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Kloosterman Integrals for $GL(2, \mathbb{R})$

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1 Introduction

We denote by G the group of invertible 2×2 matrices and by N the subgroup of matrices of the form

$$n = \begin{pmatrix} 1 & \bullet \\ 0 & 1 \end{pmatrix}.$$

The group $N(\mathbb{R}) \times N(\mathbb{R})$ operates on $GL(2, \mathbb{R})$ and $M(2 \times 2, \mathbb{R})$ by

$$s \mapsto {}^t n_1 s n_2 .$$

We say that an element s or its orbit is **relevant** if

$$\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} s \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} = s \Rightarrow x_1 + x_2 = 0 .$$

A system of representatives for the relevant orbits in $M(2 \times 2, \mathbb{R})$ are the diagonal matrices

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \; a_1 \neq 0 ,$$

and the matrices

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \; a \neq 0 .$$

We set

$$w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that the previous matrix can be written wa.

For a 2×2 matrix

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we set $\Delta_1(m) = a$, $\Delta_2(m) = \det m$. They are invariants of the action of $N \times N$.

We let $\psi_{\mathbb{R}}$ or simply ψ be a non trivial additive character of $\mathbb{R}$. We define **the orbital integrals** of a Schwartz function Φ on $M(2 \times 2, \mathbb{R})$: for $a_1 \neq 0$,

$$\Omega[\Phi, \psi : a_1, a_2] :=$$
$$\int \Phi \left[\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \right] \psi(x_1 + x_2) dx_1 dx_2$$

and, for $a \neq 0$,

$$\Omega[\Phi, \psi : wa] := \int \Phi \left[wa \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right] \psi(x) dx$$

$$= \int \Phi \left[a \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \right] \psi(x) dx \, .$$

Most of the time, we will assume that Φ is in fact a smooth function of compact support on $GL(2,\mathbb{R})$. Our purpose is to study the asymptotic of these integrals.

Similarly, we denote by $M_h(2\times 2, \mathbb{C}/\mathbb{R})$ the space of 2×2 Hermitian matrices. The group $N(\mathbb{C})$ operates by

$$s \mapsto {}^t\overline{n} s n \, .$$

We say that an element s or its orbit is **relevant** if

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} s \begin{pmatrix} 1 & \overline{z} \\ 0 & 1 \end{pmatrix} = s \Rightarrow z + \overline{z} = 0 \, .$$

The previous matrices are also a set of representatives for the relevant orbits. We define **the orbital integrals** of a function $\Psi \in \mathcal{S}(M_h(2\times 2, \mathbb{C}/\mathbb{R}))$: for $a_1 \neq 0$,

$$\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2] :=$$

$$\int_{\mathbb{C}} \Psi \left[\begin{pmatrix} 1 & 0 \\ \overline{z} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right] \psi(z+\overline{z}) dz$$

and, for $a \neq 0$,

$$\Omega\left[\Psi, \mathbb{C}/\mathbb{R}, \psi : aw\right]$$

$$:= \int_{\mathbb{R}} \Psi \left[a \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \right] \psi(x) dx \, .$$

We set $H := GL(2,\mathbb{C}) \cap M_h(2\times 2, \mathbb{C}/\mathbb{R})$. We often write

$$\psi_{\mathbb{C}}(z) = \psi_{\mathbb{R}}(z+\overline{z}) \, .$$

Most of the time we will assume that Ψ is in fact a smooth function of compact support on H.

We want to study the asymptotic of these new integrals and show that, apart from a sign, they have the same asymptotic as the previous integrals.

We will not discus here the motivation for the study of these integrals. See the references [10], [5], [6]. In fact the integrals at hand are already discussed in [10] (1). The novelty here is the introduction of the Casimir operator and the use of a partial Fourier transform of the functions at hand. Indeed, we write the orbital integrals as the integrals of a partial Fourier transform of Φ or Ψ. In the end both kind of orbital integrals are written as the integral over $\mathbb{R}$ of a Schwartz function against an oscillatory factor, the **same** in both cases. Moreover, the results and the methods are likely to generalize to the case of $GL(n)$. Indeed our use of the Fourier transform is inspired by the fact that the orbital integral of a function Φ or Ψ and the orbital integral of its full Fourier transform are related by a simple integral transform. This relation holds in the context of $GL(n)$ ([5], (4)).

Analogous integrals and dual *Bessel distributions* have been studied by a number of authors, specially Baruch ([1]) and Baruch/Mao ([2]). For the relation with the classical literature and an exhaustive list of references see [3]. The idea of introducing a partial Fourier transform already occurs in [2].

2 Stationary phase

We recall and extend somewhat classical results on the stationary phase method ([9] is a convenient reference.). We recall the elementary formula

$$\int_{\mathbb{R}} \phi(y)\psi\left(\frac{y^2}{2x}\right) dy = |x|^{1/2}\gamma(x,\psi)\int_{\mathbb{R}} \hat{\phi}(y)\psi\left(-\frac{xy^2}{2}\right) dy\,, \tag{1}$$

where ϕ is a Schwartz function on $\mathbb{R}$ and $\hat{\phi}$ denotes its Fourier transform

$$\hat{\phi}(x) = \int_{\mathbb{R}} \phi(y)\psi(-yx)dy\,.$$

The factor $\gamma(x,\psi)$ is an eighth root of 1 dpending only on the sign of x.

Proposition 1 *Let $\phi(y,x)$ be a Schwartz function on $\mathbb{R}^2$. Assume that the support ϕ is contained in a set*

$$\{(y,x) : |y| \geq C_1\,,\ |x| \leq C_2\}$$

where $C_1 > 0, C_2 > 0$. Consider the integral

$$\int \phi(y,x)\psi_{\mathbb{R}}\left(\frac{y+\frac{1}{y}}{x}\right) dy\,.$$

There are two smooth functions of compact support on $\mathbb{R}$, θ_ϵ, $\epsilon = \pm 1$, such that the integral is equal to the sum

$$\sum_{\epsilon=\pm 1} \psi\left(\frac{2\epsilon}{x}\right)\gamma(\epsilon x,\psi)|x|^{1/2}2^{-1/2}\theta_\epsilon(x)\,. \tag{2}$$

Any two such functions satisfy

$$\theta_\epsilon(0) = \phi(\epsilon,0)\,. \tag{3}$$

PROOF: Consider first the case where $C_1 > 1$. Set

$$t = y + \frac{1}{y}\,.$$

Then, on the support of ϕ, $|y| \asymp |t|$. Moreover

$$\frac{dt}{dy} = 1 - \frac{1}{y^2}$$

so that, on the support of ϕ

$$1 \geq \frac{dt}{dy} \geq 1 - \frac{1}{C_1^2} > 0\,.$$

Thus we may view y as a function of t. Then

$$\frac{d^n y}{dt^n} = \frac{P_n(y)}{(y^2-1)^{2n}}\;\frac{dy}{dt}\,,$$

where P_n is a polynomial. This is bounded by a polynomial in $|t|$. Now regard the function

$$\phi(y(t), x)$$

as a function of (t, x). Then any partial derivative

$$\frac{\partial^{n+s}\phi}{\partial t^n \partial^s x}$$

can be computed as a linear combination of terms of the form

$$\frac{\partial^{m+s}\phi}{\partial y^m \partial x^s}$$

with coefficients in the ring $\mathbb{C}[\frac{dy}{dt}]$. and is thus rapidly decreasing for t large. Thus $\phi(y(t), x)$ is a Schwartz function of (t, x). Using t is a variable we get

$$\int \psi\left(\frac{t}{x}\right)\phi(y(t), x)\frac{dy}{dt}dt\,.$$

If we set

$$\phi_1(t, x) = \phi(y(t), x)\frac{dy}{dt}$$

we see that ϕ_1 is a Schwartz function and the integral is the partial Fourier transform of $\phi_1(t, x)$ evaluated at (x^{-1}, x). This is a smooth function $\theta(x)$ of compact support on $\mathbb{R}$ with the additional property that

$$\frac{\partial^m \theta}{\partial x^m}(0) = 0$$

for all m. We can rewrite θ in the prescribed form with $\theta_\epsilon(0) = 0 = \phi(\epsilon, 0)$.

