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The Projective Tangent Bundles of a Complex Three-Fold

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*Dedicated to my friend Armand Borel
in memory of the golden fifties in Princeton*

§1. Introduction and statement of results.

When I visited the Institute for Advanced Study for its 75th anniversary in March 2005, I lectured about my joint work with Armand in the fifties [BH] and reported in particular about an example we studied in §13.9 and §24.11. This concerns the 10-dimensional flag manifold

$$X = \mathbf{U}(4)/\mathbf{U}(2) \times \mathbf{U}(1) \times \mathbf{U}(1).$$

The points of X are the ordered triples of one 2-dimensional and two 1-dimensional linear subspaces of the standard hermitian space $\mathbb{C}^4$ which are pairwise hermitian orthogonal. The manifold X carries two homogenous complex structures, namely the 5-dimensional complex manifold X_1 consisting of the flags in $\mathbb{C}^4$ of type $(0) \subset (1) \subset (3) \subset (4)$, i. e. the origin contained in a one-dimensional linear subspace, contained in a three-dimensional linear subspace, contained in $\mathbb{C}^4$, and the complex manifold X_2 consisting of flags of type $(0) \subset (1) \subset (2) \subset (4)$. Borel and I prove for the first Chern class c_1 and the Chern number c_1^5

$$\text{(1)} \qquad \begin{array}{l} c_1(X_1) \text{ is divisible by } 3 \\ c_1(X_2) \text{ is not divisible by } 3 \end{array}$$

$$\text{(2)} \qquad \begin{array}{l} c_1^5[X_1] = 3^5 \cdot 20 = 4860 \\ c_1^5[X_2] = 4500 \end{array}$$

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Borel and I did this as an example to our general method to study characteristic classes of homogeneous spaces by using the root systems of Lie groups. As written in [BH], §24.11, "... we get an example of two 5-dimensional algebraic varieties which are C^∞-differentiably homeomorphic, but have different Chern numbers".

E. Calabi attended my Princeton talk and mentioned to me afterwards that X_1 is the projective covariant tangent bundle of the projective space $\mathbb{P}_3(\mathbb{C})$, and X_2 is the projective contravariant tangent bundle of $\mathbb{P}_3(\mathbb{C})$. Indeed, the points of the projective covariant bundle consist of a point x in $\mathbb{P}_3(\mathbb{C})$ and a plane going through x which as a tangent plane in x is annihilated by a covariant tangent vector in x. This corresponds to the inclusion (1) $\subset$ (3). Similarly (1) $\subset$ (2) describes the projective contravariant tangent bundle.

Motivated by the remark of Calabi I studied the following problem:

Let B be a compact complex manifold of dimension 3 and $c = 1+c_1+c_2+c_3$ its total Chern class ($c_i \in H^{2i}(B, \mathbb{Z})$). Let X_1, X_2 be the projective covariant tangent bundle of B and the projective contravariant tangent bundle respectively. Then X_1, X_2 are 5-dimensional compact complex manifolds fibred over B with the projective plane as fibre. After introducing a hermitian metric in B the covariant and contravariant tangent bundles with structural group $\mathbf{U}(3)$ are anti-isomorphic by complex conjugation in each fibre. Therefore X_1 and X_2 are diffeomorphic. The cohomology ring $H^*(B, \mathbb{Z})$ maps isomorphically into the cohomology ring of X_1 and X_2 respectively and $H^*(X_i, \mathbb{Z})$ is an extension of $H^*(B, \mathbb{Z})$ by the first Chern class η of the tautological line bundle over X_i. The total Chern classes of X_1, X_2 will be denoted by

$$d = 1 + d_1 + d_2 + d_3 + d_4 + d_5 \text{ where } d_j \in H^2(X_i, \mathbb{Z})$$

Each Chern number of X_1, X_2 can be calculated as a linear combination of the Chern numbers c_1^3, c_1c_2, c_3 of B. I did not carry this out completely. I formulate here only the following result for d_1 and the Chern number d_1^5.

