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CM Number Fields and Modular Forms

Tonghai Yang

In memory of Armand Borel

Introduction

In 1969, Siegel ([14]) constructed modular forms from arithmetic of totally real number fields as follows.

Theorem 1.1. *(Siegel) Let F be a totally real number field of degree d, and let ∂_F be the different of F. Then for every even integer $k \ge 2$, ($e(\tau) = e^{2\pi i \tau}$)*

$$g_k(\tau) = \zeta_F(1-k) + 2^{d-1} \sum_{m \ge 1} a_m(F, k) e(n\tau)$$

is a holomorphic elliptic modular form of weight dk for $\mathrm{SL}_2(\mathbb{Z})$ (except for the case $(d, k) = (1, 2)$), where

$$a_m(F, k) = \sum_{x \in \partial_F^{-1,+}, \mathrm{tr}_{F/\mathbb{Q}} x = m} \sum_{x\partial_F \subset \mathfrak{a} \subset \mathcal{O}_F} (N\mathfrak{a})^{k-1}.$$

Here the superscript '$+$' stands for totally positive elements.

Siegel further derived from this a simpler proof of his famous theorem that $\zeta_F(1-k)$ is rational for all $k \ge 1$.

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Siegel's construction is based on the simple observation that a Hilbert modular form becomes an elliptic modular form when restricting diagonally to the upper half plane. Indeed, Hecke constructed and proved in 1924 that

$$E_k(\tau) = \zeta_F(1-k) + 2^{d-1} \sum_{x \in \partial_F^{-1,+}} \sigma_{k-1}(x\partial_F) e(\mathrm{tr} x\tau) \tag{1.1}$$

is a Hilbert modular form of weight k for $\mathrm{SL}_2(\mathcal{O}_F)$. Here

$$\sigma_{k-1}(\mathfrak{b}) = \sum_{\mathfrak{b} \subset \mathfrak{a} \subset \mathcal{O}_F} (N\mathfrak{a})^{k-1}$$

and $\mathrm{tr} x\tau = \sum_{j=1}^d \sigma_j(x)\tau_j$ for the real embeddings σ_j. It is easy to see that $g_k(\tau) = E_k(\tau, \cdots, \tau)$.

In fact, Hecke also gave similar construction for odd k together with some ideal class character as long as there are no obvious cancellations. The case $k = 1$ is particularly interesting, where he concentrated on the real quadratic fields to avoid complication. Indeed, let $F = \mathbb{Q}(\sqrt{D})$ and assume that $D = d_1 d_2$ such that $d_1, d_2 < 0$ are fundamental discriminants of imaginary quadratic fields with $(d_1, d_2) = 1$. Let χ be the genus character of F associated to the genus field $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Then Hecke proved in the same paper (Hecke's trick) that

$$\begin{aligned} E_{1,\chi}(\tau, \tau', s) = &\sum_{[\mathfrak{a}] \in \mathrm{CL}^+(F)} \chi(\mathfrak{a}) (N\mathfrak{a})^{1+2s} \\ &\cdot \sum_{0 \neq (m,n) \in \mathfrak{a}^2 / \mathcal{O}_F^{*,+}} \frac{v^s v'^{,s}}{(m\tau + n)(m'\tau' + n')|m\tau + n|^{2s} |m'\tau' + n'|^{2s}} \end{aligned} \tag{1.2}$$

is a (non-holomorphic) Hilbert modular form of weight 1 for $\mathrm{SL}_2(\mathcal{O}_F)$, and is holomorphic at $s = 0$. So $E_{1,\chi}(\tau, \tau', 0)$ is a holomorphic Hilbert modular form of weight 1. He further computed the Fourier expansion of this holomorphic modular form, which is very similar to (1.2). Unfortunately, he messed up a sign in the calculation, and it turns out that $E_{1,\chi}(\tau, \tau', 0) \equiv 0$ identically. It should be mentioned that Gross and Zagier took advantage of this fact to compute its central derivative at $s = 0$ and use it to compute the factorization of the singular moduli ([6]).

Hecke was unfortunate in another sense. If Hecke had used a quartic totally real field or injected some ramification in his example, he would have produced honest Hilbert modular forms of weight 1. This is one of the main purposes of this paper. Indeed, we will prove in Section 3 the following theorem, after local preparation in Section 2.

Theorem 1.2. *Let F be a totally real number field of degree d with different ∂_F, and let K be a totally imaginary quadratic extension of F with relative discriminant $d_{K/F}$. Let $\chi = \chi_{K/F}$ be the quadratic Hecke character of F associated to K/F. Let $\alpha = (\alpha_v) \in \prod_{v|d_{K/F}} F_v^*$ with $\mathrm{ord}_v \alpha_v = \mathrm{ord}_v \partial_F$, and let $\mathcal{N}$ be a square-free integral ideal of F such that all its prime factors are inert in K. Then there is a function on $\mathbb{H}^d \times \mathbb{C}$, denoted by $E^*(\tau, s, \Phi^{\alpha,\mathcal{N}})$, such that*

(1) As a function of s, $E^*(\tau, s, \Phi^{\alpha,\mathcal{N}})$ is meromorphic with possibly finitely many poles and is holomorphic along the unitary line Re $s = 0$. It has a simple functional equation

$$\prod_{v|\mathcal{N}}(|\mathcal{N}|_v^{\frac{1+s}{2}}+|\mathcal{N}|_v^{-\frac{1+s}{2}})E^*(\tau,s,\Phi^{\alpha,\mathcal{N}}) = \epsilon(\alpha,\mathcal{N})\prod_{v|\mathcal{N}}(|\mathcal{N}|_v^{\frac{1-s}{2}}+|N|_v^{-\frac{1-s}{2}})E^*(\tau,-s,\Phi^{\alpha,\mathcal{N}}), \tag{1.3}$$

where

$$\begin{aligned}\epsilon(\alpha,\mathcal{N}) &= (-1)^{o(\mathcal{N})} i^d \prod_{v|d_{K/F}} \chi_v(\alpha_v)\epsilon(\chi_v,\psi_v) \\ &= (-1)^{o(\mathcal{N})} \prod_{v|d_{K/F}} \chi_v(\alpha_v) \prod_{v|\partial_F, v\nmid d_{K/F}} \chi_v(\partial_F).\end{aligned} \tag{1.4}$$

Here $o(\mathcal{N})$ is the number of prime factors of $\mathcal{N}$, $\epsilon(\chi_v, \psi_v)$ is Tate's local root number (ψ_v is to be defined later), and $\chi_v(\mathfrak{a}) = \chi_v(\varpi_v)^{\mathrm{ord}_v \mathfrak{a}}$ is independent of the choice of a uniformizer ϖ_v when χ_v is unramified.

(2) As a function of $\tau = (\tau_1, \cdots, \tau_d) \in \mathbb{H}^d$, $E^*(\tau, s, \Phi^{\alpha,\mathcal{N}})$ is a Hilbert (non-holomorphic) modular form of weight 1, level $d_{K/F}\mathcal{N}$, and character χ, where χ stands for

$$\chi:\ (\mathcal{O}_F/d_{K/F}\mathcal{N})^* \twoheadrightarrow (\mathcal{O}_F/d_{K/F})^* \longrightarrow \{\pm1\},\ \chi(a) = \prod_{v|d_{K/F}} \chi_v(a).$$

(3) The central value $E^*(\tau, 0, \Phi^{\alpha,\mathcal{N}}) \neq 0$ if and only if $\epsilon(\alpha,\mathcal{N}) = 1$. The central value $E^*(\tau, 0, \Phi^{\alpha,\mathcal{N}})$ is a holomorphic modular form and has Fourier expansion

$$\begin{aligned}E^*(\tau,0,\Phi^{\alpha,\mathcal{N}}) &= (1+\epsilon(\alpha,\mathcal{N}))L(0,\chi_{K/F}) \\ &\quad + 2^d\epsilon(\alpha,\mathcal{N}) \sum_{t\in(\partial_F^{-1}\mathcal{N})^+} \delta(\alpha t)\rho_{K/F}(t\partial_F\mathcal{N}^{-1})e(\mathrm{tr} t\tau).\end{aligned}$$

Here $\mathrm{tr}(t\tau) = \sum_i \sigma_i(t)\tau_i$ for the real embeddings $\{\sigma_1, \cdots, \sigma_d\}$ of F,

$$\delta(\alpha t) = \prod_{v|d_{K/F}} (1 + \chi_v(\alpha_v t)), \tag{1.5}$$

and

$$\rho_{K/F}(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_K : N_{K/F}\mathfrak{A} = \mathfrak{a}\}. \tag{1.6}$$

We gave two formulae for $\epsilon(\alpha, \mathcal{N})$ in Theorem 1.2 on purpose. From the second formula, it is clear $\epsilon(\alpha, \mathcal{N}) = \pm 1$, so the first formula implies $d_{K/F} \neq \mathcal{O}_F$ when $d = [F : \mathbb{Q}]$ is odd. That is,

Corollary 1.3. *Let F be a totally real number field of odd degree, and let K be a totally imaginary quadratic extension of F. Then K/F is ramified at some finite prime.*

This fact was also observed recently by Gross and McMullen ([5], Proposition 3.1). By looking at the sign $\epsilon(\alpha, \mathcal{N})$ in the special case where K/F is unramified at every finite prime and $\mathcal{N} = \mathcal{O}_F$, one also obtains the following corollary, which explains Hecke's misfortune.

Corollary 1.4. *Assume that $d = [F : \mathbb{Q}]$ is even, and that K/F is unramified at every finite prime. Then the 'spherical Eisenstein series' $E^*(\tau, 0, \Phi^{1,\mathcal{O}_F}) = 0$ if and only if $d \equiv 2 \mod 4$. Moreover, when $d \equiv 2 \mod 4$, for every $t \in \partial_F^{-1,+}$, there is no ideal $\mathfrak{A}$ of K with relative norm $t\partial_F$.*

In particular, when $4|d$, the spherical Eisenstein series give holomorphic Hilbert modular forms of weight one of $\mathrm{SL}_2(\mathcal{O}_F)$ Hecke tried to construct in 1924. Even for degree 2, our construction gives holomorphic Hilbert modular forms of weight one with small level and trivial character (see Theorems 5.1 and 5.3)

Following Siegel ([14]) and restricting the function diagonally to $(\tau, \cdots, \tau)$, one obtains

Theorem 1.5. *Let the notation be as in Theorem 1.2. with $\epsilon(\alpha, \mathcal{N}) = 1$. Then*

$$f_{\alpha,\mathcal{N}}(\tau) = L(0, \chi) + 2^{d-1} \sum_{m=1}^{\infty} a_m(\alpha, \mathcal{N}) e(m\tau)$$

is a holomorphic elliptic modular form of weight d, level N, and Nebentypus character $\tilde{\chi}$. Here $N > 0$ is given by $N\mathbb{Z} = d_{K/F}\mathcal{N} \cap \mathbb{Z}$, and $\tilde{\chi}$ is the composition of the embedding $(\mathbb{Z}/N)^* \hookrightarrow (\mathcal{O}_F/d_{K/F}\mathcal{N})^*$ with χ, i.e.,

$$\tilde{\chi}(a) = \prod_{v | d_{K/F}} \chi_v(a). \tag{1.7}$$

Finally,

$$a_m(\alpha, \mathcal{N}) = \sum_{\substack{t \in (\partial_F^{-1}\mathcal{N})^+ \\ \mathrm{tr}_{F/\mathbb{Q}} t = m}} \delta(\alpha t) \rho_{K/F}(t \partial_F \mathcal{N}^{-1}). \tag{1.8}$$

Theorem 1.5 has at least three types of potential applications:

(1) One can use it to compute the L-value $L(0, \chi_{K/F})$, or equivalently the relative class number of K/F.

(2) One can use it to construct a lot of (infinitely many, in fact) holomorphic modular forms of some fixed weight, level, and quadratic Nebentypus character.

(3) Since the space of holomorphic modular forms of a fixed weight, level, and Nebentypus character is finite, the infinitely many modular forms constructed in (2) have to have some relations. They should be reflected on the arithmetic of the chosen number fields.

We don't address these applications fully in this paper. Instead, we focus on some interesting examples in Sections 4 and 5. Section 4 deals with unramified extensions and Section 5 deals with real quadratic fields and its totally imaginary quadratic extensions, both biquadratic and non-biquadratic. It turns out that biquadratic and non-biquadratic fields have slightly different flavors (see for example Corollaries 5.5 and 5.8). We record two simple examples here to give the reader a flavor and refer to these two sections for other examples. Notice that both examples are slight variants of Hecke's original example.

