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On the Kähler-Ricci Flow on Complex Surfaces

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1 Introduction

One of the most important properties of a geometric flow is whether it preserves the positivity of various notions of curvature. In the case of the Kähler-Ricci flow, the positivity of the curvature operator (Hamilton [7]), the positivity of the biholomorphic sectional curvature (Bando [1], Mok[8]), and the positivity of the scalar curvature (Hamilton [4]) are all preserved. However, whether the positivity of the Ricci curvature is preserved is still not known. As stressed for example in Chen-Tian [3], this is central to the problem of convergence of the Kähler-Ricci flow on Kähler-Einstein manifolds of positive curvature. The existence of Kähler-Einstein metrics has been conjectured by S.T.Yau [10] to be equivalent to stability in geometric invariant theory, and there is strong interest in relating these notions to the behavior of the Kähler-Ricci flow.

In this note, we show that the positivity of the Ricci curvature is preserved on compact complex surfaces, under the additional assumption that the sum of any two eigenvalues of the traceless curvature operator on traceless $(1,1)$-forms is non-negative.

2 The curvature operator in the Kähler case

Let X be an n-dimensional compact complex manifold, with a Kähler metric $ds^2 = g_{\bar{k}j} dz^j d\bar{z}^k$. The Kähler-Ricci flow is the flow $\dot{g}_{\bar{k}j} = -R_{\bar{k}j} + \mu g_{\bar{k}j}$, where $n\mu$

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is the average scalar curvature, and $R_{\bar{k}j}$ is the Ricci curvature. By differentiating the defining relation $[\nabla_j, \nabla_{\bar{k}}]V^p = R_{\bar{k}j}{}^p{}_q V^q$, we obtain the corresponding flows for the Riemann curvature tensor $R_{\bar{k}j}{}^p{}_q$, the Ricci curvature $R_{\bar{k}j} = R_{\bar{k}j}{}^p{}_p$, and the scalar curvature $R = g^{j\bar{k}} R_{\bar{k}j}$:

$$\begin{aligned}
\dot{R} &= \Delta R - \mu R + R^{\bar{k}j} R_{\bar{k}j} \\
\dot{R}_{\bar{k}j} &= \Delta R_{\bar{k}j} + R_{\bar{k}l}{}^m{}_j R^l{}_m - R_{\bar{k}}{}^{\bar{m}} R_{\bar{m}j} \\
\dot{R}_{\bar{q}j\bar{l}m} &= \Delta R_{\bar{q}j\bar{l}m} + \mu R_{\bar{q}j\bar{l}m} - R_{\bar{l}}{}^{\bar{r}} R_{\bar{q}j\bar{r}m} - R_{\bar{q}}{}^{\bar{r}} R_{\bar{l}j\bar{r}m} + R_{\bar{r}}{}^{\bar{p}}{}_{\bar{q}j} R_{\bar{p}}{}^{\bar{r}}{}_{\bar{l}m} \\
&\qquad + R_{\bar{q}}{}^{\bar{p}}{}_{\bar{r}m} R_{\bar{p}}{}^{\bar{r}}{}_{\bar{l}j} - R_{\bar{q}}{}^{\bar{p}}{}_{\bar{l}}{}^{\bar{r}} R_{\bar{p}j\bar{r}m}
\end{aligned} \tag{2.1}$$

Here $\Delta = \nabla_l \nabla^l = g^{l\bar{k}} \nabla_l \nabla_{\bar{k}}$ is the complex Laplacian. It is easily seen that the flows of R and $R_{\bar{k}j}$ can be written in the same form with Δ replaced by $\bar{\Delta} = \nabla_{\bar{l}} \nabla^{\bar{l}} = \nabla^l \nabla_l$, and hence with $\frac{1}{2}\Delta_{\mathbf{R}} = \frac{1}{2}(\Delta + \bar{\Delta})$. On the other hand, the flow for the Riemann curvature tensor becomes, when written with $\bar{\Delta}$

$$\begin{aligned}
\dot{R}_{\bar{q}j\bar{l}m} = &\bar{\Delta} R_{\bar{q}j\bar{l}m} + \mu R_{\bar{q}j\bar{l}m} - R^r{}_m R_{\bar{q}j\bar{l}r} - R^r{}_j R_{\bar{q}m\bar{l}r} + R^p{}_{r\bar{q}j} R^r{}_{p\bar{l}m} \\
&+ R^p{}_{j\bar{l}r} R^r{}_{p\bar{q}m} - R^p{}_j{}^r{}_m R_{\bar{q}p\bar{l}r}
\end{aligned} \tag{2.2}$$