Proposition 1. *We have*

$$\begin{aligned} d_1(X_1) &= -3\eta \\ d_1(X_2) &= 2c_1 - 3\eta \end{aligned} \tag{3}$$

$$\begin{aligned} d_1^5[X_1] &= 3^5(c_1^3 - 2c_1c_2 + c_3)[B] \\ d_1^5[X_2] &= 9(23c_1^3 - 36c_1c_2 - 27c_3)[B] \end{aligned} \tag{4}$$

Of course, (1) and (2) follow from (3) and (4). Let $g \in H^2(\mathbb{P}_3(\mathbb{C}), \mathbb{Z})$ be the positive generator of the cohomology ring. The total Chern class of $\mathbb{P}_3(\mathbb{C})$ is

$(1+g)^4$, and the Chern numbers are

$$c_1^3 = 64, c_1c_2 = 24, c_3 = 4,$$

and (4) gives the values 4860 and 4500 respectively.

We now specialize to the case that the complex 3-fold B is a Calabi-Yau manifold, i. e. we assume that the first Chern class of B vanishes. Then we only have the Chern number $c_3[B]$ which is the Euler number of B and all Chern numbers of X_1 and X_2 are multiples of $c_3[B]$. We have the following table for X_1, X_2 and the Cartesian product $B \times \mathbb{P}_2(\mathbb{C})$.

(5)

	X_1	X_2	$B \times \mathbb{P}_2(\mathbb{C})$
d_5	$3c_3$	$3c_3$	$3c_3$
d_4d_1	$9c_3$	$9c_3$	$9c_3$
d_3d_2	$3c_3$	$3c_3$	$3c_3$
$d_3d_1^2$	$9c_3$	$9c_3$	$9c_3$
$d_2^2d_1$	$27c_3$	$-27c_3$	0
$d_2d_1^3$	$81c_3$	$-81c_3$	0
d_1^5	$243c_3$	$-243c_3$	0

If the Euler number $c_3[B]$ is not zero, then X_1, X_2 have different Chern numbers, therefore they are not biholomorphically equivalent. The complex cobordism class of a compact complex manifold is determined by its Chern numbers [M]. Therefore the complex cobordism classes of X_1 and X_2 are equal if and only if $c_3[B] = 0$. From table (5) we obtain

Proposition 2. *For the projective covariant or contravariant tangent bundles X_1, X_2 of the Calabi-Yau manifold B we have in the complex cobordism ring the equation*

$$X_1 + X_2 = 2B \times \mathbb{P}_2(\mathbb{C}) \tag{6}$$

Remarks:

(1) The results (5) and (6) are true if we only assume that the Chern numbers c_1^3 and c_1c_2 of B vanish.

(2) The Todd genus is multiplicative in projective bundles ([H1] and [H3]). The Todd genus of B equals $\frac{1}{24}c_1c_2[B] = 0$. Hence the Todd genus of X_1 and X_2 vanishes. For a 5-fold Y with Chern classes d_i the Todd genus is given by ([H3], §1)

$$T(Y) = \frac{1}{1440}(-d_4d_1 + d_3d_1^2 + 3d_2^2d_1 - d_2d_1^3)[Y]$$

which indeed vanishes for X_1, X_2 and $B \times \mathbb{P}_2(\mathbb{C})$, see table (5).

§2. Proofs

Let E be a complex vector bundle (fibre $\mathbb{C}^n$, structural group $\mathbf{U}(n)$) over the compact manifold B with total Chern class

$$1 + a_1 + ... + a_n \quad , \quad a_i \in H^{2i}(B, \mathbb{Z}).$$

Consider the associated projective bundle X over B with fibre $\mathbb{P}_{n-1}(\mathbb{C})$ and projection $\pi : X \to B$. We have

$$\pi^*E = L \oplus \bar{E}$$

where L is the tautological line bundle over X and $\bar{E}$ (fibre $\mathbb{C}^{n-1}$) its hermitian orthogonal complement. Then $\bar{E} \otimes L^{-1}$ is the tangential vector bundle T along the fibres of X (see [H3], §13) and

