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A Generalization of the Kuga-Satake Construction

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A la mémoire d'Armand Borel

0. Introduction

The Kuga-Satake construction [3] associates to a polarized (effective) Hodge structure H of weight 2 with $h^{2,0} = 1$ an abelian variety A which satisfies the property that H is a sub-Hodge structure of the weight 2 Hodge structure $Hom\,(H_1(A), H_1(A))(1)$. The construction is very tricky and intriguing geometrically: one first associates to the lattice $(H, <,>)$ its Clifford algebra $C(H)$, which is again a lattice. Then one constructs a complex structure on $C(H) \otimes \mathbb{R}$, using the rank 1 subspace $H^{2,0} \subset H \otimes \mathbb{C}$ defining the Hodge structure on H. Thus the quotient

$$\frac{C(H) \otimes \mathbb{R}}{C(H)}$$

is endowed with the structure of a complex torus, and with some more work, on can show that it is in fact an abelian variety. This abelian variety A has by definition $H_1(A, \mathbb{Z}) = C(H)$ and the morphism of weight 2 Hodge structures

$$H \to End\,(H_1(A, \mathbb{Z}))(1)$$

is given by Clifford multiplication on the left acting on $C(H)$: $H \to End\,(C(H))$.

In [2], Deligne proved that a general weight 2 polarized Hodge structure coming from geometry, is not a quotient of a Hodge structure of the form $K \otimes L$, where K and L are weight 1 polarized Hodge structures. His argument is that the Mumford-Tate group [7] of a Hodge structure $K \otimes L$ has a very restricted form, while the Mumford-Tate group for general weight 2 Hodge structures coming from geometry is very large, as it contains a subgroup of finite index in the monodromy group, and thus can be in some cases the whole orthogonal group (cf [1]).

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In the papers [6], [10], [8], some constructions are given, which realize geometrically for $K3$-surfaces with large Picard number the Kuga-Satake construction, as expected from the Hodge conjecture: the inclusion

$$H \subset Hom\,(H_1(A), H_1(A))(1) \cong H^1(A) \otimes H^1(A)$$

of weight 2 rational Hodge structures, where we used the isomorphism of weight 1 Hodge structures

$$H_1(A)(1) \cong H^1(A) = H_1(A)^*$$

given by the polarization, can be understood as a degree 4 Hodge class in $H^2(S) \otimes H^2(A \times A)$, and thus should correspond to a codimension 2 cycle

$$Z \subset S \times A \times A$$

with rational coefficients, such that the inclusion above is given by

$$[Z]_* : H^2(S) \to H^2(A \times A).$$

Morrison [5] proves that if S is a Kummer surface, that is the minimal desingularization of the quotient of an abelian surface by the $-Id$ involution:

$$S = \widetilde{T/\pm 1},$$

then its Kuga-Satake variety A is a sum of copies of T. Paranjape [6] solves the problem for the members of a certain family of $K3$-surfaces with Picard number 16, by proving that such a $K3$ surface S is dominated by the self-product of curve C, such that the Kuga-Satake variety of S is a sum of copies of an abelian subvariety of $J(C)$. Other families of examples are studied in [10].

For general $K3$ surfaces, there is however no geometric understanding of the Kuga-Satake construction, while combined with the Hodge conjecture and the Bloch conjecture, it predicts a lot concerning the geometry of algebraic $K3$-surfaces : first of all, as mentioned above, the Hodge conjecture predicts the existence of a codimension 2-cycle Γ in some product $S \times \Sigma$, where $\Sigma \subset B$ is a surface in an abelian variety, such that the induced map

$$[\Gamma]_* : H^2(S, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$$

is injective and takes value in $Im\,(H^2(A, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}))$. Equivalently, the map

$$[\Gamma]^* : H^2(\Sigma, \mathbb{Q}) \to H^2(S, \mathbb{Q})$$

should be surjective when restricted to $Im\,(\bigwedge^2 H^1(\Sigma, \mathbb{Q}) \to H^2(\Sigma, \mathbb{Q}))$. If Bloch's conjecture ([11] II, chapter 11) is true, this implies that the map induced by Γ at the level of 0-cycles of degree 0 :

$$\Gamma_* : CH_0(\Sigma)_0 \to CH_0(S)_0$$

is surjective when restricted to products of 1-cycles homologous to 0 on Σ. None of these statements seems to have any natural approach for general $K3$-surfaces.

In [10], we observed that the key property needed in order to construct the Kuga-Satake variety of S, namely the fact that $h^{2,0}(S) = 1$, is also reflected by the fact that the weight $2i$ Hodge structures on the exterior powers $\bigwedge^i H^2(S,\mathbb{Z})$ have Hodge level 2. We then started to investigate geometrically the properties concerning the geometry of the self-products S^i, predicted as a consequence of this last fact, by the Hodge-Grothendieck conjecture and the Bloch conjecture.

In this paper, we show indeed that the weight 2 Hodge structure on the exterior algebra $\bigwedge^* H^2(S,\mathbb{Z})$ is actually the crucial ingredient to construct the Kuga-Satake variety of S, when combined with the Clifford algebra structure (rather than the exterior algebra structure) on $\bigwedge^* H^2(S,\mathbb{Z})$. Both algebra structures are compatible with the Hodge structure (cf Remark 2), but the Clifford algebra has furthermore the property that it is stable under adjunction with respect to the pairing.

Our goal in this paper is to investigate this second aspect, and to present the Kuga-Satake construction from a more general point of view, namely that of weight 2 polarized Hodge structures endowed with an associative algebra structure, compatible with the Hodge structure and the pairing in the way described in 1, 2 below.

Let $H^2 = (H, H_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2})$ be a non trivial polarized integral Hodge structure of weight 2. (In the sequel, we shall use the superscripts to denote the weights of the considered Hodge structures. This is needed as we will have a lattice endowed with a weight 2 and a weight 1 Hodge structure.)

Assume there is a (associative, unitary) ring structure on the underlying lattice H, satisfying the following conditions (*):

(1) The product $H \otimes H \to H$ is a morphism of Hodge structures $H^2 \otimes H^2 \to H^2(1)$.
(2) There is an (involutive) endomorphism $t : H \to H$ such that for any $a \in H$, multiplication (on left and right) by $t(a)$ is adjoint to multiplication (on left and right) by a with respect to the intersection form $<,>$ which gives the polarization.
(3) The t-invariant part of the center of H is $\mathbb{Z}$.

We then prove:

Theorem 1. *Under these assumptions, there exists a unique weight* 1 *Hodge structure $H^1 = (H, H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1})$ on H which is polarizable, (or equivalently an abelian variety A with $H = H_1(A,\mathbb{Z})(1)$,) such that multiplication on the left*

$$H \to Hom\,(H,H)$$

is a morphism of Hodge structures

$$H^2 \to Hom\,(H^1,H^1)(1).$$

We explain this construction in section 1. In section 2, we show that under the assumptions above, any simple abelian variety B such that there is an inclusion of weight 2 Hodge structures, which is also the inclusion of a subring stable under some Rosati involution,

$$H^2 \subset Hom\,(H_1(B), H_1(B))(1)$$

must be a quotient of the abelian variety A associated to H^2 by Theorem 1. We also compare this construction with the Kuga-Satake construction.

In the final section, we turn to the problem of removing the assumption on the center. Passing to rational coefficients, we analyse the structure of the center K of an algebra satisfying properties 1, 2 above. We show that it is a product of number fields K_i which are either totally real fields or a quadratic extension of totally real fields, and that unless some corresponding factor H_i of H is a simple central algebra over K_i the same conclusion as in Theorem 1 holds. We finally study the last case. We then show that we always have existence of an abelian variety A as in Theorem 1 but not uniqueness.