Combining the flows with Δ and $\bar{\Delta}$, we obtain the flow with *real* Laplacian:

$$\begin{aligned}
\dot{R}_{\bar{q}j\bar{l}m} = &\frac{1}{2}\Delta_{\mathbf{R}} R_{\bar{q}j\bar{l}m} + \mu R_{\bar{q}j\bar{l}m} - \frac{1}{2}(R^r{}_m R_{\bar{q}j\bar{l}r} + R^r{}_j R_{\bar{q}m\bar{l}r} + R_{\bar{l}}{}^{\bar{r}} R_{\bar{q}j\bar{r}m} + R_{\bar{q}}{}^{\bar{r}} R_{\bar{l}j\bar{r}m}) \\
&+ R^p{}_{r\bar{q}j} R^r{}_{p\bar{l}m} + R^p{}_{r\bar{l}j} R^r{}_{p\bar{q}m} - R_{\bar{q}}{}^{\bar{p}}{}_{\bar{l}}{}^{\bar{r}} R_{\bar{p}j\bar{r}m}
\end{aligned} \tag{2.3}$$

As in the Riemannian case [6,7], the flow for the Riemann curvature operator simplifies considerably in the formalism of frames. Let $e_a = \frac{\partial}{\partial z^j} e^j{}_a$, $e_{\bar{a}} = e_{\bar{a}}{}^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}$ be an orthonormal frame at time $t = 0$, i.e., $e_{\bar{b}}{}^{\bar{k}} g_{\bar{k}j} e^j{}_a = \delta_{\bar{b}a}$, $e^j{}_a e^a{}_k = \delta^j{}_k$, and $e^j{}_a \delta^{a\bar{b}} e_{\bar{b}}{}^{\bar{k}} = g^{j\bar{k}}$. Let $g_{\bar{k}j}$ flow by $\dot{g}_{\bar{k}j}$. We want to flow $e^j{}_a$ so that it remains an orthonormal frame with time. Thus we impose $0 = (g_{\bar{k}j} e^j{}_a e_{\bar{b}}{}^{\bar{k}})^{\cdot} = \dot{g}_{\bar{k}j} e^j{}_a e_{\bar{b}}{}^{\bar{k}} + g_{\bar{k}j} \dot{e}^j{}_a e_{\bar{b}}{}^{\bar{k}} + g_{\bar{k}j} e^j{}_a \dot{e}_{\bar{b}}{}^{\bar{k}}$. For this to hold, it suffices to set

$$\dot{e}^j{}_a = -\frac{1}{2} g^{j\bar{r}} \dot{g}_{\bar{r}s} e^s{}_a = -\frac{1}{2} g^{j\bar{r}} (-R_{\bar{r}s} + \mu g_{\bar{r}s}) e^s{}_a \tag{2.4}$$

from which it follows that $\dot{e}_{\bar{b}}{}^{\bar{k}} = \frac{1}{2} R_{\bar{b}}{}^{\bar{k}} - \frac{1}{2}\mu e^{\bar{k}}{}_{\bar{b}}$, $\dot{e}^b{}_k = -\frac{1}{2} R^b{}_k + \frac{1}{2}\mu e^b{}_k$, where in general, we can go back and forth between middle Latin indices (j,k,l...) and early Latin indices (a,b,c...) by using frames, e.g. $V^a = e^a{}_j V^j$, $R_{\bar{a}b\bar{c}d} = e_{\bar{a}}{}^{\bar{j}} e^k{}_b e_{\bar{c}}{}^{\bar{l}} e^m{}_d R_{\bar{j}k\bar{l}m}$.