Theorem 1.6. *(Theorem 4.1, Corollary 4.7) Let F be a totally real number field of degree d divisible by 4 and let K be a totally imaginary quadratic extension of F unramified at all finite primes. Then*

$$f_{K/F}(\tau) = L(0, \chi_{K/F}) + 2^{d-1} \sum_{m=1}^{\infty} a_m(K/F) q^m$$

is a holomorphic modular form of weight d for $\mathrm{SL}_2(\mathbb{Z})$, where

$$a_m(K/F) = \sum_{t \in \partial_F^{-1,+}, \mathrm{tr}_{F/\mathbb{Q}} t = m} \rho_{K/F}(t\partial_F).$$

Moreover,

(1) If $d = [F : \mathbb{Q}] = 4$, then

$$L(0, \chi_{K/F}) = \frac{1}{30} \sum_{\substack{t \in \partial_F^{-1,+} \\ \mathrm{tr}_{F/\mathbb{Q}} t = 1}} \rho_{K/F}(t\partial_F),$$

and the ratio $\frac{a_m(K/F)}{a_1(K/F)}$ is independent of F or K, and is non-zero.

(2) *If* $d = [F : \mathbb{Q}] = 8$, *then*

$$L(0, \chi_{K/F}) = \frac{4}{15} \sum_{\substack{t \in \partial_F^{-1,+} \\ \mathrm{tr}_{F/\mathbb{Q}} t = 1}} \rho_{K/F}(t\partial_F),$$

and the ratio $\frac{a_m(K/F)}{a_1(K/F)}$ *is independent of* F *or* K, *and is non-zero.*

Theorem 1.7. *(Theorem 5.1) Let* N *be a square-free positive integer. Let* $d_1, d_2 < 0$ *be two fundamental discriminants of imaginary quadratic fields, and let* $F = \mathbb{Q}(\sqrt{D})$ *with* $D = d_1 d_2 > 0$, *and let* $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. *Assume that*

$$(d_1, d_2) = 1, \text{ and } (\frac{d_1}{p}) = (\frac{d_2}{p}) = -1 \text{ for every } p|N. \tag{1.9}$$

So every prime $p|N$ *splits in* F *and every prime of* F *above* N *is inert in* K. *Let* $\mathcal{N}$ *be an integral ideal of* F *with odd number of prime factors in* F *such that* $\mathcal{N} \cap \mathbb{Z} = N\mathbb{Z}$. *Then*

$$f_{d_1,d_2,\mathcal{N}}(\tau) = L(0, \chi_{K/F}) + 2 \sum_{m=1}^{\infty} a_m(d_1, d_2, \mathcal{N}) e(m\tau) \tag{1.10}$$

is a holomorphic (elliptic) modular form of weight 2 *for* $\Gamma_0(N)$ *with trivial Nebentypus character, where*

$$a_m(d_1, d_2, \mathcal{N}) = \sum_{t = \frac{a + m\sqrt{D}}{2} \in \mathcal{N}, |a| < m\sqrt{D}} \rho_{K/F}(t\mathcal{N}^{-1}). \tag{1.11}$$

The case $\epsilon(\alpha, \mathcal{N}) = -1$ is even more interesting as first demonstrated by Gross and Zagier ([6], [7], see also [10] and [17]). In ([3]), Bruinier and the author compute the central derivative in one of the special case (see Theorem 5.7) when $\epsilon(\alpha, N) = -1$, and use it to generalize the work of Gross and Zagier on singular moduli ([6]) to a family of Hilbert modular form (the Borcherds forms on a Hilbert modular surface) valued a CM 0-cycle associated to a non-biquadratic quartic field.

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Notation Let F be a totally real number field, and let $\psi = \psi_{\mathbb{Q}} \circ \mathrm{tr}_{F/\mathbb{Q}}$ be the additive character of F used in this paper, where $\psi_{\mathbb{Q}}$ is the 'canonical' additive character of $\mathbb{Q}_{\mathbb{A}}$ such that $\psi_{\mathbb{R}}(x) = e(x)$. Let $K = F(\sqrt{\Delta})$ be a totally imaginary quadratic extension of F, and let $\chi = (\Delta,)_{\mathbb{A}}$ be the quadratic Hecke character of F associated to K/F. Let ∂_F and $\partial_{K/F}$ be the different of F and relative different of K/F respectively, and let $d_F = N_{F/\mathbb{Q}}\partial_F$ and $d_{K/F} = N_{K/F}\partial_{K/F}$ be the discriminant and relative discriminant respectively. Let $I(s,\chi) = \otimes' I(s,\chi_v)$ be the induced representation of $\mathrm{SL}_2(F_{\mathbb{A}})$, consisting of Schwartz functions $\Phi(g,s)$ on $\mathrm{SL}_2(F_{\mathbb{A}})$ such that

$$\Phi(n(b)m(a)g,s) = \chi(a)|a|^{s+1}\Phi(g,s), \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tag{1.12}$$

For a factorizable section $\Phi = \prod_v \Phi_v \in I(s,\chi)$ which is standard in the sense that $\Phi_v|_{\mathrm{SL}_2(\mathcal{O}_v)}$ is independent of s for $v < \infty$, the Eisenstein series

$$E(g,s,\Phi) = \sum_{\gamma \in B\backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s) \tag{1.13}$$

is absolutely convergent when $\mathrm{Re}(s) >> 0$ and has a meromorphic continuation to the whole complex s-plane with finitely many poles and is holomorphic on the unitary line $\mathrm{Re}(s) = 0$. Moreover, it satisfies a functional equation

$$E(g,s,\Phi) = E(g,-s,M(s)\Phi), \tag{1.14}$$

where

$$M(s)\Phi(g,s) = \int_{F_{\mathbb{A}}} \Phi(wn(b)g,s)\,db, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.15}$$

is an intertwining operator from $I(s,\chi)$ to $I(-s,\chi)$, The Eisenstein series $E(g,s,\Phi)$ has the Fourier expansion

$$E(g,s,\Phi) = E_0(g,s,\Phi) + \sum_{t \in F^*} E_t(g,s,\Phi) \tag{1.16}$$

where, for $t \in F^*$

$$E_t(g,s,\Phi) = \prod_v W_{t,v}(g,s,\Phi_v) \tag{1.17}$$

with

$$W_{t,v}(g,s,\Phi_v) = \int_{F_v} \Phi_v(wn(b)g,s)\psi_v(-tb)\,db \tag{1.18}$$

Here db is the Haar measure on F_v with respect to the character ψ_v. The constant term

$$E_0(g,s,\Phi) = \Phi(g,s) + W_0(g,s,\Phi) = \Phi(g,s) + M(s)\Phi(g,s). \tag{1.19}$$

We normalize

$$W^*_{t,v}(g,s,\Phi) = L(s+1,\chi_v)W_{t,v}(g,s,\Phi) \tag{1.20}$$

and

$$E^*(g,s,\Phi) = \Lambda(s+1,\chi)E(g,s,\Phi) \tag{1.21}$$

with

$$\Lambda(s,\chi) = A^{\frac{s}{2}}\prod_v L(s,\chi_v) = A^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s+1)^d L(s,\chi), \quad A = N_{F/\mathbb{Q}}(\partial_F d_{K/F}). \tag{1.22}$$

Here

$$L(s,\chi_v) = \Gamma_{\mathbb{R}}(s+1) = \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})$$

for $v|\infty$. Notice that the normalized L-function satisfies

$$\Lambda(s,\chi) = \Lambda(1-s,\chi), \quad \Lambda(0,\chi) = L(0,\chi). \tag{1.23}$$

Finally, when $\Phi_v = \Phi^k_{\mathbb{R}}$ is the eigenfunction of $\mathrm{SO}_2(\mathbb{R})$ of 'weight' k for every $v|\infty$, i.e,

$$\Phi_v(gk_\theta, s) = e^{ik\theta}\Phi_v(g,s), \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

we define for $\tau = (\tau_1, \tau_2, \cdots, \tau_d) \in \mathbb{H}^d$

$$E^*(\tau,s,\Phi) = \prod_{j=1}^d v_j^{-\frac{k}{2}} E^*(\prod_j g_{\tau_j}, s, \Phi), \tag{1.24}$$

where $\tau_j = u_j + iv_j$ and $g_{\tau_j} = n(u_j)m(\sqrt{v_j}) \in \mathrm{SL}_2(F_{\sigma_j})$ with $g_{\tau_j}(i) = \tau_j$. Here $\{\sigma_1, \cdots, \sigma_j\}$ are real embeddings of F.

The proof of Theorem 1.2 is simply to choose a proper section $\Phi^{\alpha,\mathcal{N}} \in I(s,\chi)$ with given data and compute the Fourier expansion of $E^*(\tau,s,\Phi^{\alpha,\mathcal{N}})$ via (1.16)-(1.19). This is a purely local calculation. In Section 2, we collect and expand results of ([11]) on local Whittaker functions needed for the proof of Theorem 1.2. They should be of independent interest.

2 Local results

In this section, we extend the local results of [11] to a general p-adic local field, which is needed in the proof of Theorem 1.2 and should be of independent interest. In this section, F stands for a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$ and a uniformizer ϖ. Let K be a quadratic extension of F, including $F \oplus F$. Let

χ be the quadratic character of F^* associated to K/F and let ψ be an unramified additive character of F in the sense

$$n(\psi) := \min\{n : \psi|_{\varpi^n \mathcal{O}_F} = 1\} = 0.$$

For $\alpha \in F^*$, let $V_\alpha = K$ with quadratic form $Q_\alpha(z) = \alpha z\bar{z}$. This gives a Weil representation $\omega_\alpha = \omega_{V_\alpha,\psi}$ of $G = \mathrm{SL}_2(F)$ on $S(K)$, and the map

$$\lambda_\alpha : S(K) \longrightarrow I(0,\chi), \quad \lambda_\alpha(\phi)(g) = \omega_\alpha(g)\phi(0),$$

is G-equivariant. Here we remark that $\chi_{V_\alpha} = (-\det V_\alpha,)_F = \chi_{K/F}$ is independent of the choice of α. Let $R(V_\alpha)$ be the image of λ_α. Then ω_α and $R(V_\alpha)$ only depends on $\alpha \in F^*/NK^*$ (up to isomorphism). In this paper, we denote V^+ for V_α with $\alpha = 1$. When K/F is inert, we fix a choice for $V^- = V_\alpha$ with $\alpha \in \varpi\mathcal{O}_F^*$. When K/F is ramified, we fix a choice for $\alpha \in \mathcal{O}_F^*$ such that $\chi(\alpha) = -1$. It is well-known that

$$I(0,\chi) = \oplus_{\alpha\in F^*/NK^*} R(V_\alpha) = \begin{cases} R(V^+) & \text{if } K/F \text{ split,} \\ R(V^+)\oplus R(V^-) & \text{if } K/F \text{ non-split.}\end{cases} \tag{2.1}$$

Let $\phi^0 = \mathrm{char}(\mathcal{O}_K)$, and let $\Phi_\alpha \in I(s,\chi)$ be the standard sections such that

$$\Phi_\alpha(g,0) = \lambda_\alpha(\phi^0).$$

Actually, Φ_α depends only on the choices of α modulo $N\mathcal{O}_K^*$. We denote Φ^+ for $\Phi_1 \in R(V^+)$. When K/F is non-split, we denote $\Phi^- = \Phi_\alpha$ for the prefixed α above. We remark that Φ_α (thus $\Phi^\pm$) depend on the choice of ψ. In fact, the section Φ_α with respect to $\beta\psi$ is the same as $\Phi_{\alpha\beta}$ with respect to ψ. Then the following is well-known and is easy to check.

Proposition 2.1. *Assume that K/F is unramified. Let $X = |\varpi|^s$.*

(1) One has $\omega^+(k)\phi^0 = \phi^0$ for all $k \in K = \mathrm{SL}_2(\mathcal{O}_F)$. Φ^+ is the unique eigenfunction of $K = \mathrm{SL}_2(\mathcal{O}_F)$ with trivial eigencharacter such that $\Phi^+(1,s) = 1$.

(2) one has

$$W_t^*(1,s,\Phi^+) = \mathrm{char}(\mathcal{O}_F)(t) \sum_{0\le r\le \mathrm{ord}_F t} (\chi(\varpi)X)^r.$$

(3) One has for $t \in \mathcal{O}_F$

$$W_t^*(1,0,\Phi^+) = \begin{cases} \mathrm{ord}_F t + 1 & \textit{if } K/F \textit{ split,} \\ \frac{1+(-1)^{\mathrm{ord}_F t}}{2} & \textit{if } K/F \textit{ inert.}\end{cases}$$

In particular, $W_t^*(1,0,\Phi^+)=0$ *if and only if* K/F *is inert and* $\mathrm{ord}_F t$ *is odd. In such a case,*

$$W_t^{*,\prime}(1,0,\Phi^+)=\frac{1}{2}(\mathrm{ord}_F t+1)\log|\varpi|^{-1}$$

(4) $M^*(s)\Phi^+(g,s)=L(s,\chi)\Phi^+(g,-s)$. *In particular,* $W_0^*(1,s,\Phi^+)=$

$M^*(s)\Phi^+(1,s)=L(s,\chi)$. *Here* $L(s,\chi)=(1-\chi(\varpi)|\varpi|^s)^{-1}$ *is the local L-function of* χ.