This result leads to the question of understanding geometrically or motivically our construction. Indeed, we could assume that the data of the weight 2 Hodge structure together with its algebra structure are geometric, namely, one is given a surface Σ (or more generally a motive (Σ, p)), and a 3-cycle $\Gamma \in \Sigma \times \Sigma \times \Sigma$ inducing our associative product

$$\Gamma_* : H^2(\Sigma, \mathbb{Z}) \otimes H^2(\Sigma, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}).$$

We could even assume that the associativity property is realized geometrically, as follows: the associativity property can be seen as an equality of cohomology classes

$$\gamma \circ (Id \otimes \gamma) = \gamma \circ (\gamma \otimes Id),$$

(where $\gamma = [\Gamma]$ and we see γ as an element of $Hom\,(H^2(\Sigma, \mathbb{Z}) \otimes H^2(\Sigma, \mathbb{Z}), H^2(\Sigma, \mathbb{Z}))$,) in

$$Hom\,(H^6(\Sigma \times \Sigma \times \Sigma, \mathbb{Z}), H^2(\Sigma, \mathbb{Z})) \subset H^8(\Sigma \times \Sigma \times \Sigma \times \Sigma, \mathbb{Z}).$$

Then we could assume it is given as a consequence of an equality of cycles modulo rational equivalence

$$\Gamma \circ (\Delta \times \Gamma) = \Gamma \circ (\Gamma \times \Delta),$$

in $CH^4(\Sigma \times \Sigma \times \Sigma \times \Sigma)$, where Δ is the diagonal of Σ.

The question would be now to understand starting from such a Γ the abelian variety constructed in this paper in terms of cycles or may be higher cycles on Σ and its self-products.

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1. The case of trivial center

We start with a polarized integral Hodge structure H^2 of weight 2, with $H^{2,0} \neq 0$. Thus we have a lattice H, endowed with a decomposition

$$H_{\mathbb{C}} := H \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$
$$H^{0,2} = \overline{H^{2,0}},\, H^{1,1} = \overline{H^{1,1}},$$

and an integral symmetric bilinear form $<,>$ satisfying the following properties (Hodge-Riemann bilinear relations):

i) The Hodge decomposition is orthogonal with respect to the Hermitian intersection pairing $h(\alpha,\beta) =< \alpha, \overline{\beta} >$ on $H \otimes \mathbb{C}$.

ii) The Hermitian form h is positive definite on $H^{2,0}$ and $H^{0,2}$, negative definite on $H^{1,1}$.

The first condition just says that the morphism

$$H \to H^*$$

given by $<\,,\,>$ is a morphism of Hodge structures

$$H^2 \to (H^2)^*(2).$$

We assume furthermore that H has the structure of an unitary associative ring, and that conditions 1, 2 of (*) are satisfied as in Theorem 1. For K, L two vector subspaces of $H_{\mathbb{C}}$, KL will be the subspace generated by products kl of elements of K and L. Condition 1 means concretely that

$$H^{2,0}H^{2,0} = 0,\, H^{2,0}H^{1,1} \subset H^{2,0},\, H^{1,1}H^{2,0} \subset H^{2,0},\, H^{1,1}H^{1,1} \subset H^{1,1},$$
$$H^{0,2}H^{0,2} = 0,\, H^{1,1}H^{0,2} \subset H^{0,2},\, H^{0,2}H^{1,1} \subset H^{0,2}.$$

Condition 2 means that the intersection form $<,>$ can be written as

$$< a, b >=< at(b), 1 >=< t(b)a, 1 >,\ \forall a,\, b \in H.$$

We observe first of all that the map t (which is a morphism of rings $H \to H^{op}$) has to be also a morphism of Hodge structures $t : H^2 \to H^2$, that is

$$t(H^{p,q}) \subset H^{p,q}.$$

Indeed, this follows immediately from the fact that the product map and the polarization are compatible with the Hodge decomposition.

We consider the subspace

$$W = H^{2,0}H_{\mathbb{C}} \subset H_{\mathbb{C}}.$$

We have the following Proposition.

Proposition 1. *i) We have $W \cap \overline{W} = 0$.*

ii) The sum $W \oplus \overline{W}$ is a two-sided ideal of $H_{\mathbb{C}}$ which is stable under the Hodge decomposition and under t.

Proof. i) Note first that $W \subset F^1 H^2_{\mathbb{C}} := H^{2,0} \oplus H^{1,1}$ and similarly $\overline{W} \subset H^{1,1} \oplus H^{0,2}$. Thus

$$W \cap \overline{W} \subset H^{1,1}.$$

Let $x \in W \cap \overline{W}$. By definition, this x can be written as

$$x = \sum_i \alpha_i \beta_i,$$

with $\alpha_i \in H^{2,0}$ and $\beta_i \in H^{0,2}$, and as

$$x = \sum_j \gamma_j \delta_j,$$

with $\gamma_j \in H^{0,2}$ and $\delta_j \in H^{2,0}$. Applying $t(\bar{\cdot})$ to the second expression, we get

$$t(\overline{x}) = \sum_j t(\overline{\delta_j}) t(\overline{\gamma_j}),$$

with $t(\overline{\delta_j}) \in H^{0,2}$, $t(\overline{\gamma_j}) \in H^{2,0}$. Thus, as $H^{2,0}H^{2,0} = 0$, we conclude that

$$t(\overline{x})x = 0.$$

It follows that

$$< t(\overline{x})x, 1 >= 0 =< x, \overline{x} >,$$

and as $x \in H^{1,1}$, this implies $x = 0$ by the condition ii) above satisfied by the polarization.

ii) By definition, W and $\overline{W}$ are stable under right multiplication. Next, if $a \in H$ and $b = \eta w \in W$, with $\eta \in H^{2,0}$, then we can write $a = a_1 + a_2$, where

$$a_1 \in F^1 H^2_{\mathbb{C}},\ a_2 \in H^{0,2}.$$

We then have

$$ab = a_1 \eta w + a_2 b,$$

and as $F^1 H^2_{\mathbb{C}}\ H^{2,0} \subset H^{2,0}$, we have $a_1 \eta \in H^{2,0}$. Thus the first term belongs to W and the second belongs to $\overline{W} = H^{0,2} H_{\mathbb{C}}$. This shows that $W \oplus \overline{W}$ is a two-sided ideal.

Finally, note that, because H is an unitary ring, W contains $H^{2,0}$, and thus can be written as

$$W = H^{2,0} \oplus H^{2,0} H^{0,2},$$

because the second space is contained in $H^{1,1}$. This shows that W and hence also $W \oplus \overline{W}$, are stable under Hodge decomposition. As the space $H^{2,0}$ is stable under t, we have

$$t(H^{2,0}H^{0,2}) = H^{0,2}H^{2,0} = \overline{H^{2,0}}\,\overline{H^{0,2}} \subset \overline{W},$$

hence we get $t(W) \subset W \oplus \overline{W}$, and thus $t(W \oplus \overline{W}) \subset W \oplus \overline{W}$.

■

Corollary 1. *We have a decomposition of the* $\mathbb{C}$*-algebra* $H_{\mathbb{C}}$ *as a direct sum*

$$H_{\mathbb{C}} = (W \oplus \overline{W}) \oplus M,$$

where M *is defined as the orthogonal of* $W \oplus \overline{W}$ *with respect to* $<,>$.