The flow of the frame gets rid of all the terms mixing the Ricci tensor and the curvature tensor in the flow of $R_{\bar{a}b\bar{c}d}$. Indeed, the cancellation mechanism is very simple:

$$\begin{aligned}\dot{R}_{\bar{a}b\bar{c}d} = &\dot{e}_{\bar{a}}{}^{\bar{j}} e^k{}_b e_{\bar{c}}{}^{\bar{l}} e^m{}_d R_{\bar{j}k\bar{l}m} + e_{\bar{a}}{}^{\bar{j}} \dot{e}^k{}_b e_{\bar{c}}{}^{\bar{l}} e^m{}_d R_{\bar{j}k\bar{l}m} + e_{\bar{a}}{}^{\bar{j}} e^k{}_b \dot{e}_{\bar{c}}{}^{\bar{l}} e^m{}_d R_{\bar{j}k\bar{l}m} \\ &+ e_{\bar{a}}{}^{\bar{j}} e^k{}_b e_{\bar{c}}{}^{\bar{l}} \dot{e}^m{}_d R_{\bar{j}k\bar{l}m} + e_{\bar{a}}{}^{\bar{j}} e^k{}_b e_{\bar{c}}{}^{\bar{l}} e^m{}_d \dot{R}_{\bar{j}k\bar{l}m} \end{aligned} \tag{2.5}$$

We have for example $\dot{e}_{\bar{a}}{}^{\bar{j}} e^k{}_b e_{\bar{c}}{}^{\bar{l}} e^m{}_d R_{\bar{j}k\bar{l}m} = \frac{1}{2} R_{\bar{a}}{}^{\bar{q}} R_{\bar{q}b\bar{c}d} - \frac{1}{2}\mu R_{\bar{a}b\bar{c}d}$, and the first term on the right hand side cancels with one of the terms in the flow with *real* Laplacian. Altogether, we obtain the equation

$$\dot{R}_{\bar{a}b\bar{c}d} = \frac{1}{2}\Delta_{\mathbf{R}} R_{\bar{a}b\bar{c}d} - \mu R_{\bar{a}b\bar{c}d} + R_{\bar{p}r\bar{a}b} R_{\bar{r}p\bar{c}d} + R_{\bar{a}p\bar{r}d} R_{\bar{p}b\bar{c}r} - R_{\bar{a}p\bar{c}r} R_{\bar{p}b\bar{r}d} \tag{2.6}$$

Similarly, the same simplification occurs for the flow of the Ricci curvature, written in a frame. Differentiating the equation $R_{\bar{a}b} = e_{\bar{a}}{}^{\bar{q}} e^j{}_b R_{\bar{q}j}$, we see, not surprisingly, that the term involving the square of the Ricci curvature cancels

$$\dot{R}_{\bar{a}b} = \frac{1}{2}\Delta_{\mathbf{R}} R_{\bar{a}b} - \mu R_{\bar{a}b} + R_{\bar{a}b}{}^p{}_r R^r{}_p. \tag{2.7}$$

2.1 The traceless curvature operator $S_{\bar{a}b\bar{c}d}$

To analyze the flow of the Riemannian curvature tensor in the operator case, it is convenient to separate out the traces. Thus set

$$\begin{aligned} S_{\bar{a}b} &= R_{\bar{a}b} - \frac{1}{n} R\delta_{\bar{a}b} \\ S_{\bar{a}b\bar{c}d} &= R_{\bar{a}b\bar{c}d} - \frac{1}{n}(R_{\bar{a}b}\delta_{\bar{c}d} + R_{\bar{c}d}\delta_{\bar{a}b}) + \frac{1}{n^2} R\delta_{\bar{a}b}\delta_{\bar{c}d} \end{aligned} \tag{2.8}$$

Then $S_{\bar{a}a} = 0$, $S_{\bar{a}a\bar{c}d} = 0 = S_{\bar{a}b\bar{c}c}$, and a straightforward calculation shows that the flows for R, $R_{\bar{a}b}$, $R_{\bar{a}b\bar{c}d}$ are equivalent to the following flows for R, $S_{\bar{a}b}$, $S_{\bar{a}b\bar{c}d}$

$$\begin{aligned} \dot{R} &= \frac{1}{2}\Delta_{\mathbf{R}} R + S_{\bar{p}r} S_{\bar{r}p} + \frac{1}{n} R(R - \mu n) \\ \dot{S}_{\bar{a}b} &= \frac{1}{2}\Delta_{\mathbf{R}} S_{\bar{a}b} + \frac{1}{n}(R - \mu n) S_{\bar{a}b} + S_{\bar{a}b\bar{c}d} S_{\bar{d}c} \\ \dot{S}_{\bar{a}b\bar{c}d} &= \frac{1}{2}\Delta_{\mathbf{R}} S_{\bar{a}b\bar{c}d} - \mu S_{\bar{a}b\bar{c}d} + S_{\bar{p}r\bar{a}b} S_{\bar{r}p\bar{c}d} + S_{\bar{a}p\bar{r}d} S_{\bar{p}b\bar{c}r} - S_{\bar{a}p\bar{c}r} S_{\bar{p}b\bar{r}d} + \frac{1}{n} S_{\bar{a}b} S_{\bar{c}d} \end{aligned} \tag{2.9}$$