Denote

$$K_0(\varpi^n)=\{g=\begin{pmatrix}a\,b\\ c\,d\end{pmatrix}\in \mathrm{SL}_2(\mathcal{O}_F):\ c\equiv 0 \mod \varpi^n\} \tag{2.2}$$

It is easy to check that

$$\mathrm{SL}_2(F)=BK_0(\varpi^n)\bigcup BwK_0(\varpi^n)\bigcup(\bigcup_{0<\mathrm{ord}_F c<n} Bn^-(c)K_0(\varpi^n)). \tag{2.3}$$

Here $n^-(c)=\begin{pmatrix}1\,0\\ c\,1\end{pmatrix}$.

Proposition 2.2. *Assume that* K/F *is inert.*

(1) One has $\omega^-(k)\phi^0=\phi^0$ *for every* $k\in K_0(\varpi)$. Φ^- *is an eigenfunction of* $K_0(\varpi)$ *with the trivial eigencharacter such that* $\Phi^-(1)=1$ *and* $\Phi^-(w)=-|\varpi|$.

(2) One has

$$\begin{aligned}W_t^*(1,s,\Phi^-)&=\mathrm{char}(\mathcal{O}_F)(t)\left(-\frac{1+|\varpi|}{1+|\varpi|X}+\sum_{n=0}^{\mathrm{ord}_F t}(-X)^n\right)\\&=-\frac{1+|\varpi|}{1+|\varpi|X}\mathrm{char}(\mathcal{O}_F)(t)+W_t^*(1,s,\Phi^+),\end{aligned}$$

and

$$W_t^*(w,s,\Phi^-)=-|\varpi|\mathrm{char}(\varpi^{-1}\mathcal{O}_F)(t)\left(-\frac{1+|\varpi|}{1+|\varpi|X}+\sum_{n=0}^{\mathrm{ord}_F t}(-X)^n\right).$$

(3) For $t\in\mathcal{O}_F$,$W_t^*(1,0,\Phi^-)=-\frac{1-(-1)^{\mathrm{ord}_F t}}{2}=0\Leftrightarrow \mathrm{ord}_F t\equiv 0 \mod 2$. *For* $t\in\varpi^{-1}\mathcal{O}_F$, $W_t(w,0,\Phi^-)=-\frac{1-(-1)^{\mathrm{ord}_F t}}{2}=0\Leftrightarrow \mathrm{ord}_F t\equiv 0 \mod 2$. *When* $\mathrm{ord}_F t\ge 0$ *is even, one has*

$$W_t^{*,\prime}(1,0,\Phi^-)=-|\varpi|^{-1}W_t^{*,\prime}(w,0,\Phi^-)=(\frac{|\varpi|}{1+|\varpi|}+\frac{\mathrm{ord}_F t}{2})\log|\varpi|.$$

(4) $M^*(s)\Phi^- = -|\varpi|\frac{L(s+1,\chi)L(s,\chi)}{L(s-1,\chi)}\Phi^-(-s).$

Proof. We first recall a general fact about Weil representation, which holds for any quadratic extension K ([9]). That is,

$$\begin{aligned} \omega_\alpha(n(b))\phi(x) &= \psi(b\alpha x\bar{x})\phi(x), \\ \omega_\alpha(m(a))\phi(x) &= \chi(a)|a|\phi(xa), \\ \omega_\alpha(w)\phi(x) &= \gamma(V_\alpha)\hat{\phi}(x) = \gamma(V_\alpha)\int_F \phi(y)\psi(-\alpha \mathrm{tr}_{K/F}(x\bar{y}))d_\psi y \end{aligned} \tag{2.4}$$

Here

$$\gamma(V_\alpha) = (\epsilon(V_\alpha)\gamma(\psi)^2\gamma(\det V_\alpha, \psi))^{-1}$$

and $d_\psi y$ is the self-dual Haar measure on K with respect to $(x, y) \mapsto \alpha\psi(\mathrm{tr}_{K/F} x\bar{y})$. $\gamma(\psi)$ and $\gamma(\alpha, \psi)$ are Weil's local indices ([13]). So

$$d_\psi y = |\partial_{K/F}|_K^{\frac{1}{2}} dy \tag{2.5}$$

where $\mathrm{vol}(\mathcal{O}_K, dy) = 1$. Since the matrix of V_α with respect to the basis $\{1, \sqrt{\Delta}\}$ is $\mathrm{diag}(\alpha, -\alpha\Delta)$, one has

$$\begin{aligned} \gamma(V_\alpha)^{-1} &= (\alpha, -\alpha\Delta)\gamma(\psi)^2\gamma(-\Delta, \psi) \\ &= \chi(\alpha)\gamma(-1, \psi)^{-1}\gamma(-\Delta, \psi) \\ &= \chi(\alpha)\frac{(-1, -\Delta)\gamma(\Delta, \psi)}{(-1, -1)} \\ &= \chi(\alpha)(-1, \Delta)\gamma(\Delta, \psi) \\ &= \chi(\alpha)\gamma(\Delta, \psi)^{-1}. \end{aligned}$$

So

$$\gamma(V_\alpha) = \chi(\alpha)\gamma(\Delta, \psi). \tag{2.6}$$

Recall also that $\Phi^- = \Phi_\alpha$ for $\alpha \in \varpi\mathcal{O}_F^*$. It is clear from (2.4) that

$$\begin{aligned} \omega_\alpha(n(b))\phi^0 &= \phi^0, \quad b \in \mathcal{O}_F, \\ \omega_\alpha(m(a))\phi^0(z) &= \chi(a)|a|\phi^0(za) = \phi^0(z), \quad z \in \mathcal{O}_F^*, \\ \omega_\alpha(w)\phi^0(z) &= \gamma(V_\alpha)\int_{\mathcal{O}_K} \psi(\alpha \mathrm{tr} z\bar{y})d_\alpha y \\ &= \gamma(V_\alpha)|\varpi|\mathrm{char}(\varpi^{-1}\mathcal{O}_K)(z). \end{aligned}$$

Here $d_\alpha y = |\varpi| dy$ is the Haar measure on V_α self-dual with respect to the bi-character $(x, y) \mapsto \psi(\alpha \mathrm{tr} x \bar{y})$. So $(c, \alpha \in \varpi \mathcal{O}_F)$

$$\begin{aligned}\omega_\alpha(n(-c)w)\phi^0(z) &= \gamma(V_\alpha)|\varpi|\psi(-c\alpha z\bar{z})\mathrm{char}(\varpi^{-1}\mathcal{O}_K)(z)\\ &= \gamma(V_\alpha)|\varpi|\mathrm{char}(\varpi^{-1}\mathcal{O}_K)(z)\\ &= \omega_\alpha(w)\phi^0(z).\end{aligned}$$

Therefore,

$$\omega_\alpha(n^-(c))\phi^0 = \omega_\alpha(w^{-1})\omega_\alpha(n(-c)w)\phi^0 = \omega_\alpha(w^{-1}w)\phi^0 = \phi^0.$$

So $\omega_\alpha(k)\phi^0 = \phi^0$ for every $k \in K_0(\varpi)$, and $\Phi^- = \Phi_\alpha$ is an eigenfunction of $K_0(\varpi)$ with trivial eigencharacter. It is clear

$$\Phi^-(1) = \phi^0(0) = 1, \quad \Phi^-(w) = \gamma(V_\alpha)|\varpi|.$$

This proves (1). Next, for $b \notin \mathcal{O}_F$, one has

$$wn(b) = \begin{pmatrix} b^{-1} & -1 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b^{-1} & 1 \end{pmatrix},$$

and $\Phi^-(wn(b)) = \chi(b^{-1})|b|^{-(s+1)}\Phi^-(1)$. Since χ is unramified and $\chi(\varpi) = -1$, one has

$$\begin{aligned}W_t(1, s, \Phi^-) &= \int_{\mathcal{O}_F} \Phi^-(wn(b))\psi(-tb) + \sum_{n\ge1}\int_{\varpi^{-n}\mathcal{O}_F^*} \Phi^-(wn(b))\psi(-tb)db\\ &= \Phi^-(w)\mathrm{char}(\mathcal{O}_F)(t) + \sum_{n\ge1}|\varpi|^{ns}\chi(\varpi)^n\int_{\mathcal{O}_F^*}\psi(-\frac{t}{\varpi^n}b)db\\ &= \gamma(V_\alpha)|\varpi|\mathrm{char}(\mathcal{O}_F)(t)\\ &\qquad + \sum_{n\ge1}(-X)^n\left(\mathrm{char}(\varpi^n\mathcal{O}_F)(t) - |\varpi|\mathrm{char}(\varpi^{n-1}\mathcal{O}_F)(t)\right).\end{aligned}$$

So $W_t(1, s, \Phi^-) = 0$ unless $t \in \mathcal{O}_F$. When $k = \mathrm{ord}_F t \ge 0$, one has

$$\begin{aligned}W_t(1, s, \Phi^-) &= \gamma(V_\alpha)|\varpi| + (1 - |\varpi|)\sum_{n=1}^{k}(-X)^n - |\varpi|(-X)^{k+1}\\ &= -1 + \gamma(V_\alpha)|\varpi| + (1 + |\varpi|X)\sum_{n=0}^{k}(-X)^n.\end{aligned}$$

So $W_t(1, 0, \Phi^-) = -1 + \gamma(V_\alpha)|\varpi| + (1 + |\varpi|)\frac{1+(-1)^k}{2}$. By [10] Proposition 1.4, one has $W_t(1, 0, \Phi^-) = 0$ if $\chi(t) \ne \chi(\alpha) = -1$, i.e., k is even. This implies

$$\gamma(\Delta, \psi) = 1 \tag{2.7}$$

and $\gamma(V_\alpha) = -1$. So

$$W_t^*(1, s, \Phi^-) = L(s+1, \chi) W_t(1, s, \Phi^-) = -\frac{1+|\varpi|}{1+|\varpi|X} + \sum_{n=0}^{k} (-X)^n$$

as claimed. The proof for $W_t^*(w, s, \Phi^-)$ is similar and left to the reader. (3) follows from (2) easily. To prove (4), notice that $\mathrm{SL}_2(F) = BK_0(\varpi) \bigcup BwK_0(\varpi)$, so the $K_0(\varpi)$-invariant space $I(s,\chi)^{K_0(\varpi)}$ is two dimensional and has a basis $\{\Phi^+, \Phi^-\}$ by Proposition 2.1 and (1). So

$$M(s)\Phi^- = a\Phi^+(-s) + b\Phi^-(-s).$$

On the other hand, $M(0)$ clearly preserves $R(V^\pm)$. So we have to have $a = 0$, and $M(s)\Phi^- = b\Phi^-(-s)$, with

$$b = M(s)\Phi^-(1, s) = W_0(1, s, \Phi^-) = -|\varpi| \frac{L(s,\chi)}{L(s-1,\chi)}.$$

□

Next, we assume that $K = F(\sqrt{\Delta})$ is ramified over F with $\mathrm{ord}_F \Delta = 0$ or 1. Let $d_{K/F} = \varpi^f \mathcal{O}_F$ be the relative discriminant, and $\partial = \partial_{K/F} = \varpi_K^f \mathcal{O}_K$ be the relative different. Notice that $f = 1$ when $p \neq 2$. There is not much difference between $V^\pm$, so we use V_α and Φ_α and so on with $\alpha \in \mathcal{O}_F^*$.

