^Q is not equal in general to the subgroup $A_{P,Q}$ defined in [11] and that the decomposition (7) is different from $A_P = A_Q \times A_{P,Q}$ of [11, 4.3(3)].

The quotient of (5) by ${}^0P(\mathbb{R})$ yields an isomorphism

$$\mathcal{A}_P^G \cong \mathcal{A}_Q^G \times \mathcal{A}_P^Q \tag{8}$$

of $(A_Q^G \times A_P^Q)$-homogeneous spaces, where $\mathcal{A}_P^Q$ is defined as ${}^0P(\mathbb{R})\backslash e_Q = A_Q^G\backslash\mathcal{A}_P^G$. The quotient of $D \cong \mathcal{A}_P^G \times e_P$ by A_Q^G yields

$$e_Q \cong \mathcal{A}_P^Q \times e_P. \tag{9}$$

3.3. **Partial Compactifications.** There is an isomorphism

$$A_Q^G \cong (\mathbb{R}^{>0})^{\Delta_Q}, \qquad a \longmapsto (a^\alpha)_{\alpha\in\Delta_Q},$$

and we partially compactify by allowing these root coordinates to attain infinity,

$$\bar{A}_Q^G \cong (\mathbb{R}^{>0} \cup \{\infty\})^{\Delta_Q}.$$

For all $R \geq Q$, let $o_R \in \bar{A}_Q^G$ denote the point defined by

$$o_R^\alpha = \begin{cases} \infty & \text{for } \alpha \in \Delta_Q \setminus \Delta_Q^R, \\ 1 & \text{for } \alpha \in \Delta_Q^R. \end{cases}$$

Then there is a stratification

$$\bar{A}_Q^G = \coprod_{R\geq Q} A_Q^G \cdot o_R = \coprod_{R \geq Q} A_Q^R \cdot o_R. \tag{10}$$

We sometimes identify A_Q^R with the stratum $A_Q^R \cdot o_R$.

Set $D(Q) = D \times_{A_Q} \bar{A}_Q^G$; the isomorphism (5) extends to

$$D(Q) \cong \bar{\mathcal{A}}_Q^G \times e_Q, \tag{11}$$

where $\bar{\mathcal{A}}_Q^G = \mathcal{A}_Q^G \times_{A_Q} \bar{A}_Q^G$. The point $o_Q \in \bar{A}_Q^G$ determines a well-defined point in $\bar{\mathcal{A}}_Q^G$ which we also denote o_Q and in general (10) induces a stratification of $\bar{\mathcal{A}}_Q^G$.

In general the product decomposition $A_P^G = A_Q^G \times A_P^Q$ does *not* extend to a product decomposition of $\bar{A}_P^G$.[3] However if $A_P^G(Q) = \{\, a \in \bar{A}_P^G \mid a^\alpha < \infty \text{ for all } \alpha \in \Delta_P^Q \}$ then [32, Lemma 3.6]

$$\bar{A}_Q^G \times A_P^Q \cong A_P^G(Q) \subseteq \bar{A}_P^G. \tag{12}$$

It follows that there is an open inclusion

$$\begin{aligned} D(Q) = D \times_{A_Q} \bar{A}_Q^G &= D \times_{A_Q \times A_P^Q} (\bar{A}_Q^G \times A_P^Q) \\ &\subseteq D \times_{A_P} \bar{A}_P = D(P). \end{aligned} \tag{13}$$

Alternatively, (8) and (12) yield

$$\bar{\mathcal{A}}_Q^G \times \mathcal{A}_P^Q \subseteq \bar{\mathcal{A}}_P^G \tag{14}$$

[3] However the product decomposition $A_P^G = A_Q^G \times A_{P,Q}^G$ from [11, 4.3(3)] does extend to $\bar{A}_P^G$.

and then by (9) and (11) we obtain the inclusion

$$D(Q) \cong \bar{\mathcal{A}}_Q^G \times e_Q \cong \bar{\mathcal{A}}_Q^G \times \mathcal{A}_P^Q \times e_P \subseteq \bar{\mathcal{A}}_P^G \times e_P \cong D(P). \tag{15}$$

3.4. **Borel-Serre Compactification.** Set

$$\overline{D} = \bigcup_Q D(Q) \tag{16}$$

where Q ranges over all parabolic $\mathbb{Q}$-subgroups of G and we identify $D(Q)$ with an open subset of $D(P)$ when $P \subseteq Q$. We identify e_Q with the subset $\{o_Q\} \times e_Q$ of $D(Q)$ (see (11)) and hence obtain a stratification $\overline{D} = \coprod_Q e_Q$.

The group of rational points $G(\mathbb{Q})$ acts on $\overline{D}$. The arithmetic quotient $\overline{X} = \Gamma\backslash\overline{D}$ is a compact Hausdorff space called the *Borel-Serre compactification* of X. The normalizer in Γ of a stratum e_Q of $\overline{D}$ is $\Gamma_Q = \Gamma \cap Q$ and the corresponding stratum of $\overline{X}$ is $Y_Q = \Gamma_Q\backslash e_Q$. The strata are indexed by the finite set $\mathcal{P}$ of Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G. To avoid overburdening the notation, we will denote the Γ-conjugacy class of Q again simply by Q. With this convention, Y_P is contained in $\operatorname{cl}(Y_Q)$ if and only if there exists $\gamma \in \Gamma$ such that $\gamma P \gamma^{-1} \subseteq Q$; in this case we will write $P \leq Q$ and this defines a partial order on $\mathcal{P}$.

By reduction theory every point of e_Q has a neighborhood in $\overline{D}$ on which the equivalence relation induced by Γ is the same as the equivalence relation induced by Γ_Q. However since Γ_Q acts on (11) only through the second factor, we obtain

$$\Gamma_Q\backslash D(Q) \cong \bar{\mathcal{A}}_Q^G \times Y_Q. \tag{17}$$

Thus every point $y \in Y_Q$ has a neighborhood in $\overline{X}$ (in fact a basis of neighborhoods) for which (17) induces a decomposition. Specifically, for $b \in \mathcal{A}_Q^G$ set

$$\bar{\mathcal{A}}_Q^G(b) = \operatorname{cl}(\exp(\mathfrak{a}_Q^{G+}) \cdot b). \tag{18}$$

If O_Q is a relatively compact neighborhood of y in Y_Q then for b sufficiently close to o_Q,

$$\bar{\mathcal{A}}_Q^G(b) \times O_Q \subseteq \Gamma_Q\backslash D(Q) \tag{19}$$

descends to a neighborhood of y in $\overline{X}$. As O_Q shrinks and $b \to o_Q$, we obtain a basis of neighborhoods of y in $\overline{X}$.

3.5. **Reductive Borel-Serre Compactification.** For a parabolic $\mathbb{Q}$-subgroup Q, the quotient $D_Q = N_Q(\mathbb{R})\backslash e_Q$ is the symmetric space associated to L_Q. On the other hand, the geodesic action of $L_Q(\mathbb{R})$ on D descends to an action on e_Q

and the quotient $\mathcal{N}_Q(\mathbb{R}) = {}^0L_Q(\mathbb{R})\backslash e_Q$ is a principal $N_Q(\mathbb{R})$-homogeneous space. The projections yield a canonical decomposition

$$e_Q \cong \mathcal{N}_Q(\mathbb{R}) \times D_Q, \tag{20}$$

an isomorphism of $N_Q(\mathbb{Q}) \times {}^0L_Q(\mathbb{R})$-homogeneous spaces [33, §1.4].

The quotient of (20) by $\Gamma_{N_Q} = \Gamma \cap N_Q$ yields $\Gamma_{N_Q}\backslash e_Q \cong \mathcal{N}_Q(\mathbb{R})' \times D_Q$, where $\mathcal{N}_Q(\mathbb{R})' = \Gamma_{N_Q}\backslash\mathcal{N}_Q(\mathbb{R})$. This is a trivial $\mathcal{N}_Q(\mathbb{R})'$-bundle over D_Q. The further quotient by $\Gamma_{L_Q} = \Gamma_Q/\Gamma_{N_Q}$ yields a flat $\mathcal{N}_Q(\mathbb{R})'$-bundle

$$p_Q \colon Y_Q \longrightarrow X_Q, \tag{21}$$

where $X_Q = \Gamma_{L_Q}\backslash D_Q$ is again a locally symmetric space. This is called the *nilmanifold fibration*.

Define $\widehat{X} = \coprod_{Q\in\mathcal{P}} X_Q$ and equip it with the quotient topology from the natural map $p = \coprod_Q p_Q \colon \overline{X} \to \widehat{X}$. This is the *reductive Borel-Serre compactification* which was introduced by Zucker [42]. Zucker proves [43] that any Satake compactification X^* [40] arises as a quotient

$$\pi \colon \widehat{X} \to X^* \tag{22}$$

of the reductive Borel-Serre compactification.

The reductive Borel-Serre compactification is stratified by X_Q for $Q \in \mathcal{P}$. The construction of $\widehat{X}$ is hereditary in the sense that the closure of X_Q in $\widehat{X}$ is itself the reductive Borel-Serre compactification $\widehat{X}_Q$ of X_Q. We let i_Q (resp. $\hat{\imath}_Q$) denote the inclusion map of X_Q (resp. $\widehat{X}_Q$) into $\widehat{X}$.

For $x \in X_Q$, let O_Q be a relatively compact neighborhood in Y_Q of $p^{-1}(x)$ which is a union of $\mathcal{N}_Q(\mathbb{R})'$-fibers. If $b \in \mathcal{A}_Q^G$ is sufficiently close to o_Q as in (19), the image $V = p(\bar{\mathcal{A}}_Q^G(b) \times O_Q)$ will be called a *special neighborhood* of x in $\widehat{X}$. Note that

$$V \cap X \cong \mathcal{A}_Q^G(b) \times O_Q. \tag{23}$$

Again as O_Q shrinks and $b \to o_Q$, we obtain a basis of neighborhoods of y in $\widehat{X}$.

3.6. L^2-cohomology Sheaf on $\widehat{X}$.

Let E be a regular G-module equipped with an admissible inner product. Then in §2.3 we have defined the L^2-cohomology $H_{(2)}(X;\mathbb{E})$. By the construction in §2.2 we obtain a sheaf $\Omega_{(2)}(\widehat{X};\mathbb{E})$ on the reductive Borel-Serre compactification. Zucker [42] shows that this sheaf is fine so its hypercohomology represents $H_{(2)}(X;\mathbb{E})$.

4. Special Differential Forms

We recall the the sheaf of *special differential forms* $\Omega_{\mathrm{sp}}(\widehat{X};\mathbb{E})$ due to Goresky, Harder, and MacPherson [19, (13.2)]. These will be used in the next section to realize an $\mathcal{L}$-module as a complex of sheaves on $\widehat{X}$.

Let $\mathbb{E}$ be a locally constant sheaf on X (for example, $\mathbb{E}$ could be induced from a regular representation E of G). For any submanifold $O_Q \subseteq Y_Q = \Gamma_Q\backslash e_Q$ which is a union of $\mathcal{N}_Q(\mathbb{R})'$-fibers, define a form $\omega \in A(O_Q;\mathbb{E})$ to be $N_Q(\mathbb{R})$-*invariant* if its lift to e_Q is $N_Q(\mathbb{R})$-invariant; denote the subcomplex of such forms by $A_{\mathrm{inv}}(O_Q;\mathbb{E})$.

Define the sheaf $\Omega_{\mathrm{sp}}(\widehat{X};\mathbb{E})$ to have sections over $U \subseteq \widehat{X}$ consisting of elements $\eta \in A(U\cap X;\mathbb{E})$ satisfying the following condition:

(24) For every boundary point $x \in U \cap X_Q$, there exists a special neighborhood $V = p(\bar{A}^G_Q(b) \times O_Q) \subseteq U$ of x (see (23)) such that $\eta|_{V\cap X} = (\mathrm{pr}^Q)^*\omega$ with $\omega \in A_{\mathrm{inv}}(O_Q;\mathbb{E})$.

(Without the $N_Q(\mathbb{R})$-invariance condition, these are the forms "locally lifted from the boundary" introduced by Borel [3, §8.1].) This condition is stable under exterior differentiation so we obtain a complex of sheaves $\Omega_{\mathrm{sp}}(\widehat{X};\mathbb{E})$ which is fine by [19, (13.4)]. There are natural quasi-isomorphisms

$$Ri_{G*}\mathbb{E} \xrightarrow{\sim} \Omega_{\mathrm{sp}}(\widehat{X};\mathbb{E}) \xrightarrow{\sim} i_{G*}\Omega(X;\mathbb{E}) \tag{25}$$

induced by inclusion [19, (13.6)]. The local normal triviality of $\widehat{X}$ then implies that $\Omega_{\mathrm{sp}}(\widehat{X};\mathbb{E})$ is constructible. (Recall that a complex of sheaves $\mathcal{S}$ on $\widehat{X}$ is called *constructible* if $H(i_P^*\mathcal{S})$ is locally constant for all $P \in \mathcal{P}$.)

The above construction may be applied to each $\widehat{X}_R$ for $R \in \mathcal{P}$. We obtain for each R a functor

$$\mathfrak{Mod}(L_R) \longrightarrow \mathbf{C}_{\mathcal{X}}(\widehat{X}_R) \qquad \text{given by} \qquad E_R \longmapsto \Omega_{\mathrm{sp}}(\widehat{X}_R;\mathbb{E}_R), \tag{26}$$

where $\mathbf{C}_{\mathcal{X}}(\widehat{X}_R)$ denotes the category of constructible complexes of sheaves on $\widehat{X}_R$.

For $P \leq Q \in \mathcal{P}$, define the principle $N_P^Q(\mathbb{R})$-homogeneous space $\mathcal{N}_P^Q(\mathbb{R}) = N_Q(\mathbb{R})\backslash\mathcal{N}_P(\mathbb{R})$. Its arithmetic quotient $\mathcal{N}_P^Q(\mathbb{R})' = (\Gamma_{N_P}/\Gamma_{N_Q})\backslash\mathcal{N}_P^Q(\mathbb{R})$ is the fiber of $\overline{X}_Q \to \widehat{X}_Q$ over $X_P \subseteq \widehat{X}_Q$. Let $A_{\mathrm{inv}}(\mathcal{N}_P^Q(\mathbb{R})';\mathbb{E}_Q)$ denote the Γ_{L_P}-module of $N_P(\mathbb{R})$-invariant forms on $\mathcal{N}_P^Q(\mathbb{R})'$ and let $\mathbb{A}_{\mathrm{inv}}(\mathcal{N}_P^Q(\mathbb{R})';\mathbb{E}_Q)$ denote the corresponding locally constant sheaf on X_P. There is a well-defined restriction map

$$\Omega_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q) \to \hat{\imath}_{P*}\Omega_{\mathrm{sp}}(\widehat{X}_P;\mathbb{A}_{\mathrm{inv}}(\mathcal{N}_P^Q(\mathbb{R})';\mathbb{E}_Q)), \qquad \omega \mapsto \omega|_{\widehat{X}_P}.$$

However there is a natural quasi-isomorphism [19, (12.15)], [33, Lem. 4.7]

$$h_{PQ}\colon \mathbb{A}_{\mathrm{inv}}(\mathcal{N}_P^Q(\mathbb{R})'; \mathbb{E}_Q) \to \mathbb{H}(\mathfrak{n}_P^Q; E_Q) \tag{27}$$

obtained by evaluating a form at a basepoint and applying harmonic projection. Thus there is a natural morphism [33, Cor. 4.8]

$$k_{PQ} : \mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_Q; \mathbb{E}_Q) \longrightarrow \hat{\imath}_{P*}\mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_P; \mathbb{H}(\mathfrak{n}_P^Q; E_Q)) \tag{28}$$

defined by $k_{PQ}(\omega) = \hat{\imath}_{P*}\mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_P; h_{PQ})(\omega|_{\widehat{X}_P})$. The morphism $\hat{\imath}_P^*(k_{PQ})$ induced on the restriction of these sheaves to $\widehat{X}_P$ is a quasi-isomorphism.

Kostant's theorem [24] implies that there exists a natural isomorphism

$$\kappa_P^Q\colon H(\mathfrak{n}_P^Q; H(\mathfrak{n}_Q^R; E_R)) \cong H(\mathfrak{n}_P^R; E_R) \tag{29}$$

when $P \le Q \le R$ (see for example, [33, §0.10.20]); this isomorphism satisfies

$$\kappa_P^Q \circ H(\mathfrak{n}_P^Q; \kappa_Q^R) = \kappa_P^R \circ \kappa_P^Q. \tag{30}$$

We will often make use of this isomorphism tacitly.

One may check for $P \le Q \le R$ that

$$\hat{\imath}_{Q*}(k_{PQ}) \circ k_{QR} = k_{PR}. \tag{31}$$

5. $\mathcal{L}$-modules

An $\mathcal{L}$-module is a combinatorial analogue of a constructible complex of sheaves on $\widehat{X}$; it has proved useful in studying various cohomology groups associated to $\widehat{X}$. We recall the definitions following [36], [33].