Proof. Indeed, we know that $W \oplus \overline{W}$ is stable under the Hodge decomposition and under complex conjugation. It follows that the intersection form $<,>$ is non degenerate on $W \oplus \overline{W}$, because the Hermitian form h is non degenerate on each of its (p,q)-pieces and the Hodge decomposition is orthogonal for h. Thus we have a orthogonal decomposition

$$H_{\mathbb{C}} = (W \oplus \overline{W}) \oplus M. \tag{1.1}$$

As $W \oplus \overline{W}$ is a two-sided ideal which is stable under t, it follows that the same is true for its orthogonal M. Thus we must have $wm = 0$ for $w \in (W \oplus \overline{W})$, $m \in M$, which shows that (1.1) is a decomposition of the algebra as a direct sum. ■

Corollary 2. *If the* t*-invariant part of the center of* H *is equal to* $\mathbb{Z}$*, then* $M = 0$*. Thus*

$$H_{\mathbb{C}} = W \oplus \overline{W} \tag{1.2}$$

which defines a weight 1 *Hodge structure* H^1 *on* H*, with* $H^{1,0} = W$*.*

Proof. Indeed, under this assumption, we also get that the t-invariant part of the center of the $\mathbb{C}$-algebra $H_{\mathbb{C}}$ is $\mathbb{C}$. But as the decomposition (1.1) is orthogonal and is a decomposition into t-invariant subspaces, the associated idempotents $e_{W \oplus \overline{W}}$ and e_M are t-invariant and of course they are central. Thus, under the assumption of Corollary 2, we have either $e_{W \oplus \overline{W}} = 0$ or $e_M = 0$. As $H^{2,0} \neq 0$, we have $W \neq 0$ and thus $e_{W \oplus \overline{W}} \neq 0$, hence $e_M = 0$ and $M = 0$.

■

With the definition of H^1 as in Corollary 2, it is clear, by construction and using the compatibility of the Hodge structure on H^2 with the product on H, that multiplication on the left $H \to End\, H$ is a morphism of Hodge structures

$$H^2 \to Hom\,(H^1, H^1)(1).$$

Thus Corollary 2 proves a good part of the existence statement of Theorem 1. In order to have the complete proof of existence, we need to show that the Hodge structure H^1 of weight 1 on H defined by the decomposition 1.2, that is

$$W = H^{1,0}, \overline{W} = H^{0,1},$$

is polarizable (or equivalently, that the corresponding complex torus

$$A = H_{\mathbb{C}}/(W \oplus H)$$

is an abelian variety). This is done as follows: Let $a \in H$ be such that $t(a) = -a$. Consider the skew pairing

$$\omega_a(w, w') =< w, w'a >$$

on H and denote in the same way its $\mathbb{C}$-linear extension to $H_{\mathbb{C}}$.

Lemma 1. *The subspace $W \subset H_{\mathbb{C}}$ (that is $H^{1,0} \subset H^1_{\mathbb{C}}$) is totally isotropic with respect to ω_a.*

Proof. let $w = \eta m$, $w' = \eta' m' \in W$, with $\eta \in H^{2,0}$, $\eta' \in H^{2,0}$. Then

$$\omega_a(w, w') =< w, w'a >=< t(a)t(w')w, 1 >=< t(a)t(m')t(\eta')\eta m, 1 >,$$

and this is 0 because $t(\eta')\eta = 0$. ∎

Let us now show the following

Proposition 2. *For an adequate choice of a, ω_a polarizes the weight 1 Hodge structure H^1.*

Proof. By lemma 1, in order that ω_a defines a polarization, it only needs to satisfy the property that the Hermitian form h_a defined by

$$h_a(w, w') = i\omega_a(w, \overline{w'})$$

is positive definite on $W = H^{1,0}$. As rational elements $a \in H_{\mathbb{Q}}$ are dense in $H_{\mathbb{R}}$, it suffices to show that for some $a \in H_{\mathbb{R}}$, this property is satisfied. Let us take a to be a sum of terms of the following form:

$$a^- = -i(\eta\overline{\eta} - \overline{\eta}\eta),$$

where $\eta \in H^{2,0}$ satisfies $t(\eta) = -\eta$ and

$$a^+ = i(\eta\overline{\eta} - \overline{\eta}\eta),$$

where $\eta \in H^{2,0}$ satisfies $t(\eta) = \eta$. Then a is real and $t(a) = -a$.

We have to compute the sign of the Hermitian form h_a on $W = H^{2,0} \oplus H^{2,0}H^{0,2}$. Note that for $\mu \in H^{2,0}$ and $\nu \in H^{2,0}H^{0,2}$, we have

$$h_{a^+}(\mu, \nu) = i < \mu, \overline{\nu}a^+ >= - < \mu, \overline{\nu}(\eta\overline{\eta} - \overline{\eta}\eta) >$$
$$=< \mu, \overline{\nu}\,\overline{\eta}\eta >

because $\overline{\nu}\eta = 0$, $\forall \eta \in H^{2,0}$. The last term is equal to

$$< \mu t(\eta), \overline{\nu}\,\overline{\eta} >$$

which is also 0 because $\mu t(\eta) = 0$, $\forall \eta \in H^{2,0}$.

The same computation works with h_{a^-} and it thus follows that $H^{2,0}$ and $H^{2,0}H^{0,2}$ are perpendicular with respect to h_a. Hence it suffices to compute the signs of h_a on each term $H^{2,0}$ and $H^{2,0}H^{0,2}$.

If now $\mu \in H^{2,0}$, we have

$$h_{a^\pm}(\mu,\mu) = i < \mu, \overline{\mu}a^\pm >= -\pm < \mu, \overline{\mu}(\eta\overline{\eta} - \overline{\eta}\eta) >$$
$$= -\pm < \mu, \overline{\mu}\eta\overline{\eta} >,$$

because $\overline{\mu}\,\overline{\eta} = 0$, for all $\eta \in H^{2,0}$, and this is equal to

$$-\pm < \mu t(\overline{\eta}), \overline{\mu}\eta > . \tag{1.3}$$

In the case of a^+ where $t(\eta) = \eta$, and in the case of a^- where $t(\eta) = -\eta$, we get

$$- < \mu\overline{\eta}, \overline{\mu\overline{\eta}} > .$$

But $\mu\overline{\eta} \in H^{1,1}$ and the Hermitian form $< \alpha, \overline{\beta} >$ is negative on $H^{1,1}$. Thus (1.3) is ≥ 0.

Let us now consider the case of $\nu \in H^{2,0}H^{0,2}$. In this case we have

$$h_{a^\pm}(\nu,\nu) = i < \nu, \overline{\nu}a^\pm >= -\pm < \nu, \overline{\nu}(\eta\overline{\eta} - \overline{\eta}\eta) >$$
$$= \pm < \nu, \overline{\nu}\,\overline{\eta}\eta >,$$

because $\overline{\nu}\eta = 0$, $\forall \eta \in H^{2,0}$. This is also equal to

$$\pm < \nu t(\eta), \overline{\nu}\,\overline{\eta} > . \tag{1.4}$$

In the case of a^+ where $t(\eta) = \eta$, and in the case of a^- where $t(\eta) = -\eta$, we get

$$< \nu\eta, \overline{\nu\eta} > .$$

But $\nu\eta \in H^{2,0}$ and the Hermitian form $< \alpha, \overline{\beta} >$ is positive on $H^{2,0}$. Thus (1.4) is also ≥ 0.

To conclude, it remains to show that for a a generic sum of terms a^+, a^- as above, h_a is non degenerate, or equivalently does not vanish on $W \setminus \{0\}$. But the computation above shows that a null-vector μ of a generic sum $\sum h_{a^\pm}$ in $H^{2,0}$ has to satisfy

$$\mu\overline{\eta} = 0, \ \forall \eta \in H^{2,0}$$

and that a null-vector ν of a generic sum of $h_{a^\pm}$ in $H^{2,0}H^{0,2}$ has to satisfy

$$\nu\eta = 0, \ \forall \eta \in H^{2,0}.$$

In the first case, we get $\mu t(\overline{\mu}) = 0$, which implies that

$$< \mu t(\overline{\mu}), 1 >=< \mu, \overline{\mu} >= 0.$$

As $\mu \in H^{2,0}$, this implies by the second Hodge-Riemann bilinear relations that $\mu = 0$.