Now we may assume the projection of the support of ϕ on the first factor is concentrated on small neighborhoods of ± 1. For y close to 1 we set

$$v = \frac{y-1}{y^{1/2}}\,.$$

Then

$$v(1) = 0\,,\; y + \frac{1}{y} = 2 + v^2\,,\; \frac{dv}{dy}(1) = 1\,.$$

For y close to -1 we set

$$v = \frac{y+1}{(-y)^{1/2}}\,.$$

Then

$$v(-1) = 0\,,\; y + \frac{1}{y} = -2 - v^2\,,\; \frac{dv}{dy}(-1) = 1\,.$$

Then our integral becomes

$$\sum_{\epsilon=\pm 1} \int \phi(y,x)\psi\left(\frac{\epsilon(2+v^2)}{x}\right)\frac{dy}{dv}\; dv$$

$$= \sum_{\epsilon} \psi\left(\frac{2\epsilon}{x}\right)\gamma(\epsilon x,\psi)|x|^{1/2}2^{-1/2}\int \phi_1(u,x)\psi\left(-\frac{\epsilon u^2 x}{4}\right) du\,,$$

where we have set

$$\phi_1(u,x) = \int \phi(y,x)\frac{dy}{dv}\; \psi(-vu)dv\,.$$

Hence the original integral has the required form with

$$\theta_\epsilon(x) = \int \phi_1(u,x)\psi\left(-\frac{\epsilon u^2 x}{4}\right) du\,.$$

In addition

$$\theta_\epsilon(0) = \int \phi_1(u,0)du = \phi(y,0)\frac{dy}{dv}\bigg|_{v=0} = \phi(\epsilon,0)\,.$$

The functions θ_ϵ are not unique but let us show that, as claimed, their values at 0 are unique. Indeed, suppose that we have a relation of the form

$$\psi\left(\frac{2}{x}\right)\phi_1(x) + \psi\left(\frac{-2}{x}\right)\phi_{-1}(x) = 0$$

valid for $x > 0$ sufficiently small, where ϕ_1 and ϕ_{-1} are continuous at 0. We have to see that

$$\phi_1(0) = \phi_{-1}(0) = 0\,.$$

If say $\phi_1(0) \neq 0$ then we can write

$$\psi\left(\frac{4}{x}\right) = \frac{\phi_{-1}(x)}{\phi_1(x)}\,.$$

It follows that $\psi\left(\frac{4}{x}\right)$ has a limit as $x \to 0^+$, a contradiction. Our conclusion follows. □

REMARK: In the previous Proposition, the values of the derivatives $\frac{d^n\theta_\epsilon}{dx^n}$ at $x = 0$ are also uniquely determined by the partial derivatives of the function $\phi(y, x)$ at the point $(\epsilon, 0)$. In particular the derivatives of θ_ϵ are arbitrary.

Proposition 2 *Let $\phi(y, x)$ be a Schwartz function on $\mathbb{R}^2$. Assume that the support ϕ is contained in a set*

$$\{(y, x) : |y| \geq C_1\,,\ |x| \leq C_2\}$$

where $C_1 > 0, C_2 > 0$. There is a smooth function of compact support θ on $\mathbb{R}$ with

$$\frac{d^m\theta}{dx^m}(0) = 0$$

for all m such that

$$\int \phi(y, x)\psi_{\mathbb{R}}\left(\frac{y - \frac{1}{y}}{x}\right) dy = \theta(x)\,.$$

PROOF: We set

$$t = y - \frac{1}{y}\,.$$

Then

$$\frac{dt}{dy} = 1 + \frac{1}{y^2} > 0\,.$$

Thus we can use t has a variable of integration and write the integral

$$\int \psi\left(\frac{t}{x}\right)\phi(y(t), x)\frac{dy}{dt}dt\,.$$

As before, if we set

$$\phi_1(t, x) = \int \phi(y(t), x)\frac{dy}{dt}dt$$

then ϕ_1 is a Schwartz function and the integral is the partial Fourier transform of ϕ_1 evaluated at (x^{-1}, x). Our assertion follows. □

3 Orbital integrals for $GL(2,\mathbb{R})$.

In this section we will study the orbital integral of a smooth function of compact support Φ on $GL(2,\mathbb{R})$. Thus we may regard Φ as a Schwartz function, in fact a function of compact support on $M(2\times 2,\mathbb{R})$, which vanishes on singular matrices. Our method is to compute the orbital integral as the integral of a partial Fourier transform of Φ against an oscillatory factor.

We first discuss the asymptotic of the integral for $a_1a_2<0$. Our goal in this section is to prove the following result.

Proposition 3 *Let Φ be a smooth function of compact support on $G(\mathbb{R})$. Then, for $b>0, c>0, \epsilon_1=\pm 1$,*

$$\Omega[\Phi,\psi:\epsilon_1 bc,-\epsilon_1 b^{-1}c]=$$

$$\sum_{\epsilon=\pm 1} 2^{-1/2}b^{-1/2}\psi\left(\frac{2\epsilon\epsilon_1}{b}\right)\gamma(\epsilon\epsilon_1,\psi)\theta_\epsilon(\epsilon_1 b,c) \tag{4}$$

where the functions $\theta_\epsilon(x,y)$ are smooth functions of compact support on $\mathbb{R}\times\mathbb{R}_+^\times$. Any two such functions verify

$$\theta_\epsilon(0,c)=\Omega[\Phi,\psi:c\epsilon w]\,. \tag{5}$$

Proof: Since Φ has compact support, in the orbital integral $\Omega[\Phi,\psi:a_1,a_2]$ the product $\Delta_2=a_1a_2$ remains in a fixed compact set of $\mathbb{R}^\times$. We first introduce the partial Fourier transform

$$\Phi_1\begin{pmatrix} a\, b\\ c\, t\end{pmatrix}:=\int\Phi\begin{pmatrix} a\, b\\ c\, y\end{pmatrix}\psi(-yt)dy\,.$$

Then by Fourier inversion formula we find, after a change of variables,

$$\Omega[\Phi,\psi:a_1,a_2]=$$

$$|a_1|^{-2}\int\Phi_1\begin{pmatrix} a_1\; x_1\\ x_2\;\; y\end{pmatrix}\psi\left(\frac{x_1+x_2+y(\Delta_2+x_1x_2)}{a_1}\right)dx_1dx_2dy\,.$$

We first consider a smooth partition of unity on $\mathbb{R}$

$$\phi_1+\phi_2=1\,,$$

where ϕ_1 is supported on a neighborhood of 0 and and is one in a smaller neighborhood of zero. We will choose ϕ_1 in a moment. The orbital integral is then the sum of two integrals

$$\Omega_i[\Phi,\psi:a_1,a_2]:=$$

$$|a_1|^{-2}\int \Phi_1\begin{pmatrix} a_1 & x_1 \\ x_2 & y\end{pmatrix}\psi\left(\frac{x_1+x_2+y(\Delta_2+x_1x_2)}{a_1}\right)\phi_i(y)dx_1dx_2dy\,,$$

with $i=1,2$. Since Δ_2 remains in a compact set of $\mathbb{R}^\times$ we may assume that the support of ϕ_1 is so small that

$$\phi_2\left(\pm\frac{1}{\sqrt{-\Delta_2}}\right)=1\,.$$

In addition, we choose the support of ϕ_1 so small that in the integral Ω_1 the quantity $1+x_2y$ remains in a compact set of $\mathbb{R}^\times_+$. We then use new variables:

$$X_1=x_1(1+x_2y)\,,\; X_2=x_2\,,\; Y=y\,.$$

The Jacobian matrix is

$$\begin{pmatrix} 1+x_2y & x_1y & x_1x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}.$$

Its determinant remains in a compact set of $\mathbb{R}^\times_+$. Thus

$$\Omega_1[\Phi,\psi:a_1,a_2]=\int\phi(X_1,X_2,Y)\psi\left(\frac{X_1+X_2+Y\Delta_2}{a_1}\right)dX_1dX_2dY$$

where ϕ is a compactly supported function. This has the form

$$f\left(\frac{1}{a_1},\frac{1}{a_1},\frac{\Delta_2}{a_1}\right)$$

where f is a Schwartz function. Thus it has the form specified in the Proposition with $\theta_\epsilon(0)=0$.

We now introduce another partial Fourier transform

$$\Phi_2\begin{pmatrix} a & u_1 \\ u_2 & y\end{pmatrix}:=\int\int\Phi_1\begin{pmatrix} a & x_1 \\ x_2 & y\end{pmatrix}\psi(-x_2u_1-x_1u_2)dx_1dx_2\,.$$

We use the elementary formula

$$\int\phi(x_1,x_2)\psi(tx_1x_2)dx_1dx_2=|t|^{-1}\int\hat\phi(x_1,x_2)\psi(-t^{-1}x_1x_2)dx_1dx_2\,.$$

We find

$$\Omega_2[\Phi,\psi:a_1,a_2]=$$

$$|a_1|^{-1}\int\Phi_2\begin{pmatrix} a_1 & u_1-\frac{1}{a_1} \\ u_2-\frac{1}{a_1} & y\end{pmatrix}\psi\left(\frac{ya_1a_2}{a_1}-\frac{a_1u_1u_2}{y}\right)\times$$

$$du_1 du_2 \frac{\phi_2(y)dy}{|y|}$$

or, after a change of variables,

$$|a_1|^{-1} \int \Phi_2 \begin{pmatrix} a_1 & u_1 \\ u_2 & y \end{pmatrix} \psi \left(\frac{ya_1a_2 - \frac{1}{y}}{a_1} - \frac{u_1 + u_2 + a_1u_1u_2}{y} \right) \times$$

$$du_1 du_2 \phi_2(y) \frac{dy}{|y|} .$$

We set

$$\phi(y, a_1) :=$$

$$\int \int \Phi_2 \begin{pmatrix} a_1 & u_1 \\ u_2 & y \end{pmatrix} \psi \left(-\frac{u_1 + u_2 + a_1u_1u_2}{y} \right) du_1 du_2 \frac{\phi_2(y)}{|y|} .$$