5.1. The Category of $\mathcal{L}$-modules.

Recall that $\mathcal{P}$ is the partially ordered finite set of Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G. Let $\mathcal{Q} \subseteq \mathcal{P}$ be a subset satisfying

$$\text{if } P \le R \le Q \text{ where } P,\ Q \in \mathcal{Q} \text{ and } R \in \mathcal{P}, \text{ then } R \in \mathcal{Q}. \tag{32}$$

An $\mathcal{L}$-*module* $\mathcal{M}$ *on* $\mathcal{Q}$ is a pair consisting of

(i) a graded regular L_P-module E_P for all $P \in \mathcal{Q}$ and
(ii) an L_P-morphism $f_{PQ}\colon H(\mathfrak{n}_P^Q; E_Q) \to E_P[1]$ for all $P \le Q \in \mathcal{Q}$

which satisfy the condition that for all $P \le R \in \mathcal{Q}$,

$$\sum_{P \le Q \le R} f_{PQ} \circ H(\mathfrak{n}_P^Q; f_{QR}) = 0. \tag{33}$$

Note that as indicated in §1.3 we are implicitly extending the functor $E \mapsto H(n_P^Q; E)$ from $\mathfrak{Mod}(L_Q)$ to $\mathrm{Gr}(\mathfrak{Mod}(L_Q))$ by setting $H(\mathfrak{n}_P^Q; E_Q) = \bigoplus_k H(\mathfrak{n}_P^Q; E_Q^k)[-k]$.

Let $\mathcal{M} = (E_\cdot, f_{\cdot\cdot})$ and $\mathcal{M}' = (E'_\cdot, f'_{\cdot\cdot})$ be $\mathcal{L}$-modules over $\mathcal{Q}$. A *morphism* $\phi\colon \mathcal{M} \to \mathcal{M}'$ is a collection of L_P-module maps $\phi_{PQ}\colon H(\mathfrak{n}_P^Q; E_Q) \to E'_P$ for all $P \le Q \in \mathcal{Q}$ such that for all $P \le R \in \mathcal{Q}$,

$$\sum_{P \le Q \le R} \phi_{PQ} \circ H(\mathfrak{n}_P^Q; f_{QR}) = \sum_{P \le Q \le R} f'_{PQ} \circ H(\mathfrak{n}_P^Q; \phi_{QR}).$$

The composition $\phi' \circ \phi$ of morphisms $\phi : \mathcal{M} \to \mathcal{M}'$ and $\phi' : \mathcal{M}' \to \mathcal{M}''$ is defined by

$$(\phi' \circ \phi)_{PR} = \sum_{P \le Q \le R} \phi'_{PQ} \circ H(\mathfrak{n}_P^Q; \phi_{QR})$$

for all $P \le R \in \mathcal{Q}$. We thus obtain a category $\mathfrak{Mod}(\mathcal{L}_\mathcal{Q})$ of $\mathcal{L}$-modules over $\mathcal{Q}$.

Given an $\mathcal{L}$-module $\mathcal{M}$ over $\mathcal{Q}$ and $P \in \mathcal{Q}$, equation (33) (applied with $P = R$) implies that $f_{PP} \circ f_{PP} = 0$ and thus that (E_P, f_{PP}) is a complex. We obtain a functor $\mathcal{M} \mapsto i_P^! \mathcal{M} = (E_P, f_{PP})$ from $\mathfrak{Mod}(\mathcal{L}_\mathcal{Q})$ to $\mathbf{C}^b(\mathfrak{Mod}(L_P))$, the category of bounded complexes of regular L_P-modules. In the case that $\mathcal{Q} = \{P\}$ this is an equivalence of categories. Other complexes contructed from the data of an $\mathcal{L}$-module by means of (33) will be introduced in §5.3.

5.2. **Realization Functor.** A subset W of $\widehat{X}$ will be called *constructible* if it is a union of strata. Since $\mathcal{P}$ parametrizes the strata of $\widehat{X}$, there is a one-to-one correspondence between constructible subsets of $\widehat{X}$ and subsets of $\mathcal{P}$, namely $W \mapsto \mathcal{P}(W) \equiv \{\, P \in \mathcal{P} \mid X_P \subseteq W \,\}$. Assume that W is locally closed; this is equivalent to having $\mathcal{P}(W)$ satisfy (32). By abuse of notation, we will speak of the category of $\mathcal{L}$-modules on W (instead of on $\mathcal{P}(W)$), and write $\mathfrak{Mod}(\mathcal{L}_W)$.

For a locally closed constructible set W and $P \in \mathcal{P}(W)$, we have inclusions $i_P\colon X_P \hookrightarrow W$, $\hat{\imath}_P\colon \widehat{X}_P \cap W \hookrightarrow W$, $j_P\colon W \setminus X_P \hookrightarrow W$, and $\hat{\jmath}_P\colon W \setminus (\widehat{X}_P \cap W) \hookrightarrow W$. Let $\mathbf{D}_\mathcal{X}(W)$ denote the derived category of constructible complexes of sheaves on W. We define a *realization* functor

$$\mathcal{S}_W\colon \mathfrak{Mod}(\mathcal{L}_W) \longrightarrow \mathbf{D}_\mathcal{X}(W)$$

by setting

$$(34) \qquad \begin{cases} \mathcal{S}_W(\mathcal{M}) = \displaystyle\bigoplus_{P \in \mathcal{P}(W)} \hat{\imath}_{P*} \Omega_{\mathrm{sp}}(\widehat{X}_P \cap W; \mathbb{E}_P), \\ d_{\mathcal{S}_W(\mathcal{M})} = \displaystyle\sum_{P \in \mathcal{P}(W)} d_P + \sum_{P \le Q \in \mathcal{P}(W)} \Omega_{\mathrm{sp}}(\widehat{X}_P \cap W; f_{PQ}) \circ k_{PQ}, \end{cases}$$

where d_P is the differential (exterior differentiation) of $\Omega_{\mathrm{sp}}(\widehat{X}_P; \mathbb{E}_P)$. The cohomology $H(W; \mathcal{M})$ of an $\mathcal{L}$-module over W is defined as the hypercohomology $H(W; \mathcal{S}_W(\mathcal{M}))$. In fact the sheaf $\mathcal{S}_W(\mathcal{M})$ is fine so $H(W; \mathcal{M}) \cong H(\mathcal{S}_W(\mathcal{M})(W))$, the cohomology of global sections of $\mathcal{S}_W(\mathcal{M})$.

5.3. **Functors.** Let W be a locally closed constructible subset of $\widehat{X}$ and let $\mathcal{M} \in \mathfrak{Mod}(\mathcal{L}_W)$. For $P \in \mathcal{P}(W)$ we define:

$$i_P^! \mathcal{M} = (E_P, f_{PP}), \tag{35}$$

$$i_P^* \mathcal{M} = \Big(\bigoplus_{P \leq R} H(\mathfrak{n}_P^R; E_R), \sum_{P \leq R \leq S} H(\mathfrak{n}_P^R; f_{RS}) \Big), \tag{36}$$

$$i_P^* j_{P*} j_P^* \mathcal{M} = \Big(\bigoplus_{P < R} H(\mathfrak{n}_P^R; E_R), \sum_{P < R \leq S} H(\mathfrak{n}_P^R; f_{RS}) \Big). \tag{37}$$

(For the individual definitions of j_{P*} and j_P^* and more general functors, see [33, §3.4].) By virtue of (33) these are complexes and their cohomology groups are called the *local cohomology supported on* P, the *local cohomology at* P, and the *link cohomology at* P respectively. The association $\mathcal{M} \mapsto i_P^* \mathcal{M}$ is a functor

$$\mathfrak{Mod}(\mathcal{L}_W) \longrightarrow \mathfrak{Mod}(\mathcal{L}_{X_P}) \cong \mathbf{C}^b(\mathfrak{Mod}(L_P))$$

compatible with i_P^* on the realization, that is, $\mathcal{S}_{X_P}(i_P^* \mathcal{M}) \cong i_P^* \mathcal{S}_W(\mathcal{M})$, and similarly for the others [33, §§3.4, 4.1].

We have short exact sequences corresponding to distinguished triangles in the derived category

$$0 \longrightarrow i_P^! \mathcal{M} \longrightarrow i_P^* \mathcal{M} \longrightarrow i_P^* j_{P*} j_P^* \mathcal{M} \longrightarrow 0 \ . \tag{38}$$

More generally, for $P \leq Q \in \mathcal{P}(W)$ define

$$i_P^* \hat{\imath}_Q^! \mathcal{M} = \Big(\bigoplus_{P \leq R \leq Q} H(\mathfrak{n}_P^R; E_R), \sum_{P \leq R \leq S \leq Q} H(\mathfrak{n}_P^R; f_{RS}) \Big) \tag{39}$$

and

$$i_P^* \hat{\jmath}_{Q*} \hat{\jmath}_Q^* \mathcal{M} = \Big(\bigoplus_{P < R \not\leq Q} H(\mathfrak{n}_P^R; E_R), \sum_{P < R \leq S \not\leq Q} H(\mathfrak{n}_P^R; f_{RS}) \Big). \tag{40}$$

Again there is a short exact sequence

$$0 \longrightarrow i_P^* \hat{\imath}_Q^! \mathcal{M} \longrightarrow i_P^* \mathcal{M} \longrightarrow i_P^* \hat{\jmath}_{Q*} \hat{\jmath}_Q^* \mathcal{M} \longrightarrow 0 \tag{41}$$

which agrees with (38) when $Q = P$.

Finally for $P \leq Q \in \mathcal{P}(W)$ define

$$\begin{aligned} i_P^* i_{Q*} i_Q^* \mathcal{M} &= \Big(\bigoplus_{P \leq Q \leq R} H(\mathfrak{n}_P^R; E_R), \sum_{P \leq Q \leq R \leq S} H(\mathfrak{n}_P^R; f_{RS}) \Big) \\ &= H(\mathfrak{n}_P^Q; i_Q^* \mathcal{M}). \end{aligned} \tag{42}$$

There is a natural surjection

$$i_P^* \mathcal{M} \longrightarrow i_P^* i_{Q*} i_Q^* \mathcal{M} \longrightarrow 0 \ . \tag{43}$$

6. Micro-support

Let $W \subseteq \widehat{X}_R$ be an open constructible subset, where $R \in \mathcal{P}$. Let V be an irreducible L_P-module for some $P \in \mathcal{P}(W)$ and recall from §1.2 that ξ_V denotes the character by which S_P acts on V. Define $Q'^R_V \geq Q^R_V \geq P$ such that

$$\Delta_P^{Q_V^R} = \{\, \alpha \in \Delta_P^R \mid \langle \xi_V + \rho_P, \alpha^\vee \rangle < 0 \,\},$$
$$\Delta_P^{Q'^R_V} = \{\, \alpha \in \Delta_P^R \mid \langle \xi_V + \rho_P, \alpha^\vee \rangle \leq 0 \,\} \tag{44}$$

where ρ_P is as in §1.1. Let $[Q^R_V, Q'^R_V]$ be the interval $\{\, Q \in \mathcal{P}(W) \mid Q^R_V \leq Q \leq Q'^R_V \,\}$ in $\mathcal{P}(W)$.

The *micro-support* $\mathrm{SS}(\mathcal{M})$ of $\mathcal{M} \in \mathfrak{Mod}(\mathcal{L}_W)$ is the set of all irreducible L_P-modules V for $P \in \mathcal{P}(W)$ satisfying

(i) $(V|_{{}^0L_P})^* \cong \overline{V|_{{}^0L_P}}$, and
(ii) there exists $Q \in [Q^R_V, Q'^R_V]$ such that

$$\mathrm{Hom}_{L_P}(V, H(i_P^* \hat{\imath}_Q^! \mathcal{M})) \neq 0. \tag{45}$$

The group in (45) is the *type* and is denoted $\mathrm{Type}_{Q,V}(\mathcal{M})$. It is sometimes helpful to also consider the *weak micro-support* $\mathrm{SS}_w(\mathcal{M})$ for which condition (i) is omitted.

Now assume $W = \widehat{X}_R$. The micro-support of $\mathcal{M}$ and the associated types control the non-vanishing of $H(\widehat{X}_R; \mathcal{M})$. Specifically, let $c(V;\mathcal{M}) \leq d(V;\mathcal{M})$ denote the least and greatest degrees in which $\mathrm{Type}_{Q,V}(\mathcal{M})$ is nonzero for any Q as above. Let μ be the highest weight of $V|_{{}^0L_P}$ for a fundamental Cartan subalgebra of ${}^0\mathfrak{l}_P$ and a choice of θ-stable ordering. Let $D_P(V)$ be the symmetric space of the centralizer in ${}^0L_P$ of μ. The space $D_P(V)$ depends on the choice of ordering and we choose the ordering to maximize the dimension of $D_P(V)$.

Theorem 1 ([33, Theorem 10.4])**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-module on $\widehat{X}_R$ where $R \in \mathcal{P}$ and set*

$$c(\mathcal{M}) = \inf_{V \in \mathrm{SS}(\mathcal{M})} \tfrac{1}{2}(\dim D_P - \dim D_P(V)) + c(V;\mathcal{M}), \text{ and} \tag{46}$$

$$d(\mathcal{M}) = \sup_{V \in \mathrm{SS}(\mathcal{M})} \tfrac{1}{2}(\dim D_P + \dim D_P(V)) + d(V;\mathcal{M}). \tag{47}$$

Then $H^i(\widehat{X}_R; \mathcal{M}) = 0$ for $i \notin [c(\mathcal{M}), d(\mathcal{M})]$. In particular, if $\mathrm{SS}(\mathcal{M}) = \emptyset$, then $H(\widehat{X}_R; \mathcal{M}) = 0$.

We can assume $R = G$. The proof in [33] uses the natural diffeomorphism $s\colon X \to X_0$ constructed in [32] onto a relatively compact open domain $X_0 \subseteq X$. (Although s is obviously not a retraction, the domain X_0 is indeed a deformation retract of X.) The map s extends to a homeomorphism $\hat{s}\colon \widehat{X} \to \widehat{X}_0$, where $\widehat{X}_0$ is constructed analogously to $\widehat{X}$. The advantage of X_0 over X is that its induced Riemannian metric extends nondegenerately to the compactification $\overline{X}_0$ and similarly for the various strata of $\widehat{X}_0$. Thus we can define a Hodge norm on sections of $\hat{s}_*\mathcal{S}_{\widehat{X}}(\mathcal{M})$ and to prove the desired vanishing of cohomology it suffices to establish an estimate

$$\|d\omega\|^2 + \|d^*\omega\|^2 \geq C\|\omega\|^2 \tag{48}$$

for global sections ω of $\hat{s}_*\mathcal{S}_{\widehat{X}}(\mathcal{M})$ in the appropriate degrees. A combinatorial generalization of the analytic arguments in [37] localizes the problem to one for each $P \in \mathcal{P}$. This local problem is treated by an L^2-vanishing theorem [33, Theorem 14.1] originally due to Raghunathan [28], [29] (and also deducible from [41]).

7. Locally Regular $\mathcal{L}$-modules

We need to generalize slightly the notion of an $\mathcal{L}$-module in order to apply the theory to $\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})$.

A (possibly infinite dimensional) representation E of a connected reductive $\mathbb{Q}$-group L is called *locally regular* [18, §1.1.3] if any finite-dimensional subspace F_0 is contained within a regular subrepresentation $F \subseteq E$. Equivalently, E is locally regular if it is the direct limit of its regular subrepresentations, $E = \varinjlim F$. Let $\mathfrak{Mod}_{\mathrm{lr}}(L)$ denote the category of locally regular L-modules. A locally regular L-module E has a canonical isotypical decomposition $\bigoplus_V V \otimes \mathrm{Hom}_L(V,E)$, where V ranges over the irreducible regular L-modules and $\mathrm{Hom}_L(V,E)$ is a (possibly infinite dimensional) vector space on which L acts trivially [18, §12.1.1].

The functor $H(\mathfrak{n}_P^Q;\cdot)$ extends to a functor from graded locally regular L_Q-modules to graded locally regular L_P-modules. We can thus define the category $\mathfrak{Mod}_{\mathrm{lr}}(\mathcal{L}_W)$ of *locally regular $\mathcal{L}$-modules over W* simply by replacing "regular" with "locally regular" in §5.

The realization functor of a locally regular $\mathcal{L}$-module can be defined as before provided we specify what is meant by the sheaf of special differential forms $\mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q)$ when E_Q is a locally regular L_Q-module. We define a function from a manifold into the locally regular representation E_Q to be *smooth* if locally its image lies in a finite dimensional subspace F_0 and the map to F_0 is smooth. This allows one to define $\mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q)$ as the direct limit of the special differential forms associated to regular subrepresentations F_Q,

$$\mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q) = \varinjlim \mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{F}_Q). \tag{49}$$

The micro-support $\mathrm{SS}(\mathcal{M})$ of a locally regular $\mathcal{L}$-module is defined as before and we have the

Proposition 2. *Theorem* 1 *remains true for locally regular* $\mathcal{L}$*-modules.*

Proof. Let $\mathcal{M}$ be a locally regular $\mathcal{L}$-module on W. Even though the category $\mathfrak{Mod}_{\mathrm{lr}}(\mathcal{L}_W)$ is not abelian it makes sense to write $\widetilde{\mathcal{M}} \subseteq \mathcal{M}$ if

$$\widetilde{E}_P \subseteq E_P, \tag{50}$$

$$f_{PQ}(H(\mathfrak{n}_P^Q; \widetilde{E}_Q)) \subseteq \widetilde{E}_P[1], \text{ and} \tag{51}$$

$$\widetilde{f}_{PQ} = f_{PQ}|_{H(\mathfrak{n}_P^Q;\widetilde{E}_Q)}, \tag{52}$$

for all $P \leq Q \in \mathcal{P}(W)$. In this case, the inclusion map defines a morphism of $\mathcal{L}$-modules.