In the second case, as $t(\overline{\nu}) \in H^{2,0}H^{0,2}$, we conclude that $\nu t(\overline{\nu}) = 0$, which implies that

$$< \nu t(\overline{\nu}), 1 >=< \nu, \overline{\nu} >= 0.$$

As $\nu \in H^{1,1}$, this implies by the second Hodge-Riemann bilinear relations that $\nu = 0$.

∎

In order to conclude the proof of Theorem 1, it suffices to prove the uniqueness statement. But this is clear, because if a weight 1 Hodge structure H^1 on H given by a decomposition

$$H_{\mathbb{C}} = W' \oplus \overline{W'} \tag{1.5}$$

satisfies the property that left multiplication

$$H \to Hom\,(H, H)$$

is a morphism of weight 2 Hodge structures

$$H^2 \to Hom\,(H^1, H^1)(1),$$

then we must have

$$H^{2,0}H_{\mathbb{C}} \subset W'.$$

Indeed, the $(2,0)$-piece of $Hom\,(H^1, H^1)(1)$ for the weight 1 Hodge structure on H given by (1.5) is equal to $Hom\,(\overline{W'}, W')$ and thus is contained in $Hom_{\mathbb{C}}\,(H_{\mathbb{C}}, W')$.

Thus by definition of W, we must have $W \subset W'$, and then equality for dimension reasons.

∎

2. General properties, examples

2.1. **General properties.** We first start with the proof of the following:

Proposition 3. *Let B be a simple abelian variety, and let*

$$H \subset End\,(H_1(B, \mathbb{Z}))$$

be a subring which is also a sub-Hodge structure, satisfying the conditions of Theorem 1. Then B is a quotient of the abelian variety A constructed in Theorem 1.

(Actually, only conditions 2 and 3 are to be verified. 2 means that H has to be stable under one Rosatti involution.)

Proof. We want to show equivalently that there exists a non trivial morphism of Hodge structure

$$\alpha : H^1 \to H_1(B, \mathbb{Z})(1)$$

where on the left, H^1 is H endowed with the weight 1 Hodge structure given in Theorem 1. Indeed, as B is simple, the induced non trivial morphism of abelian varieties

$$\alpha : A \to B$$

has to be surjective.

Let $\beta \in H_1(B, \mathbb{Z})$ and consider the map

$$e_\beta : H \to H_1(B, \mathbb{Z}),$$

$$h \mapsto h(\beta).$$

Certainly this map is non zero for at least one β.

We claim that this is a morphism of weight 1 Hodge structures

$$H^1 \to H_1(B, \mathbb{Z})(1).$$

Indeed, we only have to show that $e_\beta(W) \subset H_{1,0}(B)$. But, as H^2 is a sub-Hodge structure of $End\, H_1(B)(1)$,

$$H^{2,0} \subset H^{2,0}(End_{\mathbb{C}}\,(H_1(B, \mathbb{C})(1)) \subset Hom_{\mathbb{C}}\,(H_1(B, \mathbb{C}), H_{1,0}(B)).$$

Thus $W = H^{2,0}H_{\mathbb{C}} \subset Hom_{\mathbb{C}}\,(H_1(B, \mathbb{C}), H_{1,0}(B))$, which proves the claim.

∎

Remark 1. Without the assumption that B is simple, we still get the following statement : if the map $H \otimes H_1(B, \mathbb{Q}) \to H_1(B, \mathbb{Q})$, $h \otimes \beta \mapsto h(\beta)$, is surjective, then B is a quotient of a sum of copies of A.

Indeed, choosing a basis β_i, $i = 1, \dots, n$ of $H_1(B, \mathbb{Z})$, the morphism

$$\sum_i e_{\beta_i} : A^n \to B$$

is then surjective. ∎

Finally, observe that the argument above can be reversed to show the following:

Proposition 4. *Let B be an abelian variety, and let*

$$H \subset End\,(H_1(B, \mathbb{Z}))$$

be a subring which is also a sub-Hodge structure, satisfying the conditions of Theorem 1. Then the associated abelian variety A is isogenous to an abelian subvariety of a sum of copies of B.

Indeed, we use the morphisms of Hodge structures e_β, which give morphisms of abelian varieties $A \to B$. As $H \subset End(H^1(B,\mathbb{Z}))$, for β_i, $i = 1, \ldots, n$, running over a basis of $H_1(B,\mathbb{Z})$, this gives a morphism of abelian varieties

$$(e_{\beta_i}) : A \to B^n$$

which has a finite kernel. ∎

2.2. **The Kuga-Satake construction.** We start from a polarized Hodge structure $(H, <,>)$ of weight 2 with $h^{2,0} = 1$. Consider the Clifford algebra $C(H)$ which is the quotient of the tensor algebra $\bigotimes H$ by the ideal generated by the relations

$$h \otimes h = - < h, h > 1.$$

For sign reasons, it is better to work with the even part $C^+(H)$ generated by products of an even number of elements of H, but we won't do this, as it makes computations more complicated. As a lattice, $C(H)$ is canonically isomorphic to the exterior algebra $\bigwedge H$. Furthermore, $C(H)$ has a natural intersection form induced by $<,>$, also denoted by $<,>$, and possesses the involution

$$t : C(H) \to C(H),$$

$$h_1 \ldots h_k \mapsto h_k \ldots h_1,\ h_i \in H.$$

For $v \in C(H)$, $t(v)$ is the adjoint of the multiplication on the left or on the right by v. Consider now $H_{\mathbb{C}}$, with its Hodge decomposition

$$H_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,1}.$$

Since the rank of $H^{2,0}$ is 1, the dimension over $\mathbb{R}$ of the real vector space

$$(H^{2,0} \oplus H^{0,2}) \cap H_{\mathbb{R}}$$

is 2. By the second Hodge-Riemann bilinear relations, the intersection form $<,>$ is positive definite on this 2-plane. Furthermore this 2-plane is canonically oriented because it is canonically isomorphic to the complex line $H^{2,0}$ via the map Re. Choose an oriented orthonormal basis e_1, e_2 of this 2-plane. Then

$$e := e_2 e_1 \in C(H_{\mathbb{R}})$$

does not depend on the choice of the basis. Furthermore we have

$$e^2 = e_2 e_1 e_2 e_1,$$

with

$$e_1 e_2 + e_2 e_1 = 0,\ e_1^2 = -1,\ e_2^2 = -1.$$

Thus $e^2 = -1$ and left multiplication by e defines a complex structure on $C(H_{\mathbb{R}})$. This provides a complex torus

$$K(H) = C(H_{\mathbb{R}})/C(H),$$

which is the Kuga-Satake variety of H.

We want now to present this construction from our point of view of weight 2 Hodge structures endowed with a compatible ring structure.

We noticed already that as vector spaces, we have a canonical identification

$$C(H_{\mathbb{C}}) = \bigwedge H_{\mathbb{C}}.$$

The right hand side is a direct sum

$$\bigoplus_k \bigwedge^k H_{\mathbb{C}}.$$

On $\bigwedge^k H_{\mathbb{C}}$, there is a weight $2k$ Hodge decomposition induced by the weight 2 Hodge decomposition on $H_{\mathbb{C}}$, with $H^{p,q}$ term given by

$$\bigoplus_{(r,s,t),2r+s=p,2t+s=q} \bigwedge^r H^{2,0} \otimes \bigwedge^s H^{1,1} \otimes \bigwedge^t H^{0,2}.$$

As $rk\, H^{2,0} = 1$, this decomposition has in fact only three terms, according to the value 0 or 1 given to r and s, noticing that for $(r,s) = (1,1)$ and $(r,s) = (0,0)$, we are in $H^{k,k}$. In other words, the induced Hodge structure on $\bigwedge^k H$ has Hodge level 2, that is, the weight 2 Hodge structure $\bigwedge^k H(-k+1)$ is effective. By definition, in the weight 2 Hodge structure $\bigwedge^k H(-k+1)$, $H^{2,0} \wedge \bigwedge^{k-1} H^{1,1}$ is assigned type $(2,0)$, $H^{0,2} \wedge \bigwedge^{k-1} H^{1,1}$ is assigned type $(0,2)$, while $\bigwedge^k H^{1,1}$ and $H^{2,0} \wedge H^{0,2} \wedge \bigwedge^{k-2} H^{1,1}$ are assigned type $(1,1)$.

