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Equivariant sheaves on loop spaces

Sergey Arkhipov and Sebastian Ørsted

Let X be an affine, smooth, and Noetherian scheme over $\mathbb{C}$ acted on by an affine algebraic group G. Applying the technique developed in [3, 4], we define a dg-model for the derived category of dg-modules over the dg-algebra of differential forms Ω_X on X equivariant with respect to the action of a derived group scheme (G, Ω_G). We compare the obtained dg-category with the one considered in [2] given by coherent sheaves on the derived Hamiltonian reduction of T^*X.

1. Introduction

Let G be an affine algebraic group acting on an affine, smooth, and Noetherian scheme X over $\mathbb{C}$. In the recent paper [2], the first author jointly studied the category of quasi-coherent sheaves on the derived Hamiltonian reduction of T^*X with respect to G. As stated in the introduction to that paper, this category is supposed to play the role of a category of equivariant sheaves on the derived loop space of the scheme X. The present paper is a step towards making this intuition into a precise construction.

Omitting equivariance, the picture is well understood. Namely, in homotopy theory, the free loop space of an object X (say, in a model category) is the homotopy fibre product of X with itself over $X \times X$. Thus, for regular schemes, the derived loop space of a scheme X is given by the sheafified Hochschild homology dg-algebra for $\mathcal{O}_X$.

The famous Hochschild–Kostant–Rosenberg theorem states that the latter is formal: it is quasi-isomorphic to its cohomology given by the sheaf of dg-algebras of differential forms on X with zero differential. Denote it by Ω_X. It follows that talking about sheaves on the derived loop space of X is essentially considering quasi-coherent sheaves of dg-modules over Ω_X. One of the goals of the present paper is to make precise sense of the derived category of Ω_X-dg-modules equivariant with respect to the action of the derived group scheme (G, Ω_G).

Our strategy is as follows. We consider the simplicial derived scheme given by the nerve of the action groupoid for (G, Ω_G) acting on (X, Ω_X).

This way, we obtain a cosimplicial diagram of dg-derived categories. We consider the homotopy totalization of this diagram. More precisely, we follow the strategy of [5] generalized and extended in our previous papers [3, 4]. We obtain a category of A_∞-comodules over a certain dg-coalgebra in the category of Ω_X-dg-modules. This category becomes our model for the derived category of Ω_X-dg-modules equivariant with respect to Ω_G.

Applying linear Koszul duality in the spirit of the papers of Mirkovic and Riche, we pass from Ω_X-dg-modules to the "even" side of the duality. We consider the sheaf of dg-algebras on X given by $\mathrm{Sym}_{\mathcal{O}_X}(T_X[-2])$ and quasi-coherent dg-modules over it. We call the obtained category the derived category of quasi-coherent sheaves on the *homologically shifted* cotangent bundle of X.

After applying Koszul duality, the (G, Ω_G)-equivariance becomes a more subtle structure. Notice that (G, Ω_G) contains $(G, \mathcal{O}_G)$ both as a subgroup and as a quotient group. The Koszul duality construction respects the $(G, \mathcal{O}_G)$-action. The remaining coaction of the Hopf dg-algebra of left invariant differential forms on G becomes the following structure. Denote the Lie algebra of G by $\mathfrak{g}$. The moment map for the G-action on T^*X provides a G-equivariant map $\mathfrak{g} \otimes \mathcal{O}_X[-1] \to T_X$. Consider the free graded commutative algebra generated by the two term complex

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{g} \otimes \mathcal{O}_X[-1] \to T_X[-2]).$$

Notice that up to the homological shift, this dg-algebra is a model for functions on the derived preimage of 0 under the moment map $\mu\colon T^*X \to \mathfrak{g}^*$. We come to the central statement of the present paper:

5.0.2. Theorem. *Let X ba an affine, smooth, and Noetherian scheme over $\mathbb{C}$ acted on by an affine group scheme G. The derived category of (G, Ω_G)-equivariant Ω_X-dg-modules is equivalent to the triangulated subcategory*

$$\langle \mathcal{O}_X \rangle \subset \mathcal{D}\big(\mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{g} \otimes \mathcal{O}_X[-1] \to T_X[-2])\text{-dgmod}\big)^G$$

of the derived category of G-equivariant dg-modules generated by $\mathcal{O}_X$ and closed under small coproducts.

Notice that the case of X equal to a point is of interest. In this case, the statement, combined with the usual Koszul duality between $\mathrm{Sym}(\mathfrak{g}[-1])$ and $\mathrm{Sym}(\mathfrak{g}^*)$, reads as follows:

Corollary. *The derived category of representations for the group* (G, Ω_G) *is equivalent to the derived category of* G*-equivariant quasicoherent sheaves on* $\mathfrak{g}$ *topologically supported at* 0.

2. Derived categories and A_∞-modules

In this chapter, we discuss and compare different candidates for derived categories of dg-modules over a dg-algebra A.

2.1. The dg-derived category

Let k be a field of characteristic zero. If A is a dg-algebra over k, recall that its **derived dg-category** is defined to be the Drinfeld localization $\mathcal{D}(A\text{-dgmod}) \cong A\text{-dgmod}[W^{-1}]$ at the class W of quasi-isomorphisms (see [6]). Its associated homotopy category $H^0\mathcal{D}(A\text{-dgmod})$ recovers the conventional derived category obtained via the machinery of triangulated categories.

Let M be a dg-module over the dg-algebra A. We say that it is **graded projective** if it is projective as a graded module over the graded k-algebra A. The full dg-subcategory of graded projective modules is denoted $\mathrm{Proj}(A) \subset A\text{-dgmod}$. The dg-module M is called **homotopy-projective** (or simply **h-projective**) if for any exact dg-module X over A, the k-complex $\mathrm{Hom}^\bullet_A(M, X)$ is exact. The full dg-subcategory of such is denoted $\text{H-Proj}(A) \subset A\text{-dgmod}$. A dg-module M is called **semifree** if it admits an ascending, bounded below, exhaustive filtration $0 = F^0M \subset F^1M \subset F^2M \subset \cdots \subset M$ such that each graded piece $\mathrm{gr}^n M$ is the direct sum of shifts of copies of A. We denote the full subcategory of such by $\mathrm{SF}(A) \subset A\text{-dgmod}$. The dg-subcategory of **quasi-free** dg-modules is defined to be the dg-subcategory $\mathrm{QF}(A) = \mathrm{Proj}(A) \cap \text{H-Proj}(A)$ of dg-modules which are simultaneously graded projective and h-projective. Note that any semifree dg-modules is quasi-free, so $\mathrm{SF}(A) \subset \mathrm{QF}(A)$.

2.1.1. Proposition. *Each map in the composition*

$$\mathrm{SF}(A) \subset \mathrm{QF}(A) \subset \text{H-Proj}(A) \to \mathcal{D}(A\text{-dgmod})$$

is a quasi-equivalence of dg-categories. In particular, the first three all present the dg-derived category of A-dgmod.