The function ϕ is a Schwartz function on $\mathbb{R} \times \mathbb{R}$ with support in a set

$$|y| \geq C_1 , |a_1| \leq C_2 .$$

In addition

$$\begin{aligned} \phi(y,0) &= \int \Phi_2 \begin{pmatrix} 0 & u_1 \\ u_2 & y \end{pmatrix} \psi \left(-\frac{u_1 + u_2}{y} \right) du_1 du_2 \frac{\phi_2(y)}{|y|} \\ &= |y|^{-1} \phi_2(y) \Phi_1 \begin{pmatrix} 0 & -y^{-1} \\ -y^{-1} & y \end{pmatrix} \\ &= |y|^{-1} \phi_2(y) \int \Phi \begin{pmatrix} 0 & -y^{-1} \\ -y^{-1} & x \end{pmatrix} \psi(-xy) dx \\ &= |y|^{-2} \phi_2(y) \Omega[\Phi, \psi : -y^{-1} w] \end{aligned}$$

Recall we assume $a_1 a_2 < 0$. We set

$$a_1 = \epsilon_1 bc , \ a_2 = -\epsilon_1 b^{-1} c$$

with $b > 0, c > 0$ and $\epsilon_1 = \pm 1$. Then

$$\Omega_2[\Phi, \psi : \epsilon_1 bc, -\epsilon_1 b^{-1} c]$$

$$= (bc)^{-1} \int \phi(y, \epsilon_1 bc) \psi \left(\frac{-c^2 y - \frac{1}{y}}{\epsilon_1 bc} \right) \ dy$$

$$= b^{-1} c^{-2} \int \phi(-c^{-1} y, \epsilon_1 bc) \psi \left(\frac{y + \frac{1}{y}}{\epsilon_1 b} \right) \ dy .$$

Now we apply Proposition (1), or rather a variant of the Proposition with c in a compact set of $\mathbb{R}^\times$, as a parameter. This can be written as

$$c^{-2}2^{-1/2}b^{-1/2}\sum_{\epsilon=\pm1}\psi\left(\frac{2\epsilon\epsilon_1}{b}\right)\gamma(\epsilon\epsilon_1,\psi)\theta_\epsilon(\epsilon_1 b,c)$$

where θ_ϵ are smooth functions of compact support on $\mathbb{R}\times\mathbb{R}_+^\times$ such that

$$\theta_\epsilon(0,c)=\phi(-c^{-1}\epsilon,0)=c^2\phi_2(-c^{-1}\epsilon)\Omega[\Phi,\psi:c\epsilon w]=c^2\Omega[\Phi,\psi:c\epsilon w]\,.$$

After a change of notations we arrive at the Proposition. □

4 Action of the Casimir Operator

In this section we show that the derivatives of the functions θ_ϵ at 0 can be computed in terms of the orbital integrals of the functions $\Omega[\rho(C)^n\Phi,\psi:wz]$ where C is the Casimir operator. Recall

$$C=\frac{H^2}{2}+X_-X_++X_+X_-=\frac{H^2}{2}+H+2X_-X_+ \tag{6}$$

where

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\,,\;X_+=\begin{pmatrix}0&1\\0&0\end{pmatrix}\,,\;X_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}\,. \tag{7}$$

For X in the enveloping algebra of $GL(2,\mathbb{R})$ we denote by $\rho(X)$ the corresponding left invariant differential operator. Thus if X is in the Lie algebra then

$$\rho(X)\Phi(g)=\left.\frac{d\Phi(g\exp(tX))}{dt}\right|_{t=0}\,.$$

We assume

$$\psi(x)=\exp(2i\pi\eta x)\,,\;\eta=\pm1\,.$$

Let Φ_1 be the function defined on the open set $\{g|\Delta_1(g)\neq0\}$ by

$$\Phi_1(g):=\int\Phi\left[\begin{pmatrix}1&0\\y&1\end{pmatrix}g\begin{pmatrix}1&x\\0&1\end{pmatrix}\right]\psi(x+y)dxdy\,.$$

Since C is in the center of the enveloping algebra we have

$$\Omega[\rho(C)\Phi,\psi:a_1,a_2]=(\rho(C)\Phi_1)\,(\mathrm{diag}(a_1,a_2))\,.$$

Now

$$\left(\rho(\frac{H^2}{2}+H)\Phi_1\right)(\mathrm{diag}(\epsilon_1 bc, -\epsilon_1 b^{-1}c)$$
$$=\left(\frac{1}{2}(b\frac{d}{db})^2+b\frac{d}{db}\right)\left(\Phi_1(\mathrm{diag}(\epsilon_1 bc, -\epsilon_1 b^{-1}c))\right)$$
$$=\left(\frac{1}{2}(b\frac{d}{db})^2+b\frac{d}{db}\right)\left(\Omega[\Phi,\psi:\epsilon_1 bc, -\epsilon_1 b^{-1}c]\right)\,.$$

Likewise

$$(\rho(X_-X_+))\,\Phi_1(\mathrm{diag}(\epsilon_1 bc, -\epsilon_1 b^{-1}c))$$
$$=\frac{\partial^2}{\partial s\partial t}\Phi_1\left[\begin{pmatrix}\epsilon_1 bc & 0\\ 0 & -\epsilon_1 b^{-1}c\end{pmatrix}\begin{pmatrix}1\,0\\ s\,1\end{pmatrix}\begin{pmatrix}1\;t\\ 0\,1\end{pmatrix}\right]\Bigg|_{s=t=0}$$
$$=\frac{\partial^2}{\partial s\partial t}\Phi_1\left[\begin{pmatrix}1 & 0\\ -sb^{-2} & 1\end{pmatrix}\begin{pmatrix}\epsilon_1 bc & 0\\ 0 & -\epsilon_1 b^{-1}c\end{pmatrix}\begin{pmatrix}1\;t\\ 0\,1\end{pmatrix}\right]\Bigg|_{s=t=0}$$
$$=\frac{\partial^2}{\partial s\partial t}e^{-2i\pi\eta b^{-2}s}e^{2i\pi\eta t}\Bigg|_{s=t=0}\Phi_1\left[\begin{pmatrix}\epsilon_1 bc & 0\\ 0 & -\epsilon_1 b^{-1}c\end{pmatrix}\right]$$
$$=\frac{4\pi^2}{b^2}\Omega[\Phi,\psi:\epsilon_1 bc, -\epsilon_1 b^{-1}c]$$

Thus

$$\Omega[\rho(C)\Phi,\psi:\epsilon_1 bc, -\epsilon_1 b^{-1}c]=$$
$$\left(\frac{1}{2}(b\frac{d}{db})^2+b\frac{d}{db}+\frac{8\pi^2}{b^2}\right)\Omega[\Phi,\psi:\epsilon_1 bc, -\epsilon_1 b^{-1}c]\,. \tag{8}$$

We remark that in terms of the coordinates (a_1, a_2) the action of the Casimir operator is given by

$$\Omega[\rho(C)\Phi,\psi:a_1,a_2]=$$
$$\left[\frac{1}{2}(a_1\partial_{a_1}-a_2\partial_{a_2})^2+(a_1\partial_{a_1}-a_2\partial_{a_2})-8\pi^2\frac{a_2}{a_1}\right]\Omega[\Phi,\psi:a_1,a_2]\,. \tag{9}$$

To continue we write

$$\Omega[\Phi,\psi:\epsilon_1 bc, -\epsilon_1 b^{-1}c]=b^{-1/2}\sum_{\epsilon}\phi_\epsilon(b,c)\,.$$

where we have set

$$\phi_\epsilon(b,c)=2^{-1/2}\gamma(\epsilon\epsilon_1,\psi)\exp(\frac{4i\pi\epsilon\epsilon_1\eta}{b})\theta_\epsilon(\epsilon_1 b,c)\,.$$

We get

$$\left(\frac{1}{2}(b\frac{d}{db})^2+b\frac{d}{db}+\frac{8\pi^2}{b^2}\right)\Omega[\Phi,\psi:\epsilon_1 bc, -\epsilon_1 b^{-1}c]$$

$$= \sum_{\epsilon} b^{-1/2}\left(\frac{1}{2}(b\frac{d}{db})^2 + \frac{1}{2}b\frac{d}{db} - \frac{3}{8} + \frac{8\pi^2}{b^2}\right)\phi_\epsilon(b,c)\,.$$

or

$$\Omega[(\rho(C) + \frac{3}{8})\Phi : \epsilon_1 bc, -\epsilon_1 bc^{-1}]$$

$$= \sum_{\epsilon} b^{-1/2}\left(\frac{1}{2}(b\frac{d}{db})^2 + \frac{1}{2}\frac{d}{db} + \frac{8\pi^2}{b^2}\right)\phi_\epsilon(b,c)\,.$$