We will show that

(i) $H(W;\mathcal{M}) = \varinjlim H(W;\widetilde{\mathcal{M}})$,
(ii) $c(\mathcal{M}) = \varinjlim c(\widetilde{\mathcal{M}})$, and $d(\mathcal{M}) = \varinjlim d(\widetilde{\mathcal{M}})$,

where the limits are taken over all regular $\mathcal{L}$-modules $\widetilde{\mathcal{M}} \subseteq \mathcal{M}$ and in (ii) the limits are taken in the sense of nets. If we set $W = \widehat{X}_R$, this suffices to prove the proposition since Theorem 1 applies to all such $\widetilde{\mathcal{M}}$.

Let ϕ be a global section of $\mathcal{S}_W(\mathcal{M})$. In view of (49), there exist regular subrepresentations $F_P \subseteq E_P$ for all $P \in \mathcal{P}(W)$ such that ϕ is a section of $\bigoplus_P \Omega_{\mathrm{sp}}(\widehat{X}_P \cap W; \mathbb{F}_P)$. It is easy to find regular representations $\widetilde{E}_P \supseteq F_P$ for all $P \in \mathcal{P}(W)$ satisfying (50) and (51). In fact, for $P = R$ maximal define $\widetilde{E}_R = F_R + f_{RR}(F_R)$; in general, assume that $\widetilde{E}_Q$ has been defined for $Q > P$ and define $\widetilde{E}_P = F_P + f_{PP}(F_P) + \sum_{Q>P} f_{PQ}(H(\mathfrak{n}_P^Q; \widetilde{E}_Q))$. Thus there exists a regular $\mathcal{L}$-module $\widetilde{\mathcal{M}} \subseteq \mathcal{M}$ such that ϕ is a global section of $\mathcal{S}_W(\widetilde{\mathcal{M}})$, which shows that

$$\mathcal{S}_W(\mathcal{M})(W) = \varinjlim \mathcal{S}_W(\widetilde{\mathcal{M}})(W).$$

Assertion (i) follows since cohomology commutes with direct limits. To prove assertion (ii) one can similarly show for all $P \leq Q \in \mathcal{P}(W)$ that

$$H(i_P^* \hat{\imath}_Q^! \mathcal{M}) = \varinjlim H(i_P^* \hat{\imath}_Q^! \widetilde{\mathcal{M}}). \qquad \square$$

Henceforth we will refer to a locally regular $\mathcal{L}$-module simply as an $\mathcal{L}$-module.

8. The L^2-cohomology $\mathcal{L}$-module

8.1. For any $P \in \mathcal{P}$, recall from §3.2 that $\mathcal{A}_P^G$ denotes the principal A_P^G-homogeneous space ${}^0P(\mathbb{R})\backslash D$ and let $i_G\colon \mathcal{A}_P^G \to \bar{\mathcal{A}}_P^G$ denote the inclusion into its partial compactification. For any graded complex vector space E, let $\mathbb{E}$ denote the corresponding constant sheaf on $\mathcal{A}_P^G$ and let $\Omega(\mathcal{A}_P^G;\mathbb{E}) = E \otimes \Omega(\mathcal{A}_P^G)$ be the complex of sheaves of $\mathbb{E}$-valued differential forms (with our usual convention (2) concerning graded coefficients); denote the differential $d_{\mathcal{A}_P^G}$. Consider the pushforward sheaf $i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{E})$ on $\bar{\mathcal{A}}_P^G$ and let $i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{E})_\infty$ denote its stalk at o_P. This is likewise a complex. If E has a locally regular action of a reductive group L, we view $i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{E})_\infty$ as a locally regular L-module by allowing L to act solely on the coefficients.

If $P \le Q \le R$ and E is a locally regular L_R-module, we have an L_Q-module $H(\mathfrak{n}_Q^R; i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{E})_\infty)$. We view this as a complex under the differential $H(\mathfrak{n}_Q^R; d_{\mathcal{A}_P^G})$. On the other hand we also have the complex $i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{H}(\mathfrak{n}_Q^R;E))_\infty$. Both of these complexes are naturally isomorphic as L_Q-modules to $H(\mathfrak{n}_Q^R;E)\otimes i_{G*}\Omega(\mathcal{A}_P^G)_\infty$ but the differentials differ by a sign. We obtain an isomorphism of complexes

$$\lambda_Q^R\colon H(\mathfrak{n}_Q^R; i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{E})_\infty) \to i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{H}(\mathfrak{n}_Q^R;E))_\infty. \tag{53}$$

by multiplying the $H^p(\mathfrak{n}_Q^R;E^q)\otimes i_{G*}\Omega^r(\mathcal{A}_P^G)_\infty$ term by $(-1)^{pr}$.

Assume now that E is a regular A_P^G-module. Given the choice of an inner product on E and a basepoint $x_0 \in \mathcal{A}_P^G$, we define a fiber metric on $\mathbb{E}$ and a weight function h_P by

$$|(a\cdot x_0, v)|_{\mathbb{E}} = |a^{-1}v|_E \qquad (a \in A_P^G), \tag{54}$$

$$h_P(a\cdot x_0) = \rho_P(a)^{-1} \qquad (a \in A_P^G). \tag{55}$$

Note the difference of (54) from equation (3) of §2.3. When E is isotypical and A_P^G acts via a character ξ_E then $|(a\cdot x_0, v)|_{\mathbb{E}} = \xi_E(a)^{-1}|v|_E$. The space $\mathcal{A}_P^G$ inherits an invariant Riemannian metric from that of D and hence we obtain a sheaf of weighted L^2-complexes $\Omega_{(2)}(\bar{\mathcal{A}}_P^G;\mathbb{E},h_P)$ as in §2.

More generally, for any $Q \ge P$ define a subcomplex

$$\Omega_{(2),Q}(\bar{\mathcal{A}}_P^G;\mathbb{E},h_P) \subseteq i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{E})$$

as the sheaf associated to the subpresheaf

$$U \longmapsto \{\, \psi \mid \psi|_{U\cap(\mathcal{A}_Q^G\times V)},\ d\psi|_{U\cap(\mathcal{A}_Q^G\times V)} \in A_{(2)}(U\cap(\mathcal{A}_Q^G\times V),\mathbb{E},h_P) \\ \text{for all } V \subseteq \mathcal{A}_P^Q \text{ relatively compact}\,\}; \tag{56}$$

here we use the decomposition $\mathcal{A}_P^G \cong \mathcal{A}_Q^G \times \mathcal{A}_P^Q$ from (8). For $Q = G$ this equals $i_{G*}\Omega(\mathcal{A}_P^G;\mathbb{E})$ while for $Q = P$ we recover $\Omega_{(2)}(\bar{\mathcal{A}}_P^G;\mathbb{E},h_P)$. Again we let $\Omega_{(2),Q}(\bar{\mathcal{A}}_P^G;\mathbb{E},h_P)_\infty$ denote the stalk at o_P.

8.2. Let E be a regular G-module with an admissible inner product. Let $x_0 \in \mathcal{A}_P^G$ be the projection of the basepoint of D for which the inner product is admissible. Apply the considerations of the previous subsection to the regular L_P-module $H(\mathfrak{n}_P; E)$ with its induced admissible inner product.

Lemma 3. *For any $P \leq P' \leq Q' \leq Q$ there exists a natural morphism of complexes of L_P-modules*

(57)
$$g_{PQ,P'Q'} \colon H(\mathfrak{n}_P^{P'}; \Omega_{(2),Q'}(\bar{\mathcal{A}}_{P'}^G; \mathbb{H}(\mathfrak{n}_{P'}; E), h_{P'})_\infty) \longrightarrow \Omega_{(2),Q}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty$$

which is induced by $\operatorname{pr}_{P'}^*$. *For $P' \leq P'' \leq Q'' \leq Q'$ these morphisms satisfy*

$$g_{PQ,P'Q'} \circ H(\mathfrak{n}_P^{P'}; g_{P'Q',P''Q''}) = g_{PQ,P''Q''} \circ \kappa_P^{P'}. \tag{58}$$

Proof. The desired morphism will be induced from the diagram

$$\begin{array}{ccc}
H(\mathfrak{n}_P^{P'}; \Omega_{(2),Q'}(\bar{\mathcal{A}}_{P'}^G; \mathbb{H}(\mathfrak{n}_{P'}; E), h_{P'})_\infty) & \hookrightarrow & H(\mathfrak{n}_P^{P'}; i_{G*}\Omega(\mathcal{A}_{P'}^G; \mathbb{H}(\mathfrak{n}_{P'}; E))_\infty) \\
\vdots & & \downarrow {\scriptstyle \Omega(\mathcal{A}_{P'}^G; \kappa_P^{P'}) \circ \lambda_P^{P'}} \\
g_{PQ,P'Q'} \vdots & & i_{G*}\Omega(\mathcal{A}_{P'}^G; \mathbb{H}(\mathfrak{n}_P; E))_\infty \\
\vdots & & \downarrow {\scriptstyle \operatorname{pr}_{P'}^*} \\
\Omega_{(2),Q}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty & \hookrightarrow & i_{G*}\Omega(\mathcal{A}_P^G; \mathbb{H}(\mathfrak{n}_P; E))_\infty.
\end{array}$$

Note that although a neighborhood of o_P does not in general decompose under (14), the inverse image under $\operatorname{pr}_{P'}$ of a neighborhood of $o_{P'} \in \bar{\mathcal{A}}_{P'}^G$ does indeed contain a neighborhood of $o_P \in \bar{\mathcal{A}}_P^G$; thus $\operatorname{pr}_{P'}^*$ above is well-defined. All the vertical maps on the right-hand side are cochain morphisms, so $g_{PQ,P'Q'}$ will be as well. Equation (58) is clear from (30) (note that $g_{PQ,P'Q'}$ involves $\kappa_P^{P'}$). It remains to check that the L^2-condition is preserved. Since (8) is quasi-isometric to a Riemannian product, the only issue is in the coefficients. For elements of $H(\mathfrak{n}_P^{P'}; \Omega_{(2),Q'}(\bar{\mathcal{A}}_{P'}^G; \mathbb{W}, h_{P'})_\infty)$, where $W \subseteq H(\mathfrak{n}_{P'}; E)$ is an irreducible $L_{P'}$-module, the coefficients contribute the function $\xi_W(a)^{-2}$ to the L^2-integral by (54). On the other hand, the image under $g_{PQ,P'Q'}$ has coefficients in the various irreducible L_P-modules $Z \subseteq H(\mathfrak{n}_P^{P'}; W)$ which contribute $\xi_Z(a)^{-2}$ to the L^2-integral. The L^2-condition is then preserved since $\xi_Z|_{\mathcal{A}_{P'}^G} = \xi_W$ and $h_P|_{\mathcal{A}_{P'}^G \times V} \sim \operatorname{pr}_{P'}^* h_{P'}$ for $V \subseteq \mathcal{A}_P^{P'}$ relatively compact. $\square$

8.3. For $\mathcal{R} = \{R_0 < \cdots < R_k\} \subseteq \mathcal{P}$, a totally ordered nonempty subset, set

$$F_{\mathcal{R}} = \Omega_{(2),R_k}(\bar{\mathcal{A}}_{R_0}^G; \mathbb{H}(\mathfrak{n}_{R_0}; E), h_{R_0})_\infty[-k]; \tag{59}$$

we view $F_{\mathcal{R}}$ (which actually only depends on R_0 and R_k) simply as a graded locally regular L_{R_0}-module, not as a complex. Define

$$u_{\mathcal{R}\mathcal{R}} = (-1)^k d_{\mathcal{A}^G_{R_0}} : F_{\mathcal{R}} \longrightarrow F_{\mathcal{R}}[1] \tag{60}$$

and for $\mathcal{R}' = \mathcal{R} \setminus \{R_\ell\} \neq \emptyset$, where $0 \leq \ell \leq k$, define

$$u_{\mathcal{R}\mathcal{R}'} = (-1)^\sigma g_{R_0 R_k, R'_0 R'_{k-1}} : H(\mathfrak{n}^{R'_0}_{R_0}; F_{\mathcal{R}'}) \longrightarrow F_{\mathcal{R}}[1], \tag{61}$$

where

$$\sigma = \sigma_{\mathcal{R}\mathcal{R}'} = \begin{cases} k & \text{if } \ell = k \text{ and } R_k = G, \\ \ell + 1 & \text{otherwise.} \end{cases}$$

Define the L^2*-cohomology* $\mathcal{L}$*-module with coefficients* E by

$$\Omega_{(2)}(E) = \begin{cases} E_P = \displaystyle\bigoplus_{\substack{\mathcal{R} \\ R_0 = P}} F_{\mathcal{R}}, \\ f_{PQ} = \displaystyle\sum_{\substack{\mathcal{R} \supseteq \mathcal{R}' \\ R_0 = P, R'_0 = Q \\ \#(\mathcal{R} \setminus \mathcal{R}') \leq 1}} u_{\mathcal{R}\mathcal{R}'}. \end{cases} \tag{62}$$

In order to verify condition (33) is satisfied with this definition of f_{PQ}, we must consider $\mathcal{R} \supseteq \mathcal{R}''$ with $\#(\mathcal{R} \setminus \mathcal{R}'') \leq 2$. The equations

$$\begin{cases} u_{\mathcal{R}\mathcal{R}} \circ u_{\mathcal{R}\mathcal{R}} = 0 & \text{if } \#(\mathcal{R} \setminus \mathcal{R}'') = 0, \\ u_{\mathcal{R}\mathcal{R}''} \circ H(\mathfrak{n}^{R''_0}_{R_0}; u_{\mathcal{R}''\mathcal{R}''}) + u_{\mathcal{R}\mathcal{R}} \circ u_{\mathcal{R}\mathcal{R}''} = 0 & \text{if } \#(\mathcal{R} \setminus \mathcal{R}'') = 1, \\ \sum_{\mathcal{R} \supset \mathcal{R}' \supset \mathcal{R}''} u_{\mathcal{R}\mathcal{R}'} \circ H(\mathfrak{n}^{R'_0}_{R_0}; u_{\mathcal{R}'\mathcal{R}''}) = 0 & \text{if } \#(\mathcal{R} \setminus \mathcal{R}'') = 2 \end{cases} \tag{63}$$

follow from $d^2_{\mathcal{A}^G_{R_0}} = 0$, the fact that $g_{PQ,P'Q'}$ is a morphism of complexes, and (58) respectively. (Note that the last sum in (63) has only two terms.) Condition (33) follows.

We have used similar notation for the L^2-cohomology $\mathcal{L}$*-module* $\Omega_{(2)}(E)$ and the L^2-cohomology *sheaf* $\Omega_{(2)}(\widehat{X}; \mathbb{E})$. This is justified by the

Theorem 4. *There is a natural isomorphism in the derived category*

$$\mathcal{S}_{\widehat{X}}(\Omega_{(2)}(E)) \cong \Omega_{(2)}(\widehat{X}; \mathbb{E}).$$

The proof will appear later in §11.

9. Local L^2-cohomology

We begin by calculating the local cohomology of the $\mathcal{L}$-module $\Omega_{(2)}(E)$ and verifying it agrees with the local cohomology of $\Omega_{(2)}(\widehat{X}; \mathbb{E})$. Strictly speaking this is not needed for the proof of Theorem 4, but it will serve as a model for similar arguments in §11. In addition, the results here will be used in the calculation of micro-support in §12.