Proof. The fact that $\mathrm{SF}(A)$ and $\mathrm{H\text{-}Proj}(A)$ present the derived category is classical, see e.g. [6]. Since $\mathrm{QF}(A)$ sits between them as a full dg-subcategory, this implies that it, too, presents $\mathcal{D}(A\text{-dgmod})$. □

Suppose that R is another dg-algebra over k. A **dg-algebra** over R is a dg-algebra A over k together with a map of k-dg-algebras $R \to A$ (note that we do *not* assume that its image is contained in the centre in any way). Note that this is equivalent to A being a unital algebra object in the monoidal dg-category $(R\text{-mod-}R, \otimes_R)$. A map of dg-algebras over R is a map preserving this structure.

2.1.2. Lemma. *Let A be a dg-algebra over the dg-algebra R which is projective (resp. h-projective resp. quasi-free) as a left R-dg-module. Then restriction*

$$A\text{-dgmod} \to R\text{-dgmod}$$

takes projective (resp. h-projective resp. quasi-free) A-dg-modules to projective (resp. h-projective resp. quasi-free) R-dg-modules.

Proof. For $M \in A$-dgmod, this follows from the observation that

$$\mathrm{Hom}_R(M|_R, -) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_R(A, -)).$$

□

2.1.3. Lemma. *If A is a dg-algebra, then tensor products of projective (resp. h-projective resp. quasi-free) modules are projective (resp. h-projective resp. quasi-free).*

Proof. Apply the adjunction statement

$$\mathrm{Hom}_A(M \otimes_A N, -) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, -)).$$

□

2.1.4. Lemma. *If $A \to B$ is a map of dg-algebras, then scalar extension of projective (resp. h-projective resp. quasi-free) B-dg-modules produces projective (resp. h-projective resp. quasi-free) A-dg-modules.*

Proof. Follows from $\mathrm{Hom}_A(A \otimes_B M, N) \cong \mathrm{Hom}_B(M, N|_B)$. □

Our main tool will be the dg-category $\mathrm{QF}(A)$. The convenience of working with this comes from the following observation:

2.1.5. Proposition. *The dg-category* $\mathrm{QF}(A)$ *of quasi-free dg-modules is closed under* ***filtered extensions****. This means that if* $M \in A$-dgmod *admits a bounded below and exhaustive filtration* $0 = F^0M \subset F^1M \subset F^2M \subset \cdots \subset M = \varinjlim F^nM$ *such that each graded piece* $\mathrm{gr}^n M \in \mathrm{QF}(A)$*, then* $M \in \mathrm{QF}(A)$ *as well.*

Proof. Each F^nM is graded projective, so the inclusions $F^{n-1}M \hookrightarrow F^nM$ split as maps of graded modules. This means that $M \cong \mathrm{gr}^{\bullet} M$ as graded modules, so as a graded module, M is the direct sum of projectives and is hence graded projecitve. To prove that it is also h-projective, we let X be an exact dg-modules over A. The short exact sequence $0 \to F^{n-1}M \to F^nM \to \mathrm{gr}^n M \to 0$ splits as a short exact sequence of graded modules. Applying the graded hom, we obtain that the sequence

$$0 \longrightarrow \mathrm{Hom}^{\bullet}_A(\mathrm{gr}^n M, X) \longrightarrow \mathrm{Hom}^{\bullet}_A(F^nM, X) \longrightarrow \mathrm{Hom}^{\bullet}_A(F^{n-1}M, X) \longrightarrow 0$$

is also exact. Now the first term of the sequence is exact by assumption, while the last term is exact by induction. Therefore, the complex $\mathrm{Hom}^{\bullet}_A(F^nM, X)$ is exact. The sequence also shows that the map $\mathrm{Hom}^{\bullet}_A(F^nM, X) \to \mathrm{Hom}^{\bullet}_A(F^{n-1}M, X)$ is surjective, hence the inverse system $\mathrm{Hom}^{\bullet}_A(F^nM, X)$ satisfies Mittag–Leffler. In conclusion, the complex $\mathrm{Hom}^{\bullet}_A(M, X) = \varprojlim \mathrm{Hom}^{\bullet}_A(F^nM, X)$ is exact, so M is h-projective. (Alternatively, if one wants to avoid Mittag–Leffler, one may write M as the the last term of a short exact sequence $0 \longrightarrow \bigoplus F^nM \xrightarrow{\mathrm{id}-s} \bigoplus F^nM \longrightarrow M \longrightarrow 0$ where s is the sum of the embeddings $F^nM \hookrightarrow F^{n+1}M$. By graded projectivity, this sequence splits as a short exact sequence of graded modules, so as before, we obtain that the sequence $0 \to \mathrm{Hom}^{\bullet}_A(M, X) \to \mathrm{Hom}^{\bullet}_A(\bigoplus F^nM, X) \to \mathrm{Hom}^{\bullet}_A(\bigoplus F^nM, X) \to 0$ is exact. As the last two terms are exact complexes, so is the first.) $\square$

2.2. The dg-category of $\mathbf{A}_\infty$-modules

We continue to work over a base which is a dg-algebra R over a field k. Let

$$\mathrm{Bar}(A) = \bigoplus_{n \geq 0} A[1]^{\otimes n} = \bigoplus_{n \geq 0} A \otimes_R A \otimes_R \cdots \otimes_R A[n]$$

be the **bar construction** of A, the coalgebra with cofree comultiplication

$$\Delta(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n).$$

There is another natural candidate for a derived category of A-dgmod, obtained using the notion of an $\mathbf{A_\infty}$**-module**. An A_∞-module over a dg-algebra A is a dg-comodule over $\mathrm{Bar}(A)$ whose underlying graded comodule is cofree. In other words, it has the form $\mathrm{Bar}_A(M) = \mathrm{Bar}(A) \otimes_R M$ for some graded R-module M. By cofreeness, the differential $d\colon \mathrm{Bar}_A(M) \to \mathrm{Bar}_A(M)[1]$ is given by a collection of maps $\mathrm{ac}_n\colon A^{\otimes(n-1)} \otimes_R M \to M[2-n]$ for $n \geq 1$. each of which we shall refer to as the nth action map. A map $f\colon M \to N$ of A_∞-modules is a map of the corresponding dg-comodules, and it boils down to a collection of maps $f_n\colon A^{\otimes n-1} \otimes_R M \to N[1-n]$ for all $n \geq 1$ satisfying a technical condition corresponding to f commuting with d. We write $A\text{-mod}_\infty^{\mathrm{nu}}$ for the dg-category of (non-uniltal) A_∞-modules over A. This is a full dg-subcategory of the dg-category of dg-comodules over the bar construction. We may write $A\text{-mod}_\infty^{\mathrm{nu}}(R\text{-dgmod})$ if we want to stress that the A_∞-modules are R-linear.

