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Remarks on the Satake Compactifications

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Abstract: This article has three independent parts. The first one is a simplification, using some old results of the author, of a construction of the compactifications recently given by A. Borel and L. Ji. The second one is another construction by directly describing the relative neighborhood systems of the boundary points . The third is a realization of the space and its compactification as a bounded domain and its closure in a vector space.

Introduction

This article is about various constructions and realizations of the Satake compactifications of Riemannian symmetric spaces. After a section on preliminaries, it consists of three logically independent parts.

The first part (Section 2) is inspired by the article of A. Borel and L. Ji on a uniform method of constructing all the known compactifications. Basic to this part are the following two observations. First, the "generalized Siegel sets" introduced and used in [1] are almost the same as the "admissible domains" introduced earlier in [10] in connection with boundary convergence of Poisson integrals on a symmetric space. Second, the proof of the "strong separation theorem" which plays an essential role in [1] can be considerably simplified by making use of the Bruhat Lemma. Based on these observations, Section 2 contains a modification of the Borel-Ji construction, making it simpler and exploiting its connections with the ideas of [10] where the possibility of a similar construction had already been sketched.

The construction of Section 2 starts with an intrinsic definition of the admissible domains, which are generalizations of non-tangential neighborhoods of the

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boundary points of the unit disc, then proceeds to the definition of the boundary points, and finally to the definition of the topology of the extended space. The question naturally arises whether it is possible to avoid the admisisble domains and start with giving a definition of the relative neighborhood systems of the prospective boundary points. Section 3 is devoted to showing that this can be done and leads fairly easily to a full independent construction of the Satake compactifications.

The last part, Section 4, gives realizations of symmetric spaces as bounded domains in a vector space in such a way that the ordinary closures are just the Satake compactifications. The Harish-Chandra realization of a Hermitian symmetric space is a special instance of this. The possibility of such realizations in the general case was suggested to me by the articles of L. Ji [9] and W. Casselman [3] where similar realizations are given for the maximal flat subspaces of a symmetric space. But actually the proofs in Section 4 are largely independent of [9] and [3].

I want to express my thanks to J. Faraut for his permission to use an important idea of his (for details see Section 4), and to L. Ji for many useful discussions and for the invitation to write this article.

1. Preliminaries and Notation

Since many of the definitions and ideas in this paper are from [10] our notations will be compatible with those of [10]. This section contains matters (sometimes slightly differently worded) that can also be found in [10]; much of this is completely standard. Some further details and references are in [10], but the present paper can be read independently.

$X \cong G/K$ will be a Riemannian symmetric space of non-compact type with G connected, semi-simple and having finite center. We denote the base point of X, stabilized by K, by o. In G, we denote conjugation as $g^h = hgh^{-1}$ (note the difference with [1]). We will use the self-explanatory notations U^h, U^V for sets U, V.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ will be a fixed Cartan decomposition, θ the Cartan involution, $\mathfrak{a}$ a maximal abelian subspace in $\mathfrak{p}$. By *roots* we will always mean $\mathfrak{a}$ - roots, $\mathfrak{g}_\lambda$ will denote the root space corresponding to the root λ. We choose an ordering and denote by Π the system of simple roots.

For any subset $E \subseteq \Pi$, we define the subalgebras $\mathfrak{a}(E) = \{H \in \mathfrak{a} : \lambda(H) = 0, \forall\lambda \in E\}$ and $\mathfrak{n}(E) = \sum \mathfrak{g}_\lambda$ with the sum over those $\lambda > 0$ that do not vanish on $\mathfrak{a}(E)$. We define $\mathfrak{n}^E = \sum \mathfrak{g}_\lambda$ with those $\lambda > 0$ that vanish on $\mathfrak{a}(E)$, and write $\mathfrak{a}^E$ for the orthogonal complement of $\mathfrak{a}(E)$ in $\mathfrak{a}$ with respect to the Killing form. We set $\bar{\mathfrak{n}}(E) = \theta\mathfrak{n}(E)$, $\bar{\mathfrak{n}}^E = \theta\mathfrak{n}^E$ and we write $\mathfrak{n}$, $\bar{\mathfrak{n}}$ for $\mathfrak{n}(\emptyset)$, $\bar{\mathfrak{n}}(\emptyset)$. The analytic subgroups of G corresponding to all these subalgebras will be denoted by the corresponding Roman capitals. They are all closed. We note that $N = N^E N(E)$ is a semidirect product.

We let $M_K(E)$ be the centralizer of $\mathfrak{a}(E)$ in K. Then $M(E) = M_K(E)A^E N^E$ (also $= M_K(E)A^E \bar{N}^E$) is a reductive subgroup of G given in its Iwasawa decomposition. $B(E)$, the normalizer of $N(E)$ (and of itself) is a parabolic subgroup, $B(E) = M(E)A(E)N(E)$ is its Langlands decomposition. Conjugates of $B(E)$ $(E \subseteq \Pi)$ give all the parabolic subgroups of G. The Lie algebras of the subgroups here introduced will be denoted $\mathfrak{m}_K(E)$, $\mathfrak{m}(E)$, $\mathfrak{b}(E)$. There is an induced Cartan decomposition $\mathfrak{m}(E) = \mathfrak{m}_K(E) + \mathfrak{p}^E$.

By the Iwasawa decomposition we have $X = G \cdot o = \bar{N}A \cdot o$, and the expression $\bar{n}a \cdot o$ $(\bar{n} \in \bar{N}, a \in A)$ of a point is unique. (There is also a unique expression $a\bar{n}' \cdot o$, here $\bar{n}' = \bar{n}^{a^{-1}}$ but a is the same.) Similarly we have $X^E = M(E) \cdot o = \bar{N}^E A^E \cdot o$, a totally geodesic subspace of X. Now, $X = \bar{N}(E)A(E) \cdot X^E$ is again a unique decomposition.

We denote by $\mathfrak{a}^+$ the open positive Weyl chamber in $\mathfrak{a}$ and we set $\mathfrak{a}(E)^+ = \{H \in \mathfrak{a}(E) : \lambda(H) > 0, \forall \lambda \in \Pi \setminus E\}$ (a face of the chamber $\mathfrak{a}^+$). For $T \in \mathfrak{a}(E)$ we set

$$A(E)^T = (\exp T)\overline{A(E)}^+,$$

where $A(E)^+ = \exp \mathfrak{a}(E)^+$. This set can also be written $\{a \in A(E) : \log a \geq T\}$ where $\log a \geq T$ means $\lambda(\log a) \geq \lambda(T)$ for all $\lambda \in \Pi \setminus E$. We say "$T$ is large in $\mathfrak{a}(E)$", "$\log a \to \infty$ in $\mathfrak{a}(E)$" if $\lambda(T)$ is large, resp. $\lambda(\log a) \to \infty$ for all $\lambda \in \Pi \setminus E$.

Given $T \in \mathfrak{a}(E)^+$, for an element g or a subset U of G we will write in short, g^T, U^T for the sets of conjugates $A^{A(E)^T}, U^{A(E)^T}$, and will write g^{-T}, U^{-T} for the sets $g^{(A(E)^T)^{-1}}$, $U^{(A(E)^T)^{-1}}$. It is an obvious but important fact that if U is compact in $\bar{N}(E)$ then for large T in $\mathfrak{a}(E)$ the set U^T is small (it is contained in any given neighborhood of e in $\bar{N}(E)$ provided T is large enough). If U is compact in $N(E)$, the same is true for U^{-T}.