Taking the direct sum over integers k, we get a weight 2 Hodge structure on $C(H) = \bigwedge H$.

Lemma 2. *This Hodge structure is compatible with the product on $C(H)$, that is satisfies condition 1 of Theorem 1.*

Proof. We have the Clifford multiplication

$$H \otimes C(H) \to C(H).$$

If we show that this map is a morphism of Hodge structures of bidegree $(-1,-1)$, then the same will be true by iteration (because H generates $C(H)$) for the multiplication map

$$C(H) \otimes C(H) \to C(H).$$

But it is well-known (see [4], p. 25) that the Clifford multiplication by $h \in H$ acting on $C(H) \cong \bigwedge H$ identifies to

$$h \wedge -h\lrcorner,$$

where $h\lrcorner$ acts on $\bigwedge H$ via the element of H^* given by $< h, \cdot >$.

Now, let $h \in H^{2,0} = \eta\mathbb{C} \subset H_{\mathbb{C}}$. Then $h\wedge$ annihilates the $(2,0)$-part of $\bigwedge^k H_{\mathbb{C}}$ which is equal to $\eta \wedge \bigwedge^{k-1} H^{1,1}$ and $h\lrcorner$ also annihilates the $(2,0)$-part of $\bigwedge^k H_{\mathbb{C}}$ because $< H^{2,0}, F^1 H_{\mathbb{C}} >= 0$.

Next, the $(1,1)$-part of $\bigwedge^k H_{\mathbb{C}}$ is equal to

$$\eta \wedge \overline{\eta} \wedge \bigwedge^{k-2} H^{1,1} \oplus \bigwedge^{k} H^{1,1}.$$

The map $h\wedge$ annihilates the first term, and sends the second one in

$$\eta \wedge \bigwedge^{k-2} H^{1,1}$$

that is, in the $(2,0)$-part of $C(H)$. Furthermore, the map $h\lrcorner$ annihilates the second term, and sends the first one in

$$\eta \wedge \bigwedge^{k-2} H^{1,1}$$

that is, in the $(2,0)$-part of $C(H)$.

Finally, the $(0,2)$-part of $\bigwedge^k H_{\mathbb{C}}$ is equal to $\overline{\eta} \wedge \bigwedge^{k-1} H^{1,1}$, and $h\wedge$ sends it to

$$\eta \wedge \overline{\eta} \wedge \bigwedge^{k-1} H^{1,1}$$

which is contained in the $(1,1)$-part of $C(H)$, while $h\lrcorner$ sends it to $\bigwedge^{k-1} H^{1,1}$ which is contained in the $(1,1)$-part of $C(H)$.

In other words, we proved that Clifford multiplication by $h \in H^{2,0}$ shifts the Hodge decomposition on $C(H_{\mathbb{C}})$ by $(1,-1)$. One shows similarly that Clifford multiplication by $h \in H^{1,1}$ preserves the Hodge decomposition on $C(H)$, which proves the claim. ∎

It turns out that the assumption on the center used in the previous section is not always satisfied by the Clifford algebra. However, it is quite easy to see directly in this case that our definition of an associated weight 1 Hodge structure on $C(H)$ still works in this case, that is, the factor M in the decomposition (1.1) is 0. Thus we have the weight 1 Hodge decomposition

$$C(H_{\mathbb{C}}) = W \oplus \overline{W}$$

as in the previous section.

To conclude, we show the following:

Proposition 5. *The Kuga-Satake construction of a complex structure on $C(H_{\mathbb{R}})$ coincides with our construction of a weight 1 decomposition on $C(H_{\mathbb{C}})$ (or equivalently a complex structure on $C(H_{\mathbb{R}})$) above.*

Proof. Recall that our weight 1 decomposition on $C(H_{\mathbb{C}})$ is given by

$$C(H_{\mathbb{C}}) = W \oplus \overline{W},$$

with $W = C(H)^{2,0}C(H_{\mathbb{C}})$. Observe now that, still denoting by η a generator of $H^{2,0}$, we have

$$C(H)^{2,0}C(H_{\mathbb{C}}) = \eta C(H_{\mathbb{C}}).$$

Indeed, as $\eta \in C(H)^{2,0}$, the inclusion $\supset$ is clear. The reverse inclusion comes from

$$C(H)^{2,0} = \eta \wedge \bigwedge H^{1,1} = \eta \cdot \bigwedge H^{1,1},$$

where on the right, the $\cdot$ stands for Clifford multiplication rather than exterior multiplication.

Next, consider the Kuga-Satake construction: the complex structure I on $C(H_{\mathbb{R}})$ is given here by multiplication by $e = -e_1e_2$, where e_1, e_2 is an oriented orthonormal basis of the real part of $H^{2,0} \oplus H^{0,2}$. Choosing η in such a way that $< \eta, \overline{\eta} >= 2$, we may assume (because $< \eta, \eta >= 0 =< e_1, e_1 > - < e_2, e_2 > +2i < e_1, e_2 >$) that

$$e_1 = Re\, \eta,\ e_2 = Im\, \eta.$$

Furthermore, the weight 1 decomposition on $C(H_{\mathbb{C}})$ associated to the Kuga-Satake complex structure is determined by the complex subspace $W' \subset C(H_{\mathbb{C}})$

$$C(H_{\mathbb{C}}) = W' \oplus \overline{W'},$$

where W' is by definition the i-eigenspace of the complex structure operator I in $C(H_{\mathbb{C}})$, that is the subspace generated by the

$$w - iI(w),\ w \in C(H_{\mathbb{C}}).$$

As I is Clifford multiplication on the left by e, W' is also the subspace generated by the

$$(1 - ie)w,\ w \in C(H_{\mathbb{C}}).$$

On the other hand, we have

$$\eta = e_1 + ie_2$$

and thus

$$\eta\overline{\eta} = (e_1 + ie_2)(e_1 - ie_2) = -2 - 2ie_1e_2 = -2(1 - ie).$$

Hence we conclude that

$$W = Im\, \eta \subset Im\, \eta\overline{\eta} = Im\, (1 - ie) = W'.$$

By the equality of dimensions, we now conclude that we have equality. ∎

In conclusion, we have split the Kuga-Satake construction into two parts:

i) The observation (studied from the point of view of its cycle-theoretic implications in [10]) that for a $K3$-type Hodge structure, the induced Hodge structures on the exterior powers of H have level 2.

ii) The construction of Theorem 1, which works for much more general compatible ring structures on polarized weight 2 Hodge structures.

Remark 2. Lemma 2 is also true for the exterior algebra structure on $\bigwedge H$. From our point of view, the key reason for which we need the Clifford algebra, is the existence of the adjunction map t, which is not satisfied by the exterior algebra.

3. The general case

We pass now to rational coefficients. We consider as in the previous section a polarized rational Hodge structure H^2 of weight 2 on a $\mathbb{Q}$-vector space H, which is also a $\mathbb{Q}$-algebra, such that the product

$$H \otimes H \to H$$

is a morphism of Hodge structures

$$H^2 \otimes H^2 \to H^2(1),$$

and such that there exists an adjunction map

$$t : H \to H$$

such that (left or right) multiplication by $t(h)$ is adjoint to (left or right) multiplication by h, with respect to the polarization form $<,>$. Our first goal is to study the possible centers of such an algebra.