The cohomology $H^\cdot(M)$ with respect to the differential $d = \mathrm{ac}_1$ is a graded module over the graded ring $H^\cdot(A)$. The A_∞-module M is called **homotopy-unital** if $H^\cdot(M)$ is unital over $H^\cdot(A)$. The full subcategory consisting of $A\text{-mod}_\infty^{\mathrm{nu}}$ of homotopy-unital A_∞-modules is denoted $A\text{-mod}_\infty^{\mathrm{hu}}$.

An A_∞-module M is called **strictly unital** if we have $\mathrm{ac} \circ (\eta \otimes \mathrm{id}_M) = \mathrm{id}_M$ as well as $\mathrm{ac}_n \circ (\mathrm{id}_A^{\otimes i} \otimes \eta \otimes \mathrm{id}_A^{\otimes j} \otimes \mathrm{id}_M) = 0$ for all $n \neq 2$ and all i, j with $i + j + 2 = n$ (here, $\eta\colon R \to A$ is the unit map). If A is augmented, this is the same as a dg-comodule over the augmented bar construction $\mathrm{Bar}_R^+(A) = \mathrm{Bar}_R(\overline{A})$ whose underlying graded comodule is cofree. We write the corresponding dg-comodule as $\mathrm{Bar}_R^+(M)$. A strictly unital A_∞-module is in particular homotopy-unital. A dg-module M is a strictly unital A_∞-module with $\mathrm{ac}_2 = \mathrm{ac}$ and $\mathrm{ac}_1 = d_M$. A map $f\colon M \to N$ of strictly unital A_∞-modules is a map of A_∞-modules such that f_1 commutes with the unit, and such that $f_n \circ (\mathrm{id}^{\otimes i} \otimes \eta \otimes \mathrm{id}^{\otimes j} \otimes \mathrm{id}_M) = 0$ for $n > 1$ and $i + j + 2 = n$. The non-full dg-subcategory of $A\text{-mod}_\infty^{\mathrm{nu}}$ consisting of strictly unital A_∞-modules is denoted $A\text{-mod}_\infty$.

An $\mathbf{A_\infty}$**-quasi-isomorphism** is a morphism $f\colon M \to N$ with the property that $f_1\colon M \to N$ is a quasi-isomorphism of R-dg-modules, where M and N are equipped with the differential $d = \mathrm{ac}_1$. Denote by

$$A\text{-mod}_\infty(\mathrm{QF}(R)) \subset A\text{-mod}_\infty(R\text{-dgmod})$$

the full subcategory of A_∞-modules M such that (M, ac_1) is quasi-free as an R-dg-module (we use analogous notation for non-unital and homotopy-unital A_∞-modules).

We shall need adjunction statements for modules and comodules. If R is a dg-algebra over k, A a dg-algebra over R, and C a dg-coalgebra over R, then a **twisting cochain** is a closed map of R-dg-bimodules $\tau\colon C \to A$ of degree 1 such that

$$d_A\tau + \tau d_C + m(\tau \otimes \tau)\Delta = 0.$$

If $M \in C$-dgcomod, we denote by $A \otimes_R^\tau M \in A$-dgmod the dg-module whose underlying graded module is $A \otimes_R M$, but whose differential is given by

$$d_{A\otimes_R^\tau M} = d_{A\otimes_R M} + d^\tau_{A\otimes_R M}$$

where $d^\tau_{A\otimes_R M}$ is the composition

$$d^\tau_{A\otimes_R M}\colon A \otimes_R M \xrightarrow{\mathrm{id}_A\otimes \mathrm{ca}} A \otimes C \otimes M \xrightarrow{\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_M} A \otimes A \otimes M \xrightarrow{m_A\otimes\mathrm{id}_M} A \otimes M.$$

Similarly, if $N \in A$-dgmod, then we denote by $C \otimes_R^\tau N \in C$-dgcomod the comodule whose underlying graded comodule is $C \otimes_R N$, but whose differential is given by

$$d_{C\otimes_R^\tau N} = d_{C\otimes_R N} + d^\tau_{C\otimes_R N}$$

where

$$d^\tau_{C\otimes_R M}\colon C \otimes_R N \xrightarrow{\Delta\otimes_R\mathrm{id}_N} C \otimes_R C \otimes_R N \xrightarrow{\mathrm{id}_C\otimes_R\tau\otimes_R\mathrm{id}_N} C \otimes_R A \otimes_R N \xrightarrow{\mathrm{id}_C\otimes_R\mathrm{ac}} C \otimes_R N.$$

Finally, if $M \in C$-dgcomod and $N \in A$-dgmod, then we denote by $\mathrm{Hom}_R^\tau(M,N)$ the k-complex whose underlying graded k-module is $\mathrm{Hom}_R^{\boldsymbol{\cdot}}(M,N)$ and whose differential is

$$d_{\mathrm{Hom}_R^\tau(M,N)} = d_{\mathrm{Hom}_R^{\boldsymbol{\cdot}}(M,N)} + d^\tau_{\mathrm{Hom}_R^{\boldsymbol{\cdot}}(M,N)}$$

where $d^\tau_{\mathrm{Hom}_R^{\boldsymbol{\cdot}}(M,N)}$ takes $g \in \mathrm{Hom}_R^{\boldsymbol{\cdot}}(M,N)$ to

$$d^\tau_{\mathrm{Hom}_R^{\boldsymbol{\cdot}}(M,N)}(g)\colon M \xrightarrow{\mathrm{ca}} C \otimes_R M \xrightarrow{\tau\otimes g} A \otimes_R N \xrightarrow{\mathrm{ac}} N.$$

One easily checks using the condition on τ that the obvious maps are in fact isomorphisms of complexes

$$\mathrm{Hom}_A^{\boldsymbol{\cdot}}(A \otimes_R^\tau M, N) \cong \mathrm{Hom}_R^\tau(M,N) \cong \mathrm{Hom}_C^{\boldsymbol{\cdot}}(M, C \otimes_R^\tau N).$$

In particular,

$$A \otimes_R^\tau -\colon C\text{-dgcomod} \longrightarrow A\text{-dgmod}$$

is left adjoint to

$$C \otimes_R^\tau -\colon A\text{-dgmod} \longrightarrow C\text{-dgcomod}.$$

One may show that for $C = \mathrm{Bar}(A)$, the natural map $\tau\colon C \to A$ that kills everything except the $A^{\otimes 1}[1]$-component is a twisting cochain. If A is augmented, we also get a twisting cochain on $C = \mathrm{Bar}^+(A)$ by map $\tau\colon C \to A$ that takes $\overline{A}^{\otimes 1}[1]$ into A. Analogous results hold for A being the augmented or non-augmented cobar construction of a coalgebra C.