For any $E \subseteq \Pi$, X^E is the direct product of irreducible spaces X^{E_i}, the subsets E_i are called the components of E. Let $E_0 \subseteq \Pi$ be such that E_0 contains no component of Π. The following construction depends on E_0, which we now consider fixed.

A set $E \subseteq \Pi$ is said to be E_0-*connected* if no component of E is contained in E_0. For such E, we define E'' as the set of all $\lambda \in E_0$ such that $E \cup \{\lambda\}$ is not E_0-connected, and we set $E' = E \cup E''$. Then, E is the unique maximal E_0-connected subset of E' and E' is the unique maximal subset of Π with the latter property. E and E'' are disjoint, both are unions of components of E'. There is a corresponding direct product decomposition $X^{E'} = X^E \times X^{E''}$. We have $M(E') = M(E)M(E'')$ and $M(E')$ preserves the product structure of $X^{E'}$. So,

$M(E')$ acts on X^E by projecting its action on $X^{E'}$. The stabilizer for this action is $M_K(E)M(E'')$. The group $B(E') = M(E')A(E')N(E')$ can also be made to act on X^E by defining the action of $A(E')N(E')$ to be trivial. We denote the stabilizer of o for this action by $B_{\mathrm{st}}(E')$. We have $B_{\mathrm{st}}(E') = M_K(E)M(E'')A(E')N(E')$. This is the same group as in Remark 3 on page 22 of [10]; it will play an important role later on.

To each one of the subsets E_0 there belongs a Satake compactification $\bar{X}(E_0)$. These have axiomatic characterizations due to Satake ([13], or rewritten in our present notations, [10, pp. 21-22]). There are a number of ways of constructing them, i.e. to prove their existence (e.g. [13], [12], [6], [9], [1]). The present paper will add a few more.

2. A construction of the compactifications

In order to avoid many repetitions we make the following conventions. For any subgroup $H \subseteq G$ or any one of the spaces X^E, the notations V_H, V_{X^E} will automatically mean compact neighborhoods of e in H, resp. of o in X^E. Similarly, U_H, U_{X^E} will mean compact subsets of H or X^E.

Throughout this section we will be in the situation described in Section 1. A subset $E_0 \subseteq \Pi$ will be given once and for all, E will always stand for an E_0 - connected set, E', E'' etc. will be as in Section 1.

Definition 2.1. For $T \in \mathfrak{a}(E')$, $U = U_{\bar{N}(E')}$, $C = U_{M(E'')}$, $V = V_{M(E')}$ we set

$$\Gamma_{U,C}^{T,V} = A(E')^T UCV \cdot o. \tag{2.1}$$

These sets were introduced in [10] and were called *admissible domains.* they are generalizations of non-tangential angular neighborhoods of the point 1 in the complex unit disc, U and C playing the role of the opening angle. They and their translates by elements of G are used in [10] to prove a generalization of Fatou's theorem about non-tangential limits of harmonic functions.

Since U, C, V are compact and all elements of G are isometries, $\Gamma_{U,C}^{T,V}$ is a set of points within a bounded distance of $A(E')^T \cdot o$ (which is a polyhedral cone in the flat subspace $A(E') \cdot o$ of X, "far out at infinity" if T is large). This is also the crucial feature of the *generalized Siegel sets* $S_{\varepsilon,T,V}$ introduced by Borel and Ji [1]. What we want to show is that the Borel-Ji construction can be simplified by replacing the $S_{\varepsilon,T,V}$ by the $\Gamma_{U,C}^{T,V}$ and using some new arguments along part of the way.

The following is a slight strengthening of [10, Lemma 2.1].

LEMMA 2.2. *Let $\tilde{U} = U_{B_{\rm st}(E')}$ be given. Then for any U, C as above there exists U_1, C_1 such that for any T_1, V_1 there can be found T, V with the property $\tilde{U}\Gamma_{U,C}^{T,V} \subseteq \Gamma_{U_1,C_1}^{T_1,V_1}$*

Proof. It suffices to consider the four cases $\tilde{U} \subseteq A(E')$, $\tilde{U} \subseteq M(E'')$, $\tilde{U} \subseteq M_K(E)$, $\tilde{U} \subseteq N(E')$. The first case is trivial: we may take $U_1 = U$, $C_1 = C$, and then $V = V_1$, $T = T^1 + T_0$ where $-T_0$ is a "lower bound" for $\tilde{U}$, that is, $\tilde{U} \subseteq A(E')^{-T_0}$. In the second case

$$\tilde{U}\Gamma_{U,C}^{T,V} = A(E')^T U^{\tilde{U}}(\tilde{U}C)V \cdot o,$$

so $U_1 = U^{\tilde{U}}$, $C_1 = \tilde{U}C$ will work (with $T = T_1$, $V = V_1$). In the third case,

$$\tilde{U}\Gamma_{U,C}^{T,V} = A(E')^T U^{\tilde{U}} C^{\tilde{U}} V^{\tilde{U}} \cdot o$$

and the situation is again clear.

Finally, if $\tilde{U} \subseteq N(E')$, first we fix some $\tilde{T}$ and $\tilde{V}$. For large T, $\tilde{U}^{-T}UCV \cdot o$ is a small neighborhood in $X$of the set $UCV \cdot o$. Therefore (using $\tilde{U}A(E')^T = A(E')^T\tilde{U}^{-T}$) we can fix U_1, C_1 so that

$$\tilde{U}\Gamma_{U,C}^{T,V} \subseteq \tilde{U}\Gamma_{U_1,C_1}^{\tilde{T},\tilde{V}}$$

holds for all sufficiently large T and small V. With this U_1, C_1, the assertion of the Lemma holds: for any V_1 there is T large enough so that $\tilde{U}^{-T}UCV \cdot o \subseteq U_1C_1V_1 \cdot o$. □

The following is our version of the "strong separation" property, essential in the Borel-Ji construction. We will consider two E_0- connected sets E, E_1 and corresponding admissible domains. Sets without subscripts correspond to E, sets with subscript 1 to E_1

PROPOSITION 2.3. *If $E \neq E_1$ or if $E = E_1$ and $g \notin B_{\rm st}(E')$, then for any U, C, U_1, C_1, there exist T, V, T_1, V_1 and V_G such that $g\Gamma_{U,C}^{T,V}$ and $V_G\Gamma_{U_1,C_1}^{T_1,V_1}$ are disjoint.*

Proof. First we note that instead of V_G it is enough to find $V_{\bar{N}(E_1')}V_{M(E_1')}$ with the same property. In fact, we can fix an (arbitrary) $\tilde{V} = V_{B_{\rm st}(E_1')}$ and by Lemma 2.2 find $U_2 \supset U_1$, $C_2 \supset C_1$ so that $\tilde{V}\Gamma_{U_2,C_2}^{T_2,V_2} \subseteq \Gamma_{U_1,C_1}^{T_1,V_1}$ for large T_2 and small V_2. Then $V_G = \tilde{V}V_{\bar{N}(E_1')}V_{M(E_1')}$ will have the required property.