Now we use the explicit form of $\phi_\epsilon(b,c)$. After simplification we find

$$\sum_{\epsilon} b^{-1/2}2^{-1/2}\gamma(\epsilon\epsilon_1,\psi)\exp\left(\frac{4i\pi\epsilon\epsilon_1\eta}{b}\right)\times$$

$$\left\{\frac{1}{2}b^2\frac{\partial^2}{\partial x^2}\theta_\epsilon(\epsilon_1 b,c) + (\epsilon_1 b - 4\epsilon\eta i\pi)\frac{\partial}{\partial x}\theta_\epsilon(\epsilon_1 b,c)\right\}\,.$$

Let us introduce the differential operators

$$Q_{\epsilon,\psi} := \frac{1}{2}x^2\frac{\partial^2}{\partial x^2} + (x - 4\epsilon\eta i\pi)\frac{\partial}{\partial x}\,. \tag{10}$$

We have just seen that if we write

$$\Omega[\Phi,\psi : \epsilon_1 bc, -\epsilon_1 b^{-1}c] =$$

$$\sum_{\epsilon} b^{-1/2}2^{-1/2}\gamma(\epsilon\epsilon_1,\psi)\exp(\frac{4i\pi\epsilon\epsilon_1\eta}{b})\theta_\epsilon(\epsilon_1 b,c)$$

where the functions $\theta_\epsilon(x,y)$ are smooth functions of compact support on $\mathbb{R}\times\mathbb{R}_+^\times$, then

$$\Omega[(\rho(C) + \frac{3}{8})\Phi,\psi : \epsilon_1 bc, -\epsilon_1 bc^{-1}] =$$

$$\sum_{\epsilon} b^{-1/2}2^{-1/2}\gamma(\epsilon\epsilon_1,\psi)\exp(\frac{4i\pi\epsilon\epsilon_1\eta}{b})Q_{\epsilon,\psi}\theta_\epsilon(c,\epsilon_1 b)\,.$$

Thus for any integer $n \geq 0$,

$$\Omega[(\rho(C) + \frac{3}{8})^n\Phi,\psi : \epsilon_1 bc, -\epsilon_1 bc^{-1}] =$$

$$\sum_{\epsilon} b^{-1/2}2^{-1/2}\gamma(\epsilon\epsilon_1,\psi)\exp(\frac{4i\pi\epsilon\epsilon_1\eta}{b})Q^n_{\epsilon,\psi}\theta_\epsilon(c,\epsilon_1 b)\,. \tag{11}$$

We point out a simple property of the operators $Q_{\epsilon,\psi}$.

Lemma 1 *Let $k \neq 0$. Let Q_k be the differential operator*

$$Q_k := \frac{1}{2}x^2 \frac{\partial^2}{\partial x^2} + (x+k)\frac{\partial}{\partial x}\,.$$

For any $j \geq 0$,

$$\frac{\partial^j}{\partial x^j} Q_k = \frac{1}{2}x^2 \frac{\partial^{j+2}}{\partial x^{j+2}} + ((j+1)x + k)\frac{\partial^{j+1}}{\partial x^{j+1}} + \frac{j(j+1)}{2}\frac{\partial^j}{\partial x^j}\,.$$

For any C^∞ function ϕ and any $n \geq 0$ we have

$$Q_k^n \phi(0) = \sum_{1 \leq r \leq n} c_r^n k^r \frac{\partial^r \phi}{\partial x^r}(0)\,,$$

where the constants c_r^n are independent of k and $c_n^n = 1$.

Proof: The first identity is established by induction on j. The second assertion is trivial for $n = 1$. Thus we may assume $n \geq 2$ and our assertion established for $n-1$. Then

$$Q_k^n \phi(0) = Q_k^{n-1}(Q_k \phi)(0)\,.$$

By the induction hypothesis we get this is

$$\sum_{1 \leq n-1} c_r^{n-1} k^r \frac{\partial^r}{\partial x^r} Q_k \phi(0)\,.$$

Applying the first identity we get

$$\sum_{1 \leq r \leq n-1} c_r^{n-1} k^r \left(k \frac{\partial^{r+1}\phi}{\partial x^{r+1}} + \frac{r(r+1)}{2}\frac{\partial^r \phi}{\partial x^r}(0) \right)\,.$$

The assertion for n follows. $\square$.

Now by Proposition (3)

$$\Omega[(\rho(C) + \frac{3}{8})^n \Phi : w\epsilon c] = Q_{\epsilon,\psi}^n \theta_\epsilon(0, c)\,.$$

By the Lemma this is

$$(-4i\pi\epsilon\eta)^n \frac{\partial^n \theta_\epsilon(0,c)}{\partial x^n} + \sum_{1 \leq r < n} (-4i\pi\epsilon\eta)^r c_r^n \frac{\partial x^r \theta_\epsilon}{\partial^r}(0, c)\,.$$

This implies the following result.

Proposition 4 *For every $n \geq 0$, there is a polynomial P_n of degree n with leading coefficient 1 such that for $\epsilon = \pm 1$, $c > 0$,*

$$\Omega[P_n(\rho(C) + \frac{3}{8})\Phi, \psi : w\epsilon c] = (-4i\pi\epsilon\eta)^n \frac{\partial^n \theta_\epsilon}{\partial x^n}(0, c)\,.$$

Thus we see that the functions $\theta_\epsilon(x, c)$ are not unique but their derivatives

$$\frac{\partial^r \theta_\epsilon(x, c)}{\partial x^r}$$

have uniquely determined values at $x = 0$.

The following lemma implies that these derivatives are arbitrary.

Lemma 2 *Let ϕ_n, $n \geq 0$, be a sequence of functions in $\mathcal{C}_c^\infty(\mathbb{R}^\times)$. There is a function $\Phi \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ such that, for all $n \geq 0$,*

$$\Omega[\rho(C^n)\Phi, \psi : wz] = \phi_n(z)\,.$$

Proof: Let U be the open set of matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $b \neq 0$ and $\Delta_2(g) < 0$.

Every matrix $g \in U$ can be written uniquely in the form

$$g = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} p\,,\; p = \begin{pmatrix} r & 0 \\ y & r^{-1} \end{pmatrix}$$

with $r > 0$, $z = \text{sign}(b)\sqrt{-\Delta_2(g)}$. The matrix g is in S if and only if $r = 1$ and $y = 0$. Let B_1 be the group of matrices of the form

$$\begin{pmatrix} r & 0 \\ y & r^{-1} \end{pmatrix}\,,\; r > 0\,.$$

The map

$$(x, z, p) \mapsto \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} p$$

gives a diffeomorphism

$$\mathbb{R} \times \mathbb{R}^\times \times B_1 \to U\,.$$

We write

$$C = 2X_+X_- - H + \frac{H^2}{2}\,.$$

Then, for any function Φ

$$\Omega[\rho(C)\Phi, \psi : wz] = \Omega[\rho(C_1)\Phi, \psi : wz]$$

where

$$C_1 = -4i\pi X_- - H + \frac{H^2}{2} .$$

More generally, since C is in the center of the enveloping algebra,

$$\Omega[\rho(C^n)\Phi, \psi : wz] = \Omega[\rho(C_1^n)\Phi, \psi : wz]$$

Now C_1 is an element of the enveloping algebra of the group B_1. Moreover

$$C_1^n = \sum_{0\leq i\leq n, 0\leq j\leq n-1} c_{i,j} X_-^i H^j + \frac{H^{2n}}{2^n} .$$

Given a sequence $\theta_n(x, z)$, $n \geq 0$ of smooth functions of compact support on $\mathbb{R} \times \mathbb{R}^\times$, there is a smooth function Φ of compact support on U such that

$$\rho(\Omega_1^n)\Phi \left[\begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right] = \theta_n(x, z) .$$

Indeed, introducing appropriate coordinates we see that we need to find a function $\Phi(x, z, u, v)$ of compact support on $\mathbb{R} \times \mathbb{R}^\times \times \mathbb{R} \times \mathbb{R}$ such that

$$\sum_{0\leq i\leq n, 0\leq j\leq n-1} c_{i,j} \frac{\partial^{i+j}}{\partial^i u \partial^j v} \Phi(x, z, 0, 0) + \frac{1}{2^n} \frac{\partial^{2n}}{\partial^{2n} v} \Phi(x, z, 0, 0) = \theta_n(x, z) .$$

This follows from Borel's Theorem (See [4], Theorem 1.2.6, page 16.).

We take θ_n to be of the form

$$\theta_n(x, z) = \phi_n(z)\mu_n(x) , \int \mu_n(x)\psi(x)dx = 1 .$$

Then

$$\Omega[\rho(C)^n\Phi, \psi : wz] = \Omega[\rho(C_1)^n\Phi, \psi : wz] = \int \theta_n(x, z)\psi(x)dx = \phi_n(z) . \square$$

5 Flat orbital integrals

If follows from the previous section that if

$$\Omega[\rho(C)^n\Phi, \psi : wz] = 0$$

for all $n \geq 0$ and all $z \in \mathbb{R}^\times$ then the functions $\theta_\epsilon(x,c)$ have the property that, for all $n \geq 0$,

$$\left.\frac{\partial^n \theta_\epsilon(x,c)}{\partial x^n}\right|_{x=0} = 0\,.$$

The products

$$\exp\left(\frac{4i\pi\epsilon}{x}\right)\theta_\epsilon(x,c)$$

are smooth functions with the same property. We arrive at the following result.