The following proposition is well-known and classical over a field (and probably also over a ring, to the right people, but we found no reference):

2.2.1. Proposition. *If the dg-algebra A is quasi-free as an R-dg-module, and if $M, N \in A\text{-mod}_\infty(\mathrm{QF}(R))$, then an A_∞-quasi-isomorphism $M \to N$ is a homotopy equivalence.*

This shows that the category $A\text{-mod}_\infty(\mathrm{QF}(R))$ is already derived.

Proof. Let $f\colon M \to N$ be an A_∞-quasi-isomorphism. Then $f_1\colon M \to N$ is a quasi-isomorphism of h-projective R-modules and hence a homotopy equivalence. Consider the map of C-comodules $f\colon C \otimes_R^\tau M \to C \otimes_R^\tau N$. We filter the coalgebra $C = \mathrm{Bar}_R(A)$ as an R-dg-module by letting $F^n C = \bigoplus_{i \le n} A^{\otimes i}[i]$. This allows us to also filter $C \otimes_R^\tau M$ and $C \otimes_R^\tau N$ as R-dg-modules by letting $F^n(C \otimes_R^\tau M) = (F^n C) \otimes_R^\tau M$ and $F^n(C \otimes_R^\tau N) = (F^n C) \otimes_R^\tau N$. In both cases, taking associated graded kills the differential d_τ. Since A is h-projective, $\mathrm{gr}^n \varphi\colon A^{\otimes n}[n] \otimes_R M \to A^{\otimes n}[n] \otimes_R N$ is a homotopy equivalence of R-dg-modules. Thus taking the cone

$$C \otimes_R^\tau M \xrightarrow{\ f\ } C \otimes_R^\tau N \longrightarrow C(f),$$

we obtain a filtration on $C(f)$ such that each graded piece is contractible over R. The exact sequence of dg-comodules

$$C \otimes_R^\tau N \longrightarrow C(f) \longrightarrow C \otimes_R^\tau M[1]$$

splits as a short exact sequence of *graded* C-comodules. Therefore, taking the graded hom, we get a short exact sequence of complexes of vector spaces

$$\begin{aligned}&\mathrm{Hom}^\bullet_C(C\otimes^\tau_R M[1], C\otimes^\tau_R M)\\ &\qquad\to \mathrm{Hom}^\bullet_C(C(f), C\otimes^\tau_R M)\to \mathrm{Hom}^\bullet_C(C\otimes^\tau_R N, C\otimes^\tau_R M).\end{aligned}$$

This therefore leads to a long exact sequence of the cohomologies. If we can prove that the middle term $\mathrm{Hom}^\bullet_C(C(f), C\otimes^\tau_R M)$ is exact, we will get that precomposition

$$f^*\colon\ \mathrm{Hom}^\bullet_C(C\otimes^\tau_R N, C\otimes^\tau_R M)\longrightarrow \mathrm{Hom}^\bullet_C(C\otimes^\tau_R M, C\otimes^\tau_R M)$$

is a quasi-isormophism. The class $[\mathrm{id}]\in H^0\,\mathrm{Hom}^\bullet_C(C\otimes^\tau_R M, C\otimes^\tau_R M)$ will then determine an element in $H^0\,\mathrm{Hom}^\bullet_C(C\otimes^\tau_R N, C\otimes^\tau_R M)$ which will be a homotopy inverse to f.

In proving the claim that $\mathrm{Hom}^\bullet_C(C(f), C\otimes^\tau_R M)=\mathrm{Hom}^\tau_R(C(f), M)$ is exact, we use the previously mentioned exhaustive C-dg-comodule filtration $F^nC(f)$ for which the graded pieces $\mathrm{gr}^nC(f)$ are contractible over R, $F^{-1}C(f)=0$, and where each filtered and graded piece is projective as a graded R-module. Therefore, the exact sequence of C-comodules

$$0\to F^{n-1}C(f)\to F^nC(f)\to \mathrm{gr}^nC(f)\to 0$$

splits as an exact sequence of graded R-modules. Since homs commute with direct sums, this yields an exact sequence

$$\begin{aligned}&0\to \mathrm{Hom}^\tau_R(\mathrm{gr}^nC(f), M)\\ &\qquad\to \mathrm{Hom}^\tau_R(F^nC(f), M)\to \mathrm{Hom}^\tau_R(F^{n-1}C(f), M)\to 0.\end{aligned}$$

Taking the long exact sequence of cohomologies, we obtain by induction that the complex $\mathrm{Hom}^\tau_R(F^nC(f), M)$ is exact for all n. Now we obtain the $C(f)$ as the right term in the exact sequence

$$0\longrightarrow \bigoplus_{n\geq 0} F^nC(f)\xrightarrow{\mathrm{id}-s}\bigoplus_{n\geq 0} F^nC(f)\longrightarrow C(f)\longrightarrow 0$$

where s is the sum of the inclusions $F^nC(f)\hookrightarrow F^{n+1}C(f)$. As before, this sequence consists of complexes which are projective as graded modules, so the sequence splits as a short exact sequence of graded modules. As above, we get that $\mathrm{Hom}^\tau_R(C(f), M)$ is exact, as claimed.

We provide an alternative argument using Mittag–Leffler: As before, we argue that the complex $\mathrm{Hom}_R^\tau(F^n C(f), M)$ is exact for all n. As the maps $\mathrm{Hom}_R^\tau(F^n C(f), M) \to \mathrm{Hom}_R^\tau(F^{n-1} C(f), M)$ are surjective, the inverse system $\mathrm{Hom}_R^\tau(F^n C(f), M)$ satisfies Mittag–Leffler. Therefore,

$$\mathrm{Hom}_R^\tau(C(f), M) = \mathrm{Hom}_R^\tau(\varinjlim F^n C(f), M) = \varprojlim \mathrm{Hom}_R^\tau(F^n C(f), M)$$

is exact (both limits being unenriched limits evaluated in the category of graded modules). □

2.2.2. Lemma. *If A is a dg-algebra over R and $M \in A$-dgmod, then the counit of adjunction*

$$A \otimes_R^\tau \mathrm{Bar}(A) \otimes_R^\tau M \to M$$

is a quasi-isomorphism of A-dg-modules.