We express g with the aid of the Bruhat decomposition $G = \cup_w \bar{N}(E')wM(E')A(E')N(E')$; here $w \in W/W^{E'}$ with $W^{E'}$ denoting the subgroup of the Weyl group W generated by reflections coming from $M(E')$. w is represented by an element of M', the normalizer of $\mathfrak{a}$ in K. Again, by Lemma 2.2, the $A(E')N(E')$- part of g will be irrelevant for our statement, so we may assume that $g = \bar{n}wm$ ($\bar{n} \in \bar{N}(E')$), $m \in M(E')$). Now we have

$$g\Gamma_{U,C}^{T,V} = \bar{n}\big(A(E')^T\big)^w U^{wm} C^{wm} V^{wm} wm \cdot o \tag{2.2}$$

and the claim is that for appropriate choices this is disjoint from

$$V_{\bar{N}(E_1')} V_{M(E_1')} A(E_1')^{T_1} U_1 C_1 V_1 \cdot o = V_{\bar{N}(E_1')} A(E_1')^{T_1} \tilde{U}_1 V_{M(E_1')} C_1 V_1 \cdot o \tag{2.3}$$

where $\tilde{U}_1$ is still in $\bar{N}(E_1')$, since $M(E_1')$ normalizes $N(E_1')$. In both sets we write the points in the form $\bar{n}a \cdot o$ using the unique decomposition $X = \bar{N}A \cdot o$. The first step in this is to write the points of $U^{wm}C^{wm}V^{wm}wm \cdot o$ and of $\tilde{U}_1 V_{M(E_1')} C_1 V_1 \cdot o$ in such a form. These sets being compact, the A - components we obtain are bounded. If $E \neq E_1$, or $w \neq e$, the Weyl chamber faces $(\mathfrak{a}(E')^+)^w$ and $\mathfrak{a}(E)^+$ are different. Hence taking T and T_1 large we can make sure that the A-components of all elements of (2.2) and (2.3) are different, so these sets are disjoint.

If $E = E_1$ and $w = e$, but $\bar{n} \neq e$ then in (2.2) and (2.3) we have $U^{wm} \subseteq \bar{N}(E')$, furthermore, $C^{mw}V^{mw}mw \cdot o$ and $V_{M(E_1')}C_1V_1 \cdot o$ are in $X^{E'}$. If we write the elements of (2.2), (2.3) in the decomposition $X = \bar{N}(E')A(E') \cdot X^{E'}$, the $\bar{N}(E')$ component will be in $\bar{n}(U^{wm})^T$, resp. in $V_{\bar{N}(E_1')}\tilde{U}_1^{T_1}$ So, when $\bar{n} \notin V_{\bar{N}(E_1')}$ and T, T_1 are large, the two sets are disjoint.

Finally, if $E = E_1$, $w = e$ and $\bar{n} = e$, then the hypothesis implies $g = m \notin M_K(E)M(E'')$. Taking $V_{M(E_1')}$ such that $e \notin V_{M(E_1')}$, (2.2) and (2.3) will again be disjoint for large T, T_1 and small V, V_1. □

In order to compare with the Borel-Ji article we reformulate and slightly extend the fundamental Proposition 2.4 of [1] in our language:

If $E \neq E_1$ or $E = E_1$ and $g \notin B(E)$, then for any U_X, $gA(E)^T \cdot U_X$ and $V_G A(E_1)^{T_1} \cdot U_X$ are disjoint for large T, T_1, provided $g^{-1}V_G$ is disjoint from $B(E)$.

Our argument above, with unessential modifications, proves this statement. It also proves Proposition 4.1 in [1], which is a complement to Proposition 2.4 dealing with the case $E = E_1$, $g \in B(E)$. Our argument can be said to be simpler since it does not make use of Satake's idea of imbedding X into a projective space; it is also shorter. As a curiosity we might mention that there is yet another possible approach to this question: One can use the results of [10] about convergence of the Poisson integral along the sets $\Gamma_{U,C}^{T,V}$ to any given continuous boundary function to prove the desired separation. Such a proof, in the last analysis, is based on the Furstenberg-Moore embedding [5], [12] of X into a space of probability measures, just as the proof in [1] is based on Satake's imbedding.

In any case, having proposed an alternative proof of Propositions 2.4 and 4.1 of [1], we could say that the rest of the construction proceeds as in [1]. But we want to make some further remarks about combining the ideas of [1] with those of [10] in order to arrive at a construction which is perhaps the most natural one.

As in [10], for fixed E, U, C we consider the filter $\mathcal{F}^E_{U,C}$ generated by all $\Gamma^{T,V}_{U,C}$. We denote by $g\mathcal{F}^E_{U,C}$ the filter of the g-translates $g\Gamma^{T,V}_{U,C}$ and by $g\tilde{\mathcal{F}}^E$ the family of all filters $g\mathcal{F}^E_{U,C}$. We say that two families are equivalent if for every filter in one of the families there is a rougher one in the other. (This means that if a sequence converges along some filter in one family then it also converges along some filter in the other; in the present case this is generalized non-tangential convergence.) The points of the compactification $\bar{X}(E_0)$ will now be the equivalence classes of filters $g\tilde{\mathcal{F}}^E$. In this way the points to be attached to X appear together with ways to tend towards them from the interior.

Lemma 2.2 and Proposition 2.3 show that $g\tilde{\mathcal{F}}^E$ is equivalent to $\tilde{\mathcal{F}}^E$ if and only if $g \in B_{\mathrm{st}}(E')$. One can write $g \cdot \iota_E(o)$ for $g\tilde{\mathcal{F}}^E$ and define the imbedding ι_E of X^E into $\bar{X}(E_0)$ by $\iota_E(m \cdot o) = m \cdot \iota_E(o)$ for $m \in M(E)$. Then $\bar{X}(E_0)$ is disjoint union of "boundary components" $g \cdot \iota_E(X^E)$, and one has a complete description of $\bar{X}(E_0)$ as a G-set.

It remains to define the topology of $\bar{X}(E_0)$ and prove its properties. At this point we can say that the construction proceeds as in [1], by describing the class of convergent sequences and verifying the required properties. We should remark here also that a convenient neighborhood basis of $\iota_E(o)$ can be obtained as follows. A basis of relative neighborhoods with respect to X of $\iota_E(o)$ is formed by the sets

$$V_G M_K(E'') A(E)^{T'} \cdot o$$

with $T' \in \mathfrak{a}(E')$. (T' is restricted to the subset $\mathfrak{a}(E')$ of $\mathfrak{a}(E)$.) It is not hard to see that this description is equivalent to the one given in Section 5 of [1]. Similarly, full neighborhoods of $\iota_E(o)$ are unions over E_0-connected sets $D \supset E$:

$$V_G M_K(E'') \big(\cup A^D(E)^{T'_D} \cdot \iota_D(o) \big)$$

where $T'_D \in \mathfrak{a}(D')$.

3. Another Construction

In this section, which is independent of the preceding one, we describe another construction of the Satake compactifications. This will still be "from inside", that is, by attaching points to X, but without the use of Siegel domains or admissible domains. We will directly define the system of relative neighborhoods of the points to be attached, thus obtaining $\bar{X}(E_0)$ as a set. Then we will define the action of G, show that the relative closures of the relative neighborhoods give the full neighborhood system of a topology, and then verify that what we have constructed is a really the Satake compactification.