Lemma 3. *The center K is a trivial sub-Hodge structure of H.*

Proof. It is obvious that K is a sub-Hodge structure of H, because it is defined as the kernel of the map

$$H \to Hom\,(H, H),$$
$$h \mapsto (a \mapsto ha - ah),$$

and this map is a morphism of Hodge structures of bidegree $(-1,-1)$.

To see that it is a trivial Hodge structure, let $0 \neq \alpha \in K^{2,0} \subset K \otimes \mathbb{C}$. Then we have

$$u := \alpha t(\overline{\alpha}) \in H^{1,1},$$

and $u \neq 0$ because $< u, 1 >=< \alpha, \overline{\alpha} >> 0$ by the Hodge-Riemann bilinear relations. Thus we have, again by the Hodge-Riemann bilinear relations:

$$< u, \overline{u} >=< ut(\overline{u}), 1 >\neq 0.$$

Hence

$$ut(\overline{u}) = \alpha t(\overline{\alpha})\alpha t(\overline{\alpha}) \neq 0.$$

But as α is central, this is 0 because $\alpha^2 = 0$, which is a contradiction. ■

Lemma 4. *The center K of H is a product of number fields. The set K^+ of t-invariant elements of K is a product of totally real number fields K_i^+. Denote by e_i the idempotent of K^+ corresponding to K_i^+, and let $K_i := e_iK$. If $K_i^+ \neq K_i$, then K_i is a field which becomes isomorphic to $\mathbb{C}$ under any embedding of K_i^+ into $\mathbb{R}$.*

Proof. K is a commutative $\mathbb{Q}$-algebra, which is by the previous Lemma contained in $H^{1,1}_{\mathbb{R}}$. Thus the pairing $<,>$ restricts to a negative definite pairing on K. Note also that K is clearly stable under t, as t is a morphism of $\mathbb{Q}$-algebras $H \to H^{op}$. On the other hand we know that this pairing is of the form

$$< \alpha, \beta >=< \alpha t(\beta), 1 > .$$

We want to show that K does not contain nilpotent elements. If $\alpha^n = 0$, then $(\alpha t(\alpha))^n = 0$, and if $\alpha t(\alpha) = 0$, then also $\alpha = 0$, because $< \alpha, \alpha >=< \alpha t(\alpha), 1 >$.

Thus it suffices to show that $\alpha^n = 0$ implies $\alpha = 0$, when $t(\alpha) = \alpha$. But if $\alpha^n = 0$, with $\alpha = t(\alpha)$ and $n = 2m$, then also $< \alpha^m, \alpha^m >= 0$ which implies that $\alpha^m = 0$. Thus the order n of nilpotency of α cannot be even, and if it is odd, $n = 2m + 1$, we have $n \leq m + 1$, that is $m = 0$.

Thus K is a product of number fields. Note that the intersection form on K must be of the form

$$< \alpha, \beta >= tr_{K/\mathbb{Q}}(y\alpha t(\beta))$$

for some $y \in K$. (Here $y \in K$ is defined by the condition that

$$< \alpha, 1 >= Tr_{K/\mathbb{Q}}(y\alpha), \forall \alpha \in K.)$$

Thus, as $< \alpha, 1 >=< t(\alpha), 1 >$, we must have $y \in K^+$.

We show now that K^+ is a product of totally real number fields. But we know that the intersection form $< \alpha\beta, 1 >$ on K^+ is negative definite, and that it is of the form

$$< \alpha, \beta >= 2tr_{K^+/\mathbb{Q}}(y\alpha\beta)$$

for some $y \in K^+$.

The algebra $K^+ \otimes \mathbb{R}$ splits as a product of quadratic extensions of $\mathbb{R}$, and we want to show that none of these extensions can be $\mathbb{C}$. But for any non-zero $y \in \mathbb{C}$, the quadratic form

$$< \alpha, \beta >= Tr_{\mathbb{C}/\mathbb{R}}(y\alpha\beta)$$

has signature $(1, 1)$, which contradicts the fact that $<,>$ should be negative definite on any factor of $K^+ \otimes \mathbb{R}$.

To conclude, consider a component K_i^+ of K^+, given by an idempotent $e_i \in K^+$. Let $K_i := e_iK$. Choose an imbedding σ of K_i^+ into $\mathbb{R}$. As K_i is a quadratic extension of K_i^+, $K_i \otimes_{\sigma(K_i^+)} \mathbb{R}$ is a quadratic extension E of $\mathbb{R}$, which is contained in $K \otimes_{\mathbb{Q}} \mathbb{R} \subset H^{1,1}_{\mathbb{R}}$. Let this extension be given by $X^2 = \lambda$. The involution t of this quadratic extension generates its Galois group, and we have the condition that the quadratic form

$$< \alpha, \beta >= Tr_{E/\mathbb{R}}(y\alpha t(\beta))$$

for some $y \in \mathbb{R}$, is definite negative. But as $t(X) = -X$, the matrix of this quadratic form in the base $(1, X)$ is

$$\begin{pmatrix} y & 0 \\ 0 & -y\lambda \end{pmatrix}$$

and thus we must have $\lambda < 0$.

■

From now on, we assume that K is a number field. This is possible because the idempotents e_i which give the decomposition of K into a product of number fields K_i are t-invariant Hodge classes in H by Lemmas 3 and 4. Thus replacing H by e_iH, we still have a polarized Hodge structure, an adjunction map t, and the compatibility of the product with the Hodge decomposition.

We define as in the previous section

$$W := H^{2,0}H.$$

Then we have the decomposition (1.1)

$$H_{\mathbb{C}} = (W \oplus \overline{W}) \oplus M,$$

which is a orthogonal decomposition and an algebra decomposition. M is defined over $\mathbb{R}$, $M = M_{\mathbb{R}} \otimes \mathbb{C}$. Note also that $M \subset H^{1,1}$, because $H^{2,0} \subset W$, $H^{0,2} \subset \overline{W}$ and $(H^{2,0} \oplus H^{0,2})^{\perp} = H^{1,1}$. Furthermore, M is stable under t and the decomposition above is given by a central idempotent $e \in K^+ \otimes \mathbb{R}$. Thus, as K^+ is totally real, $M_{\mathbb{R}}$ must be a sum

$$M_{\mathbb{R}} = \oplus_{\sigma \in \Sigma_M} H_\sigma,\ H_\sigma := H \otimes_{\sigma(K^+)} \mathbb{R},$$

where Σ_M is a certain set of imbeddings of K^+ into $\mathbb{R}$. (Here we see $H \otimes_{\sigma(K^+)} \mathbb{R}$ as the sub-algebra of $H_{\mathbb{R}}$ defined as the image of the idempotent e_σ of $K^+ \otimes_{\mathbb{Q}} \mathbb{R}$ given by σ.)

Proposition 6. *If $M \neq 0$, then the algebra H is a simple central K-algebra.*

Proof. We want to show that if Σ_M is not empty, then H has no non trivial two-sided ideal. As $\Sigma_M \neq \emptyset$, it clearly suffices to show that the algebra H_σ has no non trivial two-sided ideal for $\sigma \in \Sigma_M$. But the $\sigma \in \Sigma_M$ are characterized by the fact that $H_\sigma \subset M$, which implies

$$H_\sigma \subset H^{1,1}.$$

Note that each H_σ is invariant under t. Furthermore the t-invariant part of the center of H_σ is equal to $\mathbb{R}$.

Let thus $I \subset H_\sigma$ be a two-sided ideal. $I \cap t(I)$ is also a two-sided ideal which is t-invariant. Furthermore, if $I \cap t(I) = 0$, then $I = 0$. Indeed, if $x \in I$, $xt(x) \in I \cap t(I)$. Thus, if $I \cap t(I) = 0$, $xt(x) = 0$. On the other hand, we have $< x, x >=< xt(x), 1 >$ which is then also 0. But as x is real of type $(1,1)$, this implies that $x = 0$ by the second Hodge-Riemann bilinear relations.