Proof. This is exactly the bar resolution of M. □

2.2.3. Lemma. *If A is an augmented, flat dg-algebra over dg-algebra R over k, and M is a strictly unital A_∞-module over A which is flat over R, then the unit of adjunction $M \to A \otimes_R^\tau \mathrm{Bar}^+(M)$ is a quasi-isormophism of A_∞-modules.*

Proof. We claim in fact that $(M, \mathrm{ac}_1) \to (A \otimes_R^\tau \mathrm{Bar}^+(M), \mathrm{ac}_1)$ is a filtered quasi-isomorphism. On both sides, filter M by the trivial filtration $F^i M = M$ for all $i \geq 0$. Filter $C = \mathrm{Bar}^+(A) = \bigoplus_n \overline{A}^{\otimes n}[n]$ by $F^i C = \bigoplus_{n \leq i} \overline{A}^{\otimes n}[n]$, and filter A by $F^0 A = R$ and $F^i A = A$ for $i > 0$. Since all the tensor factors are flat, this induces a filtration on $A \otimes_R^\tau \mathrm{Bar}^+(M)$ with $\mathrm{gr}^0(A \otimes_R^\tau \mathrm{Bar}^+(M)) = M$ and

$$\mathrm{gr}^i(A \otimes_R^\tau \mathrm{Bar}^+(M)) = (\overline{A} \otimes_R \mathrm{gr}^{i-1} C \otimes_R M) \oplus (\mathrm{gr}^i C \otimes_R M) \qquad \text{for } i > 0.$$

We claim that $\mathrm{gr}^i(A \otimes_R^\tau \mathrm{Bar}^+(M))$ is contractible for $i > 0$. Indeed, the differential is given by

$$d_{\mathrm{gr}^i(A \otimes_R^\tau \mathrm{Bar}^+(M))} = \begin{pmatrix} d_{\overline{A} \otimes_R \mathrm{gr}^{i-1} C \otimes_R M} & p \\ 0 & d_{\mathrm{gr}^i C \otimes_R M} \end{pmatrix}$$

where $p \colon \mathrm{gr}^i C \otimes_R M \to \overline{A} \otimes_R \mathrm{gr}^{i-1} C \otimes_R M$ comes from the isomorphism $\mathrm{gr}^i C \cong \overline{A}[1] \otimes_R \mathrm{gr}^{i-1} C$. Clearly, letting

$$s = \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{pmatrix},$$

we get $\mathrm{id} = ds + sd$. $\square$

2.2.4. Lemma. *Let A be a dg-algebra over a dg-algebra R over k and let $M \in \mathrm{QF}(A)$. If $A \in \mathrm{QF}(R)$, then $A \otimes_R^\tau \mathrm{Bar}(A) \otimes_R^\tau M \in \mathrm{QF}(A)$. Similarly, if A is augmented and $\overline{A} \in \mathrm{QF}(R)$, then $A \otimes_R^\tau \mathrm{Bar}^+(A) \otimes_R^\tau M \in \mathrm{QF}(A)$.*

Proof. Filter the bar constructions by the number of tensor factors to kill the bar differential and the τ-differential. Then apply Theorems 2.1.3 and 2.1.4. $\square$

2.2.5. Proposition. *Let A be an augmented dg-algebra over a dg-algebra R over k such that A is quasi-free as an R-dg-module. Then we have a quasi-equivalence of dg-categories*

$$\mathcal{D}(A\text{-dgmod}) \cong A\text{-mod}_\infty(\mathrm{QF}(R)).$$

Proof. We use the presentation $\mathcal{D}(A\text{-dgmod}) \cong \mathrm{QF}(A)$. Since M is quasi-free over A and A is quasi-free over R, we obtain from Theorem 2.1.2 that $M|_R \in \mathrm{QF}(R)$. Thus we obtain a dg-functor

$$\mathrm{QF}(A) \longrightarrow A\text{-mod}_\infty(\mathrm{QF}(R)).$$

To see that it is fully faithful, we notice that if $M, N \in A\text{-mod}_\infty(\mathrm{QF}(R))$, then

$$\begin{aligned}\mathrm{Hom}_{A\text{-mod}_\infty(\mathrm{QF}(R))}(M, N) &= \mathrm{Hom}_{\mathrm{Bar}^+(A)}(\mathrm{Bar}^+(A) \otimes_R^\tau M, \mathrm{Bar}^+(A) \otimes_R^\tau N) \\ &= \mathrm{Hom}_A(A \otimes_R^\tau \mathrm{Bar}^+(A) \otimes_R^\tau M, N) \cong \mathrm{Hom}_{\mathcal{D}(A\text{-dgmod})}(M, N)\end{aligned}$$

where the last equivalence is by Theorems 2.2.3 and 2.2.4. Quasi-essential surjectivity also follows from Theorem 2.2.3. $\square$

2.2.6. Proposition. *For any augmented dg-algebra A which is quasi-free over R, the inclusion $A\text{-mod}_\infty(\mathrm{QF}(R)) \subset A\text{-mod}_\infty^{\mathrm{hu}}(\mathrm{QF}(R))$ is a quasi-equivalence of dg-categories.*

We believe this also holds for non-augmented dg-algebras, following a proof like [1], but the proof is simpler in this special case.

Proof. We need to prove that the embedding is quasi-essentially surjective, that is, that any homotopy-unital module can be resolved by a strictly unital one. Given a module $M \in A\text{-mod}_\infty^{\mathrm{hu}}(\mathrm{QF}(R))$, we may regard M as a strictly unital $A \oplus R$-module. Using Theorem 2.2.3, we obtain a quasi-isomorphism $M \to (A \oplus R) \otimes_R^\tau \mathrm{Bar}^+_{A\oplus R}(M)$ over $A \oplus R$. Restricting by the unital dg-algebra map $A = \overline{A} \oplus R \hookrightarrow A \oplus R$, we obtain that it is also a quasi-equivalence over A. Since $(A \oplus R) \otimes_R^\tau \mathrm{Bar}^+_{A\oplus R}(M)$ is strictly unital over A, we have proved essential surjectivity. It is also quasi-free over R, as we can filter the bar construction to kill the bar differential until we are left with a tensor product of quasi-free R-modules. To prove quasi-fully faithfulness, we note that the calculation of the hom space in $\mathcal{D}(A\text{-dgmod})$ in the previous proof works no matter if we are using the augmented or non-augmented bar construction (one then applies Theorem 2.2.2 in place of Theorem 2.2.3). Therefore, hom spaces agree on H^0. By shifting, they also agree on H^i for all i. □

3. Koszul duality

The material below is a cross between three approaches to Koszul duality for triangulated categories of modules. Firstly, there is a general approach of [11]: the author considers a pair of a dg-algebra A over a ring $\mathbb{K}$ and a dg-coalgebra C given by its bar construction. The duality couples certain *exotic* derived categories of dg-modules (resp., dg-comodules).

Secondly, there is a similar approach due to [8] with one difference: we stay in the world of usual derived categories all the way. Keller's Koszul duality is a special case of a general theorem describing a subcategory in a dg-category $\mathcal{C}$ generated by a compact object $M \in \mathcal{C}$.

Lastly, there is an approach of [10]. Its advantage is that the authors work over a possibly non-affine base. Its drawback is that the authors work in categories of *graded* dg-modules over dg-algebras equipped with the second (inner) grading. We mostly retell linear Koszul duality below replacing the use of the inner grading by Keller's considerations.