In the Kähler case, the Riemann curvature tensor can be viewed as a symmetric operator $Op(R)$ on the space $\Lambda^{1,1}$ of real $(1,1)$-forms. This space itself decomposes into the line spanned by the Kähler form $\omega = \frac{\sqrt{-1}}{2} g_{\bar{k}j} dz^j \wedge d\bar{z}^k$, and its orthogonal complement, namely the space $\Lambda_0^{1,1}$ of traceless real $(1,1)$-forms. Now the term $R_{\bar p r \bar a b} R_{\bar r p \bar c d}$ can clearly be viewed as $Op(R)^2$. Similarly, the tensor $S_{\bar a b \bar c d}$ can be viewed as an operator $Op(S)$ on $\Lambda_0^{1,1}$, and we have the decomposition

$$Op(R) = \begin{pmatrix} R/2 & S \\ S^t & Op(S) \end{pmatrix} \tag{2.10}$$

The term $S_{\bar p r \bar a b} S_{\bar r p \bar c d}$ in the flow for $S_{\bar a b \bar c d}$ corresponds to $Op(S)^2$. Following Hamilton [6,7], we show that the remaining terms $S_{\bar a p \bar r d} S_{\bar p b \bar c r} - S_{\bar a p \bar c r} S_{\bar p b \bar r d}$ admit a Lie algebra interpretation. Define the Lie bracket by

$$[\phi, \psi]_{\bar a b} = \phi_{\bar a p} \psi_{\bar p b} - \psi_{\bar a p} \phi_{\bar p b} \tag{2.11}$$

Let $\phi^\alpha_{\bar a b}$ be an orthonormal basis of real traceless $(1,1)$-forms, and set $S_{\bar a b \bar c d} = \sum_{\alpha\beta} M_{\alpha\beta} \phi^\alpha_{\bar a b} \phi^\beta_{\bar c d}$. Thus $M_{\alpha\beta}$ is the matrix of $Op(S)$ in the basis $\phi^\alpha_{\bar a b}$. Then

$$\begin{aligned} S_{\bar a p \bar c r} S_{\bar p b \bar r d} - S_{\bar a p \bar r d} S_{\bar p b \bar c r} &= M_{\alpha\lambda} M_{\beta\mu} \phi^\alpha_{\bar a p} \phi^\beta_{\bar p b} (\phi^\lambda_{\bar c r} \phi^\mu_{\bar r d} - \phi^\lambda_{\bar r d} \phi^\mu_{\bar c r}) \\ &= M_{\alpha\lambda} M_{\beta\mu} \phi^\alpha_{\bar a p} \phi^\beta_{\bar p b} [\phi^\lambda, \phi^\mu]_{\bar c d} \end{aligned} \tag{2.12}$$

Set $[\phi^\lambda, \phi^\mu] = c^{\lambda\mu\rho} \phi^\rho$, where $c^{\lambda\mu\rho}$ are the structure constants of the Lie algebra. The antisymmetry of $c^{\lambda\rho\mu}$ implies

$$M_{\alpha\lambda} M_{\beta\mu} \phi^\alpha_{\bar a p} \phi^\beta_{\bar p b} c^{\lambda\mu\rho} \phi^\rho_{\bar c d} = \frac{1}{2} M_{\alpha\lambda} M_{\beta\mu} [\phi^\alpha, \phi^\beta]_{\bar a b} c^{\lambda\mu\rho} \phi^\rho_{\bar c d},$$

and thus

$$S_{\bar a p \bar c r} S_{\bar p b \bar r d} - S_{\bar a p \bar r d} S_{\bar p b \bar c r} = \frac{1}{2} M_{\alpha\lambda} M_{\beta\mu} c^{\alpha\beta\nu} \phi^\nu_{\bar a b} c^{\lambda\mu\rho} \phi^\rho_{\bar c d} \equiv \frac{1}{2} M^\#_{\nu\rho} \phi^\nu_{\bar a b} \phi^\rho_{\bar c d} \tag{2.13}$$