In order not to get too lengthy and complicated, we will carry this out in detail only for the maximal compactification $\bar{X} = \bar{X}(\emptyset)$. We will then briefly sketch the modifications and additional arguments necessary for the general case.

Definition 3.1. For $E \subseteq \Pi$, let $\mathcal{F}^E$ be the filter generated by the sets $V_G A(E)^T \cdot o$ ($V_G \subset G$, $T \in \mathfrak{a}(E)$).

Remark 1. The filters $\mathcal{F}_U^E$ of Section 2 (now without C, and with $E' = E$, since $E_0 = \emptyset$) are all finer than $\mathcal{F}^E$. In fact, given V_G and T, for any U there exist $\tilde{T}$, V such that $\Gamma_U^{\tilde{T},V} = A(E)^{\tilde{T}} UV \cdot o = U^{\tilde{T}} V A(E)^{\tilde{T}} \cdot o \subseteq V_G A(E)^T \cdot o$.

Lemma 3.2. *The sets* $V_{\bar{N}(E)} A(E)^T \cdot V_{X^E}$ *or* $V_{\bar{N}A^E} A(E)^T \cdot o$ *also generate* $\mathcal{F}^E$.

Proof. The sets written are the same by $\bar{N} = \bar{N}(E)\bar{N}^E$ and $X^E = \bar{N}^E A^E \cdot o$. We must show that for given $\tilde{V}_{\bar{N}(E)} A(E)^{\tilde{T}} \cdot V_{X^E}$ there exists some $V_G A(E)^T \cdot o$ contained in it. We look for V_G in the form

$$V_G = V_{\bar{N}(E)} V_{N(E)} V_{A(E)} V_{M(E)}.$$

We choose $V_{N(E)}$ and $V_{A(E)}$ arbitrarily and $V_{M(E)}$, $V_{\bar{N}(E)}$ such that

$$\tilde{V}_{X^E} = V_{M(E)} \cdot o \Subset V_{X^E}.$$

$$V_{\bar{N}(E)} \Subset \tilde{V}_{\bar{N}(E)}.$$

(The notation $\Subset$ means "contained in the interior of".) We take $T_0 \in \mathfrak{a}^+(E)$ such that $V_{A(E)} \subseteq A(E)^{-T_0}$.

For any choice of $\tilde{\tilde{V}}_{\bar{N}(E)}$ we will have, for sufficiently large T,

$$V_{N(E)}^{-(T-T_0)} \cdot \tilde{V}_{X^E} \subset V_{A(E)} \tilde{\tilde{V}}_{\bar{N}(E)} \cdot V_{X^E}$$

because the right hand side is a neighborhood in X of the compact set $\tilde{V}_{X^E}$.

It follows that for sufficiently large T,

$$\begin{aligned} V_{\bar{N}(E)} V_{N(E)} V_{A(E)} V_{M(E)} A(E)^T \cdot o &\subseteq V_{\bar{N}(E)} A(E)^{T-T_0} V_{N(E)}^{-(T-T_0)} \tilde{V}_{X^E} \\ &\subseteq V_{\bar{N}(E)} \tilde{\tilde{V}}_{\bar{N}(E)}^{T-2T_0} A(E)^{T-2T_0} \cdot V_{X^E} \\ &\subseteq \tilde{V}_{\bar{N}(E)} A(E)^{\tilde{T}} \cdot V_{X^E} \end{aligned}$$

finishing the proof. □

Lemma 3.3. *Given* $U = U_{\mathrm{B_{st}(E)}}$ *and any* $\tilde{V}_G$, $\tilde{T} \in \mathfrak{a}(E)$, *there exist* V_G, T *such that*

$$UV_G A(E)^T \cdot o \subset \tilde{V}_G A(E)^{\tilde{T}} \cdot o \tag{3.1}$$

Proof. It suffices to fix some $\tilde{\tilde{V}}_G \Subset \tilde{V}_G$ and find T such that

$$UA(E)^T \cdot o \subseteq \tilde{\tilde{V}}_G A(E)^{\tilde{T}} \cdot o \tag{3.2}$$

because then we can always find V_G such that $V_G^U \tilde{\tilde{V}}_G \subseteq \tilde{V}_G$, and (3.1) follows.

To prove (3.2) we distinguish cases. When $U \subseteq M_K(E)$, the statement is trivial since $M_K(E)$ commutes with $A(E)$ and fixes o. The case $U \subseteq A(E)$ is also trivial: then $UA(E)^T \subseteq A(E)^{T+T_0}$ with some $T_0 \in \mathfrak{a}(E)$. When $U \subseteq N(E)$, we choose some $V_{\bar{N}(E)} V_{M(E)} \subseteq \tilde{\tilde{V}}_G$ and some arbitrary $V_{A(E)}$. For large T we have

$$U^{-T} \cdot o \subseteq V_{\bar{N}(E)} V_{M(E)} V_{A(E)} \cdot o$$

since the right side is a neighborhood of o in X. So, for large T,

$$\begin{aligned} UA(E)^T \cdot o &= A(E)^T U^{-T} \cdot o \\ &\subseteq A(E)^T V_{\bar{N}(E)} V_{M(E)} V_{A(E)} \cdot o \\ &= V_{\bar{N}(E)}^T V_{M(E)} A(E)^T V_{A(E)} \cdot o \\ &\subseteq \tilde{\tilde{V}}_G A(E)^{\tilde{T}} \cdot o \end{aligned}$$

as was to be shown. □

One consequence of this Lemma is that $g\mathcal{F}^E = \mathcal{F}^E$ if $g \in B_{\mathrm{st}}(E)$. (By $g\mathcal{F}^E$ we mean the filter formed by the sets gS, $S \in \mathcal{F}^E$.) Writing the generators of $\mathcal{F}^E$ as in Lemma 3.2 it is immediate that this is actually "if and only if". We can now define $\bar{X} = \bar{X}(\emptyset)$ as a set by attaching all filters $g\mathcal{F}^E$ ($g \in G$, $E \subseteq \Pi$) to X; more intuitively we can say that the filter $g\mathcal{F}^E$ determines a point $g \cdot \iota_E(o)$ of $\bar{X}$. We can define the imbedding $\iota_E : X^E \to \bar{X}$ consistently by setting $\iota_E(m \cdot o) = m \cdot \iota_E(o)$ for $m \in M(E)$. It is then easy to see that each orbit $G \cdot \iota_E(o)$ is the disjoint union of *boundary components* of the form $g \cdot \iota_E(X^E)$, the family of these being parametrized by $G/B(E)$.

To complete the construction of $\bar{X}$ we must define its topology. We want this to be such that $g\mathcal{F}^E$ is exactly the system of relative X-neighborhoods of $g \cdot \iota_B(o)$. The full neighborhoods then must include the closures of the elements of $g\mathcal{F}^E$.

The closure of an element $\mathcal{N} \in \mathcal{F}^E$ must contain the *relative accumulation points*, that is, those $g \cdot \iota_D(o) \in \bar{X}$ for which every element of $g\mathcal{F}^D$ meets $\mathcal{N}$. We proceed to determine these for a family of generators of $\mathcal{F}^E$.