Proposition 2.3. *Assume that K/F is ramified, and let the notation be as above.*

(1) For any $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\varpi^f)$, one has

$$\omega_\alpha(k)\phi^0 = \chi(d)\phi^0.$$

Let $\Phi_\alpha \in I(s,\chi)$ be the standard section associated to ϕ^0 via the Weil representation $(\omega_\alpha, S(V_\alpha))$, i.e., $\Phi_\alpha(g, 0) = \omega_\alpha(g)\phi(0)$. Then Φ_α is the eigenfunction of $K_0(\varpi^f)$ with character $\chi : K_0(\varpi^f) \longrightarrow \{\pm 1\}, k \mapsto \chi(d)$ such that

$$\Phi_\alpha(1) = 1, \quad \Phi_\alpha(w) = \chi(\alpha)\gamma(\Delta,\psi)|\varpi|^{\frac{f}{2}} = \chi(-\alpha)\epsilon(\chi,\psi)|\varpi|^{\frac{f}{2}}, \tag{2.8}$$
$$\Phi_\alpha(n^-(c)) = 0 \quad \text{for } 0 < \mathrm{ord}_F c < f.$$

(2) Its Whittaker function with respect to ψ satisfies

$$W_t^*(1, s, \Phi_\alpha) = \Phi_\alpha(w)(1 + \chi(\alpha t) X^{\mathrm{ord}_F t + f}) \mathrm{char}(\mathcal{O}_F)(t),$$
$$W_t^*(w, s, \Phi_\alpha) = \Phi_\alpha(w)^2 (1 + \chi(\alpha t) X^{\mathrm{ord}_F t + f}) \mathrm{char}(d_{K/F}^{-1})(t).$$

(3) One has the functional equation

$$M^*(s)\Phi_\alpha = \Phi_\alpha(-s)\Phi_\alpha(w).$$

(4) For $t \in \mathcal{O}_F$, $W_t^*(1, 0, \Phi_\alpha) = 0$ if and only if $\chi(\alpha t) = -1$. In such a case,

$$W_t^{*,\prime}(1, 0, \Phi_\alpha) = \Phi_\alpha(w)(\mathrm{ord}_F t + f) \log |\varpi|^{-1}.$$

(5) For $t \in d_{K/F}^{-1}$, $W_t^*(w, 0, \Phi_\alpha) = 0$ if and only if $\chi(\alpha t) = -1$. In such a case,

$$W_t^{*,\prime}(w, 0, \Phi_\alpha) = \chi(-1)|\varpi|^f(\mathrm{ord}_F t + f) \log |\varpi|^{-1}.$$

Proof. This is basically Proposition 2.7 of [11] when $p \neq 2$. Notice that $W_t^*(g, s, \Phi) = W_t(g, s, \Phi)$ in this case. Recall that χ has conductor $\varpi^f \mathcal{O}_F$, and $K_0(\varpi^n)$ is generated by $m(a)$, $n(b)$, and $n^-(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $a \in \mathcal{O}_F^*$, $b \in \mathcal{O}_F$, and $c \in \varpi^n \mathcal{O}_F$.

(1) By (2.4), one clearly has $\omega_\alpha(n(b))\phi^0 = \phi^0$ and $\omega_\alpha(m(a))\phi^0 = \chi(a)\phi^0$ for $b \in \mathcal{O}_F$ and $a \in \mathcal{O}_F^*$. Next, $n^-(c) = w^{-1}n(-c)w$. First,

$$\omega_\alpha(w)\phi^0(x) = \gamma(V_\alpha) \int_{\mathcal{O}_K} \psi(-\alpha \mathrm{tr}_{K/F} x\bar{y}) d_\psi y = \gamma(V_\alpha)|\varpi|^{\frac{f}{2}} \mathrm{char}(\partial^{-1})(x).$$

Second, for $c \in \varpi^f \mathcal{O}_F$, one has then

$$\begin{aligned}\omega_\alpha(n(-c)w)\phi^0(x) &= \gamma(V_\alpha)|\varpi|^{\frac{f}{2}} \psi(-\alpha c x\bar{x}) \mathrm{char}(\partial^{-1})(x) \\ &= \gamma(V_\alpha)|\varpi|^{\frac{f}{2}} \mathrm{char}(\partial^{-1})(x) = \omega_\alpha(w)\phi^0.\end{aligned}$$

So

$$\omega_\alpha(n^-(c))\phi^0 = \omega_\alpha(w^{-1})\omega_\alpha(w)\phi^0 = \phi^0.$$

This proves the first claim of (1) and that Φ_α is an eigenfunction of $K_0(\varpi^f)$ with character χ. Clearly

$$\Phi_\alpha(1) = \omega_\alpha(1)\phi^0(0) = 1,$$

and

$$\Phi_\alpha(w) = \omega_\alpha(w)\phi^0(0) = \gamma(V_\alpha)|\varpi|^{\frac{f}{2}} = \chi(\alpha)\gamma(\Delta, \psi)|\varpi|^{\frac{f}{2}}$$

by the above calculation. When $0 < n = \mathrm{ord}_F c < f$, one has $n^-(c) = -wn(-c)w$, and thus

$$\begin{aligned}\Phi_\alpha(n^-(c)) &= \chi(-1)\omega_\alpha(wn(-c)w)\phi^0(0)\\ &= \chi(-1)\gamma(V_\alpha)\int_F \omega_\alpha(n(-c)w)\phi^0(z)dz\\ &= \chi(-1)\gamma(V_\alpha)^2\int_F \psi(-c\alpha z\bar{z})dz\int_F \phi^0(y)\psi(-\alpha\mathrm{tr}_{K/F}z\bar{y})dy\\ &= \chi(-1)\gamma(V_\alpha)^2\int_{\partial_{K/F}^{-1}} \psi(-\alpha cz\bar{z})dz\\ &= |\varpi|^{-f}\int_{\mathcal{O}_K}\psi(-\alpha c\varpi^{-f}z\bar{z})dz.\end{aligned}$$

The norm map $z \mapsto z\bar{z}$ maps $\mathcal{O}_K - \{0\}$ onto the subset A of $\mathcal{O}_F - \{0\}$ whose characteristic function is given by $\mathrm{char}(A)(x) = \frac{1+\chi(x)}{2}\mathrm{char}(\mathcal{O}_F)(x)$ (for $x \neq 0$). The kernel of the norm map is K^1, which is a compact group. So there is a constant $C > 0$ such that

$$\begin{aligned}\int_{\mathcal{O}_K}\psi(-\alpha c\varpi^{-1}z\bar{z})dz &= C\int_{\mathcal{O}_F}\psi(-\alpha c\varpi^{-1}x)\frac{1+\chi(x)}{2}dx\\ &= \frac{1}{2}\int_{\mathcal{O}_F}\psi(-\alpha c\varpi^{-f}x)dx + \frac{1}{2}\int_{\mathcal{O}_F}\psi(-\alpha c\varpi^{-f}x)\chi(x)dx\\ &= 0.\end{aligned}$$

This proves (1).

(2) For $b \notin \mathcal{O}_F$, write

$$wn(b) = \begin{pmatrix} b^{-1} & -1\\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & 0\\ b^{-1} & 1\end{pmatrix}.$$

So $\Phi_\alpha(wn(b)) = \chi^{-1}(b)|b|^{-1-s}\Phi_\alpha(n^-(b^{-1}))$ for $b \notin \mathcal{O}_F$, and

$$\begin{aligned}W_t(1,s,\Phi_\alpha) &= \int_{\mathcal{O}_F}\Phi_\alpha(wn(b))\psi(-tb)db + \sum_{n\geq 1}\int_{\varpi^{-n}\mathcal{O}_F^*}\Phi_\alpha(wn(b))\psi(-tb)db\\ &= \Phi_\alpha(w)\mathrm{char}(\mathcal{O}_F)(t) + \sum_{n\geq f}|\varpi|^{ns}\chi(\varpi)^n\int_{\mathcal{O}_F^*}\chi^{-1}(b)\psi(-\frac{tb}{\varpi^n})db.\end{aligned}$$

Here we have used the fact that $\Phi_\alpha(n^-(c)) = 0$ if $0 < \mathrm{ord}_F c < f$.

Notice that the Gaussian integral

$$\chi(\varpi)^n\int_{\mathcal{O}_F^*}\chi^{-1}(b)\psi(-\frac{tb}{\varpi^n})db = \mathrm{char}(\varpi^{n-f}\mathcal{O}_F^*)(t)\chi(-t)\epsilon(\chi,\psi)|\varpi|^{\frac{f}{2}}$$

where $\epsilon(\chi, \psi)$ is the Tate's root number. So

$$\sum_{n \geq f} |\varpi|^{ns} \chi(\varpi)^{-n} \int_{\mathcal{O}_F^*} \chi^{-1}(b) \psi(-\frac{tb}{\varpi^n}) db = \chi(-t) \epsilon(\chi, \psi) |\varpi|^{\frac{f}{2}} |\varpi|^{k+f} \mathrm{char}(\mathcal{O}_F)(t)$$

with $k = \mathrm{ord}_\varpi t \geq 0$. In summary, we have proved for $t \in \mathcal{O}_F$

$$W_t(1, s, \Phi_\alpha) = \gamma(V_\alpha) |\varpi|^{\frac{f}{2}} \left(1 + \chi(\alpha t) C X^{k+f}\right)$$

with

$$C = \epsilon(\chi, \psi) \gamma(\Delta, \psi)^{-1} \chi(-1).$$

So $W_t(1, 0, \Phi_\alpha) = 0$ if and only if $\chi(\alpha t) C = -1$. On the other hand, by Proposition 1.4(iii) of [10], $\epsilon_F(V_\alpha) = -\chi(t)$, i.e., $\chi(\alpha t) = -1$, implies $W_t(1, 0, \Phi_\alpha) = 0$. So $C = 1$, i.e.,

$$\gamma(\Delta, \psi) = \chi(-1) \epsilon(\chi, \psi). \tag{2.9}$$

Therefore

$$W_t(1, s, \Phi_\alpha) = \chi(\alpha) \gamma(\Delta, \psi) |\varpi|^{\frac{f}{2}} (1 + \chi(\alpha t) X^{\mathrm{ord}_F t + f}).$$

The formula for $W_t^*(w, s, \Phi_\alpha)$ can be proved similarly. For the purpose of proving (3), we also need to compute $W_t(n^-(c), s, \Phi_\alpha)$ when $0 < n = \mathrm{ord}_F c < f$. Similar calculation gives

$$W_t(n^-(c), s, \Phi_\alpha) = 0 \tag{2.10}$$

unless $\mathrm{ord}_F t = \mathrm{ord}_F c - f$. In such a case,

$$\begin{aligned} W_t(n^-(c), s, \Phi_\alpha) = \Phi_\alpha(w) \psi(\frac{t}{c}) |c|^{-1} \int_{1+\varpi^n \mathcal{O}_F} \chi^{-1}(b) \psi(-\frac{t}{c} b) db \\ + \chi(-c) |c|^s \int_{1+\varpi^{f-n} \mathcal{O}_F} \chi(b) \psi(-\frac{t}{c} b) db. \end{aligned} \tag{2.11}$$

Now we are ready to prove (3). It is clear that $M(s)\Phi_\alpha \in I(-s, \chi)$ is an eigenfunction of $K_0(\varpi^f)$ with eigencharacter χ since Φ_α is. So it suffices to verify (3) for $g = 1, w$, or $n^-(c)$ with $0 < n = \mathrm{ord}_F c < f$. Since

$$M(s)\Phi_\alpha(g, s) = W_0(g, s, \Phi_\alpha),$$

one sees immediately from (2)

$$M(s)\Phi_\alpha(1, s) = \Phi_\alpha(w), \quad M(s)\Phi_\alpha(w, s) = \Phi_\alpha(w)^2.$$

So $M(s)\Phi_\alpha(g, s) = \Phi_\alpha(w)\Phi_\alpha(g, -s)$ is true for $g = 1, w$. By (2.10), one has for $0 < \mathrm{ord}_F c < f$

$$M(s)\Phi_\alpha(n^-(c), s) = 0 = \Phi_\alpha(w)\Phi_\alpha(n^-(c), -s).$$

This proves (3). Claims (4) and (5) follow from (2) directly. □

Finally, we consider the case $F = \mathbb{R}$, and fix $\psi_{\mathbb{R}}(x) = e(x)$. (2.1) still holds. When $K = \mathbb{C}$, i.e., $\chi = \mathrm{sgn}$ is the sign character. Let $\phi^0 = e^{-\pi z\bar{z}}$ be the Gaussian, and let $\Phi^{\pm}$ be its associated standard section in $I(s, \chi)$ via the Weil representation $\omega_{\pm 1}$. Then $\Phi^{\pm} = \Phi_{\mathbb{R}}^{\pm 1}$ is the unique eigenfunction of $K_{\mathbb{R}} = \mathrm{SO}_2(\mathbb{R})$ of weight ± 1 defined in the introduction. The following is [11], Proposition 2.6. The sign difference is due to the fact that we use $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to define the Whittaker function while $-w$ is used in [11]. For general $\Phi_{\mathbb{R}}^n$, we refer to [12], Proposition 14.1.

Proposition 2.4. *For $\tau = u + iv$ in the upper half plane $\mathbb{H}$, let $g_\tau = n(u)m(\sqrt{v})$, and*

$$W_{t,\mathbb{R}}^*(\tau, s, \Phi_{\mathbb{R}}^1) = v^{-\frac{1}{2}} L(s+1, \chi) W_t(g_\tau, s, \Phi_{\mathbb{R}}^1),$$

where $L(s, \chi) = \Gamma_{\mathbb{R}}(s+1)$ as in the introduction. Then

(1)

$$W_{t,\mathbb{R}}^*(\tau, 0, \Phi_{\mathbb{R}}^1) = \begin{cases} -2ie(t\tau) & \text{if } t > 0, \\ -i & \text{if } t = 0, \\ 0 & \text{if } t < 0. \end{cases}$$

(2) When $t < 0$, one has

$$W_{t,\mathbb{R}}^{*,\prime}(\tau, 0, \Phi_{\mathbb{R}}^1) = -ie(t\tau)\beta_1(4\pi|t|v),$$

where

$$\beta_1(x) = \int_1^\infty e^{-ux} \frac{du}{u} = -Ei(-x), \quad x > 0$$

is a partial Gamma function.