To make $M^\#_{\nu\rho}$ explicit, we need the structure constants $c^{\alpha\beta\nu}$ of the Lie algebra of traceless $(1,1)$-forms. Choose a coordinate system centered at a point $p \in M$ such that the metric $g_{\bar k j}$ is the identity matrix at p. Then an orthogonal basis for the space of real $(1,1)$ forms is (in dimension 2 to simplify notations) $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar z_1 + dz_2 \wedge d\bar z_2)$, $\eta_1 = dx_1 \wedge dy_1 - dx_2 \wedge dy_2 = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar z_1 - dz_2 \wedge d\bar z_2)$, $\eta_2 = dx_1 \wedge dy_2 + dx_2 \wedge dy_1 = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar z_2 + dz_2 \wedge d\bar z_1)$, $\eta_3 = dx_1 \wedge dx_2 + dy_1 \wedge dy_2 = \frac{1}{2}(dz_1 \wedge d\bar z_2 - dz_2 \wedge d\bar z_1)$, with $\sqrt{2}\eta_i$ forming an orthonormal basis for $\Lambda_0^{1,1}$. Furthermore, $[\eta_2, \eta_3] = \eta_1$, $[\phi_1, \phi_2] = \phi_3$, $[\phi_3, \phi_1] = \phi_2$, which means that $\Lambda_0^{1,1}$ is $su(2)$ with structure constants

$$c^{\alpha\beta\gamma} = \sqrt{2} \epsilon^{\alpha\beta\gamma} \tag{2.14}$$

where $\epsilon^{\alpha\beta\gamma}$ is the sign of the permutation $(1,2,3) \mapsto (\alpha, \beta, \gamma)$.

2.2 Positivity of the Ricci curvature in dimension 2

We are now in position to prove the following theorem:

Theorem. *Let X be a compact Kähler manifold of dimension 2, and consider the Kähler-Ricci flow $\dot g_{\bar kj} = -R_{\bar kj} + \mu g_{\bar kj}$. If the initial metric has Ricci curvature non-negative everywhere and positive somewhere, and if the sum of the two lowest eigenvalues of the operator $S_{\bar ab\bar cd}$ on the space $\Lambda_0^{1,1}$ of traceless $(1,1)$-forms is non-negative, then both of these properties continue to hold for all time $t > 0$.*

Proof. If we view the Ricci curvature as a Hermitian form on $T^{1,0}$ vectors, its positivity is equivalent to the positivity of its trace and of its determinant. Set

$$R_{\bar ab} \;=\; A\frac{\omega}{\sqrt{-1}/2} + B_1\frac{\eta_1}{\sqrt{-1}/2} + B_2\frac{\eta_2}{\sqrt{-1}/2} + B_3\frac{\eta_3}{\sqrt{-1}/2} \tag{2.15}$$

In particular, $A = \frac{1}{2}R$ and $\frac{\sqrt2}{\sqrt{-1}}B_i$ are the components of $S_{\bar ab}$ in the orthonormal basis $\sqrt2\eta_i$ for $\Lambda_0^{1,1}$.

Claim: The Ricci curvature is non-negative if and only if $A \geq 0$ and

$$A^2 - B_1^2 - B_2^2 - B_3^2 \geq 0\;, \quad \text{i.e., } S_{\bar ab}S_{\bar ba} \leq \tfrac{1}{2}R^2 \tag{2.16}$$

To see this, we let $X = a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2}$ be an arbitrary tangent vector. Then

$$\begin{aligned} Ricci(X,\bar X) &= A(|a|^2+|b|^2) + B_1(|a|^2-|b|^2) + B_2(a\bar b + b\bar a) - \sqrt{-1}B_3(a\bar b - b\bar a)\\ &= \begin{pmatrix} a & b\end{pmatrix}\begin{pmatrix} A+B_1 & B_2-\sqrt{-1}B_3\\ B_2+\sqrt{-1}B_3 & A-B_1\end{pmatrix}\begin{pmatrix}\bar a\\ \bar b\end{pmatrix} = \begin{pmatrix} a & b\end{pmatrix} P \begin{pmatrix}\bar a\\ \bar b\end{pmatrix}\end{aligned}$$

Thus the Ricci curvature is non-negative if and only if the matrix P is non-negative. Now the trace of P is $2A$ and the determinant is $A^2 - B_1^2 - B_2^2 - B_3^2$. This proves the claim.