$$\pi^*E \otimes L^{-1} = T \oplus 1 \tag{1}$$

where 1 denotes the trivial line bundle. We always use tacitly that π^* maps the integral cohomology ring of B isomorphically into $H^*(X, \mathbb{Z})$ and usually omit π^* in the notation. The ring $H^*(X, \mathbb{Z})$ is an extension of $H^*(B, \mathbb{Z})$ by the first Chern class $\eta \in H^2(X, \mathbb{Z})$ of the tautological line bundle L over X. The total Chern class of π^*E can be split formally (or by lifting E to the associated bundle with $\mathbf{U}(n)/\mathbf{U}(1)^n$ as fibre)

$$1 + a_1 + ... + a_n = (1 + x_1)...(1 + x_n) \quad , \quad x_i \in H^2$$

and (1) implies that the total Chern class of T equals

$$(1 + x_1 - \eta)...(1 + x_n - \eta) = \sum_{i=0}^{n}(1 - \eta)^{n-i}a_i \tag{2}$$

and, since the n-th Chern class of T is zero,

$$\sum_{i=0}^{n}(-\eta)^{n-i}a_i = 0 \quad \text{(formula of Guy Hirsch)} \tag{3}$$

The ring $H^*(X,\mathbb{Z})$ is the extension of $H^*(B,\mathbb{Z})$ by η with the relation (3). Thus $H^*(X,\mathbb{Z})$ is a free module over $H^*(B,\mathbb{Z})$ with base 1, η, ..., η^{n-1}.

All this is already mentioned in [H1] and in the following paper [H2] where the total Chern class of the flag manifold $\mathbf{U}(n)/\mathbf{U}(1)^n$ is calculated. This is the beginning of the joint work with A. Borel. When I told him in 1953 about the formula for $\mathbf{U}(n)/\mathbf{U}(1)^n$ he pointed out that this can be generalized to G/T where G is a compact connected Lie group and T a maximal torus of G (see [H2], remark at the end of §7).

If B is a complex manifold of dimension m with total Chern class

$$c(B) = 1 + c_1 + c_2 + ...c_m$$

then

$$c(X) = c(B) \cdot \sum_{i=0}^{n}(1-\eta)^{n-i}a_i \tag{4}$$

To calculate the Chern numbers of X we use that $-\eta$ when restricted to the fibre $\mathbb{P}_{n-1}(\mathbb{C})$ over X is the positive generator of $H^2(\mathbb{P}_{n-1}(\mathbb{C}),\mathbb{Z}) \simeq \mathbb{Z}$ represented by a hyperplane.

We denote the total Chern class $c(X)$ by d

$$d = 1 + d_1 + ... + d_{m+n-1}$$

A Chern number $d_{r_1}d_{r_2}...d_{r_s}(r_1 + r_2 + ... + r_s = m + n - 1)$ equals by (3) and (4) a linear combination of monomials of dimension m in the a_i and c_j multiplied with $(-\eta)^{n-1}$. Since $(-\eta)^{n-1}$ gives 1 when evaluated on the fibre, the Chern number $d_{r_1}d_{r_2}...d_{r_s}[X]$ equals the above polynomial in a_i and c_j of dimension m evaluated on B.

We now carry this out in the cases we are interested in.

We specialize to $m = n = 3$. The Hirsch relation (3) gives

$$\eta^3 = \eta^2 a_1 - \eta a_2 + a_3 \tag{5}$$

and by repeated applications of (5)

(5')
$$\eta^4 = \eta^2(a_1^2 - a_2) + \eta(-a_1 a_2 + a_3)$$

(5")
$$\eta^5 = \eta^2(a_1^3 - 2a_1 a_2 + a_3)$$

We now consider the covariant tangent bundle of the 3-dimensional complex manifold B with Chern classes c_1, c_2, c_3. The Chern classes of the covariant tangent bundle of B are $-c_1, c_2, -c_3$. According to (4), the total Chern class of the projective covariant tangent bundle X_1 equals

(6)
$$d = (1 + c_1 + c_2 + c_3)((1-\eta)^3 - (1-\eta)^2 c_1 + (1-\eta)c_2 - c_3)$$

$$d_1 = -3\eta$$

We have by (5")

(7)
$$d_1^5 = -3^5\eta^2(-c_1^3 + 2c_1 c_2 - c_3)$$

Thus the first formula in (4) of §1 is proved.