Proposition 5. *The projection* $i_P^*\mathit{\Omega}_{(2)}(E) \to \mathit{\Omega}_{(2)}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty$ *is a quasi-isomorphism.*

Proof. By (36) and (62) write

$$i_P^*\mathit{\Omega}_{(2)}(E) = \Big(\bigoplus_{R \geq P} H(\mathfrak{n}_P^R; E_R), \sum_{S \geq R \geq P} H(\mathfrak{n}_P^R; f_{RS}) \Big) \tag{64}$$

$$= \Big(\bigoplus_{\substack{\mathcal{R} \\ R_0 \geq P}} H(\mathfrak{n}_P^{R_0}; F_{\mathcal{R}}), \sum_{\substack{\mathcal{R} \supseteq \mathcal{R}' \\ R_0' \geq R_0 \geq P \\ \#(\mathcal{R} \setminus \mathcal{R}') \leq 1}} H(\mathfrak{n}_P^{R_0}; u_{\mathcal{R}\mathcal{R}'}) \Big)$$

where $F_{\mathcal{R}}$ and $u_{\mathcal{R}\mathcal{R}'}$ are given by (59)–(61). Define a decreasing filtration by setting

$$F^p i_P^*\mathit{\Omega}_{(2)}(E) = \bigoplus_{\substack{\mathcal{R} \\ R_0 \geq P \\ \#(\mathcal{R} \setminus \{P\}) \geq p}} H(\mathfrak{n}_P^{R_0}; F_{\mathcal{R}}) \tag{65}$$

with the induced differential. The associated graded complex for $p = 0$ is

$$\operatorname{Gr}_F^0 i_P^*\mathit{\Omega}_{(2)}(E) = (F_{\{P\}}, d_{\mathcal{A}_P^G}) = \mathit{\Omega}_{(2)}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty, \tag{66}$$

while for $p > 0$ it is a direct sum

$$\operatorname{Gr}_F^p i_P^*\mathit{\Omega}_{(2)}(E) = \bigoplus_{\substack{\mathcal{R} \\ R_0 > P \\ \#\mathcal{R} = p}} M(-u_{\{P\} \cup \mathcal{R}, \mathcal{R}})[-1] \tag{67}$$

of the shifted mapping cones (see §1.3) associated to the chain morphisms

$$-u_{\{P\}\cup\mathcal{R},\mathcal{R}} \colon (H(\mathfrak{n}_P^{R_0}; F_{\mathcal{R}}), H(\mathfrak{n}_P^{R_0}; u_{\mathcal{R}\mathcal{R}})) \longrightarrow (F_{\{P\}\cup\mathcal{R}}, u_{\{P\}\cup\mathcal{R},\{P\}\cup\mathcal{R}})[1]. \tag{68}$$

Since $u_{\{P\}\cup\mathcal{R},\mathcal{R}} = \pm g_{PR_{p-1},R_0R_{p-1}}$, Lemma 6 below completes the proof. □

Lemma 6. *For* $P \leq P' \leq Q$, *the morphism* $g_{PQ,P'Q}$

$$H(\mathfrak{n}_P^{P'}; \mathit{\Omega}_{(2),Q}(\bar{\mathcal{A}}_{P'}^G; \mathbb{H}(\mathfrak{n}_{P'}; E), h_{P'})_\infty) \longrightarrow \mathit{\Omega}_{(2),Q}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty \tag{69}$$

from Lemma 3 *is a quasi-isomorphism.*

Proof. From the proof of Lemma 3 one sees that $g_{PQ,P'Q} = \operatorname{pr}_{P'}^* \circ \mathit{\Omega}(\mathcal{A}_{P'}^G; \kappa_P^{P'}) \circ \lambda_P^{P'}$, where $\operatorname{pr}_{P'} \colon \mathcal{A}_P^G = \mathcal{A}_{P'}^G \times \mathcal{A}_P^{P'} \to \mathcal{A}_{P'}^G$ and the last two factors are isomorphisms by (29) and (53). Thus it suffices to show

$$\operatorname{pr}_{P'}^* \colon \mathit{\Omega}_{(2),Q}(\bar{\mathcal{A}}_{P'}^G; \mathbb{H}(\mathfrak{n}_P; E), h_{P'})_\infty \longrightarrow \mathit{\Omega}_{(2),Q}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty$$

is a quasi-isomorphism. Without the L^2-conditions this result is standard: one shows that $\psi \mapsto \psi|_{\mathcal{A}^G_{P'} \times \{c\}}$ is a homotopy inverse to $\mathrm{pr}^*_{P'}$ by defining a cochain homotopy operator that integrates in the $\mathcal{A}^{P'}_P$-factor from a point $c \in \mathcal{A}^{P'}_P$. Since the L^2-conditions in (56) are only imposed on subsets with relatively compact projection to $\mathcal{A}^{P'}_P$, these conditions are preserved by the homotopy operator. □

The homotopy operators used above will be discussed in more detail in §§10.3—10.4; in §10.6 a related but more subtle homotopy formula will be established on $\Omega_{(2)}(\bar{\mathcal{A}}^G_P; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty$.

Corollary 7. *For $P \in \mathcal{P}$ and $x \in X_P$,*

$$H(i^*_P \Omega_{(2)}(E)) \cong H(\Omega_{(2)}(\widehat{X}; \mathbb{E})_x).$$

Proof. Apply Proposition 5 and Zucker's calculation [42, (4.24)]. □

Corollary 8. *For $P \leq Q$, the natural morphism (see (42) and (43))*

$$H(i^*_P \Omega_{(2)}(E)) \longrightarrow H(i^*_P i_{Q*} i^*_Q \Omega_{(2)}(E)) = H(\mathfrak{n}^Q_P; H(i^*_Q \Omega_{(2)}(E))) \tag{70}$$

corresponds under Proposition 5 to the composition

(71)

$$H(\Omega_{(2)}(\bar{\mathcal{A}}^G_P; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty) \xrightarrow{(-1)^\sigma (g_{PQ,PP})_*} H(\Omega_{(2),Q}(\bar{\mathcal{A}}^G_P; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty) \xrightarrow[\cong]{(g_{PQ,QQ})^{-1}_*} H(\mathfrak{n}^Q_P; H(\Omega_{(2)}(\bar{\mathcal{A}}^G_Q; \mathbb{H}(\mathfrak{n}_Q; E), h_Q)_\infty))$$

where $\sigma = 1$ if $Q = G$ and $\sigma = 0$ otherwise.

Proof. We can assume $P < Q$. Let $\Phi = (\phi_{\mathcal{R}})_{\mathcal{R}} \in i^*_P \Omega_{(2)}(E)$ represent an element of $H(i^*_P \Omega_{(2)}(E))$; the image $\Phi' \in i^*_P i_{Q*} i^*_Q \Omega_{(2)}(E)$ of Φ under the natural morphism (43) is obtained by including only those $\mathcal{R}$ with $R_0 \geq Q$. The projection of Proposition 5 sends Φ to $\phi_{\{P\}}$ and sends Φ' to $\phi_{\{Q\}}$. However the $F_{\{P<Q\}}$-component of the equation $d\Phi = 0$ (computed using (64)) yields

$$u_{\{P<Q\},\{P\}}(\phi_{\{P\}}) + u_{\{P<Q\},\{Q\}}(\phi_{\{Q\}}) + u_{\{P<Q\},\{P<Q\}}(\phi_{\{P<Q\}}) = 0.$$

The corollary follows by applying (60) and (61) to yield

$$(-1)^\sigma g_{PQ,PP}(\phi_{\{P\}}) - g_{PQ,QQ}(\phi_{\{Q\}}) - d_{\mathcal{A}^G_P}(\phi_{\{P<Q\}}) = 0.$$ □

10. Quasi-special Differential Forms

Proposition 5 of the preceding section does not imply Theorem 4 of §8 since it only provides a local quasi-isomorphism which may not arise from a global morphism $\mathcal{S}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E)) \to \mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})$. A global quasi-isomorphism, albeit in the opposite direction, will be constructed in §11. However this morphism can only be defined if we replace $\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})$ by a subcomplex whose forms have well-defined restrictions to boundary strata. The special differential forms considered in §4 have restrictions to boundary strata but they are not sufficient to represent L^2-cohomology. Instead we define in this section a functor of *quasi-special differential forms*,

$$\mathfrak{Mod}_{\mathrm{lr}}(L_R) \longrightarrow \mathbf{C}_{\mathcal{X}}(\widehat{X}_R), \qquad E_R \longmapsto \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_R;\mathbb{E}_R), \tag{72}$$

for each $R \in \mathcal{P}$ and prove it has the desired properties. The definition will be in §10.8; before that we define and study quasi-special differential forms $\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^R;\mathbb{E}_P)$ on $\bar{\mathcal{A}}_P^R$.

10.1. Stratification of $\bar{\mathcal{A}}_P^G$. Recall that $\bar{\mathcal{A}}_P^G$ is stratified by its A_P^G-orbits and these are indexed by those $Q \in \mathcal{P}$ satisfying $P \le Q \le G$. By (14), the product decomposition $\mathcal{A}_Q^G \times \mathcal{A}_P^Q$ of the dense stratum $\mathcal{A}_P^G$ extends to a decomposition $\bar{\mathcal{A}}_Q^G \times \mathcal{A}_P^Q$ of the open star neighborhood of the stratum associated to Q; in view of this we denote the Q-stratum $\{o_Q\} \times \mathcal{A}_P^Q$ or sometimes simply $\mathcal{A}_P^Q$. Define a *special neighborhood* of a boundary point $z \in \{o_Q\} \times \mathcal{A}_P^Q$ to be any set of the form $\bar{\mathcal{A}}_Q^G(b) \times V^Q$ where V^Q is a relatively compact neighborhood of z in $\mathcal{A}_P^Q$.

10.2. Quasi-special Differential Forms on $\bar{\mathcal{A}}_P^R$. Let E_P be a locally regular L_P-module. In order to define $\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^R;\mathbb{E}_P)$ we will use induction on $\#\Delta_P^R$. If $R = P$ set $\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^P;\mathbb{E}_P) = E_P$. For general R, we may assume by induction that $\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q;\mathbb{E}_P)$ has been defined for all $P \le Q < R$. We will also for simplicity of notation assume that $R = G$.

Define the sheaf $\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P)$ to be the subcomplex of $i_{G*}\mathit{\Omega}(\mathcal{A}_P^G;\mathbb{E}_P)$ whose sections over $U \subseteq \bar{\mathcal{A}}_P^G$ are those forms ψ on $U \cap \mathcal{A}_P^G$ which satisfy the following two conditions for all $P \le Q < G$:

(73a) For every boundary point $z \in U \cap (\{o_Q\} \times \mathcal{A}_P^Q)$, there exists a special neighborhood $V = \bar{\mathcal{A}}_Q^G(b) \times V^Q$ such that $\psi|_{V\cap\mathcal{A}_P^G} = \sum_i \mathrm{pr}_Q^* \psi_{Q,i} \wedge (\mathrm{pr}^Q)^*\psi_i^Q$, where $\psi_{Q,i} \in A(\mathcal{A}_Q^G(b);\mathbb{E}_P)$ and $\psi_i^Q \in A(V^Q)$.

Given (73a), the "restriction" $\widetilde{m}_{QG}(\psi)$ of ψ to $\{o_Q\} \times \mathcal{A}_P^Q$ may defined locally as $\sum_i (\psi_{Q,i})_\infty \otimes \psi_i^Q$, where $(\psi_{Q,i})_\infty \in i_{G*}\mathit{\Omega}(\mathcal{A}_Q^G;\mathbb{E}_P)_\infty$ denotes the associated germ.

This is a section of $i_{Q*}\mathit{\Omega}(\mathcal{A}_P^Q; i_{G*}\mathit{\Omega}(\mathcal{A}_Q^G;\mathbb{E}_P)_\infty)$. Now require:

(73b) $\widetilde{m}_{QG}(\psi)$ is a section of $\hat{\imath}_{Q*}\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q; i_{G*}\mathit{\Omega}(\mathcal{A}_Q^G;\mathbb{E}_P)_\infty)$.

Condition (73b) is asserting that $\widetilde{m}_{QG}(\psi)$ is a quasi-special form on $\bar{\mathcal{A}}_P^Q$ (this notion is well-defined by our inductive hypothesis).

Remark. If in (73a) we required that $\psi_{Q,i}$ be constant, we obtain the forms locally lifted from the boundary (see following (24)); these are the analogue of special differential forms in the current context. If this were satisfied for all Q then (73b) is automatic. Thus the forms which are locally lifted from the boundary form a subcomplex of the quasi-special differential forms on $\bar{\mathcal{A}}_P^G$.

For a given $Q \geq P$, it is not difficult to see that if (73a) and (73b) are satisfied for all $Q' \geq Q$, then we can arrange that $(\psi_{Q,i})_\infty \in \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_Q^G;\mathbb{E}_P)_\infty$. We thus have a morphism

$$\widetilde{m}_{QG}\colon \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P) \longrightarrow \hat{\imath}_{Q*}\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q;\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_Q^G;\mathbb{E}_P)_\infty) \tag{74}$$

such that $i_Q^*(\widetilde{m}_{QG})$ is an isomorphism. Note that the analogue of (31) does not hold.

If E_P is a regular L_P-module, define $\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P,h_P) = \mathit{\Omega}_{(2)}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P,h_P) \cap \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P)$. The mapping $\widetilde{m}_{QG}$ of (74) restricts to

$$\widetilde{m}_{QG}\colon \widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P,h_P) \longrightarrow \hat{\imath}_{Q*}\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q;\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_Q^G;\mathbb{E}_P,h_Q)_\infty) \tag{75}$$

since $h_P|_{\mathcal{A}_Q^G\times V^Q} \sim \mathrm{pr}_Q^* h_Q$ and again $i_Q^*(\widetilde{m}_{QG})$ is an isomorphism. For general $R \neq G$, $\widetilde{m}_{QR}$ and its restriction to L^2-forms are defined analogously to (74) and (75).

Proposition 9. *Let E_P be a locally regular L_P-module.*

(i) *The inclusion map is a quasi-isomorphism*

$$\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P)_\infty \longrightarrow i_{G*}\mathit{\Omega}(\mathcal{A}_P^G;\mathbb{E}_P)_\infty.$$

(ii) *If E_P is a regular L_P-module and $\mathbb{E}_P$ is given the fiber metric coming from an admissible inner product, the inclusion map is a quasi-isomorphism*

$$\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P,h_P)_\infty \longrightarrow \mathit{\Omega}_{(2)}(\bar{\mathcal{A}}_P^G;\mathbb{E}_P,h_P)_\infty.$$

Remark. If E_P is irreducible and $\xi_{E_P} + \rho_P$ as in (44) is either regular or nondominant with respect to Δ_P, the local L^2-cohomology in part (ii) is finite dimensional and is represented by constant forms [42, (4.51)]. We need to consider the general situation however where the local L^2-cohomology may be infinite dimensional.

The proof will appear below in §10.7 after some preliminaries on homotopy operators. In order to use coordinates defined by roots, we assume a basepoint has been chosen and identify $\bar{\mathcal{A}}_P^G$ with $\bar{A}_P^G$ for the remainder of the section.

10.3. Basic Homotopy Operator. From now through §10.6 we consider a fixed $Q \geq P$. We would like to have a homotopy formula between a form ψ on A_P^G and a form that satisfies (73a) and (73b) near the Q-stratum. We write $a = a_Q a^Q \in A_P^G$ according to the decomposition $A_P^G = A_Q^G \times A_P^Q$.

Fix $c \in A_P^Q$ and let $\pi_c \colon A_P^G \to A_Q^G \times \{c\} \subseteq A_P^G$ be the projection $a_Q a^Q \mapsto a_Q c$. Order $\Delta_P^Q = \{\alpha_1, \dots, \alpha_n\}$ and let $x^i(a) = \log a^{\alpha_i}$ be the corresponding coordinates on the A_P^Q factor. Let $c^i = x^i(c)$ for $1 \leq i \leq n$ and set $(A_P^G)_{c,i} = \{\, a \in A_P^G \mid x^j(a) = c^j \text{ for } 1 \leq j \leq i \,\}$. Let $\pi_{c,i} \colon A_P^G \to (A_P^G)_{c,i} \subseteq A_P^G$ be the coordinate projection; note that $\pi_{c,0} = \mathrm{id}_{A_P^G}$ and $\pi_{c,n} = \pi_c$. As in [14] we have homotopy and projection operators

$$H_{Q,c}\psi = \sum_{i=1}^{n} \int_{c^i}^{x^i} \iota_{\frac{\partial}{\partial x^i}} \pi_{c,i-1}^* \psi \qquad \text{and} \qquad P_{Q,c}\psi = \pi_c^* \psi, \tag{76}$$

where the integral is integrating with respect to the i^{th}-coordinate. One calculates that

$$dH_{Q,c}\psi + H_{Q,c}d\psi = \psi - P_{Q,c}\psi. \tag{77}$$

10.4. Homotopy of Forms on Neighborhoods. We wish to apply $H_{Q,c}$ to forms on $U \cap A_P^G$, where U belongs to a fundamental system of neighborhoods of $o_P \in \bar{A}_P^G$. We will use the standard fundamental system of neighborhoods given by

$$\bar{A}_P^G(s) = \{\, a \in \bar{A}_P^G \mid a^\alpha > s \text{ for all } \alpha \in \Delta_P \,\} \tag{78}$$

for $s \geq 1$; we set $A_P^G(s) = \bar{A}_P^G(s) \cap A_P^G$. The difficulty is that no matter what $c \in A_P^Q(s)$ is chosen (unless Δ_P^Q is orthogonal to $\Delta_P \setminus \Delta_P^Q$) there will exist points $a \in A_P^G(s)$ such that $\pi_c(a) \notin A_P^G(s)$. Thus if ψ is a form on $A_P^G(s)$, equation (76) does not define $H_{Q,c}\psi$ and $P_{Q,c}\psi$ as forms on $A_P^G(s)$. Nonetheless we have the following lemma:

Lemma 10. *For all $s \geq 1$ and for all $c \in A_P^Q(s)$, equation* (76) *defines operators*

$$H_{Q,c} \colon A(A_P^G(s); \mathbb{E}_P) \to A(A_P^G(m_c s); \mathbb{E}_P)[-1]$$

and

$$P_{Q,c} \colon A(A_P^G(s); \mathbb{E}_P) \to A(A_P^G(m_c s); \mathbb{E}_P),$$

where $m_c = \max_{\alpha \in \Delta_P^Q} c^\alpha / s > 1$, and the homotopy formula (77) *holds.*

Proof. We need to verify that $H_{Q,c}\psi$ and $P_{Q,c}\psi$ are defined at $a = a_Q a^Q \in A_P^G(m_c s)$; note that in this case $c^\alpha < (a^Q)^\alpha$ for all $\alpha \in \Delta_P^Q$. It suffices to check that if $b \in A_P^Q$ is any element such that $c^\alpha \leq b^\alpha \leq (a^Q)^\alpha$ for all $\alpha \in \Delta_P^Q$, then $a_Q b \in A_P^G(s)$. For $\alpha \in \Delta_P^Q$, $(a_Q b)^\alpha = b^\alpha \geq c^\alpha > s$. On the other hand, for $\delta \in \Delta_P \setminus \Delta_P^Q$, write $\delta|_{A_P^Q} = \sum_{\alpha\in\Delta_P^Q} \langle \delta, \beta_\alpha^{Q\vee}\rangle \alpha$ where $\langle \delta, \beta_\alpha^{Q\vee}\rangle \leq 0$. Then $(a_Q b)^\delta > m_c s (b/a^Q)^\delta = m_c s \prod_{\alpha\in\Delta_P^Q} (b/a^Q)^{\langle \delta,\beta_\alpha^{Q\vee}\rangle\alpha} \geq m_c s > s$. □

10.5. Cutoff Homotopy Operator. If a form ψ on $A_P^G(s)$ already satisfies (73a) and (73b) near the R-stratum with $R \neq Q$, the same may not be true for $H_{Q,c}\psi$ and $P_{Q,c}\psi$. We will remedy this by multiplying $H_{Q,c}$ by a certain cutoff function in order to restrict its effect to points near the Q-stratum. For later use in §10.6 we need to allow some flexibility in the cutoff function.