Set $|S|^2 = S_{\bar pr}S_{\bar rp}$. Using the flow for $S_{\bar pr}$, we find

$$(|S|^2)^{\cdot} = \frac{1}{2}\Delta_{\mathbf{R}}|S|^2 - (\nabla^l S_{\bar pr}\nabla_l S_{\bar rp} + \nabla_l S_{\bar pr}\nabla^l S_{\bar rp}) + \frac{2}{n}(R-\mu n)|S|^2 + 2\,S_{\bar ba}S_{\bar ab\bar cd}S_{\bar dc} \tag{2.17}$$

Combining with the flow for R

$$\dot R = \frac{1}{2}\Delta_{\mathbf{R}}R + |S|^2 + \frac{1}{n}R(R-\mu n), \tag{2.18}$$

we obtain the flow for the determinant of the Ricci curvature

$$(\frac{1}{2}R^2 - |S|^2)^{\cdot} = \frac{1}{2}\Delta_{\mathbf{R}}(\frac{1}{2}R^2 - |S|^2) - \nabla_l R\nabla^l R + (\nabla^l S_{\bar{p}r}\nabla_l S_{\bar{r}p} + \nabla_l S_{\bar{p}r}\nabla^l S_{\bar{r}p})$$
$$+\frac{2}{n}(R-\mu n)(\frac{1}{2}R^2 - |S|^2) + R|S|^2 - 2\, S_{\bar{b}a}S_{\bar{a}b\bar{c}d}S_{\bar{d}c} \qquad (2.19)$$

We shall abbreviate this equation by

$$(\frac{1}{2}R^2 - |S|^2)^{\cdot} = \frac{1}{2}\Delta_{\mathbf{R}}(\frac{1}{2}R^2 - |S|^2) - |\nabla R|^2 + (\nabla S\bar{\nabla}\bar{S} + \bar{\nabla}S\nabla\bar{S})$$
$$+\frac{2}{n}(R-\mu n)(\frac{1}{2}R^2 - |S|^2) + R|S|^2 - 2\langle S\, Op(S)\, S\rangle \qquad (2.20)$$

We examine the non-negativity of the expression $\frac{1}{2}R^2 - |S|^2$, assuming that it is non-negative at initial time. Consider then the first time when $\min(\frac{1}{2}R^2 - |S|^2) = 0$, and consider a minimum point. At this point, by the maximum principle, we have

$$(\frac{1}{2}R^2 - |S|^2)^{\cdot} \geq -|\nabla R|^2 + (\nabla S\bar{\nabla}\bar{S} + \bar{\nabla}S\nabla\bar{S}) + R|S|^2 - 2\langle S\, Op(S)\, S\rangle \qquad (2.21)$$

On the other hand, at a minimum, the derivatives of $\frac{1}{2}R^2 - |S|^2$ all vanish. Thus we have

$$\nabla_l R = \frac{1}{R}(\nabla_l S\cdot\bar{S} + S\cdot\nabla_l\bar{S}) \qquad (2.22)$$

and hence

$$|\nabla_l R| \leq \frac{1}{R}(|\nabla_l S|\cdot|\bar{S}| + |S|\cdot|\nabla\bar{S}|) = \frac{1}{\sqrt{2}}(|\nabla_l S| + |\nabla_l\bar{S}|) \qquad (2.23)$$

since $\frac{1}{2}R^2 - |S|^2 = 0$. But then

$$\sum_l |\nabla_l R|^2 \leq \frac{1}{2}\sum_l(|\nabla_l S| + |\nabla_l\bar{S}|)^2 \leq \sum_l(|\nabla_l S|^2 + |\nabla_l\bar{S}|^2) \qquad (2.24)$$

(In the preceding argument, we have assumed that $R > 0$, which follows from the strong maximum principle if $t > 0$. If $R = 0$ and $t = 0$, then we are at a minimum of R, and $\nabla_l R = 0$, so that the above inequality holds trivially). Thus the inequality from the maximum principle reduces to

$$(\frac{1}{2}R^2 - |S|^2)^{\cdot} \geq R|S|^2 - 2\langle S\, Op(S)\, S\rangle \qquad (2.25)$$

In an orthonormal basis $\phi^\alpha_{\bar a b}$ for the space of traceless $(1,1)$-forms where the operator $S_{\bar a b \bar c d}$ is diagonal, with eigenvalues m_1, m_2, m_3, the preceding inequality can be rewritten as

$$(\frac{1}{2}R^2 - |S|^2)^{\cdot} \geq 2\sum_{\alpha=1}^{3}(\frac{1}{2}R - m_\alpha)|s_\alpha|^2 \tag{2.26}$$

where we have denoted by $s_\alpha \in \mathbf{R}$ the components of $S_{\bar a b}$ in that basis:

$$S_{\bar a b} = \sum_{\alpha=1}^{3} s_\alpha \phi^\alpha_{\bar a b} \tag{2.27}$$