(3) $M^*(s)\Phi_{\mathbb{R}}^1 = -iL(s, \chi)\Phi_{\mathbb{R}}^1(-s)$. Here $M^*(s) = L(s+1, \chi)M(s)$ is the normalized intertwining operator from $I(s, \chi)$ to $I(-s, \chi)$.

We end this section with a useful fact relating the local Weil index with the local root number.

Corollary 2.5. *Let F be a local field, and let ψ be a non-trivial additive character of F. Let $\Delta \in F^*$. Then*

$$\gamma(\Delta, \psi)\epsilon(\chi_\Delta, \psi) = 1.$$

Here $\chi_\Delta = (\Delta,)_F$ is the quadratic character of F^* associated to $F(\sqrt{\Delta})$.

Proof. First notice that

$$\gamma(\Delta, a\psi) = \chi_\Delta(a)\gamma(\Delta, \psi), \quad \epsilon(\chi_\Delta, a\psi) = \chi_\Delta(a)\epsilon(\chi_\Delta, \psi).$$

So the identity does not depends on the choice of ψ. Next, if $\Delta \in F^{*,2}$, then $\gamma(\Delta, \psi) = \epsilon(\chi_\Delta, \psi) = 1$. So we can assume $\Delta \notin F^{*,2}$. When $F = \mathbb{R}$, $\Delta < 0$, and thus

$$\gamma(\Delta, \psi_{\mathbb{R}}) = i^{-1} = \epsilon(\mathrm{sgn}, \psi_{\mathbb{R}})^{-1}.$$

When F is a p-adic field, we take ψ to be unramified. If $K = F(\sqrt{\Delta})$ is unramified over F, then one has by (2.7)

$$\gamma(\Delta, \psi) = 1 = \epsilon(\chi_\Delta, \psi)^{-1}.$$

If $K = F(\sqrt{\Delta})$ is ramified, then one has by (2.9)

$$\gamma(\Delta, \psi) = \chi_\Delta(-1)\epsilon(\chi_\Delta, \psi) = \epsilon(\chi_\Delta, \psi)^{-1}.$$

The case of non-archimedean field of positive character is the same.

□

3 The main formula

Now we are back to the global situation and let the notation be as in the introduction. In particular, F is a totally real number field of degree d, and $K = F(\sqrt{\Delta})$ is a totally imaginary quadratic extension of F. Recall that $\chi = \chi_{K/F} = (\Delta, \ )_{\mathbb{A}}$ be the quadratic Hecke character of F associated to K/F.

For $\alpha = (\alpha_v) \in \prod_{v|d_{K/F}} F_v^*$ with $\mathrm{ord}_v \alpha_v = \mathrm{ord}_v \partial_F$ and a square-free integral ideal $\mathcal{N}$ of F prime to $d_{K/F}$ as in Theorem 1.2, we choose a standard section $\Phi = \Phi^{\alpha, \mathcal{N}} = \prod_v' \Phi_v \in I(s, \chi)$ as follows.

When $v|\infty$, we choose $\Phi_v = \Phi_{\mathbb{R}}^1 \in I(s, \chi_v)$ be the unique eigenfunction of $\mathrm{SO}_2(\mathbb{R})$ of weight one as in the introduction.

When $v|d_{K/F}$, choose $\Phi_v = \Phi_v^+$ with respect to the unramified additive character $\psi_v^0 = \alpha_v^{-1}\psi_v$ as in Section 2.

When $v|\mathcal{N}$, choose $\Phi_v = \Phi_v^-$ with respect to a unramified additive character ψ_v^0, say $\psi_v^0 = \alpha_v^{-1}\psi_v$ for some $\alpha_v \in \partial_F \mathcal{O}_v^*$ as in Section 2.

When $v \nmid \mathcal{N} d_{K/F}\infty$, choose $\Phi_v = \Phi_v^+$ with respect to a unramified additive character ψ_v^0, say $\psi_v^0 = \alpha_v^{-1}\psi_v$ for some $\alpha_v \in \partial_F \mathcal{O}_v^*$ as in Section 2.

We remark that Φ_v is independent of the choice of ψ_v^0 or equivalently α_v^{-1} when $v \nmid d_{K/F}\infty$. The purpose of this section is to prove

Theorem 3.1. *Let the notation be as above. Then the Eisenstein series $E^*(\tau, s, \Phi^{\alpha, \mathcal{N}})$ satisfies all the properties in Theorem 1.2.*

Proof. We first make a remark on the Fourier expansion and the local Whittaker function with respect to ψ. When $t \neq 0$, the t-th Fourier coefficient of an Eisenstein series $E(g, s, \Phi)$ is

$$\begin{aligned} E_t(g, s, \Phi) &= \int_{F \backslash F_{\mathbb{A}}} E(n(b)g, s, \Phi)\psi(-tb)d_{\psi}b \\ &= \prod_v \int_v \Phi_v(wn(b)g_v)\psi_v(-tb)d_{\psi_v}b \\ &= \prod_v W_{t,v}^{\psi_v}(g_v, s, \Phi_v). \end{aligned}$$

Here the Haar measure $d_{\psi_v}b$ on F_v is chosen to be self-dual with respect to ψ_v. The Haar measure db on F_v used in Section 2 is chosen to be self-dual with respect to an unramified additive character, say $\psi_v^0 = \alpha_v^{-1}\psi_v$. So we have (we set $\alpha_v = 1$ for $v|\infty$) $d_{\psi_v}b = |\alpha_v|_v^{\frac{1}{2}} db$. So

$$W_{t,v}^{\psi_v}(g, v, \Phi_v) = |\alpha_v|_v^{\frac{1}{2}} W_{t\alpha_v,v}^{\psi_v^0}(g, s, \Phi_v). \tag{3.1}$$

Everything in (1) follows from the general theory of Eisenstein series except for the function equation, where one has

$$E(g, s, \Phi) = E(g, -s, M(s)\Phi).$$

For the rest of the proof, we denote $\Phi = \Phi^{\alpha,\mathcal{N}}$. By the results in Section 2, and the above remark one has

$$M_v^*(s)\Phi_v = |\alpha_v|_v^{\frac{1}{2}} M_v^{*,\psi_v^0}\Phi_v = \Phi_v(-s) \begin{cases} -iL(s, \chi_v) & \text{if } v|\infty, \\ |d_{K/F}|_v^{\frac{1}{2}} \chi_v(-\alpha_v)\epsilon(\chi_v, \psi_v) & \text{if } v|d_{K/F}, \\ -\frac{|\varpi_v|+|\varpi_v|^s}{1+|\varpi_v|^{1+s}} L(s, \chi_v) & \text{if } v|\mathcal{N}, \\ |\partial_F|_v^{\frac{1}{2}} L(s, \chi_v) & \text{otherwise.} \end{cases}$$

It is easy to see

$$\prod_{v|d_{K/F}} |d_{K/F}|_v^{\frac{1}{2}} \prod_{v|\partial_F} |\partial_F|_v^{\frac{1}{2}} = A^{-\frac{1}{2}}, \quad A = N_{F/\mathbb{Q}}(\partial_F d_{K/F}),$$

and

$$\prod_{v|\mathcal{N}} \frac{|\varpi_v| + |\varpi_v|^s}{1 + |\varpi_v|^{1+s}} = \prod_{v|\mathcal{N}} \frac{|\mathcal{N}|_v^{\frac{1-s}{2}} + |\mathcal{N}|_v^{-\frac{1-s}{2}}}{|\mathcal{N}|_v^{\frac{1+s}{2}} + |\mathcal{N}|_v^{-\frac{1+s}{2}}}.$$

So

$$M^*(s)\Phi = |A|^{\frac{s+1}{2}} \prod_v M_v^*(s)\Phi_v = \epsilon(\alpha, \mathcal{N})\Lambda(s,\chi)\Phi(-s) \prod_{v|\mathcal{N}} \frac{|\mathcal{N}|_v^{\frac{1-s}{2}} + |\mathcal{N}|_v^{-\frac{1-s}{2}}}{|\mathcal{N}|_v^{\frac{1+s}{2}} + |\mathcal{N}|_v^{-\frac{1+s}{2}}},$$

with

$$\epsilon(\alpha, \mathcal{N}) = (-1)^{o(\mathcal{N})}(-i)^d \prod_{v|d_{K/F}} \chi_v(-\alpha_v)\epsilon(\chi_v, \psi_v) = -1)^{o(\mathcal{N})} i^d \prod_{v|d_{K/F}} \chi_v(\alpha_v)\epsilon(\chi_v, \psi_v)$$

being as in the first formula for $\epsilon(\alpha, \mathcal{N})$ in Theorem 1.2. Here we used the fact that

$$(-1)^d \prod_{v|d_{K/F}} \chi_v(-1) = \prod_v \chi_v(-1) = 1.$$

So

$$\begin{aligned} E^*(\tau, s, \Phi) &= E(\tau, -s, M^*(s)\Phi) \\ &= \epsilon(\alpha, \mathcal{N})\Lambda(1-s,\chi)E(\tau, -s, \Phi) \prod_{v|\mathcal{N}} \frac{|\mathcal{N}|_v^{\frac{1-s}{2}} + |\mathcal{N}|_v^{-\frac{1-s}{2}}}{|\mathcal{N}|_v^{\frac{1+s}{2}} + |\mathcal{N}|_v^{-\frac{1+s}{2}}} \\ &= \epsilon(\alpha, \mathcal{N})E^*(\tau, -s, \Phi) \prod_{v|\mathcal{N}} \frac{|\mathcal{N}|_v^{\frac{1-s}{2}} + |\mathcal{N}|_v^{-\frac{1-s}{2}}}{|\mathcal{N}|_v^{\frac{1+s}{2}} + |\mathcal{N}|_v^{-\frac{1+s}{2}}}. \end{aligned}$$

This verifies the functional equation. The constant term of $E^*(\tau, s, \Phi)$ is

$$\begin{aligned} E_0^*(\tau, s, \Phi) &= \Lambda(s+1, \chi)(\prod_{1 \le i \le d} v_i)^{\frac{s}{2}} + M^*(s)\Phi(g_\tau, s)(\prod_{1 \le i \le d} v_i)^{-\frac{1}{2}} \\ &= \Lambda(-s, \chi)(\prod_{1 \le i \le d} v_i)^{\frac{s}{2}} + \epsilon(\alpha, \mathcal{N})\Lambda(s, \chi)(\prod_{1 \le i \le d} v_i)^{-\frac{s}{2}}. \end{aligned}$$

Here $v_i = \mathrm{Im}(\tau_i)$. In particular,

$$E_0^*(\tau, 0, \Phi) = (1 + \epsilon(\alpha, \mathcal{N}))\Lambda(0, \chi) = (1 + \epsilon(\alpha, \mathcal{N}))L(0, \chi).$$

This verifies (3) for the constant term. Moreover, if $\epsilon(\alpha, \mathcal{N}) = 1$, then $E_0^*(\tau, 0, \Phi) = 2L(0, \chi) \neq 0$. If $\epsilon(\alpha, \mathcal{N}) = -1$, then $E^*(\tau, 0, \Phi) = 0$ by the functional equation proved in (1). So

$$E^*(\tau, 0, \Phi) = 0 \Leftrightarrow \epsilon(\alpha, \mathcal{N}) = -1.$$

For $t \in F^*$, the t-th Fourier coefficient of $E^*(\tau, 0, \Phi)$ is given by

$$\begin{aligned} E_t^*(\tau, 0, \Phi) &= A^{\frac{1}{2}} \prod_{1 \le i \le d} W_{t,\sigma_i}^{*,\psi_{\mathbb{R}}}(\tau_i, 0, \Phi_{\mathbb{R}}^{\frac{1}{}}) \prod_{v|d_{K/F}} W_{t,v}^{*,\psi_v}(1, 0, \Phi_v) \prod_{v \nmid d_{K/F}\infty} W_{t,v}^{*,\psi_v}(1, 0, \Phi_v) \\ &= 0 \end{aligned}$$

unless $t >> 0$ and $t \in \partial_F^{-1}$, i.e., $t \in \partial_F^{-1,+}$, by the results in Section 2. In such a case, one has by the same results in Section 2.