We denote by pr^D the projection of $\mathfrak{a}(E)$ onto $\mathfrak{a}^D(E)$ (which is the joint 0-space of E in $\mathfrak{a}^D$) along the decomposition $\mathfrak{a}^D(E) \oplus \mathfrak{a}(D)$.

PROPOSITION 3.4. *The relative accumulation points of* $\mathcal{N} = V_{\bar{N}A^E} A(E)^T \cdot o$ *are the points* $g \cdot \iota_D(o)$ *with* $D \supset E$, $g \in V_{\bar{N}A^E} A^D(E)^{\mathrm{pr}^D T}$.

Proof. For any $D \subseteq \Pi$ we have

$$G = \cup_{w \in W/W^D} \bar{N}(D) w B(D)$$

by the Bruhat Lemma. Together with Lemma 3.3 this shows that all the points of $\bar{X} = \bar{X}(\emptyset)$ can be uniquely written as $g \cdot \iota_D(o)$ with $g = \bar{n}_D w \bar{n}^D a^D$ ($\bar{n}_D \in \bar{N}(D)$, $w \in W/W^D$, $\bar{n}^D \in \bar{N}^D$, $a^D \in A^D$). We must determine for which ones of these does $g\tilde{\mathcal{N}} = g\tilde{V}_{\bar{N}A^D} A(D)^{\tilde{T}} \cdot o$ meet $\mathcal{N}$ for all $\tilde{V}_{\bar{N}A^D}$ and all $\tilde{T} \in \mathfrak{a}(D)$.

We consider first the case where $w = e$. If $E \not\subseteq D$, there is a root $\lambda \in E \setminus D$. For $\bar{n}a \cdot o \in g\tilde{\mathcal{N}}$ (with $\bar{n} \in \bar{N}$, $a \in A$), then $\lambda(\log a)$ will be very large for large $\tilde{T}$ while $\lambda(\log a)$ is bounded on $\mathcal{N}$. So, $\mathcal{N}$ can't meet every $g\tilde{\mathcal{N}}$.

Suppose now that $D \supset E$. Then we can rewrite g as $g = \bar{n} a^E a^D_E$ (with $\bar{n} = \bar{n}_D \bar{n}^D$, $a^D = a^E a^D_E$) using the decomposition $A^D = A^E A^D(E)$.

If $g \in V_{\bar{N}A^E} A^D(E)^{\mathrm{pr}^D T} \cdot o$ then $\bar{n}a^E \in V_{\bar{N}A^E}$ and $\log a^D_E \geq \mathrm{pr}^D T$ in $\mathfrak{a}^D(E)$. Taking any $\tilde{\mathcal{N}}$, the set $g\tilde{\mathcal{N}}$ contains $\bar{n}a^E a^D_E \tilde{a}_D \cdot o$ for all suffciently large $\tilde{a}_D$ in $A(D)$. But among these there are points which are also in $\mathcal{N}$. Indeed, for $\lambda \in D \setminus E$ we have $\lambda(\log(a^D_E \tilde{a}_D)) = \lambda(\log a^D_E) \geq \lambda(\mathrm{pr}^D T) = \lambda(T)$, and for $\lambda \in \Pi \setminus D$, $\lambda(\log(a^D_E \tilde{a}_D)) = \lambda(\log a^D_E) + \lambda(\log \tilde{a}_D)$ will be larger than $\lambda(T)$ for large $\tilde{a}_D$, showing that $a^D_E \tilde{a}_D \in A(E)^T$.

In the contrary case either $\bar{n}a^E \notin V_{\bar{N}A^E}$ or $\log a^D_E \not\geq \mathrm{pr}^D T$. In either case, for sufficiently small $\tilde{V}_{\bar{N}A^D}$, $g\tilde{\mathcal{N}}$ will be disjoint from $\mathcal{N}$, as can be seen by looking at the $\bar{N}A^E$ - part resp. the $A^D(E)$ - part in the unique representation of X as $\bar{N}A \cdot 0$.

Now we consider the case where $w \neq e$ in W/W^D. So, now $g = \bar{n}_D w \bar{n}^D a^D$ ($w \notin W^D$) and we ask the question: Is there such a g with the property that for all $\tilde{V}_G$, $\tilde{T}$, the set $g\tilde{\mathcal{N}} = g\tilde{V}_G A(D)^{\tilde{T}} \cdot o$ meets $\mathcal{N}$?

Since $g\tilde{V}_G = \tilde{V}^g_G g$ the question amounts to whether $gA(D)^{\tilde{T}} \cdot o$ meets $\left(\tilde{V}^g_G\right)^{-1}\mathcal{N}$ for all $\tilde{V}_G$, $\tilde{T}$. The first one of these sets can be written as $\bar{n}_D (A(D)^{\tilde{T}})^w U_{\bar{N}A} \cdot o$. For the second set we can use Lemma 3.3 to see that, for small $\tilde{V}_G$, it is contained in some $\tilde{U}_{\bar{N}A} A(E)^{\tilde{\tilde{T}}} \cdot o$. Writing the points of X in the form $\bar{n}a \cdot o$ ($\bar{n} \in \bar{N}$, $a \in A$) we see that on the second set $\lambda(\log a)$ is bounded below for every $\lambda \in \Pi$. Since $w \notin W^D$, some λ will be negative on $(\mathfrak{a}(D)^+)^w$. Hence for large $\tilde{T}$, $\lambda(\log a)$ will be negative with large absolute value on the first set. It follows that for large $\tilde{T}$ there is no intersection. □

By this Proposition if we attach to $\mathcal{N}$ its relative accumulation points, we obtain the set

$$V_{\bar{N}A^E} \cdot \left(\cup_{D \supset E} \iota_D(A^D(E)^{\mathrm{pr}^D T} \cdot o) \right). \tag{3.3}$$

Every neighborhood of $\iota_E(0)$ in the topology we are trying to construct must contain such a set. Now it is possible to check that these sets and their translates by elements of G already form the basis of a topology which we then define to be the topology of $\bar{X}(\emptyset)$. One way to do this checking is to verify that the "closure" $\bar{S}$ of sets $S \subseteq \bar{X}(\emptyset)$ determined by our prospective neighborhoods (3.3) (and their translates) satisfies the Kuratowski axioms: $\bar{\emptyset} = \emptyset$, $S \subseteq \bar{S}$, $\bar{\bar{S}} = \bar{S}$, $\overline{S \cup T} = \bar{S} \cup \bar{T}$.

It is immediate that in (3.3) we can replace $V_{\bar{N}A^E}$ by V_G. The continuity of the G-action follows from Lemma 3.3. From the axiomatic characterization of the compactification ([13], or rewritten in the present notation [10, pp. 22-23]) it is clear that the topology we have constructed is indeed the Satake topology. This can also be seen by comparing (3.3) with the neighborhood basis given in Section 4 of [1].