3.1. Koszul complex

In what follows we develop a story in the spirit of quadratic-linear-scalar duality due to Positselski.

Recall the most classical setting for it. Let $\mathbb{K}$ be a field of characteristic zero, and fix a commutative $\mathbb{K}$-algeba S and a finite rank free complex of S-modules M. Denote $M^* = \mathrm{Hom}_S(M, S)$. Consider the pair of quadratic dual dg-algebras $A = \mathrm{Sym}_S(M[1])$ and $A^! = \mathrm{Sym}_S(M^*[-2])$, with differentials obtained from the one in M by the Leibniz rule. Notice that symmetric algebra in the definition is understood in the graded sense. Letting $\tau\colon A^* \to A^![1]$

be the twisting cochain given by the identity on M^*, it is known that $K = (A^* \otimes_S^\tau A^!, d_{A^* \otimes_S^\tau A^!})$ has a structure of an A–$A^!$-dg-bimodule quasi-isomorphic to the trivial module S both as an A- and as an $A^!$-module. The bimodule K is called the Koszul complex.

3.1.1. Remark. One way to find cohomology of K is as follows. Consider the complex of S-modules $C_M = C(\mathrm{id}_M)[1]$. The Koszul complex K can be realized as $\mathrm{Sym}_S(C_M)$ as a graded S-module. Evidently, it is quasi-isomorphic to S. The commuting actions of A and $A^!$ are recovered in the following way. Consider the complex of S-modules $C_M \oplus C_M^*$ equipped with the canonical non-degenerate, graded skew-symmetric bilinear form. This allows us to define the **Heisenberg algebra** $\mathrm{Heis}_S(C_M \oplus C_M^*)$ associated to $C_M \oplus C_M^*$ as the free algebra with generators $C_M \oplus C_M^*$ and relations $ab - (-1)^{|a||b|}ba = \langle a, b\rangle$. It can be rewritten as

$$\mathrm{Heis}_S(C_M \oplus C_M^*) = \mathrm{Heis}_S(M[2] \oplus M^*[-1]) \otimes_S \mathrm{Heis}_S(M[1] \oplus M^*[-2])$$

and maps naturally into the endomorphism algebra of the S-module K. It contains the commuting subalgebras $A = \mathrm{Sym}_S(M[1])$ and $A^! = \mathrm{Sym}_S(M^*[-2])$. One checks directly that $\mathrm{Heis}_S(C_M \oplus C_M^*)$ is a dg-subalgebra in $\mathrm{End}_S(K)$ and both subalgebras A and $A^!$ are dg-subalgebras in $\mathrm{Heis}_S(C_M \oplus C_M^*)$.

3.2. Twisted case

Next we replace the ring S with a graded algebra and make its interaction with A and $A^!$ more complicated. Fix a commutative ring $\mathbb{K}$ of characteristic zero. Take two free $\mathbb{K}$-modules of finite rank M and N. Let $\varphi\colon M \to N^*$ be a $\mathbb{K}$-linear map. We consider the dg-algebras over $\mathbb{K}$ given by

$$\begin{aligned} A &= \mathrm{Heis}_{\mathbb{K}}(M[1] \oplus N[-1]), \\ A^! &= \mathrm{Sym}_{\mathbb{K}}(N[-1] \xrightarrow{\varphi^*} M^*[-2]). \end{aligned}$$

The first algebra is the Heisenberg algebra on $M[1] \oplus N[-1]$ with respect to the canonical, graded skew-symmetric bilinear form induced by φ.

3.2.1. Remark. Clearly this form restricts trivially both to $M[1]$ and to $N[-1]$. It follows that $S = \mathrm{Sym}_{\mathbb{K}}(N[-1])$ is a subalgebra in both A and $A^!$. However, S is not a dg-subalgebra in $A^!$. Also, A is not a free graded skew-commutative algebra over S (in fact, S is not even central in A). In this sense, the previous case is a toy example for the present one.

We generalize the construction of the Koszul complex. Consider the complex of $\mathbb{K}$-modules:

$$L = \big((M^* \oplus N)[-1] \xrightarrow{(-\mathrm{id}_{M^*},\varphi^*)} M^*[-2]\big).$$

Clearly it is quasi-isomoprhic to $N[-1]$. Consider $K = \mathrm{Sym}_{\mathbb{K}}(L)$ with the differential extended to symmetric powers by the Leibniz rule. It is quasi-isomorphic to $\mathrm{Sym}_{\mathbb{K}}(N[-1])$ as a complex of $\mathbb{K}$-modules. Generalizing the toy example, we construct two commuting actions of the dg-algebras A and $A^!$ on K explicitly.

Consider the Heisenberg algebra of the complex $L \oplus L^*$, where we write $L^* = \mathrm{Hom}_S(L, S)$:

$$\mathrm{Heis}_{\mathbb{K}}(L \oplus L^*) = \mathrm{Heis}_{\mathbb{K}}\big(M[2] \oplus N[1] \oplus M[1] \oplus M^*[-1] \oplus N[-1] \oplus M^*[-2]\big).$$

It embeds into the dg-algebra of endomorphisms of $\mathrm{Sym}_{\mathbb{K}}(L)$ and acts naturally on K.

Notice that the graded algebra $A = \mathrm{Heis}_{\mathbb{K}}(M[1] \oplus N[-1])$ embeds into $\mathrm{Heis}_{\mathbb{K}}(L \oplus L^*)$ in the following way: While the embedding of $M[1]$ is the obvious one, $N[-1]$ maps to $M^*[-1] \oplus N[-1]$ via (φ^*, id). One checks directly that both the relations and the differentials match.

Notice that the dg-algebra $A^! = \mathrm{Sym}_{\mathbb{K}}(N[-1] \to M^*[-2])$ also embeds naturally into $\mathrm{Heis}_{\mathbb{K}}(L \oplus L^*)$. Moreover its image belongs to the centralizer of the image of A. We proved the following statement.

3.2.2. Lemma. *The complex K has a natural structure of an A–$A^!$-dg-bimodule quasi-isomorphic to $S = \mathrm{Sym}_{\mathbb{K}}(N[-1])$ both as an A-module and as an $A^!$-module.*

3.2.3. Remark. In fact, it is not hard to see that $K \cong A^! \otimes^\tau A^*$, where the twisting cochain $\tau\colon A^* \to A^![1]$ is given by the identity on $M^*[-1]$. Note that by filtering A^* by the number of tensor factors, we kill the twisted differential, and hence we obtain a filtration of K by a finite number of free, finitely generated $A^!$-modules. Therefore, K is semifree over S (and hence quasi-free). Also, K is free of finite sank as a graded $A^!$-module and therefore compact and h-projective in $A^!$-dgmod.