Proposition 5 *Suppose that*

$$\Omega[\rho(C)^n\Phi, \psi : wz] = 0$$

for all $n \geq 0$ and all $z \in \mathbb{R}^\times$. Then there is a smooth function of compact support θ on $\mathbb{R} \times \mathbb{R}_+^\times$ such that

$$\frac{\partial^r \theta}{\partial x^r}(0,c) = 0$$

for all $c > 0$ and all $r \geq 0$, and

$$\Omega[\Phi, \psi : \epsilon_1 bc, -\epsilon_1 b^{-1} c] = \theta(\epsilon_1 b, c)$$

for $c > 0$, $b > 0$, $\epsilon_1 = \pm 1$.

We have a converse.

Proposition 6 *Let $\theta(x,c)$ be a smooth function of compact support on $\mathbb{R} \times \mathbb{R}_+^\times$ such that*

$$\frac{\partial^n \theta}{\partial^n x}(0,c) = 0$$

for all $c > 0$ and all $n \geq 0$. Then there is a smooth function of compact support Φ such that

$$\Omega[\Phi, \psi : xc, -x^{-1}c] = \theta(x,c)\,.$$

Proof: Assuming Φ supported on U we set

$$\Phi_1(g) = \int \Phi\left[g\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]\psi(x)dx\,.$$

Then

$$\Phi_1\left[g\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right] = \Phi_1(g)\psi(-x)\,.$$

Moreover

$$(a_1, a_2, x) \mapsto \Phi_1 \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a_2 \\ a_1 & 0 \end{pmatrix} \right]$$

is a smooth function of compact support on $\mathbb{R}^\times \times \mathbb{R}^\times \times \mathbb{R}$. Then

$$\Omega \left[\Phi, \psi : \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right] = \int \Phi_1 \left[\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right] \psi(y) dy \, .$$

This becomes:

$$\int \Phi_1 \left[\begin{pmatrix} 1 & \frac{1}{y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{a_2}{y} \\ y a_1 & 0 \end{pmatrix} \right] \psi \left(y - \frac{a_2}{y a_1} \right) dy \, .$$

or

$$\Omega \left[\Phi, \psi : \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right] =$$

$$|a_1|^{-1} \int \Phi_1 \left[\begin{pmatrix} a_1 & \frac{-\Delta_2}{y} \\ y & 0 \end{pmatrix} \right] \psi \left(\frac{y - \frac{\Delta_2}{y}}{a_1} \right) dy \, .$$

With $a_1 = xc, a_2 = -x^{-1} c, c > 0$ we want

$$|xc|^{-1} \int \Phi_1 \left[\begin{pmatrix} xc & \frac{c^2}{y} \\ y & 0 \end{pmatrix} \right] \psi \left(\frac{y + \frac{c^2}{y}}{xc} \right) dy = \theta(x, c) \, .$$

After a change of variables this becomes

$$\int \Phi_1 \left[\begin{pmatrix} x & \frac{1}{y} \\ y & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right] \psi \left(\frac{y + \frac{1}{y}}{x} \right) dy = |x| \theta(x, c) \, .$$

Note that

$$(x, y, c) \mapsto \Phi_1 \left[\begin{pmatrix} x & \frac{1}{y} \\ y & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right]$$

is an arbitrary smooth function of compact support on

$$\mathbb{R} \times \mathbb{R}^\times \times \mathbb{R}_+^\times \, .$$

Let $\phi_0(y)$ be a smooth function of compact support on $\mathbb{R}^\times$ such that

$$\int \phi_0(y) dy = 1 \, .$$

We take

$$\Phi_1 \left[\begin{pmatrix} x & \frac{1}{y} \\ y & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right]$$

$$= |x|\theta(x,c)\psi\left(-\frac{y+\frac{1}{y}}{x}\right)\phi_0(y)\,.$$

Because of the condition the function $\theta(x,c)$ is divisible by x^n for every $n > 0$. Thus this is indeed a smooth function which has the required property. □

REMARK: it is easy to obtain the asymptotic of the orbital integral of a function supported on U. Since $G(\mathbb{R})$ is a finite union of sets ${}^t n_i U$ with $n_i \in N(\mathbb{R})$ this gives another proof of Proposition (3).

6 The case $\Delta_2 > 0$

We simply record the result. The proof follows easily from Proposition (2).

Proposition 7 *Suppose Φ is a smooth function with compact support contained in the set $\Delta_2 > 0$. Then there is a smooth function of compact support θ on $\mathbb{R} \times \mathbb{R}_+^\times$ such that*

$$\frac{\partial^r}{\partial x^r}\theta(0,c) = 0$$

for all $c > 0$ and all $r \geq 0$ and

$$\Omega[\Phi,\psi : \epsilon_1 bc, \epsilon_1 b^{-1} c] = \theta(\epsilon_1 b, c)$$

for $c > 0$, $b > 0$, $\epsilon_1 = \pm 1$. Conversely, if θ is such a function, then there is a smooth function Φ with compact support contained in the set $\Delta_2 > 0$ such that the above relation holds.

7 Orbital integrals for H

Let Ψ be a smooth function of compact support on H. Then

$$\Omega[\Psi,\mathbb{C}/\mathbb{R},\psi : a_1,a_2] :=$$

$$\int \Psi\left[\begin{pmatrix}1 & 0\\ \overline{z} & 1\end{pmatrix}\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right]\psi_{\mathbb{R}}(z+\overline{z})dz\,.$$

We will study the integrals for $a_1a_2 < 0$. Our goal in this section is the following result.

Proposition 8 *For any smooth function of compact support on H*

$$\Omega[\Psi,\psi : \epsilon_1 bc, -\epsilon_1 b^{-1} c] =$$

$$\epsilon_1 b^{-1/2} 2^{-1/2} \sum_{\epsilon \pm 1} \psi\left(\frac{2\epsilon\epsilon_1}{b}\right) \gamma(\epsilon\epsilon_1, \psi)\theta_\epsilon(\epsilon_1 b, c) \tag{12}$$

where θ_ϵ are smooth functions of compact support on $\mathbb{R} \times \mathbb{R}_+^\times$. For any two such functions

$$\theta_\epsilon(0, c) = -\epsilon\Omega[\Psi, \psi : wc\epsilon]\,.$$

Proof: We introduce the partial Fourier transform

$$\Psi_1 \begin{pmatrix} a & z \\ \overline{z} & b \end{pmatrix} := \int \Psi \begin{pmatrix} a & z \\ \overline{z} & y \end{pmatrix} \psi_{\mathbb{R}}(-yb)dy$$

Then by Fourier inversion formula we get

$$\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2] =$$

$$|a_1|^{-2} \int\int \Psi_1 \begin{pmatrix} a_1 & z \\ \overline{z} & y \end{pmatrix} \psi\left(\frac{z + \overline{z} + y(\Delta_2 + z\overline{z})}{a_1}\right) dydz\,.$$

As before, we choose a partition of unity

$$\phi_1 + \phi_2 = 1$$

on $\mathbb{R}$ with ϕ_1 supported on a small neighborhood of 0. We assume that $\phi_2\left(\pm\frac{1}{\sqrt{-\Delta_2}}\right) = 1$.

Then $\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2]$ is the sum of two integrals $i = 1, 2$

$$\Omega_i[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2] =$$

$$|a_1|^{-2} \int\int \Psi_1 \begin{pmatrix} a_1 & z \\ \overline{z} & y \end{pmatrix} \psi\left(\frac{z + \overline{z} + y(\Delta_2 + z\overline{z})}{a_1}\right) \phi_i(y)dydz\,.$$

We choose the support of ϕ_1 so small that, in the first integral, writing $z = u + iv$, $1 + yu$ remains in a compact set of $\mathbb{R}^\times$. We then use the following variables of integration:

$$U = u + \frac{y(u^2 + v^2)}{2}, V = v, Z = U + iV, Y = y\,.$$

Indeed

$$\frac{\partial U}{\partial u} = 1 + yu \neq 0$$

Thus the Jacobian determinant $\frac{\partial(U,V,Y)}{\partial(u,v,y)}$ is non-zero. Moreover, we can compute u, v, y in terms of U, V, Y by

$$u = \frac{-1 + \sqrt{1 + Y(2U - YV^2)}}{Y}, v = V, y = Y\,.$$

Thus this is a legitimate change of coordinates. The integral takes the form

$$\int f(Z,Y)\psi\left(\frac{Z+\overline{Z}+Y\Delta_2}{a_1}\right)dZdY\,.$$

This has the required form with $\theta_\epsilon(0,c)=0$.