$$W_{t,\sigma_i}^{*,\psi_{\mathbb{R}}}(\tau_i, 0, \Phi_{\mathbb{R}}^1) = -2ie(\sigma_i(t)\tau_i),$$
$$W_{t,v}^{*,\psi_v}(1, 0, \Phi_{\alpha_v}) = |A|_v^{\frac{1}{2}}\chi_v(-\alpha_v)\epsilon(\chi_v, \psi_v)(1 + \chi_v(\alpha_v t)), \text{ if } v|d_{K/F},$$

and

$$W_{t,v}^{*,\psi_v}(1, 0, \Phi_{\alpha_v}) = |A|_v^{\frac{1}{2}}\begin{cases} -\frac{1+(-1)^{\operatorname{ord}_v \alpha_v t-1}}{2} & \text{if } v|\mathcal{N}, \\ \frac{1+(-1)^{\operatorname{ord}_v(\alpha_v t)}}{2} & \text{if } v \nmid \mathcal{N}d_{K/F}\infty \text{ is inert}, \\ (1 + \operatorname{ord}_v(\alpha_v t)) & \text{if } v \nmid \mathcal{N}d_{K/F}\infty \text{ is split}. \end{cases}$$

On the other hand, decomposing $\mathfrak{a} = \prod_v \mathfrak{p}_v^{\operatorname{ord}_v \mathfrak{a}}$, one sees immediately that

$$\rho_{K/F}(\mathfrak{a}) = \prod_{v<\infty} \rho_v(\mathfrak{a}) \tag{3.2}$$

with

$$\rho_v(\mathfrak{a}) = \begin{cases} 1 & \text{if } v \text{ is ramified in } K, \\ \frac{1+(-1)^{\operatorname{ord}_v \mathfrak{a}}}{2} & \text{if } v \text{ is inert in } K, \\ 1 + \operatorname{ord}_v \mathfrak{a} & \text{if } v \text{ is split in } K, \end{cases} \tag{3.3}$$

So

$$\prod_{v \nmid d_{K/F}\infty} W_{t,v}^{*,\psi_v}(1, 0, \Phi_v) = (-1)^{o(\mathcal{N})}\rho_{K/F}(t\partial_F\mathcal{N}^{-1}) \prod_{v \nmid d_{K/F}\infty} |A|_v^{\frac{1}{2}}. \tag{3.4}$$

Therefore, one has

$$E_t^*(\tau, 0, \Phi) = \epsilon(\alpha, \mathcal{N})2^d\delta(\alpha t)\rho_{K/F}(t\partial_F\mathcal{N}^{-1})$$

for $t \in \partial_F^{-1,+}$. This verifies (3).

To verify (2), let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(d_{K/F})$, let γ_v be its image in $\mathrm{SL}_2(F_v)$. For $v = \sigma_j|\infty$, write $\gamma_j = \sigma_j(\gamma)$ and $c_j = \sigma_j(c)$ and so on. Write $g_{\gamma_j\tau_j} = \gamma_j g_{\tau_j} k_\theta$, then

$$e^{i\theta} = \frac{c_j\tau_j + d_j}{|c_j\tau_j + d_j|}.$$

Also note that $\operatorname{Im}\gamma_j\tau_j = \frac{v_j}{|c_j\tau_j+d_j|^2}$. We also write the action of $\mathrm{SL}_2(F_{\mathbb{A}})$ on $I(s,\chi)$ as

$$r(g)\Phi(g_1) = \Phi(g_1 g).$$

The proof of (2) is a standard shifting technique between finite and infinite primes. Indeed,

$$\begin{aligned} E^*(\gamma\tau, s, \Phi) &= (\prod_j \mathrm{Im}\gamma_j\tau_j)^{-\frac{1}{2}} E^*(\prod_j g_{\gamma_j\tau_j}, s, \Phi) \\ &= \prod_j (c_j\tau_j + d_j) E^*(\prod_j \gamma_j g_{\tau_j}, s, \Phi) \\ &= \prod_j (c_j\tau_j + d_j)(v_j)^{-\frac{1}{2}} E^*(\prod_j g_{\tau_j}, s, r(\prod_{v<\infty} \gamma_v^{-1})\Phi). \end{aligned}$$

The last identity is due to the trivial but important fact that $f(\gamma g) = f(g)$ for any automorphic form f, $\gamma \in \mathrm{SL}_2(F)$, and $g \in \mathrm{SL}_2(F_{\mathbb{A}})$. Now, the results in Section 2 imply

$$r(\gamma_v^{-1})\Phi_v = \begin{cases} \Phi_v & \text{if } v \text{ unramified,} \\ \chi_v(d)\Phi_v & \text{if } v \text{ ramified.} \end{cases}$$

Therefore

$$\begin{aligned} E^*(\gamma\tau, s, \Phi) &= \prod_j (c_j\tau_j + d_j)(v_j)^{-\frac{1}{2}} E^*(\prod_j g_{\tau_j}, s, \Phi) \prod_{v \text{ ramified}} \chi_v(d) \\ &= \chi(\gamma) \prod_j (c_j\tau_j + d_j) E^*(\tau, s, \Phi). \end{aligned}$$

This verifies (2). Finally, we verify the second formula for $\epsilon(\alpha, \mathcal{N})$. It is well-known ([15] that

$$\epsilon(\chi_v, \psi_v) = \begin{cases} i & \text{if } v|\infty, \\ \chi_v(\varpi_v)^{n(\psi_v)} & \text{if } \chi \text{ is unramified at } v < \infty. \end{cases}$$

So $i^d = \prod_{v|\infty} \epsilon(\chi_v, \psi_v)$. Since the global root number of χ is

$$1 = \prod_v \epsilon(\chi_v, \psi_v),$$

a simple calculation gives then

$$\epsilon(\alpha, \mathcal{N}) = (-1)^{o(\mathcal{N})} \prod_{v|d_{K/F}} \chi_v(\alpha_v) \prod_{v|\partial_F, v\nmid d_{K/F}} \chi_v(\partial_F).$$

□

Elementary Proofs of Corollary 1.3: Since this corollary is so simple, we give two other simple proofs here. First, if K/F, i.e., χ is unramified at every finite prime, then the global root number χ is

$$1 = \prod_v \epsilon(\chi_v, \psi_v) = i^{[F:\mathbb{Q}]} \prod_{v<\infty} \chi_v(\varpi_v)^{n(\psi_v)} = \pm i$$

as in the final part of the proof of the theorem. This is impossible and so K/F is ramified at some finite prime. The second proof is even simpler and is generously shown to me by David Rohrlich. Indeed,

$$1 = \chi(-1) = \prod_v \chi_v(-1) = (-1)^{[F:\mathbb{Q}]} \prod_{v<\infty} \chi_v(-1) = -\prod_{v<\infty} \chi_v(-1).$$

So there is some finite prime v such that $\chi_v(-1) = -1$, which implies that K/F is ramified at v. Notice that the above argument actually proved a slightly stronger result: If K/F is a quadratic extension of number fields unramified at every finite prime, then the number of real primes of F which become complex in K is even.

We end this section with a simple fact complement to Corollary 1.3.

Proposition 3.2. *Let $d > 0$ be an even integer. Then there is a totally real number field F of degree d together with a totally imaginary quadratic extension K such that K/F is unramified at every finite prime.*

Proof. Let p be a prime number such that $p \equiv 1 \mod d$. Then $\mathbb{Q}^+(\zeta_p) = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$ is a cyclic totally real number field of degree $\frac{p-1}{2}$, which is divisible by $\frac{d}{2}$. So it has a unique totally real subfield F_1 of degree $\frac{d}{2}$. Let $d_1 < 0$ and $d_2 < 0$ are two fundamental discriminants of imaginary quadratic fields such that p, d_1 and d_2 are pairwise relatively prime. Then $F = F_1(\sqrt{d_1 d_2})$ is a totally real number field of degree d, and $K = F_1(\sqrt{d_1}, \sqrt{d_2})$ is a totally imaginary quadratic extension of F unramified at every finite prime of F. Incidently, F is abelian over $\mathbb{Q}$ with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/(d/2)$. □

4 Unramified cases

In this section, we assume that K/F is unramified at every finite prime and take $\mathcal{N} = \mathcal{O}_F$. So $N = 1$ in Theorem 1.5. Theorem 1.5 and Corollary 1.4 give

Theorem 4.1. *Let F be a totally real number field of degree d with $4|d$, and let K be a totally imaginary quadratic extension of F unramified at every finite prime.*

Let $\chi_{K/F}$ be the quadratic Hecke character of F associated to K/F. Then

$$f_{K/F}(\tau) = L(0, \chi_{K/F}) + 2^{d-1} \sum_{m \geq 1} a_m(K/F) q^m$$

is a holomorphic modular form of weight d for $\mathrm{SL}_2(\mathbb{Z})$. Here $q = e(\tau)$, and

$$a_m(K/F) = \sum_{\substack{t \in \partial_F^{-1,+} \\ \mathrm{tr}_{F/\mathbb{Q}} t = m}} \rho_{K/F}(t\partial_F).$$

Theorem 4.1 can be used to compute $L(0, \chi_{K/F})$ and thus the relative class number h_K/h_F. Indeed, they are related by (see for example [16])

$$L(0, \chi) = \frac{2^d}{W_K[\mathcal{O}_K^* : \mu_K \mathcal{O}_F^*]} \frac{h_K}{h_F}. \tag{4.1}$$

Here h_K and h_F are ideal class numbers of K and F respectively, and μ_K is the group of roots of unity in K and $W_K = \#\mu_K$.

Let

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \tag{4.2}$$

be the classical Eisenstein series of weight k, where B_k is the Bernoulli numbers, and

$$\sigma_k(n) = \sum_{m|n} m^k.$$

Let

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} q^n \tau(n) \tag{4.3}$$

be the cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. Siegel proved in [14] page 90 the following proposition.

Proposition 4.2. *Let*

$$r = \begin{cases} [\frac{d}{12}] & \text{if } d \equiv 2 \mod 12, \\ 1 + [\frac{d}{12}] & \text{if } d \not\equiv 2 \mod 12, \end{cases}$$

be the dimension of the space $M_d(\mathrm{SL}_2(\mathbb{Z}))$ of holomorphic modular form of weight d for $\mathrm{SL}_2(\mathbb{Z})$. Write

$$\frac{E_{12r-d+2}}{\Delta^r} = \sum_{n \geq -r} c_{d,n} q^n.$$

Then $c_{d,0} \neq 0$. *Moreover, if* $f(\tau) = \sum_{n\geq 0} a_n q^n$ *be a holomorphic modular form of weight* d *for* $\mathrm{SL}_2(\mathbb{Z})$*, then*

$$\sum_{0\leq m\leq r} c_{d,-m} a_m = 0.$$

We refer to [1] and [2] for generalization of this proposition. Combining this proposition with Theorem 4.1, one obtains

Corollary 4.3. *Let the notation and assumption be as in Theorem 4.1. Then*

$$c_{d,0} L(0, \chi_{K/F}) + \sum_{1\leq m\leq r} c_{d,-m} a_m(K/F) = 0,$$

where $c_{d,m}$ *and* r *are as in Proposition 4.2.*

Example 4.4. Let $D_1, D_2, \cdots, D_r > 0$ are fundamental discriminants of real quadratic fields such that $(D_i, D_j) = 1$ for $1 \leq i < j \leq r$. Assume $D_1 = d_1 d_2$ such that d_1 and d_2 are fundamental discriminants of imaginary quadratic fields. Let $F = \mathbb{Q}(\sqrt{D_1}, \cdots, \sqrt{D_r})$ be a totally real number field of degree 2^r, and let $K = F(\sqrt{d_1}, \sqrt{d_2})$. Then K is a totally imaginary quadratic extension of F which is unramified at every finite prime of F. So, for $r \geq 2$,

$$f_{d_1,d_2,D_2,\cdots,D_r}(\tau) = f_{K/F}(\tau)$$

is a holomorphic modular form of weight 2^r for $\mathrm{SL}_2(\mathbb{Z})$.

Example 4.5. Let d be an even positive integer, and F_1 be a totally real number field of degree $d/2$ with discriminant d_{F_1}. Let $D = d_1 d_2 > 0$ be a fundamental discriminant prime to d_{F_1} such that d_1 and d_2 are fundamental discriminants of imaginary quadratic fields. Then $F = F_1(\sqrt{D})$ is a totally real number field of degree d, and $K = F_1(\sqrt{d_1}, \sqrt{d_2})$ is a totally imaginary quadratic extension of F unramified at every finite prime. If $4|d$, then we obtain a holomorphic modular form of weight d for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$f_{F_1,d_1,d_2}(\tau) = L(0, \chi_{K/F}) + 2^{d-1} \sum_{m=1}^{\infty} a_m(K/F) q^m.$$

On the other hand, if $d \equiv 2 \mod 4$, then for every $t \in \partial_F^{-1,+} = (\sqrt{D}\partial_{F_1})^{-,+}$, the ideal $t\partial_F$ is a not a norm from K.