Let us call a smooth function $g\colon \mathbb{R} \to [1,\infty)$ *ϵ-admissible* if

(i) $g(r)$ is monotonically increasing to ∞, and
(ii) $g'(r) < \epsilon$ for all $r \in \mathbb{R}$.

Clearly ϵ-admissible functions exist for any $\epsilon > 0$. For an ϵ-admissible function g, define

$$\chi_g = \chi_g(a) = \zeta\big(\max_{\alpha\in\Delta_P^Q} \log (a^Q)^\alpha - g(\min_{\gamma\in\Delta_Q} \log a_Q^\gamma)\big) \tag{79}$$

where $\zeta(x)$ is a smooth cutoff function such that $\zeta(x) = 1$ for $x \leq -1$ and $\zeta(x) = 0$ for $x \geq 0$. Note that χ_g is not smooth; one can rectify this by replacing max and min by appropriate smooth approximations. (Alternatively one could use piecewise smooth forms and the distribution exterior derivative throughout.)

Define a cutoff homotopy operator and projection,

$$H_{Q,c,g}\psi = \chi_g H_{Q,c}\psi, \tag{80}$$

$$P_{Q,c,g}\psi = (1-\chi_g)\psi + \chi_g P_{Q,c}\psi + d\chi_g \wedge H_{Q,c}\psi, \tag{81}$$

where $H_{Q,c}$ and $P_{Q,c}$ are obtained from Lemma 10. It is straightforward to calculate from (77) that for $\psi \in A(A_P^G(s); \mathbb{E}_P)$,

$$dH_{Q,c,g}\psi + H_{Q,c,g}d\psi = \psi|_{A_P^G(m_c s)} - P_{Q,c,g}\psi. \tag{82}$$

Lemma 11. *Let $c \in A_P^Q(s)$ and let g be any ϵ-admissible function. For any $\psi \in A(A_P^G(s); \mathbb{E}_P)$, the form $P_{Q,c,g}\psi$ satisfies* (73a) *and* (73b) *near the boundary component corresponding to Q. If $\epsilon > 0$ is sufficiently small* (*depending only on G*) *and if ψ already satisfies* (73a) *and* (73b) *near the boundary component corresponding to some $R \not\leq Q$, then so do $H_{Q,c,g}\psi$ and $P_{Q,c,g}\psi$.*

Proof. In a special neighborhood $V = \bar{A}_Q^G(b) \times V^Q$ of a point on the Q-stratum, $\log a^{Q\alpha}$ is bounded for all $\alpha \in \Delta_P^Q$ and by making b sufficiently regular we can arrange that $\log a_Q^\gamma$ is arbitrarily large for all $\gamma \in \Delta_Q$. Since g tends to infinity, we can arrange that $\chi_g|_V \equiv 1$ and thus $P_{Q,c,g}\psi|_V = \pi_c^*\psi$. This proves (73a) for $P_{Q,c,g}\psi$ near the Q-stratum. Furthermore this shows that the restriction $\widetilde{m}_{QG}(P_{Q,c,g}\psi)$ is the function on $\{o_Q\} \times A_P^Q$ which has the constant value $(\psi|_{A_P^G(s)\cap(A_Q^G\times\{c\})})_\infty \in i_{G*}\Omega(\mathcal{A}_Q^G;\mathbb{E}_P)_\infty$. Thus (73b) holds as well.

Now consider $R \not< Q$ such that (73a) and (73b) hold near the R-stratum. We will prove the final assertion by induction on $\#\Delta_P^G$. We will just treat $H_{Q,c,g}\psi$ since the argument for $P_{Q,c,g}\psi$ is identical. Let $V = \bar{A}_R^G(b) \times V^R$ be a special neighborhood of a point on the R-stratum and write, as in (73a),

$$\psi|_{V\cap\mathcal{A}_P^G} = \sum_i \operatorname{pr}_R^* \psi_{R,i} \wedge (\operatorname{pr}^R)^* \psi_i^R.$$

First assume $R \geq Q$. Then we can assume that V^R decomposes into $V_Q^R \times V^Q$ according to $A_P^R = A_Q^R \times A_P^Q$ and that both factors are relatively compact. By enlarging V^Q we may also assume that $c \in V^Q$. For $a \in V \cap A_P^G$, decompose $a = a_R a_Q^R a^Q$ according to these decompositions; note that $a^R = a_Q^R a^Q$ and $a_Q = a_R a_Q^R$. Since V_Q^R is relatively compact, $(a_Q^R)^\gamma$ belongs to a relatively compact subset of $(0,\infty)$ for all $\gamma \in \Delta_Q$. Thus if $\gamma \in \Delta_Q^R$ then $a_Q^\gamma = (a_Q^R)^\gamma$ is bounded. If $\gamma \in \Delta_Q \setminus \Delta_Q^R$ then $a_Q^\gamma = a_R^\gamma (a_Q^R)^\gamma$ can be made arbitrarily large by making b more regular (since $a_R \in A_R^G(b)$). Thus $\min_{\gamma\in\Delta_Q} \log a_Q^\gamma = \min_{\gamma\in\Delta_Q^R} \log(a_Q^R)^\gamma$ and hence $\chi_g(a)$ depends only on a^R. This means that

$$(H_{Q,c,g}\psi)|_{V\cap\mathcal{A}_P^G} = \sum_i \operatorname{pr}_R^* \psi_{R,i} \wedge (\operatorname{pr}^R)^*(H_{Q,c,g}\psi_i^R).$$

Thus (73a) holds for $H_{Q,c,g}\psi$ near the R-stratum and also $\widetilde{m}_{RG}(H_{Q,c,g}\psi) = H_{Q,c,g}\widetilde{m}_{RG}(\psi)$. But the lemma applies by induction to $\widetilde{m}_{RG}(\psi)$, which is quasi-special, and implies that $H_{Q,c,g}\widetilde{m}_{RG}(\psi)$ is quasi-special. Thus (73b) holds.

Now assume $R \not\leq Q$ and $R \not\geq Q$. The first of these conditions implies that there exists $\alpha' \in \Delta_P^R$ such that $\alpha' \notin \Delta_P^Q$; set $\gamma = \alpha'|_{A_Q}$. For $a \in V \cap A_P^G$ we have $a_Q^\gamma = a^{\alpha'}(a^Q)^{-\alpha'} = (a^R)^{\alpha'}(a^Q)^{-\alpha'}$. But since $a^R \in V^R$ and V^R is relatively compact, we may estimate that

$$\begin{aligned}\log a_Q^\gamma &\leq C + \langle -\alpha', \tau_P^{Q\vee}\rangle \max_{\alpha\in\Delta_P^Q} \log(a^Q)^\alpha \\ &\leq C + M \cdot \max_{\alpha\in\Delta_P^Q} \log(a^Q)^\alpha,\end{aligned}$$

where $C > 0$ and $M = \max_{\delta\in\Delta_P\setminus\Delta_P^Q}\langle -\delta, \tau_P^{Q\vee}\rangle \geq 0$. Thus for $a \in V$,

$$g(\min_{\gamma\in\Delta_Q} \log a_Q^\gamma) \leq g(C) + \epsilon M \cdot \max_{\alpha\in\Delta_P^Q} \log(a^Q)^\alpha. \tag{83}$$

Next, the condition $R \not\geq Q$ implies that there exists $\alpha'' \in \Delta_P^Q$ such that $\alpha'' \notin \Delta_P^R$ and hence $(a^Q)^{\alpha''} = a^{\alpha''} = a_R^{\alpha''}(a^R)^{\alpha''}$. For $a \in V$ we have $a_R \in \bar{A}_R^G(b)$ and $a^R \in V^R$; since V^R is relatively compact, this implies that $(a^Q)^{\alpha''}$ can be made arbitrarily large by making b more regular. In particular we can arrange that

$$g(C) \leq \frac{1}{2} \max_{\alpha\in\Delta_P^Q} \log(a^Q)^\alpha, \qquad \text{for all } a \in V.$$

This bounds the first term of (83), while we can choose $\epsilon > 0$ to arrange that the second term of (83) is $\leq \frac{1}{2}\max_{\alpha\in\Delta_P^Q} \log(a^Q)^\alpha$ as well. Thus $\chi_g|_V \equiv 0$. Therefore $H_{Q,c,g}\psi|_V = 0$ and $P_{Q,c,g}\psi|_V = \psi$ so (73a) and (73b) trivially hold. □

10.6. Homotopy of L^2 Forms. Now assume that E_P is regular with an admissible inner product. In general the homotopy operator $H_{Q,c,g}$ for any fixed choice of g will not be bounded on L^2. We will get around this by choosing g depending on each given $\psi \in A_{(2)}(A_P^G(s); \mathbb{E}_P, h_P)$.

Set $A_P^G(s)_{c,i} = A_P^G(s) \cap (A_P^G)_{c,i} = \{\, a \in A_P^G(s) \mid x^j(a) = c^j \text{ for } 1 \leq j \leq i \,\}$ and

$$\sigma_Q = \langle \tau_Q^G, \tau_Q^{G\vee}\rangle^{-1}\tau_Q^G. \tag{84}$$

For an ϵ-admissible function g we will also denote by g the weight function $a \mapsto g(\log a_Q^{\sigma_Q})$ on $A_P^G(s)$.

Lemma 12. *Let $\{\psi_\mu\}$ be a sequence in $A(A_P^G(s)_{c,i}; \mathbb{E}_P)$, where $c \in A_P^Q(s)$ and $0 \leq i \leq n$. Assume that $\|\psi_\mu\|_{h_P} < \infty$ for all μ and that $\{\psi_\mu\}$ is convergent in this norm. For any $\epsilon > 0$ there exists an ϵ-admissible function g such that $\|\psi_\mu\|_{gh_P} < \infty$ for all μ and, after passing to a subsequence, $\{\psi_\mu\}$ is convergent in this new norm.*

Proof. We assume that $i = 0$, that is, $\{\psi_\mu\}$ is a sequence in $A(A_P^G(s); \mathbb{E}_P)$; the general case is identical. Assume the sequence starts with $\psi_0 = 0$ and let $\psi = \lim_{\mu\to\infty}\psi_\mu$. If we set $A_P^G(s)^t = \{\, a \in A_P^G(s) \mid \log a_Q^{\sigma_Q} = t \,\}$, then by Fubini's theorem

$$\|\psi_\mu - \psi\|_{h_P}^2 = \int_0^\infty \left(\int_{A_P^G(s)^t} |\psi_\mu - \psi|^2 h_P(a)^2 \, da \right) dt \tag{85}$$

where da is the induced measure on $A_P^G(s)^t$. Let $f_\mu(t)$ denote the inner integral in (85). Note that $\lim_{\mu\to\infty}\int_t^\infty f_\mu \, dt = 0$ for all t and that $\lim_{t\to\infty}\int_t^\infty f_\mu \, dt = 0$ for all μ. Thus we can find a sequence $t_1 < t_2 < \cdots < t_k \to \infty$ such that $\int_{t_k}^\infty f_\mu \, dt \leq (k+1)^{-3}$ for all μ, k and $t_{k+1} \geq t_k + (2/\epsilon)$ for all k. (Given k, the inequality is automatic for large μ and then t_k can be made sufficiently large to accommodate the rest.) It follows that $\sum_{k=1}^{\infty}(k+1)\int_{t_k}^{t_{k+1}} f_\mu\,dt < \infty$ for all μ. Let $g(t)$ be a smooth monotonic function such that $g(t)=1$ for $t \le t_1$, $g(t_k)=\sqrt{k}$ for $k \ge 1$, and $g'(t) \le \epsilon$. Then $\|\psi_\mu - \psi\|_{gh_P} < \infty$ for all μ which implies $\|\psi\|_{gh_P} < \infty$ (set $\mu = 0$) and hence $\|\psi_\mu\|_{gh_P} < \infty$ for all μ.

For the final assertion, choose a subsequence such that $\int_0^\infty f_\mu\,dt \le \mu^{-2}$ for $\mu \ge 1$ and hence $\lim_{\mu\to\infty} \mu \int_t^\infty f_\mu\,dt = 0$ for all t. We can then choose t_k in the argument above such that $\|\psi_\mu - \psi\|_{gh_P} < \mu^{-1}$ for $\mu \ge 1$. □

Remark 13. The lemma continues to hold (with the same proof) if ψ or ψ_μ are measurable, not necessarily smooth, L^2 forms. Furthermore, given a finite number of convergent sequences, a single function g can be found so that the conclusion of the lemma will hold for all the sequences. Our main interest will be when these sequences are actually constant.

Lemma 14. *Let $\psi \in A_{(2)}(A_P^G(s);\mathbb{E}_P,h_P)$. For any $\epsilon > 0$ and for almost every $c \in A_P^Q(s)$, there exists an ϵ-admissible function g such that $H_{Q,c,g}\psi$, $P_{Q,c,g}\psi \in A_{(2)}(A_P^G(m_c s);\mathbb{E}_P,h_P)$.*

Proof. Pick $c \in A_P^Q(s)$ such that for all $0 \le i \le n$ both $\psi|_{A_P^G(s)_{c,i}}$ and $d\psi|_{A_P^G(s)_{c,i}}$ are L^2 with weight h_P; this condition is satisfied by almost every c. Apply Lemma 12 (see also Remark 13) to find an ϵ-admissible function g such that all the above forms are L^2 with weight gh_P. For $a \in \operatorname{supp}\chi_g$ we have the estimate

$$x^i(a) = \log a^{\alpha_i} \le \max_{\alpha\in\Delta_P^Q} \log a^{Q\alpha} \le g(\min_{\gamma\in\Delta_Q} \log a_Q^\gamma) \le g(\log a_Q^{\sigma_Q}) \tag{86}$$

where the last inequality comes from the estimate

$$a_Q^{\tau_Q^G} = a_Q^{\sum_{\gamma\in\Delta_Q}\langle\tau_Q^G,\beta_\gamma^\vee\rangle\gamma} \ge (\min_{\gamma\in\Delta_Q} a_Q^\gamma)^{\sum_\gamma\langle\tau_Q^G,\beta_\gamma^\vee\rangle} = (\min_{\gamma\in\Delta_Q} a_Q^\gamma)^{\langle\tau_Q^G,\tau_Q^{G\vee}\rangle}.$$

On the other hand, for any $a \in A_P^G(s)$ we have the estimate

$$c^i \lesssim g(\log s^{1+\min_\delta\langle-\delta,\tau_P^{Q\vee}\rangle}) \le g(\min_{\gamma\in\Delta_Q} \log a_Q^\gamma) \le g(\log a_Q^{\sigma_Q}). \tag{87}$$

(Here and below we will use the notation $p \lesssim q$ to indicate $p \le Cq$ where $C > 0$ is a sufficiently large constant depending only on P, Q, c, g, and s.) We also observe that $h_P(a)^2 = h_P(a_Q)^2 \prod_i e^{-d_i x^i}$ where $d_i > 0$. From these facts we can estimate

$$
(88)\quad
\begin{aligned}
\|H_{Q,c,g}\psi\|^2_{h_P} &\lesssim \sum_{i=1}^{n} \int_{A_P^G(m_c s)} \Bigl(\chi_g \int_{c^i}^{x^i} |\pi^*_{c,i-1}\psi|\Bigr)^2 h_P(a)^2\, dV \\
&\lesssim \sum_{i=1}^{n} \int_{A_P^G(m_c s)\cap \operatorname{supp}\chi_g} \Bigl(\int_{c^i}^{x^i} |\pi^*_{c,i-1}\psi|^2\Bigr) g(\log a_Q^{\sigma_Q}) h_P(a)^2\, dV \\
&\lesssim \sum_{i=1}^{n} \int_{A_P^G(s)\cap \operatorname{supp}\chi_g} |\pi^*_{c,i-1}\psi|^2 g(\log a_Q^{\sigma_Q}) h_P(a)^2\, dV \\
&\lesssim \sum_{i=1}^{n} \|\psi|_{A_P^G(s)_{c,i-1}}\|^2_{g h_P} < \infty
\end{aligned}
$$

where the inner integrals are with respect to the i^{th}-coordinate and for the third line we use Fubini's theorem. We may similarly prove $\|H_{Q,c,g}d\psi\|_{h_P}$, $\|\chi_g P_{Q,c}\psi\|_{h_P}$, and $\|d\chi_g \wedge H_{Q,c}\psi\|_{h_P}$ are finite which by (81) shows that $\|P_{Q,c,g}\psi\|_{h_P} < \infty$ as well. Finally (82) shows that $\|dH_{Q,c,g}\psi\|_{h_P} < \infty$. The same argument applied to $d\psi$ shows that $\|dH_{Q,c,g}d\psi\|_{h_P} < \infty$ and hence by (82) that $\|dP_{Q,c,g}\psi\|_{h_P} < \infty$. ☐

Remark 15. Suppose that $\psi_\mu \to \psi$ in the graph norm on $A_{(2)}(A_P^G(s);\mathbb{E}_P, h_P)$. Then for all i and almost every $c \in A_P^Q(s)$, $\psi_\mu|_{A_P^G(s)_{c,i}} \to \psi|_{A_P^G(s)_{c,i}}$ in the graph norm. Hence the argument of the lemma shows that for any $\epsilon > 0$ and for almost every $c \in A_P^Q(s)$, there exists an ϵ-admissible function g such that after passing to a subsequence, $H_{Q,c,g}\psi_\mu \to H_{Q,c,g}\psi$ and $P_{Q,c,g}\psi_\mu \to P_{Q,c,g}\psi$ in the graph norm of $A_{(2)}(A_P^G(m_c s);\mathbb{E}_P, h_P)$. Again the result continues to hold for measurable forms which together with their exterior derivatives in the distribution sense are L^2.