3.3. Compact generators

Recall that an object M in a pretriangulated dg-category $\mathcal{C}$ is called **compact** if the functor $\mathrm{Hom}_{\mathcal{C}}(M, -)$ commutes with direct sums. We call M a **compact**

generator if its **right orthogonal** $M^\perp$ in the homotopy category, i.e. the full subcategory of objects annihilated by $H^0(\mathrm{Hom}_{\mathcal{C}}(M,-))$, is 0.

3.3.1. Theorem ([8, section 7.3]). *Consider a dg-algebra S over a commutative ring $\mathbb{K}$. Let A be a dg-algebra in the category of dg-bimodules over S. For $M \in A\text{-dgmod}$ a compact generator, we have a quasi-equivalence of dg-categories*

$$\mathcal{D}(A\text{-dgmod}) \xrightarrow{\sim} \mathcal{D}(E\text{-dgmod})$$

given by $N \mapsto \mathrm{RHom}_A(M,N)$. Here, E denotes the dg-algebra $E = \mathrm{RHom}_A(M,M)^{\mathrm{op}}$.

The generator condition can be omitted in the following way. Slightly abusing notation we denote the full subcategory in $\mathcal{D}(A\text{-dgmod})$ consisting of objects whose images belong to the right orthogonal to $M \in A\text{-dgmod}$ by $M^\perp$. Take the Drinfeld quotient dg-category $\mathcal{D}(A\text{-dgmod})/M^\perp$.

3.3.2. Corollary. *For a compact object $M \in A\text{-dgmod}$ we have a quasi-equivalence of dg-categories*

$$\mathcal{D}(A\text{-dgmod})/M^\perp \xrightarrow{\sim} \mathcal{D}(E\text{-dgmod})$$

given by the same functor as in the previous theorem.

3.3.3. Remark. It is convenient to identify the quotient dg-category

$$\mathcal{D}(A\text{-dgmod})/M^\perp$$

with the subcategory in $\mathcal{D}(A\text{-dgmod})$ generated by M, i.e. with the minimal pretriangulated subcategory in $\mathcal{D}(A\text{-dgmod})$ containing M that has arbitrary small coproducts. Denote the latter by $\langle M\rangle$. The previous corollary means that

$$\langle M\rangle \xrightarrow{\sim} \mathcal{D}(E\text{-dgmod})$$

for $E = \mathrm{RHom}_A(M,M)^{\mathrm{op}}$.

3.4. Geometric setting

We pass to our case of interest. Let X be a regular Noetherian affine scheme over a field of characeristic zero, and let $\mathbb{K} = \mathcal{O}_X$ be its ring of regular functions. Consider two finite rank vector bundles M and N on X and a vector bundle map $\varphi\colon M \to N^*$ like in section 3.2. We obtain a pair of dg-algebras $A^!$ and A and their Koszul complex K which is a resolution of S. In other words,

$$\begin{aligned} A &= \mathrm{Heis}_{\mathcal{O}_X}(M[1] \oplus N[-1]),\\ A^! &= \mathrm{Sym}_{\mathcal{O}_X}(N[-1] \to M^*[-2]),\\ S &= \mathrm{Sym}_{\mathcal{O}_X}(N[-1]). \end{aligned}$$

We work in the dg-category $\mathcal{C} = A^!$-dgmod. Recall from Theorem 3.2.3 that K is an h-projective and quasi-free resolution of S in $\mathcal{C}$.

Denote the dg-algebra $\mathrm{Hom}_{A^!}(K,K)^{\mathrm{op}}$ by E.

3.4.1. Corollary. *We have a quasi-equivalence of dg-categories*

$$\mathcal{D}(A^!\text{-dgmod})/S^{\perp} \xrightarrow{\sim} \mathcal{D}(E\text{-dgmod}).$$

3.4.2. Remark. Following the usual geometric intuition, we denote the triangualted subcategory $\langle S\rangle$ by $\mathcal{D}(A^!\text{-dgmod})_{\mathrm{tors}}$. This way, using Theorem 3.3.3, the quasi-equivalence from Theorem 3.3.2 reads as follows:

$$\mathcal{D}(A^!\text{-dgmod})_{\mathrm{tors}} \xrightarrow{\sim} \mathcal{D}(E\text{-dgmod}).$$

3.4.3. Lemma. *We have a quasi-isomorphism of dg-algebras* $A \xrightarrow{\sim} \mathrm{Hom}_{A^!}(K,K)^{\mathrm{op}}$.

Proof. We have a map

$$A \longrightarrow \mathrm{Hom}_S(A^*, A^*)^{\mathrm{op}} \longrightarrow \mathrm{Hom}_{A^!}(A^! \otimes_S^{\tau} A^*, A^! \otimes_S^{\tau} A^*)^{\mathrm{op}},$$

which also shows that $\mathrm{Hom}_S(K,K)^{\mathrm{op}}$ is a dg-algebra over S. To see that this map is a quasi-isomorphism, we observe that the fact that K is a quasi-free resolution of S (see Theorems 3.2.2 and 3.2.3) implies that $\mathrm{Hom}^{\cdot}_{A^!}(K,K)^{\mathrm{op}}$ calculates $\mathrm{RHom}_{A^!}(S,S)^{\mathrm{op}}$. Therefore, the augmentation $A^! \to S$ induces a quasi-isomorphism

$$\mathrm{Hom}^{\cdot}_{A^!}(K,K)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Hom}^{\cdot}_{A^!}(K,S) = \mathrm{Hom}^{\tau}_S(A^*,S) = A.$$

One checks directly that the composition $A \to \mathrm{Hom}^{\cdot}_{A^!}(K,K)^{\mathrm{op}} \to A$ is the identity. By the two-out-of-three property, the first map is a quasi-isomorphism. Therefore, as an algebra, $\mathrm{Hom}^{\cdot}_{A^!}(K,K)^{\mathrm{op}}$ is quasi-isomorphic to A. $\square$

We proved the following statement:

3.4.4. Proposition. *We have a quasi-equivalence of dg-categories*

$$\mathcal{D}(A^!\text{-dgmod})_{\mathrm{tors}} \xrightarrow{\sim} \mathcal{D}(A\text{-dgmod}).$$

4. Homotopy limits

For a general introduction to the homotopy limit machinery we shall use, we refer the reader to [3]. To us, a homotopy limit will be always understood as the derived functor of the limit functor in the model category-theoretic sense, and it serves as a concrete realization of the ∞-categorical limit functor. In particular, the homotopy limit preserves weak equivalences.

4.1. General lemmata on homotopy limits

The following section adds a few technical lemmata to [3] which we shall need.