Thus we have to study

$$|a_1|^{-2}\int\Psi_1\begin{pmatrix}a_1 & z\\ \overline{z} & y\end{pmatrix}\psi\left(\frac{z+\overline{z}+y(\Delta_2+z\overline{z})}{a_1}\right)\phi_2(y)dydz\,.$$

We recall the elementary formula

$$\int_{\mathbb{C}} f(z)\psi\left(\frac{z+\overline{z}}{x}\right)dz = x\int_{\mathbb{C}}\hat{f}(z)\psi(-x(z+\overline{z}))dz \tag{13}$$

(which can derived from (1) applied twice) where

$$\hat{f}(z)=\int_{\mathbb{C}} f(u)\psi_{\mathbb{C}}(-zu)du\,.$$

To apply it we introduce a new Fourier transform

$$\Psi_2\begin{pmatrix}a & z\\ \overline{z} & b\end{pmatrix}=\int\Psi_1\begin{pmatrix}a & u\\ \overline{u} & b\end{pmatrix}\psi_{\mathbb{C}}(-uz)du\,.$$

Then we find the previous integral is equal to

$$sign(a_1)|a_1|^{-1}\int\Psi_2\begin{pmatrix}a_1 & z-\frac{1}{a_1}\\ \overline{z}-\frac{1}{a_1} & y\end{pmatrix}\psi\left(\frac{y\Delta_2}{a_1}-\frac{a_1}{y}z\overline{z}\right)\phi_2(y)\frac{dy}{y}dz\,.$$

After a change of variables this becomes

$$sign(a_1)|a_1|^{-1}\int\Psi_2\begin{pmatrix}a_1 & z\\ \overline{z} & y\end{pmatrix}\psi\left(\frac{y\Delta_2-\frac{1}{y}}{a_1}-\frac{a_1z\overline{z}+z+\overline{z}}{y}\right)\phi_2(y)\frac{dy}{y}dz\,.$$

At this point we set

$$\phi(y,a_1):=\int\Psi_2\begin{pmatrix}a_1 & z\\ \overline{z} & y\end{pmatrix}\psi\left(-\frac{z+\overline{z}+a_1z\overline{z}}{y}\right)\frac{\phi_2(y)}{y}dz\,.$$

This is a Schwartz function supported on a set

$$|y|\geq C_1\,,\ |a_1|\leq C_2\,.$$

In addition

$$\phi(y,0)=\int_{\mathbb{C}}\Psi_2\begin{pmatrix}0 & z\\ \overline{z} & y\end{pmatrix}\psi\left(-\frac{z+\overline{z}}{y}\right)dz\frac{\phi_2(y)}{y}$$
$$=y^{-1}\phi_2(y)\Psi_1\begin{pmatrix}0 & -y^{-1}\\ -y^{-1} & y\end{pmatrix}$$
$$=y^{-1}\phi_2(y)\int\Psi\begin{pmatrix}0 & -y^{-1}\\ -y^{-1} & x\end{pmatrix}\psi(-xy)dx$$
$$=sign(y)y^{-2}\Omega[\Psi,\mathbb{C}/\mathbb{R},\psi:-y^{-1}w]\,.$$

Our integral takes the form

$$\Omega_2[\Psi,\mathbb{C}/\mathbb{R},\psi:\epsilon_1 bc,-\epsilon_1 b^{-1}c]$$
$$=\epsilon_1 b^{-1}c^{-1}\int\phi(y,\epsilon_1 bc)\psi\left(\frac{-c^2y-\frac{1}{y}}{\epsilon_1 bc}\right)dy$$
$$\epsilon_1 b^{-1}c^{-2}\int\phi(-c^{-1}y,\epsilon_1 bc)\psi\left(\frac{y+y^{-1}}{\epsilon_1 b}\right)dy$$

By Proposition (1) this has the form

$$\epsilon_1 b^{-1/2}c^{-2}2^{-1/2}\sum_{\epsilon=\pm1}\psi\left(\frac{2\epsilon\epsilon_1}{b}\right)\gamma(\epsilon\epsilon_1,\psi)\theta_\epsilon(\epsilon b,c)$$

where $\theta_\epsilon(x,y)$ are smooth functions of compact support on $\mathbb{R}\times\mathbb{R}_+^\times$ such that

$$\theta_\epsilon(0,c)=\phi(-c^{-1}\epsilon,0)=$$
$$-\epsilon c^2\phi_2(-c^{-1}\epsilon)\Omega[\Psi,\mathbb{C}/\mathbb{R},\psi:wc\epsilon]=-\epsilon c^2\Omega[\Psi,\mathbb{C}/\mathbb{R},\psi:wc\epsilon]\,.$$

After a change of notations we arrive at the Proposition. □.

8 Comparison of the action of the Casimir operators

We again assume $\psi_{\mathbb{R}}(x)=\exp(2i\pi\eta x)$, $\eta=\pm1$. Now view $\mathfrak{sl}(2,\mathbb{C})$ as real vector space. Consider the bilinear form

$$\beta(X,Y)=\mathrm{Re}(\mathrm{Tr}(XY))\,.$$

Then the dual basis of

$$H,K=:\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix},X_+,X_-,X'_+=\begin{pmatrix}0 & 0\\ i & 0\end{pmatrix},X'_-=\begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix}\tag{14}$$

is

$$\frac{H}{2}, -\frac{K}{2}, X_-, X_+, -X'_-, -X'_+ \,.$$

Thus the element

$$C_c := \frac{H^2}{2} - \frac{K^2}{2} + X_+X_- + X_-X_+ - X'_+X'_- - X'_-X'_+$$

is in the center of the enveloping algebra of $GL(2,\mathbb{C})$ viewed as a real Lie group. It can be written as

$$C_c = \frac{H^2}{2} + 2H - \frac{K^2}{2} + 2X_-X_+ - 2X'_-X'_+ \,.$$

The group $GL(2,\mathbb{C})$ operates on the space of Hermitian matrices by $s \mapsto {}^t\overline{g}sg$. We have a corresponding action of the enveloping algebra on the space of smooth functions of compact support. We denote by σ this action. Thus if X is in the Lie algebra then

$$\sigma(X)\Psi(s) = \Psi(\exp(t\,{}^t\overline{X})s\exp(tX))\big|_{t=0} \,.$$

We wish to compute

$$\Omega[\rho(C_c)\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2]$$

as the application of a differential operator to the function

$$\omega(a_1, a_2) := \Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2] \,.$$

Since C_c is in the center of the enveloping algebra, it amounts to the same to apply the left invariant differential operator $\rho(C_c)$ to the function

$$f(g) := \int \Psi[{}^t\overline{u}\,{}^t\overline{g}agu]\psi(z+\overline{z})dz$$

where

$$a = \mathrm{diag}(a_1, a_2)\,,\; u = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

and then evaluate at $g = 1$.

Applying H we get

$$\frac{d}{dt}\omega(a_1e^{2t}, a_2e^{-2t})\Big|_{t=0} = 2(a_1\partial_{a_1} - a_2\partial_{a_2})\omega(a_1, a_2) \,.$$

Now

$$\exp(t\,{}^t\overline{K})a\exp(tK) = \exp(tK)^{-1}a\exp(tK) = a \,.$$

Thus the contribution of K and K^2 is 0. Thus the contributions of the terms in C_c containing H or K is

$$4\left[\frac{1}{2}(a_1\partial_{a_1}-a_2\partial_{a_2})^2+(a_1\partial_{a_1}-a_2\partial_{a_2})\right]\omega(a_1,a_2)\,.$$

Now we compute the contribution of X_-X_+. The value of $\rho(X_-X_+)f$ at $g=e$ is obtained by differentiating

$$\int\Psi\left[{}^t\overline{u}\begin{pmatrix}1\,0\\ s\,1\end{pmatrix}\begin{pmatrix}1\,t\\ 0\,0\end{pmatrix}a\begin{pmatrix}1\,0\\ t\,1\end{pmatrix}\begin{pmatrix}1\,s\\ 0\,0\end{pmatrix}u\right]\psi(z+\overline{z})dz$$

with respect to s,t at $t=s=0$. This reduces at once to

$$-4i\pi\eta\frac{d}{dt}\int\Psi\left[{}^t\overline{u}\begin{pmatrix}1\,t\\ 0\,0\end{pmatrix}a\begin{pmatrix}1\,0\\ t\,1\end{pmatrix}u\right]\psi(z+\overline{z})dz\Big|_{t=0}\,.$$

Now we exploit the relation

$$\begin{pmatrix}1\,t\\ 0\,0\end{pmatrix}a\begin{pmatrix}1\,0\\ t\,1\end{pmatrix}=\begin{pmatrix}a_1+t^2a_2 & ta_2\\ ta_2 & a_2\end{pmatrix}=$$

$$\begin{pmatrix}1 & 0\\ \frac{ta_2}{a_1+t^2a_2} & 1\end{pmatrix}\begin{pmatrix}a_1+t^2a_2 & 0\\ 0 & \frac{a_1a_2}{a_1+t^2a_2}\end{pmatrix}\begin{pmatrix}1 & \frac{ta_2}{a_1+t^2a_2}\\ 0 & 1\end{pmatrix}.$$

We get the derivative at $t=0$ of

$$(-4i\eta\pi)\exp\left(-\frac{4i\eta\pi ta_2}{a_1+t^2a_2}\right)\int\Psi\left[{}^t\overline{u}\begin{pmatrix}a_1+t^2a_2 & 0\\ 0 & \frac{a_1a_2}{a_1+t^2a_2}\end{pmatrix}u\right]\psi(z+\overline{z})dz\,.$$

The above derivative is

$$-16\pi^2\frac{a_2}{a_1}\omega(a_1,a_2)\,.$$

On the other hand the action of $X'_-X'_+$ can be computed similarly. It is the value of $\rho(X_-X_+)f$ at $g=e$. It is is obtained by differentiating

$$\int\Psi\left[{}^t\overline{u}\begin{pmatrix}1\;0\\ is\,1\end{pmatrix}\begin{pmatrix}1\,it\\ 0\;0\end{pmatrix}a\begin{pmatrix}1\;0\\ it\,1\end{pmatrix}\begin{pmatrix}1\,is\\ 0\;0\end{pmatrix}u\right]\psi(z+\overline{z})dz$$

with respect to s,t at $t=s=0$. It is simply 0.