Now, to finish, we describe the modifications required for the general case, that is, for constructing the Satake compactifications $\bar{X}(E_0)$. In this case, we take only E_0-connected sets E, and define $\mathcal{F}^E$ to be the filter generated by the sets

$$V_G M_K(E'')A(E)^{T'} \cdot o$$

for all V_G and all $T' \in \mathfrak{a}(E')$. (Note the restriction on T'.) The general version of Lemma 3.2 says now that $\mathcal{F}^E$ is also generated by the sets $V_{\bar{N}(E')\bar{N}^E A^E} M_K(E'')$ $A(E)^{T'} \cdot o$ or $V_{\bar{N}(E')} M_K(E'')A(E)^{T'} \cdot V_{X^E}$. Lemma 3.3 remains true with $U = U_{B_{\mathrm{st}}(E')}$ for the generators of $\mathcal{F}^E$; in the proof one has to split the $A(E)^{T'}$-component into factors in $A(E')$ and $A^{E''}$ and use the Cartan decomposition of $M_K(E'')$. One defines the set $\bar{X}(E_0)$ and imbeddings ι_E as in the special case $E_0 = \emptyset$. Proposition 3.4 now says that the relative accumulation points of $\mathcal{N} = VM_K(E'')A(E)^{T'}$, where $V = V_{\bar{N}(E')\bar{N}^E A^E}$ will be the points $g \cdot \iota_D(o)$ with E_0-connected $D \supset E$ and $g \in VM_K(E'')A^D(E)^{\mathrm{pr}^D T'}$. The construction of the topology then proceeds as in the case $E_0 = \emptyset$.

4. Bounded domain realizations

In this section, which is independent of the preceding two, we will give realizations of $X \cong G/K$ as bounded domains in $\mathfrak{p}$ in such a way that the Satake compactifications can be obtained by taking the ordinary closure in $\mathfrak{p}$. Many of the elements needed for this result are in [9], [3], [6, Ch. 3], one could try to prove it by combining the results of [9] with the extension theorem of equivariant maps in [11]. We take here a different approach using much more elementary methods than [9] and getting some explicit information about the boundary structure along the way.

My original idea was to prove Proposition 4.1 by use of a monotonicity property of Ψ and then use [11] to extend Ψ from $\mathfrak{a}$ to $\mathfrak{p}$. This got essentially simplified by J. Faraut's observation that Ψ is the gradient of φ and therefore it is possible to use the method of [4, Prop 1.3.4] (which is a reproduction of a proof in [14]). This observation also makes [11] almost superfluous: we will need only a very small part of the argument of [11].

In the following we fix a set $E_0 \subseteq \Pi$ as in Section 1 and an element $H_0 \in \mathfrak{a}(E_0)^+$. The condition (assumed) that E_0 contains no component of Π means that the orbit $W \cdot H_0$ spans $\mathfrak{a}$ as a linear space. We denote by $C_\mathfrak{a}$ the convex hull of $W \cdot H_0$, by $\overset{\circ}{C}_\mathfrak{a}$ its interior. Similarly, $C_\mathfrak{p}$ will be the convex hull of $K \cdot H_0$ and $\overset{\circ}{C}_\mathfrak{p}$ its interior. We use the inner product $(\cdot|\cdot)$ given on $\mathfrak{p}$ (and $\mathfrak{a}$) by the Killing form. We denote by $\nabla_\mathfrak{p}$, $\nabla_\mathfrak{a}$ the gradient with respect to it on $\mathfrak{p}$ resp. on $\mathfrak{a}$.

On $\mathfrak{a}$ we define the function φ by

$$\varphi(H) = \sum_{s \in W} e^{(sH_0|H)} \tag{4.1}$$

and we define the map $\Psi : \mathfrak{a} \to \mathfrak{a}$ by $\Psi = \nabla_\mathfrak{a} \log \varphi$, i.e.

$$\Psi(H) = \frac{1}{\sum_{s\in W} e^{(sH_0|H)}} \sum_{s \in W} e^{(sH_0|H)} sH_0. \tag{4.2}$$

(Ψ is a direct explicit expression for the moment map also used by Ji [9].) Clearly, φ is real analytic and W-invariant. By a version of Chevalley's theorem it extends to a K-invariant real analytic function $\tilde{\varphi}$ on $\mathfrak{p}$. The gradient Ψ of $\log \varphi$ is then W-equivariant; similarly $\tilde{\Psi} = \nabla_\mathfrak{p} \log \tilde{\varphi}$ is K-equivariant as a map $\mathfrak{p} \to \mathfrak{p}$. At points of $\mathfrak{a}$, $\nabla_\mathfrak{p} \log \tilde{\varphi}$ is the same as $\nabla_\mathfrak{a} \log \varphi$ because $\log \tilde{\varphi}$ is constant on K-orbits and those are orthogonal to $\mathfrak{a}$. (With more detail this is in [11]; it is the only part of [11] we have to use.) So, we have $\tilde{\Psi}_{|\mathfrak{a}} = \Psi$, i.e. $\tilde{\Psi}$ is an extension of Ψ.

Proposition 4.1. *Ψ is a real analytic diffeomorphism of $\mathfrak{a}$ onto $\overset{\circ}{C}_\mathfrak{a}$.*

Proof. Let $H' \in \overset{\circ}{C}_\mathfrak{a}$. We show that it is the image of a unique $H \in \mathfrak{a}$ by showing that the function

$$f(H) = \log \varphi(H) - (H'|H)$$

is (i) strictly convex, and (ii) tends to ∞ as $H \to \infty$ in $\mathfrak{a}$. (Then it has a unique minimum, so there is a unique H where $\Psi(H) - H' = \nabla f(H) = 0$.)

Writing D_{H_1} for the directional derivative we have for any $H_1 \in \mathfrak{a}$,

$$\begin{aligned} D^2_{H_1} f(H) &= \frac{1}{\varphi(H)^2}\big(\varphi(H) D^2_{H_1}\varphi(H) - (D_{H_1}\varphi(H))^2\big) \\ &= \frac{1}{\varphi(H)^2}\Big(\sum_{s\in W} e^{(sH_0|H)} \sum_{s\in W} e^{(sH_0|H)} (sH_0|H_1)^2 - \big(\sum_{s\in W} e^{(sH_0|H)}(sH_0|H_1)\big)^2\Big). \end{aligned}$$

By the Schwarz inequality this is positive since the $(sH_0|H_1)$ $(s \in W)$ cannot be independent of s. This proves (i).

For (ii), when $H \neq 0$ and $t \in \mathbb{R}$ we can write

$$f(tH) = \log \sum_{s \in W} e^{t\big((sH_0|H) - (H'|H)\big)}.$$

Since H' is a strict convex combination of the sH_0 $(s \in W)$, there is some s such that $(sH_0|H) - (H'|H) > 0$. Therefore, $\lim_{t\to\infty} f(tH) = \infty$, proving (ii).

Finally we note that image of Ψ must be contained in $\overset{\circ}{C}_{\mathfrak{a}}$ by (4.2). $\square$

REMARK 2. Ψ maps $\mathfrak{a}(E)$ to $\mathfrak{a}(E)$ and $\mathfrak{a}(E)^+$ to $\mathfrak{a}(E)^+$ for every $E \subseteq \Pi$.