It follows from (2.8) that $S_{\bar a b \bar b a} = \frac{n-1}{n}R = \frac{1}{2}R$ when $n = 2$. On the other hand, since $S_{\bar a b \bar c d} = \sum_{\alpha=1}^{3} m_\alpha \phi^\alpha_{\bar a b}\phi^\alpha_{\bar c d}$ we obtain $S_{\bar a b \bar b a} = \sum_{\alpha=1}^{3} m_\alpha$. Thus the non-negativity of the determinant of the Ricci curvature will be preserved if we can show that

$$0 \leq \frac{1}{2}R - m_\alpha = \sum_{\beta \neq \alpha} m_\beta, \tag{2.28}$$

that is, the sum of any two eigenvalues of $S_{\bar a b \bar c d}$ is non-negative.

Recall that a symmetric bilinear form is 2-nonnegative if the sum of its two smallest eigenvalues is non-negative. We have assumed that the traceless curvature operator $S_{\bar a b \bar c d}$ is 2-nonnegative at initial time. It remains to show that the 2-nonnegativity of the traceless curvature operator is preserved under the Kähler-Ricci flow. Chen [2] has shown that the 2-nonnegativity of the curvature operator $Op(R)$ is preserved by the Ricci flow. Now if the Riemann curvature operator $Op(R)$ is 2-nonnegative, then so is $M = Op(S)$, but the converse does not hold, so we cannot directly quote Chen's result.

First note that if $m_1 \leq m_2 \leq m_3$ are the eigenvalues of M, then

$$m_1 + m_2 \;=\; \inf\{M(\phi,\phi) + M(\psi,\psi) : \phi,\psi \in \Lambda^{1,1}_0, |\phi| = |\psi| = 1, \phi \perp \psi\}$$

Moreover, the condition $m_1 + m_2 \geq 0$ is clearly closed and convex. The ODE associated to M from the heat flow for the system (2.9) for $R, S_{\bar a b}, S_{\bar a b \bar c d}$ is

$$\frac{dM}{dt} \;=\; -\mu M + M^2 + M^{\#} + T \tag{2.29}$$

where, in coordinates where M is diagonal, $M^{\#}_{\alpha\beta} = -2(\prod_{\gamma \neq \alpha} m_\gamma)\delta_{\alpha\beta}$ and $T_{\alpha\beta} = s_\alpha s_\beta$. To show that $m_1 + m_2 \geq 0$ is preserved, it suffices, by Hamilton's maximum principle for systems, to show that (2.29) preserves this condition. Now Lemma 3.5 of [7]implies

$$\frac{d}{dt}(m_1+m_2) \ \geq \ \inf\{\frac{dM}{dt}(\phi,\phi)+\frac{dM}{dt}(\psi,\psi)\}:$$

where ϕ,ψ range over all $\phi,\psi\in\Lambda_0^{1,1}$ such that $|\phi|=|\psi|=1,\phi\perp\psi$ and $M(\phi,\phi)+M(\psi,\psi)=m_1+m_2$. For such ϕ,ψ, we have $M^2(\phi,\phi)+M^2(\psi,\psi)=m_1^2+m_2^2$, $M^{\#}(\phi,\phi)+M^{\#}(\psi,\psi)=-2m_3(m_1+m_2)$ and $T(\phi,\phi)+T(\psi,\psi)\geq 0$, since T is a non-negative operator. Thus (2.29) implies

$$\frac{d}{dt}(m_1+m_2)\geq -\mu(m_1+m_2)+m_1^2+m_2^2-2m_3(m_1+m_2). \tag{2.30}$$

The right hand side is non-negative when m_1+m_2 becomes 0. Thus the non-negativity of m_1+m_2, and hence of $\frac{1}{2}R^2-|S|^2$ is preserved under the flow. Q.E.D.

Remark: By flowing $(\frac{1}{2}R^2-|S|^2)^{-1}$, one can show, using a similar argument, that $(\frac{1}{2}R^2-|S|^2)$ is bounded below by a positive constant, if it is positive everywhere at the initial time and if the traceless curvature operator is 2-nonnegative.

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