Altogether then we see that

$$\Omega[\sigma(\frac{C_c}{4})\Psi,\psi:a_1,a_2]=$$

$$\left[\frac{1}{2}(a_1\partial_{a_1}-a_2\partial_{a_2})^2+(a_1\partial_{a_1}-a_2\partial_{a_2})-8\pi^2\frac{a_2}{a_1}\right]\Omega[\Psi,\mathbb{C}/\mathbb{R},\psi:a_1,a_2]\,.\qquad(15)$$

The differential operator is the same as the one for $\rho(C)$ (see (9). It follows that the Proposition (4) is true for the integrals $\Omega[\Psi,\mathbb{C}/\mathbb{R},\psi:\bullet]$.

9 Matching

We say that Φ and Ψ **match** for ψ and we write $\Phi \overset{\psi}{\leftrightarrow} \Psi$ if

$$\Omega[\Phi, \psi : a_1, a_2] = sign(a_1)\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2] .$$

It follows from Propositions (3) and (7) that if $\Phi \overset{\psi}{\leftrightarrow} \Psi$ then

$$\Omega[\Phi, \psi : wz] = -sign(z)\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : wz] .$$

Also, by the previous section we see that if $\Phi \overset{\psi}{\leftrightarrow} \Psi$ then

$$\rho(C^n)\Phi \overset{\psi}{\leftrightarrow} \sigma(\frac{C_c}{4}^n)\Psi$$

for all $n \geq 0$.

Proposition 9 *For every $\Psi \in \mathcal{C}_c^\infty(H)$ there is $\Phi \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ such that $\Phi \overset{\psi}{\leftrightarrow} \Psi$*

Proof: Let us write

$$\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : \epsilon_1 bc, -\epsilon_1 b^{-1}c] =$$

$$\epsilon_1 b^{-1/2} 2^{-1/2} \sum_{\epsilon \pm 1} \psi\left(\frac{2\epsilon\epsilon_1}{b}\right) \gamma(\epsilon\epsilon_1, \psi)\theta_\epsilon(\epsilon_1 b, c) .$$

We can find $\Phi \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ such that

$$\Omega[\Phi, \psi : \epsilon_1 bc, -\epsilon_1 b^{-1}c] = b^{-1/2} 2^{-1/2} \sum_{\epsilon \pm 1} \psi\left(\frac{2\epsilon\epsilon_1}{b}\right) \gamma(\epsilon\epsilon_1, \psi)\theta^1_\epsilon(\epsilon_1 b, c) .$$

where

$$\left.\frac{\partial^n \theta^1(x,c)}{\partial x^n}\right|_{x=0} = \left.\frac{\partial^n \theta(x,c)}{\partial x^n}\right|_{x=0}$$

for all $n \geq 0$. Thus

$$\Omega[\Phi, \psi : \epsilon_1 bc, -\epsilon_1 b^{-1}c] - \epsilon_1 \Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : \epsilon_1 bc, -\epsilon_1 b^{-1}c] = \theta^2(\epsilon_1 b, c)$$

where $\theta^2(x,c)$ is a smooth function of compact support such that

$$\left.\frac{\partial^n \theta^2(x,c)}{\partial x^n}\right|_{x=0}$$

for all $n \geq 0$. There is a function Φ_1 supported on the set $\Delta_2 < 0$ such that the difference is equal to

$$\Omega[\Phi_1, \psi : \epsilon_1 bc, -\epsilon_1 b^{-1} c] \,.$$

Likewise there is a function Φ_2 supported on the set $\Delta_2 > 0$ such that

$$\Omega[\Phi, \psi : \epsilon_1 bc, \epsilon_1 b^{-1} c] - \epsilon_1 \Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : \epsilon_1 bc, \epsilon_1 b^{-1} c]$$
$$= \Omega[\Phi_2, \psi : \epsilon_1 bc, \epsilon_1 b^{-1} c] \,.$$

Our assertion follows. $\square$.

However, it is not true any Φ of compact support on $GL(2, \mathbb{R})$ matches a function Ψ of compact support on H. Indeed, consider the orbital integral of a smooth function of compact support Ψ on the space of invertible Hermitian matrices.

$$\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2] = \int \Psi \begin{bmatrix} a_1 & a_1 z \\ a_1 \overline{z} & a_2 + a_1 z \overline{z} \end{bmatrix} \psi_{\mathbb{C}}(z) dz \,.$$

Here the determinant of the matrix in the integrand is $\Delta_2 = a_1 a_2$. Thus Δ_2 remains in a compact set of $\mathbb{R}^\times$. Changing variables we get

$$|a_1|_{\mathbb{R}}^{-2} \int \Psi \begin{bmatrix} a_1 & z \\ \overline{z} & \frac{\Delta_2 + z\overline{z}}{a_1} \end{bmatrix} \psi_{\mathbb{C}} \left(\frac{z}{a_1} \right) dz \,.$$

On the support of the integrand there is $C > 0$ such that

$$|\Delta_2 + z\overline{z}| \leq C |a_1| \,.$$

But if $\Delta_2 > 0$ then

$$\Delta_2 \leq |\Delta_2 + z\overline{z}| \,.$$

Since Δ_2 is in a compact set of $\mathbb{R}^\times$ we have

$$0 < C_1 \leq \Delta_2$$

for a suitable constant C_1 Hence $|a_1| \geq C_1 C^{-1}$. Thus the orbital integral vanishes for $|a_1|$ small enough.

This is not in general the case for the orbital integral of a function Φ smooth of compact support on $GL(2, \mathbb{R})$ supported on the set $\Delta_2 > 0$. Indeed the orbital integral has the form $\theta(a_1, \Delta_2)$ where $\theta(x, y)$ is a smooth function of compact support on $\mathbb{R} \times \mathbb{R}_+^\times$ such that

$$\left. \frac{\partial^n \theta(x, y)}{\partial x^n} \right|_{x=0}$$

for all $n \geq 0$.

However, for every Φ supported on the set $\{g \in G(\mathbb{R}) : \Delta_2 < 0\}$ there is a function Ψ with matching orbital integrals. Indeed, arguing as before, it suffices to prove the following lemma.

Lemma 3 *Given a function θ, smooth of compact support on $\mathbb{R} \times \mathbb{R}_+^\times$ and such that*

$$\left.\frac{\partial^n \theta(x,y)}{\partial x^n}\right|_{x=0}$$

for all $n \geq 0$ there is a function Ψ supported on $\{s \in H : \Delta_2 s < 0\}$ such that

$$\Omega[\Psi, \mathbb{C}/\mathbb{R}, \psi : a_1, a_2] = \theta(a_1, \Delta_2)\,.$$

PROOF: Indeed the orbital integral of a function Ψ can be written

$$\int \Psi \begin{pmatrix} a_1 & x+iy \\ x-iy & \frac{x^2-(c^2-y^2)}{a_1} \end{pmatrix} \psi\left(\frac{2x}{a_1}\right) dxdy\,,$$

where $c = \sqrt{-\Delta_2}$. For a suitable choice of Ψ this can be written

$$|a_1|^{-2} \int \phi_1(a_1,x,c)\phi_2(y)\phi_3\left(\frac{x^2-(c^2-y^2)}{a_1}\right)\psi\left(\frac{2x}{a_1}\right) dxdy\,,$$

where the functions ϕ_i have compact support and the projection of the support of ϕ_1 on the last factor is a compact set of $\mathbb{R}_+^\times$. Now take

$$\phi_1(a_1,x,c) = \psi\left(-\frac{2x}{a_1}\right)\phi_4(a_1,c)\phi_5(x)$$

where the partial derivatives of ϕ_4 with respect to the first variable vanish at $(0,c)$. Then the integral takes the form

$$|a_1|^{-2} \int \phi_4(a_1,c)\phi_5(x)\phi_2(y)\phi_3\left(\frac{x^2-(c^2-y^2)}{a_1}\right) dxdy\,.$$

We take $\phi_2(y)$ supported on a small neighborhood of 0 so that $c^2 - y^2$ remains in a compact set of $\mathbb{R}_+^\times$. We also assume that $\phi_5(x)$ is supported on a compact set of $\mathbb{R}_+^\times$. We set

$$x = \sqrt{c^2 - y^2 + ta_1}\,.$$

Then the integral becomes

$$2^{-1}|a_1|^{-1} \int \phi_1(a_1,c)\phi_4(\sqrt{c^2-y^2+ta_1})\phi_2(y)\phi_3(t)\frac{1}{\sqrt{c^2-y^2+ta_1}}\, dtdy\,.$$