Proof. Writing W^E for the subgroup of W generated by the reflections in the roots $\lambda \in E$, the fixed set of W^E is exactly $\mathfrak{a}(E)$. So, for $H \in \mathfrak{a}(E)$, $\Psi(H) = \Psi(sH) = s\Psi(H)$ is in $\mathfrak{a}(E)$. Since it preserves each $\mathfrak{a}(E)$, Ψ maps Weyl chambers to Weyl chambers. If the defining element H_0 of $\Psi_{H_0} = \Psi$ is in $\mathfrak{a}^+$, then (4.2) shows that $\Psi(tH_0)$ is near H_0 for large t, by $(H_0|H_0) > (sH_0|H_0)$, $(s \neq e)$. So Ψ preserves $\mathfrak{a}^+$. Otherwise, we write $H_0 = \lim H_n$ with $H_n \in \mathfrak{a}^+$. Then $\Psi = \lim \Psi_{H_n}$ maps chambers to chambers and each Ψ_{H_n} preserves $\mathfrak{a}^+$. So, Ψ again preserves $\mathfrak{a}^+$. But then it also preserves the faces $\mathfrak{a}(E)^+$. $\square$

PROPOSITION 4.2. $\tilde{\Psi}$ *is a real analytic diffeomorphism of* $\mathfrak{p}$ *onto* $\overset{\circ}{C}_{\mathfrak{p}} = K \cdot \overset{\circ}{C}_{\mathfrak{a}}$

Proof. $\bar{\mathfrak{a}}^+$ is a fundamental domain for W in $\mathfrak{a}$ and is the disjoint union of the $\mathfrak{a}(E)^+$, $(E \subseteq \Pi)$. So, every point of $\mathfrak{p}$ has a representation $k \cdot \exp H$ with a unique H in some $\mathfrak{a}(E)^+$ and with $k \in K$ unique modulo $M_K(E)$. From the preceding Remark it is then immediate that $\tilde{\Psi}$ is one-to-one onto its image $K \cdot \overset{\circ}{C}_{\mathfrak{a}}$.

To see that $K \cdot \overset{\circ}{C}_{\mathfrak{a}} = \overset{\circ}{C}_{\mathfrak{p}}$ we note that $W \cdot H_0 \subseteq K \cdot H_0$, implies $C_{\mathfrak{a}} \subseteq C_{\mathfrak{p}}$ and hence $K \cdot \overset{\circ}{C}_{\mathfrak{a}} \subseteq \overset{\circ}{C}_{\mathfrak{p}}$. Conversely, if $k \cdot H \in C_{\mathfrak{p}}$, then $H \in C_{\mathfrak{p}}$, so $H = \int_K k \cdot H_0 d\mu(k)$ with some probability measure. Hence $H = \mathrm{pr}_{\mathfrak{a}} H = \int_K \mathrm{pr}_{\mathfrak{a}}(k \cdot H_0) d\mu(k)$, where $\mathrm{pr}_{\mathfrak{a}}$ denotes orthogonal projection onto $\mathfrak{a}$. By the easy part of Kostant's convexity theorem (see [7, p. 473]), $\mathrm{pr}_{\mathfrak{a}}(k \cdot H_0) \in C_{\mathfrak{a}}$. So, $H \in C_{\mathfrak{a}}$, $k \cdot H \in K \cdot C_{\mathfrak{a}}$, and $C_{\mathfrak{p}} \subseteq K \cdot C_{\mathfrak{a}}$. $\square$

Next we give a description of the boundary structure of $C_{\mathfrak{a}}$. Detailed proofs of our statements are in [3, Thm. 3.1], the proofs can also be based on [7, Lemma 8.3, p. 459] and the standard properties of root systems (e.g. [2, pp. 277 - 284]).

$\mathfrak{a}^+$ is a polyhedral cone whose open faces are the $\mathfrak{a}(E)^+$ $(E \subseteq \Pi)$. We denote by ${}^+\mathfrak{a}$ the open dual cone, i.e. the set $\{H \in \mathfrak{a} | (H'|H) > 0, \forall H' \in \mathfrak{a}^+\}$. In each subalgebra $\mathfrak{a}^E$ we similarly define $\mathfrak{a}^{E+}$ and ${}^+\mathfrak{a}^E$. The open faces of ${}^+\mathfrak{a}$ are the

${}^+\mathfrak{a}^E$; ${}^+\bar{\mathfrak{a}}^E$ is the convex cone spanned by the H_λ $(\lambda \in E)$, where H_λ denotes the element of $\mathfrak{a}$ such that $\lambda(H) = (H_\lambda|H)$, $\forall H \in \mathfrak{a}$. Each closed face of $C_\mathfrak{a}$ that meets $\bar{\mathfrak{a}}^+$ is contained in $H_0 - {}^+\bar{\mathfrak{a}}^E$ for some E (for a short proof see [7, p. 459]).

Let now E be E_0 - connected. We write $H_0 = H_{0,E} + H_0^E$ following the decomposition $\mathfrak{a} = \mathfrak{a}(E) + \mathfrak{a}^E$. We have $W^E \cdot H_0 = H_{0,E} + W^E \cdot H_0^E$, and $W^E \cdot H_0^E$ spans $\mathfrak{a}^E$ (since every component of E contains some $\lambda \in \Pi$ with $\lambda(H_0^E) \neq 0$ due to the E_0-connectedness of E). Note that by $W^{E'} = W^E \times W^{E''}$ we also have $H_{0,E} \in \mathfrak{a}(E')$ and $W^{E'} \cdot H_0 = W^E \cdot H_0$.

We denote by $C_\mathfrak{a}^E$ the convex hull of $W^E \cdot H_0^E$; this is a closed domain in the subspace $\mathfrak{a}^E$. The supporting hyperplane of $C_\mathfrak{a}$ orthogonal to H_0^E meets $C_\mathfrak{a}$ exactly in $H_{0,E} + C_\mathfrak{a}^E$. So the $H_{0,E} + \overset{\circ}{C}{}_\mathfrak{a}^E$ are just the open faces of $C_\mathfrak{a}$ meeting $\bar{\mathfrak{a}}^+$; the other faces are W - images of these.

We proceed to the boundary structure of $C_\mathfrak{p}$. First we note that Proposition 4.1 and 4.2 also hold for the symmetric space $X^E \cong M(E)/M_K(E)$ in place of X. So, for each E_0-connected E there is an analytic diffeomorphism $\Psi^E : \mathfrak{a}^E \to \overset{\circ}{C}{}_\mathfrak{a}^E$ which extends to $\tilde{\Psi}^E : \mathfrak{p}^E \to \overset{\circ}{C}{}_\mathfrak{p}^E$ where $\overset{\circ}{C}{}_\mathfrak{p}^E$ denotes the relative interior of the convex hull $C_\mathfrak{p}^E$ of $M_K(E) \cdot H_0^E$. This is a domain in $\mathfrak{p}^E$, and we also have $C_\mathfrak{p}^E = M_K(E) \cdot C_\mathfrak{a}^E$.

Using the decomposition $H_0 = H_{0,E} + H_0^E$ and the fact that $M_K(E)$ is a group of orthogonal linear transformations fixing $H_{0,E}$ it follows that the convex hull of $M_K(E) \cdot H_0$ is $H_{0,E} + C_\mathfrak{p}^E$ and that this is a closed face of $C_\mathfrak{p}$. The boundary of $C_\mathfrak{p}$ is partitioned into open faces which are of the form $k \cdot (H_0 + \overset{\circ}{C}{}_\mathfrak{p}^E)$ with E_0-connected E and $k \in K$ unique modulo $M_K(E')$.

After these remarks we can prove the main result of this Section.