Thus, replacing I by $I \cap t(I)$, we may assume that I is t-invariant. The orthogonal complement $J := I^\perp$ of I in H_σ with respect to $<,>$ is then also a t-invariant two-sided ideal of H_σ.

On the other hand, the intersection form $<,>$ restricted to I is non degenerate, because for $I \subset H_\sigma \subset H^{1,1}$, and $<,>$ is negative definite on $H^{1,1}$. Hence we get an orthogonal decomposition of H_σ into the sum of two two-sided ideals, or equivalently a decomposition

$$H_\sigma = I \oplus J,$$

as the direct sum of two sub-algebras. But then, as I and J are t-invariant, the idempotents associated to I and J are central and t-invariant. As the center of H_σ is $\mathbb{R}$, it follows that either I or J is 0. ∎

Corollary 3. *The conclusion of Theorem 1 holds without the hypothesis 3 on the center, unless possibly when H has a direct summand which is a simple central algebra H_i over a number field K_i, where the decomposition is also a Hodge structure decomposition, and an orthogonal decomposition.*

Example 1. We give here an example of a rational polarized weight 2 Hodge structure, which admits a $\mathbb{Q}$-algebra structure, satisfying conditions 1 and 2 of Theorem 1, with center a number field K, and for which we have both

$$M \neq 0,\ W \neq 0,$$

where W and M are defined as in section 1.

We start from a rank 2 vector space V over the number field $K := \mathbb{Q}(\sqrt{2}, i)$, that is, $K = \mathbb{Q}[x, y]/(x^2 = 2, y^2 = -1)$. Let $K^+ := \mathbb{Q}(x) = \mathbb{Q}(\sqrt{2}) \subset K$. On K, there is a unique K^+-bilinear skew-symmetric form Ω which satisfies the property that

$$\Omega(1, y) = 1.$$

On $V = K \oplus K$, consider the rational skew-symmetric bilinear form :

$$\omega := tr_{K^+/\mathbb{Q}}\Omega_1 + tr_{K^+/\mathbb{Q}}x\Omega_2,$$

where Ω_i is Ω on the i-th factor.

We want now to put a weight 1 Hodge structure on V, polarized by ω and admitting K as an endomorphism algebra. This is done as follows:

The space $V \otimes_{\mathbb{Q}} \mathbb{R}$ splits as the direct sum

$$V_{\sqrt{2}} \oplus V_{-\sqrt{2}}$$

corresponding to the two embeddings of K^+ in $\mathbb{R}$ sending x to $\sqrt{2}$ or to $-\sqrt{2}$ respectively. Each term of this decomposition admits the action of y, with $y^2 = -1$.

We want to put a complex structure on $V_{\mathbb{R}}$, given by an operator of complex structure I, which leaves invariant ω, and is such that $\omega(x, Ix) > 0$ for $0 \neq x \in V_{\mathbb{R}}$. We define for this $I = y$ on $V_{\sqrt{2}}$, while on the factor $V_{-\sqrt{2}}$, the operator I is defined as follows:

The space

$$K_{\mathbb{R}} := K \otimes \mathbb{R}$$

is equal to the quadratic extension of $K^+_{\mathbb{R}} := K^+ \otimes \mathbb{R}$ given by $y^2 = -1$. The algebra $K^+_{\mathbb{R}}$ splits as a sum of two copies of $\mathbb{R}$, $K^+_{\sqrt{2}}$ and $K^+_{-\sqrt{2}}$, where x acts by multiplication by $\sqrt{2}$ and $-\sqrt{2}$ respectively. The first copy is canonically isomorphic to $\mathbb{R}$ generated by the idempotent $1^+ := \frac{x+\sqrt{2}}{2\sqrt{2}}$ and the second copy is similarly generated by the idempotent $1^- := \frac{x-\sqrt{2}}{-2\sqrt{2}}$.

This makes (via the choice $y = i$) the algebra $K_{\mathbb{R}}$ canonically isomorphic to the sum of two copies $K_{\sqrt{2}}$, $K_{-\sqrt{2}}$ of $\mathbb{C}$, and thus $V_{\mathbb{R}} \cong K^2_{\mathbb{R}}$ is a direct sum of four copies of $\mathbb{C}$. It is not hard to compute that the extended form ω in the canonical coordinates $z_1, \ldots, z_4$ on $\mathbb{C}^4$ is equal to

$$\frac{i}{2}(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z}_2 + \sqrt{2}dz_3 \wedge d\overline{z}_3 - \sqrt{2}dz_4 \wedge d\overline{z}_4). \tag{3.6}$$

Thus it is of type $(1,1)$ for the complex structure given by y, but not positive.

We will define our I to be equal to i on the first three factors and to $-i$ on the last factor. This has the effect of exchanging the coordinates z_4 and $\overline{z}_4$, and then by formula (3.6), ω becomes positive of type $(1,1)$ with respect to I. By construction, I commutes with x and y, as it preserves their eigenspace decomposition.

Having this, we will now consider as polarized Hodge structure

$$H := End_K(V).$$

This is a sub-Hodge structure of $End\,(V)$, because K acts as endomorphisms of the weight 1 Hodge structure on V.

This is stable under adjunction with respect to ω, because the action of K on V is such that multiplication by x is self-adjoint with respect to ω, while multiplication by y is anti-self-adjoint.

On the other hand, consider the decomposition

$$H_{\mathbb{R}} = H_{\sqrt{2}} \oplus H_{-\sqrt{2}}.$$

Then $H_{\sqrt{2}} = End_{\mathbb{C}}(V_{\sqrt{2}})$ and $H_{-\sqrt{2}} = End_{\mathbb{C}}(V_{-\sqrt{2}})$, where the $\mathbb{C}$-structures are given by the action of iy on both terms. As $I = y$ on $V_{\sqrt{2}}$, the term $H_{\sqrt{2}}$ is made of endomorphisms commuting with I, thus of type $(1,1)$, while as $I \neq y$ on $V_{-\sqrt{2}}$, the term $H_{-\sqrt{2}}$ is not of type $(1,1)$. Thus in this case both $M = H_{\sqrt{2}}$ and $W \oplus \overline{W} = H_{-\sqrt{2}}$ are non zero.

We now come back to the general case of a weight 2 rational Hodge structure H^2 on H, polarized by an intersection form $<,>$, and endowed with a $\mathbb{Q}$-algebra structure, satisfying conditions 1 and 2 of Theorem 1. We also assume that the t-invariant part of the center is a number field K^+.

Our goal is to show the following:

Theorem 2. *There exists a polarized weight* 1 *Hodge structure* H^1 *on* H*, such that the multiplication on the left*

$$H \to Hom\,(H,H)$$

is a morphism of weight 2 *Hodge structures*

$$H^2 \to Hom\,(H^1, H^1)(1).$$

Remark 3. The main defect of this construction is the fact that it is not unique, and does not satisfy the universal property of Proposition 3.

Proof. Here we will see a (effective) weight 1 Hodge structure on H as a complex structure on $H_{\mathbb{R}}$.

Recall that we have the decomposition of Corollary 1

$$H_{\mathbb{R}} = (W \oplus \overline{W})_{\mathbb{R}} \oplus M_{\mathbb{R}},$$

which is a orthogonal decomposition, an algebra decomposition, and is compatible with Hodge decomposition. M is a sum of factors

$$M = \oplus_{\sigma \in \Sigma_M} H_\sigma,$$

where $H_\sigma := H \otimes_{\sigma(K^+)} \mathbb{R}$ is a subalgebra of $H_{\mathbb{R}}$. There is already a complex structure on the first term, given by the isomorphism of real vector spaces

$$Re : W \cong (W \oplus \overline{W})_{\mathbb{R}},$$

and we simply have to put a complex structure I_σ on each component H_σ for $\sigma \in \Sigma_M$.