10.7. Proof of Proposition 9. Choose a total ordering $Q_0 = P$, Q_1, Q_2, …, Q_N, $Q_{N+1} = G$ of the parabolic $\mathbb{Q}$-subgroups containing P which is compatible with the partial order; such a total ordering exists. Let $\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp},i}(\bar{A}_P^G;\mathbb{E}_P, h_P)$ be the subcomplex of $\mathit{\Omega}_{(2)}(\bar{A}_P^G;\mathbb{E}_P, h_P)$ whose sections satisfy (73a) and (73b) near points of the Q_j-strata for all $j \geq i$, and let $\widetilde{A}_{(2),\mathrm{sp},i}(U;\mathbb{E}_P, h_P)$ denote its sections over $U \subseteq \bar{A}_P^G$.

For part (ii) we will show for any $0 \leq i \leq N$ that the map of local cohomology

$$
(89)\qquad H(\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp},i}(\bar{A}_P^G;\mathbb{E}_P)_\infty) \to H(\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp},i+1}(\bar{A}_P^G;\mathbb{E}_P)_\infty)
$$

is an isomorphism. For all $s \geq 1$, set $C_{(2),i}(s) = \widetilde{A}_{(2),\mathrm{sp},i}(\bar{A}_P^G(s);\mathbb{E}_P, h_P)$ and consider

$$
\begin{array}{ccc}
C_{(2),i}(s) & \hookrightarrow & C_{(2),i+1}(s) \\
\downarrow & & \downarrow \\
C_{(2),i}(2s) & \hookrightarrow & C_{(2),i+1}(2s)
\end{array}
$$

where the vertical maps are induced by restriction. If $[\psi] \in H(C_{(2),i+1}(s))$, use Lemmas 11 and 14 (applied with $Q = Q_i$ and R ranging over Q_j for all $j > i$) and (82) to find $c \in A_P^Q(s) \cap A_P^Q(1/(2s))^{-1}$ and g such that $dH_{Q_i,c,g}\psi = \psi|_{A_P^G(2s)} - P_{Q_i,c,g}\psi$ with $P_{Q_i,c,g}\psi \in C_{(2),i}(2s)$ and $H_{Q_i,c,g}\psi \in C_{(2),i+1}(2s)$. This shows surjectivity in (89). Similarly, if $[\eta] \in H(C_{(2),i}(s))$ and $\eta = d\psi$ with $\psi \in C_{(2),i+1}(s)$, we find that $\eta|_{A_P^G(2s)} = dH_{Q_i,c,g}\eta + dP_{Q_i,c,g}\psi$ with $H_{Q_i,c,g}\eta$, $P_{Q_i,c,g}\psi \in C_{(2),i}(2s)$. This shows $[\eta] = 0$ and hence injectivity.

The proof of part (i) is simpler since any c and any ϵ-admissible g will suffice. □

10.8. Quasi-special Differential Forms on $\widehat{X}$. We now define $\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X};\mathbb{E})$, the quasi-special differential forms on $\widehat{X}$. For a locally regular representation E of G, a section of $\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X};\mathbb{E})$ over $U \subseteq \widehat{X}$ is an element $\eta \in A(U \cap X;\mathbb{E})$ satisfying the following condition for all $Q \leq G$:

(90) For every boundary point $x \in U \cap X_Q$, there exists a special neighborhood $V = p(\bar{\mathcal{A}}_Q^G(b) \times O_Q) \subseteq U$ of x (see (23)) such that $\eta|_{V\cap X}$ is a sum of terms $\mathrm{pr}_Q^*\,\psi \wedge (\mathrm{pr}^Q)^*\omega$, where $\psi \in \widetilde{A}_{\mathrm{sp}}(\bar{\mathcal{A}}_Q^G(b);\mathbb{E})$ and $\omega \in A(O_Q)$ is $N_Q(\mathbb{R})$-invariant.

A special differential form as in §5.2 satisfies this condition with ψ constant. Thus $\Omega_{\mathrm{sp}}(\widehat{X};\mathbb{E})$ is a subcomplex of $\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X};\mathbb{E})$. Set $\widetilde{\Omega}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E}) = \Omega_{(2)}(\widehat{X};\mathbb{E}) \cap \widetilde{\Omega}_{\mathrm{sp}}(\widehat{X};\mathbb{E})$.

Proposition 16. *Let E be a locally regular G-module.*

(i) *The inclusion map is a quasi-isomorphism*

$$\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X};\mathbb{E}) \xrightarrow{\sim} i_{G*}\Omega(X;\mathbb{E}).$$

(ii) *If E is a regular G-module and $\mathbb{E}$ is given the fiber metric coming from an admissible inner product, then the inclusion map is a quasi-isomorphism*

$$\widetilde{\Omega}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E}) \xrightarrow{\sim} \Omega_{(2)}(\widehat{X};\mathbb{E}).$$

Proof. Consider the diagram of maps between stalks at a point $x \in X_P$:

$$\begin{array}{ccc} \Omega_{(2)}(\bar{\mathcal{A}}_P^G;\mathbb{H}(\mathfrak{n}_P;E),h_P)_\infty & \hookrightarrow & \Omega_{(2)}(\widehat{X};\mathbb{E})_x \\ \uparrow & & \uparrow \\ \widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{H}(\mathfrak{n}_P;E),h_P)_\infty & \hookrightarrow & \widetilde{\Omega}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E})_x. \end{array}$$

Zucker [42, (4.24)] defines the top inclusion and shows that it is a quasi-isomorphism. The proof involves constructing a projection mapping in the opposite direction (the composition of harmonic projection in the O_P factor of a special neighborhood, projection to $N_P(\mathbb{R})$-invariant forms, and harmonic projection in the complex $\bigwedge \mathfrak{n}_P^* \otimes E$) and a bounded homotopy operator between it and the identity. These operators all preserve the condition that a form is quasi-special, so the bottom inclusion is also a quasi-isomorphism. The left inclusion is a quasi-isomorphism by Proposition 9(ii) and hence the right inclusion is as well. This proves part (ii); the proof of part (i) is similar using Proposition 9(i). □

10.9. The Quasi-special Differential Forms Functor. If we apply the construction of §10.8 to each $\widehat{X}_R$, we obtain the desired functors (72).

Proposition 17. *Let E_Q be a locally regular L_Q-module.*

(i) *The inclusion morphism yields a natural quasi-isomorphism*

$$\mathit{\Omega}_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q) \xrightarrow{\sim} \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q).$$

(ii) *For $P \le Q \in \mathcal{P}$, the "restriction" morphism k_{PQ} of special differential forms (see (28)) extends to a natural morphism*

$$\widetilde{k}_{PQ} : \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q) \longrightarrow \hat{\imath}_{P*}\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_P;\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q;\mathbb{H}(\mathfrak{n}_P^Q;E_Q))_\infty)$$

such that $\hat{\imath}_P^*(\widetilde{k}_{PQ})$ is a quasi-isomorphism. This morphism satisfies

$$\hat{\imath}_{Q*}(\widetilde{k}_{PQ}) \circ \widetilde{k}_{QR} = \hat{\imath}_{P*}\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_P;\widetilde{m}_{QR}) \circ \widetilde{k}_{PR}. \tag{91}$$

(iii) *Assume E_Q is a regular L_Q-module with an admissible inner product. For $P \le Q \in \mathcal{P}$, the "restriction" morphism $\widetilde{k}_{PQ}$ restricts to a natural morphism*

$$\widetilde{k}_{PQ} : \widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\widehat{X}_Q;\mathbb{E}_Q) \longrightarrow \hat{\imath}_{P*}\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_P;\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^Q;\mathbb{H}(\mathfrak{n}_P^Q;E_Q),h_P)_\infty)$$

such that $i_P^*(\widetilde{k}_{PQ})$ is a quasi-isomorphism.

Remark 18. Note that in (ii) we can conclude that $\hat{\imath}_P^*(\widetilde{k}_{PQ})$ is a quasi-isomorphism, but in (iii) we only have that $i_P^*(\widetilde{k}_{PQ})$ is a quasi-isomorphism

Proof. Part (i) follows from (25) and Proposition 16(i). In a special neighborhood $V = p(\bar{\mathcal{A}}_P^Q(b) \times O_P)$ of a point $x \in X_P$ the morphism $\widetilde{k}_{PQ}$ may be defined (in the notation of §10.8) by

$$\mathrm{pr}_P^*\,\psi \wedge (\mathrm{pr}^P)^*\omega \longmapsto \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_P;\lambda_P^Q) \circ k_{PQ}(\psi_\infty \otimes (\mathrm{pr}^P)^*\omega),$$

where $\psi \in \widetilde{A}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q(b);\mathbb{E}_Q)$ and $\omega \in A(O_P)$ is $N_P^Q(\mathbb{R})$-invariant. This indeed takes quasi-special forms to quasi-special forms as may be verified using (73b). The assertion that $\hat{\imath}_P^*(\widetilde{k}_{PQ})$ is a quasi-isomorphism follows from the corresponding assertion for k_{PQ} [33, Cor. 4.8] and the verification of (91) may be left to the reader. Part (ii) follows. Furthermore the L^2-norm near $x \in X_P$ on the left-hand side involves an integral over the $\mathcal{N}_P^Q(\mathbb{R})'$-fibers of $\mathcal{A}_P^Q(b) \times O_P \to \mathcal{A}_P^Q(b) \times p(O_P)$ which is not present in the right-hand side. Since the volume of the fiber over $(a \cdot x_0, x')$ is $\sim \rho_P(a)^{-2}$, where $a \in A_P^Q$, the integral over the fibers is accounted for in the right-hand side by the weight h_P. Part (iii) follows. $\square$

11. Proof of Theorem 4

We will prove Theorem 4 (from §8) by constructing morphisms of complexes of sheaves

$$\begin{array}{ccc} \mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E}) & & \mathcal{S}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E)) \\ \uparrow r & & \downarrow t \\ \widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E}) & \xrightarrow{s} & \widetilde{\mathcal{S}}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E)) \end{array}$$

and show that they are all quasi-isomorphisms.

The complex $\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})$ is the usual L^2-cohomology sheaf as defined in §2 and the sheaf $\widetilde{\mathit{\Omega}}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E})$ is the subcomplex obtained by intersecting that with quasi-special forms as in §10.8. The morphism r is the inclusion and it is a quasi-isomorphism by Proposition 16(ii).

The complex of sheaves $\mathcal{S}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E))$ is the usual realization, defined by applying (34) to (62), and the complex of sheaves $\widetilde{\mathcal{S}}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E))$ is defined similarly but using the quasi-special differential forms functor. Explicitly, the latter is (compare (34))

$$\tag{92} \begin{cases} \widetilde{\mathcal{S}}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E)) = \bigoplus_{P\in\mathcal{P}} \hat{\imath}_{P*}\widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_P;\mathbb{E}_P), \\ d_{\widetilde{\mathcal{S}}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E))} = \sum_{P\in\mathcal{P}} d_P + \sum_{P\le Q\in\mathcal{P}} \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\widehat{X}_P;\widetilde{f}_{PQ})\circ \widetilde{k}_{PQ}, \end{cases}$$

where $\widetilde{k}_{PQ}$ is from Proposition 17(ii) and $\widetilde{f}_{PQ}$ is defined by

$$\tag{93} \widetilde{f}_{PQ}\colon \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q;\mathbb{H}(\mathfrak{n}_P^Q;E_Q))_\infty \longrightarrow E_P[1], \qquad v\otimes\psi \longmapsto f_{PQ}(v)\wedge(\mathrm{pr}^Q)^*\psi,$$

for $v \in H(\mathfrak{n}_P^Q;E_Q)$ and $\psi \in \widetilde{\mathit{\Omega}}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^Q)_\infty$. The wedge product in (93) should be interpreted in the following way. By (53), (59), and (62), the vector v is a sum of germs of forms on $\mathcal{A}_Q^G$, each of which satisfy an L^2-condition with weight h_Q on $\mathcal{A}_R^G \times V$ for various $R \ge Q$ and all $V \subseteq \mathcal{A}_Q^R$ relatively compact. (For simplicity of notation we ignore here the finite dimensional coefficients and the shift.) By Lemma 3 the map f_{PQ} pulls these back to germs of forms on $\mathcal{A}_P^G$ satisfying a similar L^2-condition, except now $V \subseteq \mathcal{A}_P^{R'}$ for some $R' \ge R$ and the weight is h_P. The wedge product with $(\mathrm{pr}^Q)^*\psi$, the germ of a pullback of a smooth form on $\mathcal{A}_P^Q$, preserves this L^2-condition.

The morphism t is the sum of the inclusions

$$\hat{\imath}_{P*}\Omega_{\mathrm{sp}}(\widehat{X}_P;\mathbb{E}_P) \hookrightarrow \hat{\imath}_{P*}\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X}_P;\mathbb{E}_P).$$

The complexes $\mathcal{S}_{\widehat{X}}(\Omega_{(2)}(E))$ and $\widetilde{\mathcal{S}}_{\widehat{X}}(\Omega_{(2)}(E))$ are filtered by the parabolic rank of P; the morphism induced by t on the associated graded complexes is a quasi-isomorphism by Proposition 17(i) and hence so is t.

It remains to define s and prove that it is a quasi-isomorphism. For each $P \in \mathcal{P}$, the morphism

$$\widetilde{k}_{PG}\colon \widetilde{\Omega}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E}) \longrightarrow \hat{\imath}_{P*}\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X}_P;\widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{H}(\mathfrak{n}_P;E),h_P)_\infty)$$

from Proposition 17(iii) may be composed with the morphism induced by

$$\widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{H}(\mathfrak{n}_P;E),h_P)_\infty \subseteq \Omega_{(2)}(\bar{\mathcal{A}}_P^G;\mathbb{H}(\mathfrak{n}_P;E),h_P)_\infty = F_{\{P\}} \subseteq E_P.$$

We can thus define

$$s = \sum_P \widetilde{k}_{PG}\colon \widetilde{\Omega}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E}) \longrightarrow \bigoplus_{P\in\mathcal{P}} \hat{\imath}_{P*}\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X}_P;\mathbb{E}_P) = \widetilde{\mathcal{S}}_{\widehat{X}}(\Omega_{(2)}(E)). \tag{94}$$

To prove that s is a quasi-isomorphism we will fix $P \in \mathcal{P}$ and prove that $i_P^* s$ is a quasi-isomorphism. Consider the morphisms

$$\begin{array}{ccc} i_P^*\widetilde{\Omega}_{(2),\mathrm{sp}}(\widehat{X};\mathbb{E}) & \xrightarrow{\sum_{R\geq P} i_P^*(\widetilde{k}_{RG})} & \bigoplus_{R\geq P} i_P^*\widetilde{\Omega}_{\mathrm{sp}}(\widehat{X}_R;\mathbb{E}_R) \\ \downarrow{\scriptstyle i_P^*(\widetilde{k}_{PG})} & & \downarrow{\scriptstyle \sum_{R\geq P} i_P^*(\widetilde{k}_{PR})} \\ \Omega(X_P;\widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{H}(\mathfrak{n}_P;E),h_P)_\infty) & & \bigoplus_{R\geq P} \Omega(X_P;\widetilde{\Omega}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^R;\mathbb{H}(\mathfrak{n}_P^R;E_R))_\infty)\ , \end{array} \tag{95}$$

where the upper right-hand group has the differential

$$\sum_{R\geq P} d_R + \sum_{S\geq R\geq P} \widetilde{\Omega}_{\mathrm{sp}}(\widehat{X}_R;\widetilde{f}_{RS})\circ\widetilde{k}_{RS} \tag{96}$$

and the bottom right-hand group has the differential

$$\sum_{R\geq P}(d_P + d_{\mathcal{A}_P^R}) + \sum_{S\geq R\geq P} \Omega(X_P;\widetilde{\Omega}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^R;H(\mathfrak{n}_P^R;\widetilde{f}_{RS}))_\infty)\circ\Omega(X_P;\widetilde{m}_{RS}). \tag{97}$$

The top horizontal map then represents $i_P^* s$. Both the left and right vertical morphisms are quasi-isomorphisms by Proposition 17, parts (iii) and (ii) respectively. (On the right-hand side, one applies the proposition to the graded morphism associated to the double filtration by parabolic rank of R and by degree of E_R.)