THEOREM 4.3. *The map* $\exp Z \cdot o \mapsto \tilde{\Psi}(Z)$ *of* X *onto* $\overset{\circ}{C}_\mathfrak{p}$ *extends by continuity to a* K*-equivariant homeomorphism of* $\bar{X}(E_0)$ *onto* $C_\mathfrak{p}$. *This extension maps the boundary components onto the open faces of* $C_\mathfrak{p}$ *and is explicitly given by* $k \cdot \iota_E(\exp Y \cdot o) \mapsto k \cdot (H_{0,E} + \tilde{\Psi}^E(Y))$, *(*$k \in K, Y \in \mathfrak{p}^E$*).*

Proof. We take an arbitrary sequence $\{x_n\}$ in X tending to a boundary point in $\bar{X}(E_0)$, i.e. we take $x_n = k_n \exp H_n \cdot o$ $(k_n \in K, H_n \in \bar{\mathfrak{a}}^+)$ converging to some $k \cdot \iota_E(\exp H^E \cdot o)$. Here k_n can be chosen so that $\lim k_n = k$, and by K-equivariance we may assume $k = e$. The convergence means that $\lim \lambda(H_n) = \lambda(H)$ for $\lambda \in E$, and $\lim \lambda(H_n) = \infty$ for $\lambda \in \Pi - E'$ (cf. the Satake axioms).

The image of x_n is $k_n \Psi(H_n)$, we have to show that this converges to $H_{0,E} + \Psi^E(H^E)$. For this we decompose H_n as $H_{n,E} + H_n^E$ and look at the expression (4.2) of $\Psi(H_n)$. The coefficients are of the form $e^{(sH_0|H_{n,E})}e^{(sH_0|H_n^E)}$. The second factor is bounded, since $\lim H_n^E = H^E$. The first factor, for $s \in W^{E'}$, is independent of s. For $s \notin W^{E'}$ it is of smaller order of magnitude than $e^{(H_0|H_n)}$ as $n \to \infty$: In fact, $H_0 - s_0H_0 = \sum_\Pi p_\lambda H_\lambda$ with $p_\lambda \geq 0$ since it is in ${}^+\mathfrak{a}$. It is not in $\mathfrak{a}^E$, so $p_\lambda > 0$ for some $\lambda \notin E'$. This implies that $\lim(H_0 - sH_0|H_n) = \infty$.

So, in the limit all terms with $s \notin W^{E'}$ drop out and the factor $e^{(H_0|H_{n,E})}$ cancels giving

$$\frac{1}{\sum_{s\in W^{E'}} e^{(sH_0^E|H^E)}} \sum_{s\in W^{E'}} e^{(sH_0^E|H^E)}\big(H_{0,E} + sH_0^E\big).$$

Since $W^{E'} = W^E \times W^{E''}$ and $W^{E''}$ fixes H_0^E, the value of this expression remains the same if we take the sums over W^E. This proves that $\lim k_n\Psi(H_n) = H_{0,E} + \Psi^E(H^E)$.

To complete the proof of the continuity of the extension of $\tilde{\Psi}$ we also have to consider convergent sequences contained in the boundary $\bar{X}(E_0)$ and their images under the extended map. This can be done by looking at sequences in one boundary component at a time, and making use of the hereditary structure of the compactification (a boundary component of a boundary component of X is again a boundary component of X). For a sequence x_n contained in $k \cdot \iota_D(X^D)$ and converging to a point in $k' \cdot \iota_E(X^E)$, the same argument works as above. We omit the details.

At this point we have a continuous K-equivariant extension of our map. It is onto $C_\mathfrak{p}$ because $C_\mathfrak{p}$ is the disjoint union of $\overset{\circ}{C}_\mathfrak{p}$ and its open faces $k \cdot (H_{0,E}+ \overset{\circ}{C}{}_\mathfrak{p}^E)$ and all points of these arise in the form $k'\cdot(H_{0,E}+\Psi^E(H^E))$. It is also one - to - one because $\tilde{\Psi}^E$ is by (the analogue of) Proposition 4.2 and because the components $k \cdot \iota_E(X^E)$ and the faces $k \cdot (H_{0,E}\ \overset{\circ}{C}{}_\mathfrak{p}^E)$ $(k \in K/M_E(E'))$ correspond to each other in a one - to - one way. Therefore, by compactness, it is a homeomorphism. □

Remark 3. The Harish-Chandra embedding of Hermitian symmetric spaces as bounded domains in $\mathbb{C}^n$ is a special instance of our realizations $\overset{\circ}{C}_\mathfrak{p}$. In fact, assuming as we may that X is irreducible Hermitian symmetric, we see from [8, Thm. 4.10, p. 460] that $\mathfrak{a}$ has an orthonormal basis H_i $(1 \leq i \leq \ell)$ such that $H = \sum t_iH_i$ is in $\mathfrak{a}^+$ iff $t_1 > t_2 > \cdots > t_\ell > 0$, and W consists of all signed permutations of the H_i. (We write H_i for H_{γ_i} of [8].) We take H_0 on the edge $t_1 = t_2 = \cdots = t_\ell$ of $\bar{\mathfrak{a}}^+$; up to an unessential homothety we may assume $H_0 = \sum H_i$. Now the W - orbit of H_0 consists of all $\sum \varepsilon_iH_i$ with $\varepsilon_i = \pm 1$. So,

$$\varphi(H) = \sum e^{\sum \varepsilon_i t_i} = \prod \left(e^{t_i} + e^{-t_i}\right) = 2^\ell \prod \cosh t_i$$

and

$$\Psi(H) = \nabla \log \varphi(H) = \sum (\tanh t_i) H_i.$$

Comparing with [8, pp. 454 - 5] we see that $\tilde{\Psi}$ coincides with the Harish-Chandra imbedding.

References

[1] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Math. Research Letters 9 (2002), 725 - 739.

[2] N. Bourbaki, *Groupes et algèbres de Lie*, Chapters 4,5 et 6, Hermann, Paris, 1968.

[3] W. A. Casselman, *Geometric rationality of Satake compactifications*, to appear in Algebraic groups and related subjects, ed. by G. I. Lehrer et al., a volume of Australian Mathematical Society Lecture series in honor of R. W. Richardson.

[4] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford University Press, 1994.

[5] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. Math. 77 (1963), 335 - 386.

[6] Y. Guivarc'h, L. Ji and J. C. Taylor, *Compactification of symmetric spaces*, Birkhauser, 1998.

[7] S. Helgason, *Groups and geometric analysis*, Acad. press 1984, Amer. Math. Soc. 2000.

[8] S. Helgason, *Geometric analysis on symmetric spaces*, Amer. Math. Soc. 1994.

[9] L. Ji, *Satake and Martin compactifications of symmetric spaces are topological balls*, Math. Res. Letters 4 (1997), 79 - 89.

[10] A. Korányi, *Poisson integrals and boundary components of symmetric spaces*, Inv. Math. 34 (1976), 19 - 35.

[11] A. Korányi and R. Szőke, *On Weyl group equivariant maps*, to appear.

[12] C. C. Moore, *Compactifications of symmetric spaces I, II*, Amer. J. Math. 86 (1964), 201 - 218, 358 - 378.

[13] I. Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. Math. 71 (1960), 77 - 110.

[14] E. B. Vinberg, *The theory of convex homogeneous cones*, Trudy Mosk. Math. Obs. 12 (1963), 303 - 358; Trans. Moscow Math. Soc. 12 (1963), 303 - 358.

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