Note that we want the left multiplication map

$$H \to Hom\,(H,H)$$

to be a morphism of weight 2 Hodge structures $H^2 \to Hom\,(H^1,H^1)(1)$. This implies that

$$H_\sigma \to Hom\,(H_\sigma,H_\sigma)$$

has to be a morphism of real Hodge structures. But, for $\sigma \in \Sigma_M$, we know that H_σ is of type $(1,1)$. Thus multiplication on the left on H_σ by any $h \in H_\sigma$ has to be of type $(1,1)$ for the Hodge decomposition on $Hom\,(H^1,H^1)(1)$, which means that it commutes with the complex structure operator I_σ. But as our algebra has an unit, this implies in turn that I_σ has to be the multiplication on the right by some element $m_\sigma \in H_\sigma$, satisfying the condition that $m_\sigma^2 = -1$.

Furthermore, we want that our weight 1 Hodge structure is polarized, with a polarization of the form

$$\omega_a(x,y) =< x, ya >,$$

for some $a \in H$ satisfying $t(a) = -a$. (This was the form chosen for the polarization on the W-term, and since the polarization must be rational, we do not have another choice here.)

The first condition for ω_a to polarize the real Hodge structure on H_σ is the fact that ω_a is of type $(1,1)$ for the complex structure I_σ. Equivalently, for any x, $y \in H_\sigma$,

$$< x, ya >=< I_\sigma(x), I_\sigma(y)a >=< xm_\sigma, ym_\sigma a >=< x, ym_\sigma at(m_\sigma) > .$$

This implies that

$$m_\sigma at(m_\sigma) = a. \tag{3.7}$$

Let us now distinguish the cases where $K = K^+$, $K \neq K^+$.

a) *Case* $K \neq K^+$. In this case, we proved in Lemma 4 that $K_\sigma \cong \mathbb{C}$. Choosing such an isomorphism gives $i_\sigma \in K_\sigma$, with $i_\sigma^2 = -1$. Thus we have the operator of complex structure acting on H_σ by multiplication by i_σ.

As i_σ is in the center of H_σ, and satisfies $t(i_\sigma) = -i_\sigma$, the relation (3.7) for $m_\sigma = i_\sigma$ is certainly satisfied for all a.

It remains to see that with this operator of complex structure I_σ, the corresponding real weight 1 Hodge structure on H_σ is polarized by ω_a for an adequate $a \in H$ satisfying $t(a) = -a$. As we know already that ω_a is of type $(1,1)$ for I_σ, we have only to verify that

$$\omega_a(x, I_\sigma x) > 0,$$

for $0 \neq x \in H_\sigma$. By definition, this is equal to

$$< x, xi_\sigma a > .$$

For $a = i_\sigma \in H_\sigma \subset H_{\mathbb{R}}$, this is equal to $- < x, x >$ which is positive by the second Hodge-Riemann bilinear relations, because $H_\sigma \subset H^{1,1}_{\mathbb{R}}$.

Thus it follows that it remains positive for any a in a neighbourhood of i_σ in $H_{\mathbb{R}}$ satisfying $t(a) = -a$, and in particular for a rational such $a \in H \subset H_{\mathbb{R}}$.

b) *Case* $K = K^+$. Here the a will be a fixed rational element of H such that ω_a satisfies the positivity conditions on the W-components (the existence of which was shown in the previous section), and on the components H_σ of the previous type a). Note that the multiplication (on the right or on the left) by a on H is an isomorphism, because it is K-linear and for any imbedding $\tau : K \hookrightarrow \mathbb{R}$ such that $\tau \notin \Sigma_M$, it induces an automorphism of H_τ, because ω_a is non-degenerate on H_τ.

(The existence of imbeddings $\tau \notin \Sigma_M$ follows from the fact that $W \neq 0$ by assumption.)

We consider the commutative K-subalgebra of H generated by a:

$$K_a := K[a] \subset H.$$

This subalgebra is invariant under t, because $t(a) = -a$. Next, the subalgebra $K_{a,\sigma} := K_a \otimes_{\sigma(K)} \mathbb{R}$ of H_σ is contained in $H^{1,1}$, and thus satisfies the property that the intersection form

$$< x, y > = < xt(y), 1 >$$

is negative definite on $K_{a,\sigma}$.

This implies as in the proof of Proposition 4 that the t-invariant part $K^+_{a,\sigma}$ of $K_{a,\sigma}$ is a sum of copies $\mathbb{R}_\rho$ of $\mathbb{R}$, and that the corresponding decomposition of $K_{a,\sigma}$ (given by the action of the idempotents of $K^+_{a,\sigma}$) is a decomposition as a sum of copies $\mathbb{C}_\rho$ of $\mathbb{C}$, where t acts as complex conjugation on each $\mathbb{C}_\rho$. Furthermore, as $t(a) = -a$ and multiplication by a is an isomorphism on K_a, each a_ρ can be written uniquely as $\lambda_\rho i_\rho$, where λ_ρ is a positive real number, and $i_\rho \in \mathbb{C}_\rho$ satisfies $i_\rho^2 = -1_\rho$.

Let us define m_σ by

$$m_\sigma = \sum_\rho i_\rho.$$

We have $m_\sigma^2 = -\sum_\rho 1_\rho = -1$. Furthermore $t(m_\sigma) = -m_\sigma$, as $t(i_\rho) = -i_\rho$, for all ρ.

Finally, as $m_\sigma \in K_{a,\sigma}$, one has $m_\sigma a = a m_\sigma$, and combining these three facts, we conclude that (3.7) is satisfied.

Thus the multiplication on the right by m_σ defines a complex structure I_σ on H_σ, which satisfies the property that ω_a is of type $(1,1)$ for I_σ. In order to conclude that we have a real weight 1 polarized Hodge structure on H_σ, we have to check the positivity property

$$\omega_a(x, I_\sigma(x)) > 0,\ \forall 0 \neq x \in H_\sigma.$$

But as the numbers λ_ρ are positive, one has

$$a_\rho = n_\rho^2 m_\rho,$$

for some $n_\rho \in \mathbb{R}$. Thus, letting

$$n := \sum_\rho n_\rho,$$

we have

$$a_\sigma = n^2 m_\sigma,$$

with $n \in K_{a,\sigma}$, $t(n) = n$. But then we have

$$\omega_a(x, I_\sigma(x)) =< x, I_\sigma(x)a >=< x, xm_\sigma a >$$
$$=< x, xn^2 m_\sigma^2 >= - < nx, nx >,$$

where the last inequality holds because $m_\sigma^2 = -1$ and $t(n) = n$. As n is non-degenerate, the second Hodge bilinear relations show that $< nx, nx >$ is negative for all $0 \neq x \in H_\sigma$, which is what we wanted.

■

Remark 4. Assume that either Σ_M is empty, or that the center K of H satisfies $K \neq K^+$. In the first case, we constructed the weight 1 Hodge structure by defining its $H^{1,0}$-part to be $H^{2,0}H_{\mathbb{C}}$, which is a right ideal. Thus multiplication on the right by elements of H are morphisms of weight 1 Hodge structure.

In the second case, the same is true if we choose for complex structure operator on the H_σ, $\sigma \in H_\sigma$, the multiplication by $i_\sigma \in K \otimes_{\sigma(K^+)} \mathbb{R}$. Indeed, as i_σ is central, this multiplication commutes with right multiplication with elements of H.

Thus, in both cases, as in the Kuga-Satake case, we can construct an abelian variety, which admits H as a sub-Hodge structure of weight 2, and also H^{op} as a ring of endomorphisms.

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