We can complete (95) to a commutative diagram by defining the morphism

$$\Omega(X_P;\widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G;\mathbb{H}(\mathfrak{n}_P;E),h_P)_\infty) \longrightarrow \bigoplus_{R\geq P}\Omega(X_P;\widetilde{\Omega}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^R;\mathbb{H}(\mathfrak{n}_P^R;E_R))_\infty) \tag{98}$$

to be $\sum_{R\geq P} \Omega(X_P; \widetilde{m}_{RG})$. More precisely, each term in this sum represents the morphism induced by

$$\widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty \longrightarrow \widetilde{\Omega}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^R; \widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_R^G; \mathbb{H}(\mathfrak{n}_P; E), h_R)_\infty)_\infty$$

from (75), followed by the morphisms induced by

$$\widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_R^G; \mathbb{H}(\mathfrak{n}_P; E), h_R)_\infty \xrightarrow{\sim} H(\mathfrak{n}_P^R; \widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_R^G; \mathbb{H}(\mathfrak{n}_R; E), h_R)_\infty)$$

from (53) and by

$$\widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_R^G; \mathbb{H}(\mathfrak{n}_R; E), h_R)_\infty \subseteq \Omega_{(2)}(\bar{\mathcal{A}}_R^G; \mathbb{H}(\mathfrak{n}_R; E), h_R)_\infty = F_{\{R\}} \subseteq E_R.$$

We need to show that (98) is a quasi-isomorphism. The proof is parallel to that of Proposition 5 of §9 so we will be brief. Define filtrations on the two complexes: on the left-hand side of (98) use the trivial filtration in which F^0 is the entire complex and $F^p = 0$ for $p > 0$; on the right-hand side use (62) to re-express the sum over $R \geq P$ as a sum over $\mathcal{R}$ with $R_0 \geq P$ and let F^p consist of terms such that $\#(\mathcal{R} \setminus \{P\}) \geq p$.

For $p = 0$ the graded morphism associated to (98) is the inclusion

$$\Omega(X_P; \widetilde{\Omega}_{(2),\mathrm{sp}}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty) \longrightarrow \Omega(X_P; \Omega_{(2)}(\bar{\mathcal{A}}_P^G; \mathbb{H}(\mathfrak{n}_P; E), h_P)_\infty)$$

which is a quasi-isomorphism by Proposition 9(ii). For $p > 0$ it is the map of 0 into

$$\bigoplus_{\substack{\mathcal{R} | R_0 > P \\ \#\mathcal{R} = p}} \Big(\Omega(X_P; \widetilde{\Omega}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^{R_0}; \mathbb{H}(\mathfrak{n}_P^{R_0}; F_{\mathcal{R}}))_\infty) \oplus \Omega(X_P; F_{\{P\}\cup\mathcal{R}}) \Big).$$

This is a direct sum of complexes so it suffices to show that the summand for a given $\mathcal{R}$ is acyclic. This summand is a shifted mapping cone for $\Omega(X_P; \widetilde{f}_{PR_0})$:

$$\begin{array}{ccc} \Omega(X_P; \widetilde{\Omega}_{\mathrm{sp}}(\bar{\mathcal{A}}_P^{R_0}; \mathbb{H}(\mathfrak{n}_P^{R_0}; F_{\mathcal{R}}))_\infty) & \xrightarrow{\Omega(X_P; \widetilde{f}_{PR_0})} & \Omega(X_P; F_{\{P\}\cup\mathcal{R}})[1] \\ & \nwarrow \qquad\qquad \nearrow {\scriptstyle -\Omega(X_P; u_{\{P\}\cup\mathcal{R},\mathcal{R}})} & \\ & \Omega(X_P; \mathbb{H}(\mathfrak{n}_P^{R_0}; F_{\mathcal{R}}))_\infty\ . & \end{array}$$

Since Lemma 6 implies that $u_{\{P\}\cup\mathcal{R},\mathcal{R}}$ is a quasi-isomorphism and Proposition 9(i) together with the usual Poincaré lemma implies that the left diagonal map is a quasi-isomorphism, the proof is complete. □

12. The Micro-support of the L^2-cohomology $\mathcal{L}$-module

Let E be a regular G-module and let $\Omega_{(2)}(E)$ be the corresponding $\mathcal{L}$-module as in §8. The calculation of the micro-support $\mathrm{SS}(\Omega_{(2)}(E))$ that follows is similar to that for weighted cohomology in [33, §16], although more complicated by the possible presence of infinite-dimensional local cohomology; I am grateful to an anonymous referee of [33] for comments that simplified the proof.

12.1. It is helpful to use the language of lattices. Let $P \in \mathcal{P}$. The partially ordered set $[P,G] = \{\, Q \in \mathcal{P} \mid P \leq Q \leq G \,\}$ is a Boolean lattice; in fact the map $Q \mapsto \Delta_P^Q$ is an isomorphism of $[P,G]$ onto the lattice of subsets of Δ_P. Given $Q \in [P,G]$, let $(P,Q) \in [P,G]$ denote the complementary element determined by $\Delta_P^{(P,Q)} = \Delta_P \setminus \Delta_P^Q$. For Q, $R \in [P,G]$, let $Q \vee R$ and $Q \wedge R$ denote the elements of $[P,G]$ determined by $\Delta_P^Q \cup \Delta_P^R$ and $\Delta_P^Q \cap \Delta_P^R$ respectively. If $R \leq S$, the subset $[R,S] = \{\, Q \in \mathcal{P} \mid R \leq Q \leq S \,\}$ is a Boolean sublattice.

12.2. Let $P \in \mathcal{P}$ and let V be an irreducible L_P-module. Recall Zucker's calculation of the L^2-cohomology of $\mathcal{A}_P^G(b)$ [42, (4.51)]:

Lemma 19. *For any $b \in \mathcal{A}_P^G$,*

$$H_{(2)}(\mathcal{A}_P^G(b); \mathbb{V}, h_P) = \begin{cases} \mathbb{C} & \text{if } \langle \xi_V + \rho_P, \beta_\alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Delta_P, \\ 0 & \text{if } \langle \xi_V + \rho_P, \beta_\alpha^\vee \rangle < 0 \text{ for any } \alpha \in \Delta_P. \end{cases}$$

In the remaining case where $\langle \xi_V + \rho_P, \beta_\alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta_P$ and $r = \#\{\, \alpha \mid \langle \xi_V + \rho_P, \beta_\alpha^\vee \rangle = 0 \,\} > 0$, then $H_{(2)}^i(\mathcal{A}_P^G(b); \mathbb{V}, h_P)$ is nonzero only if $i \in [1, r]$ in which case it is infinite dimensional.

Define S_V, T_V, $T_V' \geq P$ by

$$\begin{aligned} \Delta_P^{S_V} &= \{\, \alpha \in \Delta_P \mid \langle \xi_V + \rho_P, \beta_\alpha^\vee \rangle > 0 \,\}, \\ \Delta_P^{T_V} &= \{\, \alpha \in \Delta_P \mid \langle \xi_V + \rho_P, \beta_\alpha^\vee \rangle < 0 \,\}, \\ \Delta_P^{T_V'} &= \{\, \alpha \in \Delta_P \mid \langle \xi_V + \rho_P, \beta_\alpha^\vee \rangle \leq 0 \,\}. \end{aligned}$$

Lemma 20.

$$H(i_P^* \Omega_{(2)}(E))_V \cong \begin{cases} H_{(2)}(\mathcal{A}_{S_V}^G(b); \mathbb{C}) \otimes H(\mathfrak{n}_P; E)_V & \text{if } T_V = P, \\ 0 & \text{otherwise,} \end{cases}$$

for any $b \in \mathcal{A}_{S_V}^G$. If $T_V = P$, the group $H_{(2)}^i(\mathcal{A}_{S_V}^G(b); \mathbb{C})$ is $\mathbb{C}$ if $S_V = G$ and otherwise is nonzero (and infinite dimensional) only for degrees in $[1, \dim A_{S_V}^G]$.

Proof. Proposition 5 implies that $H(i_P^* \Omega_{(2)}(E))_V \cong H_{(2)}(\mathcal{A}_P^G(b); \mathbb{V}, h_P) \otimes H(\mathfrak{n}_P; E)_V$ which by Lemma 19 is zero unless $T_V = P$. Zucker's Künneth formula [42, (2.34)(i)] applied to the decomposition $\mathcal{A}_P^G(b) \cong \mathcal{A}_{S_V}^G(b) \times \mathcal{A}_{T_V'}^G(1)$ from [11, 4.3(3)] (which is different from (7)) yields

$$H_{(2)}(\mathcal{A}_P^G(b); \mathbb{V}, h_P) \cong H_{(2)}(\mathcal{A}_{S_V}^G(b); \mathbb{V}, h_{S_V}) \otimes H_{(2)}(\mathcal{A}_{T_V'}^G(1); \mathbb{V}, h_{T_V'})$$

provided one of the factors is finite dimensional. Lemma 19 implies the second factor is $\mathbb{C}$ and the remaining assertions. □

Lemma 21. *For* $P \leq R$*,*

$$H(\mathfrak{n}_P^R; H(i_R^* \Omega_{(2)}(E)))_V \cong \begin{cases} H_{(2)}(\mathcal{A}_{R\vee S_V}^G(b); \mathbb{C}) \otimes H(\mathfrak{n}_P; E)_V & \text{if } T_V \leq R, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If the group is nonzero then $H(\mathfrak{n}_P^R; W)_V \neq 0$ for some irreducible regular L_R-module W occuring in $H(i_R^* \Omega_{(2)}(E))$. Let λ be the highest weight of W with respect to a Cartan subalgebra of $\mathfrak{l}_R$ and a positive system of roots which contains those in $\mathfrak{n}_P^R$. Under these conditions, Kostant's theorem [24, Theorem 5.14] implies that $H(\mathfrak{n}_P^R; W)_V = V$ and the highest weight of V is $w(\lambda+\rho)-\rho$, where ρ is one-half the sum of the positive roots of $\mathfrak{l}_R$ and w belongs to a certain subset of the Weyl group of $\mathfrak{l}_R$. Thus W is uniquely determined and $\xi_V|_{\mathfrak{a}_R} = \xi_W$. The lemma now follows from Lemma 20 (applied to $H(i_R^* \Omega_{(2)}(E))_W$), since it is clear that $T_W = R \vee T_V$ and $S_W = R \vee S_V$. □

Lemma 22. *Suppose that* $P \leq R \leq R'$ *such that* $R \vee S_V = R' \vee S_V$*. Then the natural morphism*

$$H(\mathfrak{n}_P^R; H(i_R^* \Omega_{(2)}(E)))_V \longrightarrow H(\mathfrak{n}_P^{R'}; H(i_{R'}^* \Omega_{(2)}(E)))_V$$

corresponds (*up to sign*) *to the identity under the isomorphisms of Lemma* 21*.*

Proof. The natural morphism is given by (71) in Corollary 8 (aside from the application of $H(\mathfrak{n}_P^R; \cdot)$). The lemma follows easily. □

12.3. Consider $S_1 \leq S_2$ and order $\Delta_{S_1} = \{\alpha_1, \ldots, \alpha_r\}$. Denote by $A_{(2),R}(\mathcal{A}_{S_1}^G(b); \mathbb{C})$ the sections of $\Omega_{(2),R}(\bar{\mathcal{A}}_{S_1}^G; \mathbb{C})$ over $\bar{\mathcal{A}}_{S_1}^G(b)$. Define the double complex

$$A_{(2)}(\mathcal{A}_{[S_1,S_2]}^G(b); \mathbb{C}) = \bigoplus_{S_1 \leq R \leq S_2} A_{(2),R}(\mathcal{A}_{S_1}^G(b); \mathbb{C})[-\#\Delta_{S_1}^R]$$

where the horizontal differential between the R' and R terms (when $\Delta_{S_1}^R = \Delta_{S_1}^{R'} \cup \{\alpha_i\}$) is $(-1)^i g_{S_1 R, S_1 R'}$. Since $H(A_{(2),R}(\mathcal{A}_{S_1}^G(b); \mathbb{C})) \cong H_{(2)}(\mathcal{A}_R^G(b); \mathbb{C})$ by Lemma 6, the E_1-term of the spectral sequence for the total complex is

$$\bigoplus_{S_1 \leq R \leq S_2} H_{(2)}(\mathcal{A}_R^G(b); \mathbb{C})[-\#\Delta_{S_1}^R], \tag{99}$$

where the terms of d_1 are given by Corollary 8. Note that all of these terms are infinite dimensional with the exception of the $R = S_2$ term in the case that $S_2 = G$. We denote the cohomology of the total complex by $H_{(2)}(\mathcal{A}_{[S_1,S_2]}^G(b); \mathbb{C})$. We may similarly define a complex and cohomology for open and half-open intervals such as $(S_1, S_2]$.

The cohomology $H_{(2)}(\mathcal{A}^G_{[S_1,S_2]}(b);\mathbb{C})$ is always nonzero; for example, the spectral sequence (99) shows that it does not vanish in degree $\dim \mathcal{A}^G_{S_1}$. Furthermore, unless $S_1 = S_2 = G$, the cohomology is infinite dimensional.

Proposition 23. *For $P \leq Q \in \mathcal{P}$ and an irreducible L_P-module V,*

$$H(i_P^* \hat{\imath}_Q^! \Omega_{(2)}(E))_V \cong \begin{cases} H_{(2)}(\mathcal{A}^G_{[T_V \vee S_V, (P,Q)\vee S_V]}(b);\mathbb{C}) \otimes H(\mathfrak{n}_P;E)_V[-\#\Delta_P^{T_V}] & \text{if } (P,T'_V) \leq Q \leq (P,T_V), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the short exact sequence

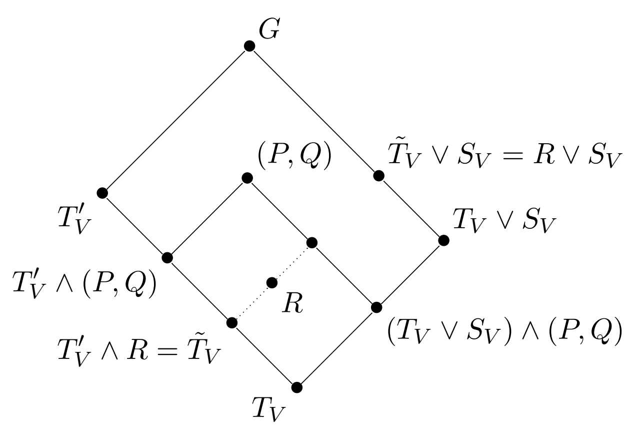
$$0 \longrightarrow i_P^* \hat{\imath}_Q^! \Omega_{(2)}(E)_V \longrightarrow i_P^* \Omega_{(2)}(E)_V \longrightarrow i_P^* \hat{\jmath}_{Q*} \hat{\jmath}_Q^* \Omega_{(2)}(E)_V \longrightarrow 0 \tag{100}$$

obtained from (41) by taking the V-isotypical component. There is a Mayer-Vietoris spectral sequence [33, Lemma 3.7] abutting to $H(i_P^* \hat{\jmath}_{Q*} \hat{\jmath}_Q^* \Omega_{(2)}(E))_V$ with

$$E_1^{p,\cdot} = \bigoplus_{\substack{P<R\leq(P,Q) \\ \#\Delta_P^R = p+1}} H(\mathfrak{n}_P^R; H(i_R^* \Omega_{(2)}(E)))_V. \tag{101}$$

By Lemma 21 the term indexed by R will vanish unless $T_V \leq R$. Thus (101) vanishes unless $T_V \leq (P,Q)$, and in this case the terms of (101) are indexed by $R \in [T_V, (P,Q)]$ (with P excluded if $T_V = P$).

In the lattice $[T_V, G]$, the complement of T'_V is $T_V \vee S_V$. Any element of $[T_V, G]$ may thus be expressed as the join of its intersection with T'_V and its intersection with $T_V \vee S_V$. We apply this to elements of the sublattice $[T_V, (P,Q)]$; if we fix the first intersection to be $\tilde{T}_V$ and let the second intersection vary, we obtain elements R on the dotted line below:



If we then vary $\tilde{T}_V$ we obtain a decomposition

$$[T_V,(P,Q)] = \coprod_{\tilde{T}_V\in[T_V,T'_V\wedge(P,Q)]} [\tilde{T}_V,(\tilde{T}_V\vee S_V)\wedge(P,Q)] \tag{102}$$

The elements R in a fixed component of (102) (say indexed by $\tilde{T}_V$) will all have the same value of $R\vee S_V$, namely $\tilde{T}_V\vee S_V$, and the same value of $T'_V\wedge R$, namely $\tilde{T}_V$. The corresponding terms of (101) will then all be isomorphic by Lemma 21.