Conjecture 4.6. *Let* $4|d$ *and let* F_1 *be a totally real number field of degree* $\frac{d}{2}$ *as in Example 4.5. The the formula forms* $f_{F_1,d_2,d_2}(\tau)$*, as* d_1 *and* d_2 *vary subject to the conditions in Example 4.5, generate the space of* $M_d(\mathrm{SL}_2(\mathbb{Z}))$ *of holomorphic modular forms of weight* d *for* $\mathrm{SL}_2(\mathbb{Z})$*.*

The small degree cases are in particular interesting. For example, since $M_4(\mathrm{SL}_2(\mathbb{Z}))$ and $M_8(\mathrm{SL}_2(\mathbb{Z}))$ are both one dimensional, we obtain immediately from Theorem 4.1

Corollary 4.7. *Let the notation and assumption be as in Theorem 4.1.*

(1) If $d = [F : \mathbb{Q}] = 4$, then

$$f_{K/F}(\tau) = L(0, \chi_{K/F})E_4(\tau),$$

and

$$L(0, \chi_{K/F}) = \frac{1}{30} \sum_{\substack{t \in \partial_F^{-1,+} \\ \mathrm{tr}_{F/\mathbb{Q}} t = 1}} \rho_{K/F}(t\partial_F).$$

(2) If $d = [F : \mathbb{Q}] = 8$, then

$$f_{K/F}(\tau) = L(0, \chi_{K/F})E_8(\tau),$$

and

$$L(0, \chi_{K/F}) = \frac{4}{15} \sum_{\substack{t \in \partial_F^{-1,+} \\ \mathrm{tr}_{F/\mathbb{Q}} t = 1}} \rho_{K/F}(t\partial_F).$$

(3) In both cases, the ratio $\frac{a_m(K/F)}{L(0,\chi_{K/F})}$ is independent of F or K.

Corollary 4.8. *(1) Let $F = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$ be a bi-quadratic totally real quartic field, and assume $D_1 = d_1 d_2$ such that d_1 and d_2 are fundamental discriminants of imaginary quadratic fields. Let $K = (\sqrt{d_1}, \sqrt{d_2}, \sqrt{D_2})$ be a totally imaginary quadratic extension of F. Then*

$$\frac{a_m(K/F)}{L(0, \chi_{K/F})}$$

is independent of the choice of d_1, d_2, or D_2.

(2) Let $F = F_1(\sqrt{D})$ and $K = F_1(\sqrt{d_1}, \sqrt{d_2})$ be as in Example 4.5, where $D_1 = d_1 d_2$, and F_1 is a totally real quartic field of degree 4 with $(d_{F_1}, D) = 1$. Then

$$\frac{a_{K/F}(m)}{L(0, \chi_{K/F})}$$

is independent of the choice of d_1, d_2 or F_1.

Similarly, we have

Corollary 4.9. *Let the notation and assumption be as in Theorem 4.1 with $d = [F : \mathbb{Q}] = 12$. Then*

$$f_{K/F}(\tau) = aE_{12}(\tau) + b\Delta(\tau)$$

with

$$a_1(K/F) = Ca + b, \quad a_2(K/F) = (1 + 2^{11})Ca - 24b, \quad C = \frac{65520}{691}. \tag{4.4}$$

Finally,

$$L(0, \chi_{K/F}) = a.$$

Example 4.10. Let $F_1 = \mathbb{Q}(\alpha)$ be a totally real cubic field, and let $D_1 = d_1 d_2 > 0$ and D_2 are two fundamental discriminants of real quadratic fields such that $(D_1, D_2) = 1$ and that d_1 and d_2 are fundamental discriminants of imaginary quadratic fields. Then $F = \mathbb{Q}(\alpha, \sqrt{D_1}, \sqrt{D_2})$ is a totally real number field of degree 12, and $K = \mathbb{Q}(\alpha, \sqrt{d_1}, \sqrt{d_2}, \sqrt{D_2})$ is a totally imaginary quadratic extension of F, and so

$$f_{\alpha,d_1,d_2,D_2}(\tau) = L(0, \chi_{K/F}) + 2^{11} \sum_{m \geq 1} a_m(K/F) q^m$$

is a holomorphic modular form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. A natural question is when two such forms are proportional (say fixing α, D_1, d_2, and varying d_1)?

Example 4.11. Let d_1, d_2 be two fundamental discriminants of imaginary quadratic fields such that $13, d_1$, and d_2 are pairwise relatively prime. Let $D = d_1 d_2$. Then $F = \mathbb{Q}(\zeta_{13} + \zeta_{13}^{-1}, \sqrt{D})$ is a totally real number field (in fact, Galois over $\mathbb{Q}$ with Galois group $\mathbb{Z}/6 \times \mathbb{Z}/2$), and $K = \mathbb{Q}(\zeta_{13} + \zeta_{13}^{-1}, \sqrt{d_1}, \sqrt{d_2})$ is a totally imaginary quadratic extension of F unramified at every finite prime. So $f_{K/F}(\tau)$ is a holomorphic modular form of weight 12.

5 Real quadratic fields

In this section, we use Theorem 1.5 to give three different ways to construct modular forms of weight 2 via CM quartic fields. Biquadratic fields give modular forms of level N with the trivial character. Non-biquadratic fields give modular forms of level N with the Dirichlet character $(\frac{}{N})$. The first construction is very close in essence to the one constructed by Hecke and is given in Theorem 1.7, which we restate here for the convenience of the reader.

Theorem 5.1. *Let N be a square-free positive integer. Let $d_1, d_2 < 0$ be two fundamental discriminants of imaginary quadratic fields, and let $F = \mathbb{Q}(\sqrt{D})$ with $D = d_1 d_2 > 0$, and let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Assume that*

$$(d_1, d_2) = 1, \text{ and } (\frac{d_1}{p}) = (\frac{d_2}{p}) = -1 \text{ for every } p|N. \tag{5.1}$$

So every prime $p|N$ splits in F and every prime of F above N is inert in K. Let $\mathcal{N}$ be an integral ideal of F with odd number of prime factors in F such that $\mathcal{N} \cap \mathbb{Z} = N\mathbb{Z}$. Then

$$f_{d_1,d_2,\mathcal{N}}(\tau) = L(0, \chi_{K/F}) + 2 \sum_{m=1}^{\infty} a_m(d_1, d_2, \mathcal{N}) e(m\tau) \tag{5.2}$$

is a holomorphic (elliptic) modular form of weight 2 for $\Gamma_0(N)$ with the trivial Nebentypus character, where

$$a_m(d_1, d_2, \mathcal{N}) = \sum_{t = \frac{a+m\sqrt{D}}{2} \in \mathcal{N}, |a| < m\sqrt{D}} \rho_{K/F}(t\mathcal{N}^{-1}). \tag{5.3}$$

Proof. Since K/F is unramified at every finite prime, α in Theorem 1.5 does not appear, we denote $\alpha = 1$ in such a case. So

$$\epsilon(1, \mathcal{N}) = i^2(-1)^{o(\mathcal{N})} = (-1)^{o(\mathcal{N})+1} = 1$$

by assumption. Notice that $\partial_F = \sqrt{D}\mathcal{O}_F$, and thus

$$t \in (\partial_F \mathcal{N})^+ \Leftrightarrow \sqrt{D}t = \frac{a + b\sqrt{D}}{2} \in \mathcal{N} \text{ with } |a| < b\sqrt{D}. \tag{5.4}$$

Now the proposition follows immediately from Theorem 1.5. □

Notice that if $\mathcal{N} = \mathcal{N}_1 d$ for some rational positive integer d then it is clear from the theorem

$$f_{d_1,d_2,\mathcal{N}} = f_{d_1,d_2,\mathcal{N}_1}(d\tau)$$

is an old form, while $f_{d_1,d_2,\mathcal{N}_1}$ is a modular form of weight two for $\Gamma_0(N_1)$ with $N_1\mathbb{Z} = \mathcal{N}_1 \cap \mathbb{Z}$. Recall that when N is a prime number, there is exactly one (up to scalar) Eisenstein series of weight 2 for $\Gamma_0(N)$, given by

$$E_{2,N}(\tau) = \frac{N-1}{24} + \sum_{n \geq 1} (\sum_{d|n, N\nmid d} d) q^n. \tag{5.5}$$

Recall also that $\dim M_2(\Gamma_0(N)) = 1$ for $N = 2, 3, 5, 7, 13$. So we have

Corollary 5.2. *Let $N = 2, 3, 5, 7$, or 13, and let d_1 and d_2 be as in Theorem 5.1. Then*

$$f_{d_1,d_2,\mathcal{N}}(\tau) = \frac{24L(0,\chi_{K/F})}{N-1}E_{2,N}(\tau),$$

and

$$L(0,\chi_{K/F}) = \frac{N-1}{24}\sum_{t=\frac{a+\sqrt{D}}{2}\in\mathcal{N},|a|<\sqrt{D}}\rho_{K/F}(t\mathcal{N}^{-1}).$$

We mention that $f_{d_1,d_2,\mathcal{N}}$ is independent of $\mathcal{N}$ when N is prime. Next we consider the case where N is a square-free and $N \equiv 3 \mod 4$ so that $\mathbb{Q}(\sqrt{-N})$ has discriminant $-N$. The following theorem gives another way to construct modular forms of weight 2 for $\Gamma_0(N)$ by biquadratic CM fields.

Theorem 5.3. *Let $K = \mathbb{Q}(\sqrt{D}, \sqrt{-N})$ be a bi-quadratic CM field with maximal totally real subfield F. Assume that $D > 0$ and $-N < 0$ are fundamental discriminants of quadratic fields such that $(2D, N) = 1$. Let $\beta = (\beta_v) \in \prod_{v|N}\mathcal{O}_{F_v}^*$ such that*

$$\chi(\beta) = \prod_{v|N}\chi_v(\beta_v) = -1.$$

Then

$$f_{D,\beta,N}(\tau) = L(0,\chi_{K/F}) + 2\sum_{m=1}^{\infty}a_m(D,\beta,N)e(m\tau)$$

is a holomorphic modular form of weight 2 for $\Gamma_0(N)$ with the trivial Nebentypus character, where

$$a_m(D,\beta,N) = \sum_{t=\frac{a+m\sqrt{D}}{2}\in O_F,|a|<m\sqrt{D}}\delta(\beta t)\rho_{K/F}(t\mathcal{O}_F).$$

Proof. In Theorem 1.5, we choose $\alpha = \sqrt{D}\beta = (\sqrt{D}\beta_v) \in \prod_{v|N}\mathcal{O}_v^*$, and $\mathcal{N} = \mathcal{O}_F$. Recall again $\partial_F = \sqrt{D}\mathcal{O}_F$. Then

$$\begin{aligned}\epsilon(\alpha,\mathcal{N}) &= \prod_{v|N}\chi_v(\alpha_v)\prod_{v|\sqrt{D}\mathcal{O}_F}\chi_v(\sqrt{D})\\ &= \prod_{v|N}\chi_v(\beta_v)\prod_{v|ND}\chi_v(\sqrt{D})\\ &= \chi(\beta)\prod_{v|\infty}\chi_v(\sqrt{D})\\ &= -\chi(\beta).\end{aligned}$$

Using (5.4), Theorem 1.5 gives in this case the modular form of weight 2, level N, and character $\tilde{\chi}$

$$f_{\alpha,\mathcal{N}}(\tau) = L(0,\chi) + 2\sum_{m\geq 1} a_m(\alpha,\mathcal{N})e(m\tau)$$

with

$$\begin{aligned} a_m(\alpha,\mathcal{N}) &= \sum_{t\in\partial_F^{-1,+},\operatorname{tr}_{F/\mathbb{Q}}t=m} \delta(\alpha t)\rho_{K/F}(t\partial_F) \\ &= \sum_{t=\frac{a+m\sqrt{D}}{2}\in\mathcal{O}_F,|a|<m\sqrt{D}} \delta(\beta t)\rho_{K/F}(t\mathcal{O}_F). \\ &= a_m(D,\beta,N). \end{aligned}$$

So $f_{D,\beta,N}$ is a modular form of weight 2, level N, and character $\tilde{\chi}$, where

$$\tilde{\chi}:\ (\mathbb{Z}/N)^* \longrightarrow \{\pm 1\}, \tilde{\chi}(a) = \prod_{v|N}\chi_v(a).$$

We need to prove $\tilde{\chi} = 1$. This follows from the following lemma. □

Lemma 5.4. *Let a be a rational integer prime to N, and let $p|N$ be a rational prime number. Then*

$$\prod_{v|p}\chi_v(a) = 1.$$

Proof. There are two cases. If $p = \mathfrak{p}\mathfrak{p}'$ is split in F. Then $F_{\mathfrak{p}} = F_{\mathfrak{p}'} = \mathbb{Q}_p$, and thus $\chi_{\mathfrak{p}}(a) = \chi_{\mathfrak{p}'}(a)$. So $\chi_{\mathfrak{p}}(a)\chi_{\mathfrak{p}'}(a) = 1$. If p is inert in F. Since p is odd, $\mathcal{O}_F/p$ is the quadratic extension of $\mathbb{Z}/p$, and every element $a \in (\mathbb{Z}/p)^*$ is a square in $\mathcal{O}_F/p$, and so $\chi_p(a) = 1$ again. □