We may choose the support of ϕ_2 and ϕ_3 to be small neighborhoods of 0 and then choose ϕ_4 so that

$$\phi_4(\sqrt{c^2 - y^2 + ta_1})\phi_2(y)\phi_3(t)\frac{1}{\sqrt{c^2 - y^2 + ta_1}} = \phi_2(y)\phi_3(t)\,.$$

Then the integral becomes

$$2^{-1}|a_1|^{-1}\phi_1(a_1, c)\int \phi_2(y)\phi_3(t)\ dtdy\,.$$

If we take

$$\phi_1(a_1, c) = 2|a_1|\theta(a_1, \Delta_2)\,, \int \phi_2(y)\ dy = 1\,, \int \phi_3(t)\ dt = 1$$

we obtain our assertion. □

10 A lemma for Bessel distributions on Hermitian matrices

If Ψ is supported on the set $\Delta_1 \neq 0$ then its orbital integral, viewed as as a function of (a_1, Δ_2), is simply a smooth function of compact support on $\mathbb{R}^\times \times \mathbb{R}^\times$. Likewise for the orbital integral of a function Φ supported on the set $\Delta_1 \neq 0$. Such functions are easily matched. Thanks to the following Proposition, for some applications, we may be able to restrict ourself to functions of this type.

If Ψ is a function on H we set, for $g \in G(\mathbb{C})$,

$$\sigma(g)\Psi(s) = \Psi({}^t\overline{g}sg)\,.$$

If

$$n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

we set

$$\theta(n) = \psi(u + \overline{u})$$

and write $n = n_u$. If Ω is an orbital integral then

$$\Omega(\sigma(n)\Psi) = \theta(n)^{-1}\Omega(\Psi)\,.$$

Proposition 10 *Suppose μ is a distribution on H such that*

$$\mu(\sigma(n)\Psi) = \theta(n)^{-1}\mu(\Psi)$$

for all f and all $n \in N(\mathbb{C})$. Suppose that

$$\mu(\sigma(C_c)f) = c\mu(f)\,.$$

If the restriction of μ to the open set $\{s \in H : \Delta_1(s) \neq 0\}$ is zero then μ is 0.

Proof: Let Z be the set $\Delta_1 = 0$. Thus Z is the subvariety of matrices of the form

$$\begin{pmatrix} 0 & b \\ \bar{b} & d \end{pmatrix}, b \in \mathbb{C}^\times, d \in \mathbb{R}\,.$$

We first prove a lemma.

Lemma 4 *Let μ be a distribution such that*

$$\mu(\sigma(n)\Psi) = \theta(n)^{-1}\mu(\Psi)$$

for all Ψ and all $n \in N(\mathbb{C})$. Suppose that μ is supported on Z. Then in fact μ is supported on the subvariety Z_1 of matrices of the form

$$\begin{pmatrix} 0 & b \\ b & d \end{pmatrix}, b \in \mathbb{R}^\times\,,\; d \in \mathbb{R}\,.$$

Proof: Indeed let B be the group of upper triangular matrices. Then Z is one orbit of the group $B(\mathbb{C})$ and $N(\mathbb{C})$ is a normal subgroup of $B(\mathbb{C})$. For any $z \in Z$ let M_z^1 be the normal tangent space to Z , that is, the quotient

$$T(H,z)/T(Z,z)\,.$$

Let $M_z^{(r)}$ the $r-$th symmetric power of M_z^1. Let N^z be the stabilizer of z in $N(\mathbb{C})$. Since N^z leaves Z invariant it operates on $T(H,z)$, $T(Z,z)$ and the quotient M_z^1. Thus it operates also on the $r-$th symmetric power $M_z^{(r)}$. Now n_u is in N^z if and only if

$$\overline{u}b + \bar{b}u = 0\,.$$

This is a real vector space of dimension 1. Calling $\phi(u)$ the action of n_u on the space of Hermitian matrices we see that $\phi(u)X$ is polynomial in (u,X). Hence the linear tangent map $d\phi(u)_z$ is a polynomial function of u and so are the linear maps induced on $T(Z,z)$, M_z^1 and M_z^r. In particular a common eigenvector in M_z^r of these linear maps is actually an invariant vector. On the other hand, by a result of Kolk and Varadarajan ([7]), the support of μ is contained in the set of z such that for some r there is a non-zero vector of M_z^r transforming under the character θ restricted to N^z. By the previous observation for such a z the restriction of θ to N^z must be trivial. This means that that

$$\overline{u}b + \overline{b}u = 0 \Rightarrow u + \overline{u} = 0\,.$$

This is equivalent to $b = \overline{b}$ which proves the lemma. $\square$.

Now we go back to the proof of the Proposition. Recall the Casimir operator is given by

$$C_c := \frac{H^2}{2} - \frac{K^2}{2} + X_+X_- + X_-X_+ - X'_+X'_- - X'_-X'_+\,.$$

It can also be written

$$C_c = \frac{H^2}{2} - 2H - \frac{K^2}{2} + 2X_+X_- - 2X'_+X'_-\,.$$

In view of the invariance property of μ we have for any function f

$$\mu(\sigma(X_+f)) = k\mu(f)$$

with $k \neq 0$ and

$$\mu(\sigma(X'_+)f) = 0\,.$$

Thus the second condition on μ reads

$$\mu\left(\left(\frac{\sigma(H)^2}{2} - 2\sigma(H) - \frac{\sigma(K)^2}{2}\right)f\right) = -k\mu\left(\sigma(X_-)f\right)\,.$$

However the vector fields $\sigma(H)$ and $\sigma(K)$ are tangential to the variety Z while the vector field $\sigma(X_-)$ is transverse. Indeed to say that a vector field Ξ is tangential to the submanifold Z means that if $f = 0$ on Z then $\Xi f = 0$ on Z. Let us look at $\sigma(H)$.

$$\sigma(H)f(s) = \frac{d}{dt}f\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} s \begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\right)\Big|_{t=0}\,.$$

In particular

$$\sigma(H)f\begin{pmatrix} 0 & a \\ \overline{a} & z\end{pmatrix} = \frac{d}{dt}f\begin{pmatrix} 0 & a \\ \overline{a} & ze^{-2t}\end{pmatrix}\Big|_{t=0}\,.$$

Thus $\sigma(H)$ is certainly tangential to Z. Likewise

$$\sigma(K)f(s) = \partial_t f\left(\begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it}\end{pmatrix} s \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{pmatrix}\right)\Big|_{t=0}\,.$$

In particular

$$\sigma(K)f\begin{pmatrix}0 & a\\ \overline{a} & z\end{pmatrix} = \frac{d}{dt}f\begin{pmatrix}0 & ae^{-2it}\\ \overline{ae^{-2it}} & z\end{pmatrix}\Bigg|_{t=0}.$$

Thus $\sigma(H)$ is certainly tangential to Z.

A vector field Ξ is transversal to Z at a point $z \in Z$ if there is a function f which vanishes on Z but $\Xi f(z)$ does not vanish at z. The vector field $\sigma(X_-)$ is transverse at any point

$$z = \begin{pmatrix}0 & b\\ \overline{b} & d\end{pmatrix}$$

such that $b + \overline{b} \neq 0$. Indeed using the coordinates a, x, y, d in

$$\begin{pmatrix}a & x+iy\\ x-iy & d\end{pmatrix}$$

we have

$$\sigma(X_-)\Psi\begin{pmatrix}0 & x+iy\\ x-iy & d\end{pmatrix} = \frac{d}{dt}\Psi\begin{pmatrix}2tx+t^2d & td+x+iy\\ td+x-iy & d\end{pmatrix}\Bigg|_{t=0}$$

$$= (2x\frac{\partial}{\partial a} + d\frac{\partial}{\partial x})\Psi\begin{pmatrix}0 & x+iy\\ x-iy & d\end{pmatrix}.$$

We can choose Ψ so that Ψ vanishes on the subvariety $a = 0$ but $\frac{\partial\Psi}{\partial a}$ does not vanish on the subvariety at the point $\begin{pmatrix}0 & x+iy\\ x-iy & d\end{pmatrix}$ with $x \neq 0$. Thus $\sigma(X_-)$ is transversal at this point.

It is an observation of Shalika ([8]) that a transverse derivative of a distribution supported on Z cannot be equal to a linear combination of tangential derivatives of the distribution. It follows that μ vanishes on the open set $x \neq 0$. It is thus supported on the closed subvariety Z_0 defined by $x = 0, a = 0$, that is the subvariety of matrices of the form

$$\begin{pmatrix}0 & iy\\ -iy & d\end{pmatrix}.$$

On the other hand, the distribution is supported on Z_1 by the lemma. Since $Z_0 \cap Z_1 = \emptyset$ the distribution is indeed 0. □

We note that the analogous resul for distributions on $GL(2,\mathbb{R})$ and more general groups has been proved by Baruch.

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