In view of the preceding discussion, filter the complex (E_1,d_1) by $\#\Delta_P^{T'_V\wedge R}$. The associated graded complex is a direct sum of complexes indexed by $\tilde{T}_V\in[T_V,T'_V\wedge(P,Q)]$. The complex associated to a given $\tilde{T}_V$ is

$$\bigoplus_{\substack{R\in[\tilde{T}_V,(\tilde{T}_V\vee S_V)\wedge(P,Q)]\\ R\neq P}} H_{(2)}(\mathcal{A}^G_{\tilde{T}_V\vee S_V}(b);\mathbb{C})\otimes H(\mathfrak{n}_P;E)_V[-\#\Delta_P^R+1] \tag{103}$$

with differential $\sum_{\#\Delta_P^{R'}=\#\Delta_P^R+1}\pm\mathrm{id}_{R,R'}$ by Lemma 22; here $\mathrm{id}_{R,R'}$ denotes the identity morphism between the R-term and the R'-term.

First assume $P<T_V$. Then Lemma 20 and the long exact sequence associated to (100) imply that $H(i_P^*\hat{i}_Q^!\mathit{\Omega}_{(2)}(E))_V\cong H(i_P^*\hat{\jmath}_{Q*}\hat{\jmath}_Q^*\mathit{\Omega}_{(2)}(E))_V[-1]$. If furthermore $(P,Q)\notin[T_V,T'_V]$, then the cohomology of (103) vanishes: aside from a shift it is the simplicial cohomology of the cone over the simplex with vertices $\Delta_{\tilde{T}_V}^{(\tilde{T}_V\vee S_V)\wedge(P,Q)}$. On the other hand, if $(P,Q)\in[T_V,T'_V]$, that is, $(P,T'_V)\le Q\le(P,T_V)$, then (103) reduces to

$$H_{(2)}(\mathcal{A}^G_{\tilde{T}_V\vee S_V}(b);\mathbb{C})\otimes H(\mathfrak{n}_P;E)_V[-\#\Delta_P^{\tilde{T}_V}+1]$$

and the spectral sequence (101) is isomorphic to the spectral sequence (99) for $H_{(2)}(\mathcal{A}^G_{[T_V\vee S_V,(P,Q)\vee S_V]}(b);\mathbb{C})$ (tensored with $H(\mathfrak{n}_P;E)_V$ and shifted by $1-\#\Delta_P^{T_V}$).

On the other hand, assume $P=T_V$. Now if $(P,Q)\notin[T_V,T'_V]$, the cohomology of (103) vanishes except for the case $\tilde{T}_V=P$, in which case it is $H_{(2)}(\mathcal{A}^G_{S_V}(b);\mathbb{C})\otimes H(\mathfrak{n}_P;E)_V$. Thus the spectral sequence (101) degenerates and its cohomology will be canceled in the long exact sequence associated to (100) by $H(i_P^*\mathit{\Omega}_{(2)}(E))_V$ (use Lemma 20). If $(P,Q)\in[T_V,T'_V]$, the above argument shows that (101) abuts to $H_{(2)}(\mathcal{A}^G_{(S_V,(P,Q)\vee S_V]}(b);\mathbb{C})\otimes H(\mathfrak{n}_P;E)_V[1]$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\longrightarrow H^{i-1}(i_P^*\hat{\jmath}_{Q*}\hat{\jmath}_Q^*\mathit{\Omega}_{(2)}(E))_V & \longrightarrow & H^i(i_P^*\hat{i}_Q^!\mathit{\Omega}_{(2)}(E))_V & \longrightarrow & H^i(i_P^*\mathit{\Omega}_{(2)}(E))_V \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\longrightarrow H^i_{(2)}(\mathcal{A}^G_{(S_V,(P,Q)\vee S_V]}(b);\mathbb{C}) & \longrightarrow & H^i_{(2)}(\mathcal{A}^G_{[S_V,(P,Q)\vee S_V]}(b);\mathbb{C}) & \longrightarrow & H^i_{(2)}(\mathcal{A}^G_{S_V}(b);\mathbb{C}) \longrightarrow
\end{array}$$

where to save space we have omitted the tensor product with $H(\mathfrak{n}_P;E)_V$ in the bottom row. We have already noted that the first vertical arrow is an isomorphism, while the last vertical arrow is an isomorphism by Lemma 20. The proposition now follows from the 5-lemma. □

Theorem 24. *For a regular G-module E, the micro-support $\mathrm{SS}(\Omega_{(2)}(E))$ consists of those irreducible L_P-modules V satisfying*

(i) $H(\mathfrak{n}_P;E)_V \neq 0$,
(ii) $(V|_{{}^0L_P})^* \cong \overline{V|_{{}^0L_P}}$, *and*
(iii) $(\xi_V+\rho_P)|_{\mathfrak{a}_P^G}=0$.

For such a V and any $Q\in[P,G]$,

$$\mathrm{Type}_{Q,V}(\Omega_{(2)}(E)) = H_{(2)}(\mathcal{A}^G_{[P,(P,Q)]}(b);\mathbb{C})\otimes H(\mathfrak{n}_P;E)_V .$$

The weak micro-support $\mathrm{SS}_w(\Omega_{(2)}(E))$ is similarly characterized by omitting condition (ii).

Proof. Let V be an irreducible L_P-module. By the definition of micro-support in §6 and Proposition 23, $V\in\mathrm{SS}(\Omega_{(2)}(E))$ if and only if conditions (i), (ii), and

$$\text{(iii)}' \qquad [Q_V,Q'_V]\cap[(P,T'_V),(P,T_V)]\neq\emptyset$$

are satisfied. Clearly (iii) implies (iii)′. Conversely we will assume (iii)′ holds and prove (iii); together with Proposition 23 this will establish the theorem.

By Langlands's "geometric lemma" [12, IV, §6.11], there exists one and only one $R\in[P,G]$ such that

$$\text{(104a)} \qquad \langle\xi_V+\rho_P,\beta_\alpha^{R\vee}\rangle\geq 0 \qquad \text{for } \alpha\in\Delta_P^R, \text{ and}$$

$$\text{(104b)} \qquad \langle\xi_V+\rho_P,\gamma_R^\vee\rangle< 0 \qquad \text{for } \gamma\in\Delta_P\setminus\Delta_P^R.$$

Since $\gamma^{\vee R}\in-\mathrm{cl}(\mathfrak{a}_P^{R*+})$ for $\gamma\in\Delta_P\setminus\Delta_P^R$, (104a) implies that $\langle\xi_V+\rho_P,\gamma^{\vee R}\rangle\leq 0$. Together with (104b), this yields $\langle\xi_V+\rho_P,\gamma^\vee\rangle<0$ for all $\gamma\in\Delta_P\setminus\Delta_P^R$, that is, $(P,R)\leq Q_V$. However (iii)′ implies that $Q_V\leq(P,T_V)$. We conclude that $\langle\xi_V+\rho_P,\beta_\gamma^\vee\rangle\geq 0$ for all $\gamma\in\Delta_P\setminus\Delta_P^R$, which means that $(\xi_V+\rho_P)_R\in\mathrm{cl}({}^+\mathfrak{a}_R^*)$. However (104b) implies that $(\xi_V+\rho_P)_R\in-\mathfrak{a}_R^{*+}$. Since $\mathrm{cl}({}^+\mathfrak{a}_R^*)\cap(-\mathfrak{a}_R^{*+})=\emptyset$ unless $R=G$, we see that

$$\text{(105)} \qquad \xi_V+\rho_P\in\mathrm{cl}({}^+\mathfrak{a}_P^*).$$

Similarly there exists a unique $R\in[P,G]$ such that

$$\text{(106a)} \qquad \langle\xi_V+\rho_P,\beta_\alpha^{R\vee}\rangle> 0 \qquad \text{for } \alpha\in\Delta_P^R, \text{ and}$$

$$\text{(106b)} \qquad \langle\xi_V+\rho_P,\gamma_R^\vee\rangle\leq 0 \qquad \text{for } \gamma\in\Delta_P\setminus\Delta_P^R.$$

Since $\beta^\vee_{\alpha R} \in \operatorname{cl}(\mathfrak{a}_R^{*+})$ for $\alpha \in \Delta_P^R$, equation (105) implies that $\langle \xi_V + \rho_P, \beta^\vee_{\alpha R}\rangle \geq 0$. Together with (106a), this yields $\langle \xi_V + \rho_P, \beta^\vee_\alpha\rangle > 0$ for all $\alpha \in \Delta_P^R$, that is, $R \leq (P, T'_V)$. However (iii)$'$ implies that $(P, T'_V) \leq Q'_V$. We conclude that $\langle \xi_V + \rho_P, \alpha^\vee\rangle \leq 0$ for all $\alpha \in \Delta_P^R$, which means that $(\xi_V + \rho_P)^R \in -\operatorname{cl}(\mathfrak{a}_P^{R*+})$. However (106a) implies that $(\xi_V + \rho_P)^R \in {}^+\mathfrak{a}_P^{R*}$. Since $(-\operatorname{cl}(\mathfrak{a}_P^{R*+})) \cap {}^+\mathfrak{a}_P^{R*} = \emptyset$ unless $R = P$, we see that $\xi_V + \rho_P \in -\operatorname{cl}(\mathfrak{a}_P^{*+})$. This together with (105) establishes (iii) since $\operatorname{cl}({}^+\mathfrak{a}_P^*) \cap (-\operatorname{cl}(\mathfrak{a}_P^{*+})) = \mathfrak{a}_G^*$. ☐

A parabolic $\mathbb{R}$-subgroup P_0 of G is called *fundamental* if $\mathfrak{p}_0$ contains a fundamental (that is, maximally compact) Cartan subalgebra of $\mathfrak{g}$.

Corollary 25. *For $P \in \mathcal{P}$, there exists an irreducible L_P-module $V \in \operatorname{SS}(\Omega_{(2)}(E))$ if and only if $(E|_{{}^0G})^* \cong \overline{E|_{{}^0G}}$ and P contains a fundamental parabolic $\mathbb{R}$-subgroup of G. The type of V is finite dimensional if and only if $P = G$.*

Proof. Apply Theorem 24, [7, 3.6(iii)(iv)] (see also [33, Lemma 8.8]) and the remark at the end of §12.3. ☐

Recall that a symmetric space $D = G(\mathbb{R})/KA_G$ is called *equal-rank* if $\mathbb{C}$-rank $G =$ rank $K +$rank A_G. Any Hermitian symmetric space is equal-rank and every equal-rank symmetric space has even dimension. The symmetric spaces associated to $G(\mathbb{R}) = \mathrm{SO}(2p, 2q+1)$ where $p > 1$ are examples of non-Hermitian equal-rank symmetric spaces.

Corollary 26. *If X is an arithmetic quotient of an equal-rank symmetric space and E is an irreducible regular G-module, then $\operatorname{SS}(\Omega_{(2)}(E)) = \{E\}$.*

Proof. Since D is equal-rank, the only fundamental parabolic $\mathbb{R}$-subgroup of G is G itself. Furthermore $(E|_{{}^0G})^* \cong \overline{E|_{{}^0G}}$ for any G-module E [7, 1.5, 1.6]. Now apply Corollary 25. ☐

13. The Conjectures of Borel and Zucker

Associated to any finite-dimensional irreducible representation σ of $G(\mathbb{R})$, Satake [39] constructs a compactification D^*_σ of D which is a disjoint union of so-called *real boundary components*; D is always a real boundary component and the others are symmetric spaces of lower rank. The group $G(\mathbb{R})$ acts on D^*_σ and those real boundary components whose normalizers are defined over $\mathbb{Q}$ are called the *rational boundary components*[4]. Under a condition on σ now known as *geometric rationality* [13], Satake [40] constructs a corresponding compactification X^*_σ of X by taking the quotient under Γ of the union of the rational boundary

[4]The actual definition is more complicated but is equivalent to what is given here under the condition of geometric rationality.

components (with a suitable topology). The compactification X^*_σ is stratified by arithmetic quotients of the rational boundary components.

An important example is when D is a Hermitian symmetric space. In this case D may be realized as a bounded symmetric domain in $\mathbb{C}^N$ for some N and one of the Satake compactifications is homeomorphic to the natural compactification $\operatorname{cl}(D) \subseteq \mathbb{C}^N$. The various real boundary components are again Hermitian symmetric spaces. Geometric rationality in this case was proved by Baily and Borel [2]; the resulting compactification X^* of X is called the *Baily-Borel-Satake compactification*. Baily and Borel prove that X^* has the structure of a normal projective algebraic variety.

More generally consider the case where D is equal-rank. A *real equal-rank Satake compactification* is a Satake compactification for which all real boundary components are equal-rank symmetric spaces. The possible real equal-rank Satake compactifications are enumerated in [44, (A.2)]. For some equal-rank symmetric spaces, such as the one associated to $G(\mathbb{R}) = \mathrm{SO}(4,4)$, such a compactification does not exist. On the other hand, if $G(\mathbb{R}) = \mathrm{SO}(2p, 2q+1)$ then a real equal-rank Satake compactification does exist (and is unique if $p > 1$).

In [35] we prove that every real equal-rank Satake compactification is geometrically rational aside from some $\mathbb{Q}$-rank 1 and 2 exceptions in which $\mathbb{Q}\text{-rank}\, G \neq \mathbb{R}\text{-rank}\, G$.. The resulting compactification X^*_σ is also called a *real equal-rank Satake compactification*; the Baily-Borel-Satake compactification is an example.

Theorem 27 (Zucker/Borel Conjecture [26], [37])**.** *Let X^*_σ be a real equal-rank Satake compactification of an equal-rank locally symmetric space $X = \Gamma\backslash D$. Then there is a natural quasi-isomorphism $\mathit{\Omega}_{(2)}(X^*_\sigma;\mathbb{E}) \cong \mathcal{I}_p\mathcal{C}(X^*_\sigma;\mathbb{E})$ where p is any middle perversity.*

Proof. (Compare the proof of the Rapoport conjecture in [33, §27] and the exposition in [34, §§19, 20].) Let $\widehat{X}$ be the reductive Borel-Serre compactification and note that $\mathit{\Omega}_{(2)}(X^*_\sigma;\mathbb{E}) \cong R\pi_*\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})$, where $\pi\colon \widehat{X} \to X^*_\sigma$ is Zucker's quotient map (22). For x in a proper stratum F of X^*_σ, let $i_x\colon \{x\} \hookrightarrow X^*_\sigma$ denote the inclusion. By the local characterization of middle perversity intersection cohomology on a space with even dimensional strata [21], [9, V, 4.2] it suffices to verify

$$H^i(i_x^* R\pi_*\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})) = 0, \qquad i \geq (1/2)\operatorname{codim} F, \tag{107}$$

$$H^i(i_x^! R\pi_*\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})) = 0, \qquad i \leq (1/2)\operatorname{codim} F. \tag{108}$$

We consider (107); the proof of (108) is similar. Let $k\colon \pi^{-1}(x) \hookrightarrow \widehat{X}$ and observe that $H(i_x^* R\pi_*\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})) \cong H(k^*\mathit{\Omega}_{(2)}(\widehat{X};\mathbb{E})) \cong H(k^*\mathcal{S}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E)))$, where for the last step we use Theorem 4. However one can show[5] that $k^*\mathcal{S}_{\widehat{X}}(\mathit{\Omega}_{(2)}(E)) \cong \mathcal{S}_{\pi^{-1}(x)}(k^*\mathit{\Omega}_{(2)}(E))$ for an $\mathfrak{L}$-module $k^*\mathit{\Omega}_{(2)}(E)$ on $\pi^{-1}(x)$ (which is itself the reductive Borel-Serre compactification of a locally symmetric space [43, (3.8), (3.10)]). Thus we are reduced to the corresponding vanishing of $H(k^*\mathit{\Omega}_{(2)}(E))$. However since $\mathrm{SS}(\mathit{\Omega}_{(2)}(E)) = \{E\}$ by Corollary 26 and since all real boundary components are equal-rank, we can use [33, Corollary 26.2] to estimate that $d(k^*\mathit{\Omega}_{(2)}(E)) < (1/2)\operatorname{codim} F$ where $d(\mathcal{M})$ is defined in (47). Now apply Theorem 1. □

Remark. Under the weaker hypothesis that all *rational* boundary components are equal-rank we can still obtain an estimate of $\mathrm{SS}(k^*\mathit{\Omega}_{(2)}(E))$ [33, Proposition 23.3]. However the precise estimate on $d(k^*\mathit{\Omega}_{(2)}(E))$ in the above proof requires the stronger hypothesis that all *real* boundary components are equal-rank [33, Corollary 25.4, Theorem 26.1].

Corollary 28. *Under the hypotheses of the theorem,* $H_{(2)}(X;\mathbb{E}) \cong I_pH(X^*_\sigma;\mathbb{E})$.

Proof. Since $\mathit{\Omega}_{(2)}(X^*_\sigma;\mathbb{E}))$ is fine [42], [44], $H(\mathit{\Omega}_{(2)}(X^*_\sigma;\mathbb{E})) \cong H_{(2)}(X;\mathbb{E})$. □

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[5] To construct $k^*\mathit{\Omega}_{(2)}(E)$ one first restricts $\mathit{\Omega}_{(2)}(E)$ to the locally closed constructible set $\pi^{-1}(F)$ as in [33, §3.4] and then to the fiber $\pi^{-1}(x)$ as in [33, §3.5].

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Leslie Saper
Department of Mathematics
Duke University
Box 90320
Durham, NC 27708
U.S.A.
E-mail: saper@math.duke.edu
Website: http://www.math.duke.edu/faculty/saper