Analogue of Corollary 5.2 holds here for $N = 3, 7$. Another interesting example is $N = 11$, where $M_2(\Gamma_0(11))$ has dimension 2 and is generated by

$$E_{2,11}(\tau) = \frac{5}{12} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \cdots. \tag{5.6}$$

and the cusp form

$$f_E = (\eta(\tau)\eta(11\tau))^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} + \cdots,$$

which is associated to the elliptic curve

$$E = X_0(11):\ y^2 - y = x^3 - x^2 - 10x - 20.$$

Here are some examples coming from Corollary 5.2 and Theorem 5.3.

$$\begin{aligned}
f_{-3,-23,\mathcal{N}_{11}} &= \frac{2}{5}(6E_{2,11} - f_E) = 1 + 2q + 8q^2 + 10q^3 + 16q^4 + \cdots ,\\
f_{-3,-31,\mathcal{N}_{11}} &= \frac{2}{5}(6E_{2,11} - f_E) = 1 + 2q + 8q^2 + 10q^3 + 16q^4 + \cdots ,\\
f_{-3,-47,\mathcal{N}_{11}} &= 4E_{2,11} = \frac{5}{3} + 4(q + 3q^2 + 4q^3 + 7q^4 \cdots),\\
f_{13,\beta,11} &= 24E_{2,11} = 10 + 24(q + 3q^2 + 4q^3 + 7q^4 \cdots),\\
f_{17,\beta,11} &= \frac{8}{5}(3E_{2,11} + 2f_E) = 2 + 8(q + q^2 + 2q^3 + 5q^4 + \cdots),\\
f_{5,\beta,11} &= \frac{8}{5}(3E_{2,11} + 2f_E) = 2 + 8(q + q^2 + 2q^3 + 5q^4 + \cdots),\\
f_{5,-\beta,11} &= \frac{48}{5}(E_{2,11} - f_E) = 4 + 48(q^2 + q^3 + q^4 + \cdots).
\end{aligned}$$

It might be worthwhile to study when two such modular forms are proportional. Notice that β does not matter in above formulae for $D = 13$ or 17, as 11 is inert in $F = \mathbb{Q}(\sqrt{D})$ in these cases, and there is basically one choice for β. When $D = 5$, 11 splits in $F = \mathbb{Q}(\sqrt{5})$ into two primes v and v'. There are two choices of β in this case. In the formula, we choose v so that $\chi_v(\sqrt{5}) = 1$, $\beta_v = -\sqrt{5}$, and $\beta_{v'} = \sqrt{5}$. Notice that $\delta(\beta t) + \delta(-\beta t) = 4$ and thus

$$f_{5,\beta,11} + f_{5,-\beta,11} = 2L(0, \chi_{K/F}) + 8\sum_{m=1}^{\infty}\Big(\sum_{t\in\frac{a+m\sqrt{5}}{2}\in\mathcal{O}_F, |a|\le m\sqrt{5}} \rho_{K/F}(t\mathcal{O}_F)\Big)q^m$$

is a modular form independent of β. In general, similar consideration gives

Corollary 5.5. *Let $N \equiv 3 \mod 4$ be a prime number and let $F = \mathbb{Q}(\sqrt{D})$ and $K = \mathbb{Q}(\sqrt{D}, \sqrt{-N})$ be as in Theorem 5.3. Then*

$$f_{D,-N} = L(0, \chi_{K/F}) + 4\sum_{m=1}^{\infty} a_m(D, -N)q^m$$

is a holomorphic modular form for $\Gamma_0(N)$ of weight 2, where

$$a_m(D, -N) = \sum_{t=\frac{a+m\sqrt{D}}{2}\in\mathcal{O}_F, |a|<m\sqrt{D}} \rho_{K/F}(t\mathcal{O}_F).$$

Finally, we come to the non-biquadratic case. We first prove a an easy lemma.

Lemma 5.6. *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, and let $K = F(\sqrt{\Delta})$ be a totally imaginary quadratic extension of F which is not biquadratic, i.e., $\tilde{F} = \mathbb{Q}(\sqrt{\Delta\Delta'})$ is also a real quadratic field. Assume that K/F is unramified at every prime above 2, and that Δ is primitive in the sense that Δ does not have rational prime factors. Then*

(1) $d_{K/F} \cap \mathbb{Z} = N_{F/\mathbb{Q}} d_{K/F} = d_{\tilde{F}}\mathbb{Z}$, where $d_{\tilde{F}}$ is the discriminant of $\tilde{F}$.

(2) The character $\tilde{\chi} : (\mathbb{Z}/d_{\tilde{F}})^* \cong (\mathcal{O}_F/d_{K/F})^* \longrightarrow \{\pm 1\}$ defined in Theorem 1.5 is the quadratic Dirichlet character associated to $\tilde{F}/\mathbb{Q}$.

Proof. By a theorem of Hilbert ([4], Theorem 17.20), one has

$$d_{K/F} = \prod_{\mathrm{ord}_{\mathfrak{p}} \Delta = \mathrm{odd}} \mathfrak{p}.$$

where $\mathfrak{p}$ runs through odd prime ideals of F. Since Δ is primitive, $p = N_{F/\mathbb{Q}}\mathfrak{p}$ is a rational prime split or ramified in F. This implies

$$N_{F/\mathbb{Q}} d_{K/F} = d_{K/F} \cap \mathbb{Z} = \prod_{\mathrm{ord}_{\mathfrak{p}} \Delta = \mathrm{odd}} p,$$

and $\Delta\Delta' = a^2 N_{F/\mathbb{Q}} d_{K/F}$. But Δ and Δ' are square modulo 4 and odd, so

$$\Delta\Delta' \equiv a^2 \equiv 1 \mod 4.$$

So $N_{F/\mathbb{Q}} d_{K/F} \equiv 1 \mod 4$ is square-free and

$$N_{F/\mathbb{Q}} d_{K/F} = d_{\tilde{F}}.$$

This proves (1). (2) follows from the fact

$$(\mathbb{Z}/d_{\tilde{F}})^* = (\mathcal{O}_F/d_{K/F})^*$$

(since every prime factor of $d_{\tilde{F}}$ is split or ramified in F) and (1) and (1.7). □

Now assume $N \equiv 1 \mod 4$ be square free and let $\epsilon_N = (\frac{}{N})$ be the quadratic Dirichlet character. Let F and K be as in Lemma 5.6 such that $\tilde{F} = \mathbb{Q}(\sqrt{N})$. Take $\alpha = (\alpha_v) \in \prod_v F_v^*$ such that $\mathrm{ord}_v \alpha_v = \mathrm{ord}_v \sqrt{D}$, and $\mathcal{N} = \mathcal{O}_F$. Then

$$\epsilon(\alpha, \mathcal{N}) = \prod_{v|\sqrt{D}\mathcal{O}_F, v\nmid d_{K/F}} \chi_v(\sqrt{D}) \prod_{v|d_{K/F}} \chi_v(-\alpha_v).$$

Here $\chi = \chi_{K/F}$. Set $\alpha = \sqrt{D}\beta$ with $\beta = (\beta_v) \in \prod_{v|d_{K/F}} \mathcal{O}_v^*$, then

$$\begin{aligned}\epsilon(\alpha, \mathcal{N}) &= \prod_{v|D, v\nmid d_{K/F}} \chi_v(\sqrt{D}) \prod_{v|d_{K/F}} \chi_v(\sqrt{D}\beta_v) \\ &= \prod_{v|d_F d_{K/F}} \chi_v(\sqrt{D}) \prod_{v|d_{K/F}} \chi_v(\beta_v) = -\prod_{v|d_{K/F}} \chi_v(\beta_v).\end{aligned}$$

The lemma above implies $\mathcal{O}_F/d_{K/F} \cong \mathbb{Z}/N$. So if we take $\beta_v = M$ for every $v|d_{K/F}$ and some fixed rational integer M prime to N, then

$$\chi(\beta) = \prod_{v|d_{K/F}} \chi_v(M) = (\frac{M}{N}). \tag{5.7}$$

So Theorem 1.5 gives

Theorem 5.7. *Let $N \equiv 1 \mod 4$ be a square-free positive integer. Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field and let $K = F(\sqrt{\Delta})$ be a CM quartic field such that $\tilde{F} = \mathbb{Q}(\sqrt{\Delta\Delta'}) = \mathbb{Q}(\sqrt{N})$. Assume that K/F is unramified at every prime above 2 and that Δ is primitive. Let M be a rational integer prime to N with $(\frac{M}{N}) = -1$. Then*

$$f_{K/F,M}(\tau) = L(0, \chi_{K/F}) + 2\sum_{m\geq 1} a_m(K/F, M)e(m\tau)$$

is a holomorphic modular form of weight 2, level N, and Nebentypus character $(\frac{}{N})$. Here

$$a_m(K/F, M) = \sum_{t=\frac{a+m\sqrt{D}}{2} \in \mathcal{O}_F, |a|<m\sqrt{D}} \delta(Mt)\rho_{K/F}(t\mathcal{O}_F).$$

The case when N is prime is in particular simple. In this case, there is a unique prime v_0 of F above N which is ramified in K/F, and $\delta(Mt) = 1 + \chi_{v_0}(Mt) = 1 - \chi_{v_0}(t) = 0$ or 2. Moreover, if $\delta(Mt) = 0$, i.e., $\chi_{v_0}(t) = 1$, then (for $t > 0 > t'$)

$$\prod_{v\nmid d_{K/F}\infty} \chi_v(t) = \chi_{v_0}(t) \prod_{v|\infty} \chi_v(t) = -1.$$

This implies $\chi_v(t) = -1$ for some inert prime v of F (in K), and so $\rho_v(t\mathcal{O}_F) = 0$. Therefore $\rho_{K/F}(t\mathcal{O}_F) = 0$ when $\delta(Mt) = 0$. So we obtain

Corollary 5.8. *Let the notation be as in Theorem 5.7. Assume further that N is a prime number. Then*

$$f_{K/F} = L(0, \chi_{K/F}) + 4 \sum_{m \ge 1} a_m(K/F) e(m\tau)$$

is a holomorphic modular form of weight 2, level N, and Nebentypus character $(\frac{}{N})$. Here

$$a_m(K/F) = \sum_{t=\frac{a+m\sqrt{D}}{2} \in \mathcal{O}_F, |a|<m\sqrt{D}} \rho_{K/F}(t\mathcal{O}_F).$$

Recall a classical result of Hecke which says that $\dim M_2(N, (\frac{}{N})) = 1$ for $N = 5, 13, 17$. Recall also that for a primitive Dirichlet character ϵ of conductor N such that $\epsilon(-1) = 1$, the Eisenstein series

$$E_{2,\epsilon}(\tau) = \frac{1}{2} L(-1, \epsilon) + \sum_{m=1}^{\infty} \sigma_{1,\epsilon}(m) q^m, \qquad q = e(\tau) \tag{5.8}$$

is a modular form of weight 2, level N with Nebentypus character ϵ. Here

$$\sigma_{1,\epsilon}(m) = \sum_{0<d|m} \epsilon(d) d.$$

Corollary 5.9. *Let the notation and assumption be as in Corollary 5.8. Then for $N = 5, 13, 17$, one has*

$$f_{K/F}(\tau) = \frac{2L(0, \chi_{K/F})}{L(-1, (\frac{}{N}))} E_{2,(\frac{}{N})}(\tau).$$

Conjecture 5.10. *Similar to Conjecture 4.6, we think the following should be true.*

(1) When $N > 0$ is square free, the modular forms $f_{d_1,d_2,\mathcal{N}}$ constructed in Theorem 5.1, as d_1, d_2, and $\mathcal{N}$ change, generate the space $M_2(\Gamma_0(N))$ of holomorphic modular forms of weight 2, level N with trivial Nebentypus character.

(2) When $N \equiv 3 \mod 4$ is square-free, the modular forms $f_{D,\beta,N}$ constructed in Theorem 5.3, as D and β change, generate the space $M_2(\Gamma_0(N))$.

(3) When $N \equiv 1 \mod 4$ is square-free, the modular forms $f_{K/F,M}$ constructed in Theorem 5.7, as K, F, and M change, generate the space $M_2(N, (\frac{}{N}))$ of holomorphic modular forms of weight 2, level N with Nebentypus character $(\frac{}{N})$.

These conjectures can easily be verified when N is small.

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Tonghai Yang
Department of Mathematics
The University of Wisconsin
Madison, WI 53706, USA
E-mail:thyang@math.wisc.edu