If we take the contravariant tangent bundle of B, then by definition its Chern classes are those of B. The total Chern class d of the associated projective bundle X_2 is now given by

(8)
$$d = (1 + c_1 + c_2 + c_3)((1-\eta)^3 + (1-\eta)^2 c_1 + (1-\eta)c_2 + c_3)$$

$$d_1 = 2c_1 - 3\eta$$

For the calculation of $d_1^5 = (2c_1 - 3\eta)^5$ we have to replace in this expression η^3, η^4, η^5 by the quadratic polynomials in η given in (5), (5'), (5") with $a_i = c_i$.

We have

$$(2c_1 - 3\eta)^5 = 9(-27\eta^5 + 90c_1\eta^4 - 120c_1^2\eta^3 + 80c_1^3\eta^2)$$

and by (5), (5'), (5")

$$\eta^5 = (c_1^3 - 2c_1c_2 + c_3)\eta^2$$

$$c_1\eta^4 = (c_1^3 - c_1c_2)\eta^2$$

$$c_1^2\eta^3 = c_1^3\eta^2$$

which gives the second formula in (4) of §1.

We now assume that B is a Calabi-Yau manifold ($c_1 = 0$). The Chern class c_2 cannot occur in the final result because $c_1c_2 = 0$. Therefore, for the calculation we put $c_2 = 0$ and have in the covariant case $\eta^3 = -c_3$ and for the total Chern class of X_1

$$d(X_1) = (1 + c_3)(1 - 3\eta + 3\eta^2) \tag{9}$$

$$d_1 = -3\eta, d_2 = 3\eta^2, d_3 = c_3, d_4 = -3c_3\eta, d_5 = 3c_3\eta^2 \tag{10}$$

The values in table (5) of §1 are now easily obtained.

For the projective contravariant bundle X_2 we have (again mod c_2) the same formulas as in (9) and (10), but in this case

$$\eta^3 = c_3$$

This checks with table 5 in §1. For the Cartesian product $B \times \mathbb{P}_2(\mathbb{C})$ again (9) and (10) hold, but $\eta^3 = 0$ and the values in table 5 result.

Remark:

Proposition 2 and table (5) of §1 can be generalized to higher dimensions. We assume that B is a compact almost complex manifold of dimension n and that all Chern numbers of B vanish except $c_n[B]$, the Euler number. Consider the projective covariant tangent bundle X_1 of B and the projective contravariant tangent bundle X_2 of B. Both X_1 and X_2 are fibred over B with $\mathbb{P}_{n-1}(\mathbb{C})$ as fibre. The Chern classes of X_1, X_2 and $B \times \mathbb{P}_{n-1}(\mathbb{C})$ are denoted d_i. Using the preceding methods, the following is proved easily.

If n is even, then X_1 and X_2 have equal Chern numbers, hence $X_1 = X_2$ in the complex cobordism ring.

If n is odd, the following holds:
If a Chern number is a monomial of dimension $2n-1$ in Chern classes d_i with all $i < n$, then for X_2 it is minus the corresponding Chern number of X_1, whereas it vanishes for $B \times \mathbb{P}_{n-1}(\mathbb{C})$. The other Chern numbers agree on X_1, X_2, and $B \times \mathbb{P}_{n-1}(\mathbb{C})$. Hence we have in the complex cobordism ring

$$X_1 + X_2 = 2B \times \mathbb{P}_{n-1}(\mathbb{C}).$$

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