4.1.1. Lemma. *Let $\mathcal{C}$ be a combinatorial model category and Γ_1, Γ_2 two categories. The homotopy limit $\underleftarrow{\mathrm{holim}}_{\Gamma_1\times\Gamma_2} F$ may be calculated componentwise, i.e.*

$$\underset{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2}{\underleftarrow{\mathrm{holim}}} F(\gamma_1,\gamma_2) = \underset{\gamma_1\in\Gamma_1}{\underleftarrow{\mathrm{holim}}}\big(\underset{\gamma_2\in\Gamma_2}{\underleftarrow{\mathrm{holim}}} F(\gamma_1,\gamma_2)\big)$$

where both homotopy limits on the right-hand side are derived functors of the pointwise limit $\underleftarrow{\lim}_{\Gamma_i} : \mathcal{C}^{\Gamma_i} \to \mathcal{C}$.

Proof. Recall from [3] that the limit functor

$$\underset{\Gamma_1\times\Gamma_2}{\underleftarrow{\lim}} : \mathcal{C}^{\Gamma_1\times\Gamma_2}_{\mathrm{inj}} \to \mathcal{C}$$

is right Quillen, where $\mathcal{C}^{\Gamma_1\times\Gamma_2}_{\mathrm{inj}}$ is the category of diagrams $\Gamma_1\times\Gamma_2 \to \mathcal{C}$ equipped with the injective model structure. Hence we have

$$\underset{\Gamma_1\times\Gamma_2}{\underleftarrow{\mathrm{holim}}} F = \underset{\Gamma_1\times\Gamma_2}{\underleftarrow{\lim}} R(F) = \underset{\gamma_1\in\Gamma_1}{\underleftarrow{\lim}}\big(\underset{\gamma_2\in\Gamma_2}{\underleftarrow{\lim}} R(F)(\gamma_1,\gamma_2)\big)$$

where $R(F)$ is a fibrant replacement of F in $\mathcal{C}^{\Gamma_1\times\Gamma_2}_{\mathrm{inj}}$. Now $\mathcal{C}^{\Gamma_1\times\Gamma_2}_{\mathrm{inj}} = (\mathcal{C}^{\Gamma_2}_{\mathrm{inj}})^{\Gamma_1}_{\mathrm{inj}}$ (since they clearly have the same cofibrations and trivial cofibrations), so $R(F)(\gamma_1,-)$ is fibrant in $\mathcal{C}^{\Gamma_2}_{\mathrm{inj}}$ for all $\gamma_1\in\Gamma_1$. This shows that the limit $\varprojlim_{\gamma_2\in\Gamma_2} R(F)(\gamma_1,\gamma_2)$ calculates $\underleftarrow{\mathrm{holim}}_{\gamma_2\in\Gamma_2} F(\gamma_1,\gamma_2)$ for all $\gamma_1\in\Gamma_1$. Furthermore, the limit functor

$$\varprojlim_{\Gamma_2}\colon (\mathcal{C}_{\mathrm{inj}}^{\Gamma_2})_{\mathrm{inj}}^{\Gamma_1} \to \mathcal{C}_{\mathrm{inj}}^{\Gamma_1}$$

is right Quillen by [9, Remark A.2.8.6]. Thus $\varprojlim_{\gamma_2\in\Gamma_2} R(F)(-,\gamma_2)$ is in fact a fibrant realization of the object $\underleftarrow{\mathrm{holim}}_{\gamma_2\in\Gamma_2} F(-,\gamma_2)$ in $\mathcal{C}_{\mathrm{inj}}^{\Gamma_1}$. Therefore,

$$\varprojlim_{\gamma_1\in\Gamma_1}\big(\varprojlim_{\gamma_2\in\Gamma_2} R(F)(\gamma_1,\gamma_2)\big) = \underleftarrow{\mathrm{holim}}_{\gamma_1\in\Gamma_1}\big(\underleftarrow{\mathrm{holim}}_{\gamma_2\in\Gamma_2} F(\gamma_1,\gamma_2)\big)$$

as claimed. □

4.1.2. Lemma. *The category* $\mathbf{\Delta}$ *is sifted, i.e. the diagonal embedding* $\mathbf{\Delta}\to\mathbf{\Delta}\times\mathbf{\Delta}$ *is homotopy-initial. In other words,*

$$\underleftarrow{\mathrm{holim}}_{[n]\in\mathbf{\Delta}} F([n],[n]) = \underleftarrow{\mathrm{holim}}_{([n],[m])\in\mathbf{\Delta}\times\mathbf{\Delta}} F([n],[m]).$$

Proof. The statement that $\mathbf{\Delta}$ is sifted is proved in [7, Example 21.5]. For a proof that homotopy-initial functors preserve homotopy limits, see e.g. [3, Theorem 6.1]. □

4.1.3. Example. Let G be a group scheme which is the semidirect product $G = N\rtimes H$ of a normal subgroup N and a subgroup H, and let it act on a scheme X. We claim that there is an isomorphism of stack quotients

$$[[X/N]/H]\cong[X/G].$$

To see this, write $\varphi_g\colon N\to N$ for the action of $g\in G$ on N. Now notice that we have an isomorphism

$$[[X/N]_n/H]_n \xrightarrow{\sim} [X/G]_n$$

given by

$$\begin{aligned}&\big(h_1,\dots,h_n,(n_1,\dots,n_n,x)\big)\\&\qquad\longmapsto \big(\varphi_{h_1\cdots h_n}(n_1)h_1,\varphi_{h_2\cdots h_n}(n_2)h_2,\dots,\varphi_{h_n}(n_n)h_n,x\big).\end{aligned}$$

Taking colimits on both sides and applying the two lemmas yields the desired result.

4.2. Homotopy limits in dg-categories

We denote by $\mathrm{dgSch}(k)$ the category of **dg-schemes** over a field k, which for us means the opposite category of the category $\mathrm{dgAlg}^{\leq 0}(k)$ of graded commutative dg-algebras over k sitting in non-positive degree. Given $X \in \mathrm{dgSch}(k)$, we denote by $A_X \in \mathrm{dgAlg}^{\leq 0}(k)$ the associated dg-algebra and call $X^\circ = \mathrm{Spec}(H^0 A_X)$ the **underlying scheme** of X. We shall often write $X = (X^\circ, A_X)$ and think of A_X as a dg-algebra over $\mathcal{O}_{X^\circ}$. Note that a morphism $f\colon (X^\circ, A_X) \to (Y^\circ, A_Y)$ of affine dg-schemes is equivalent to the data of a morphism of schemes $f\colon X^\circ \to Y^\circ$ and a comorphism $f^\#\colon A_Y \to A_X$ of $\mathcal{O}_{Y^\circ}$-dg-algebras. The dg-category of **quasi-coherent sheaves** on the dg-scheme (X°, A_X) is just $\mathrm{QCoh}(X) = A_X$-dgmod, the category of dg-modules over A_X.

An (affine) **group dg-scheme** is a group object $G = (G^\circ, A_G)$ in the category of affine dg-schemes. This means that the underlying scheme G° is an affine group scheme, and that the comorphism $A_G \to A_G \otimes_{\mathcal{O}_G} A_G$ of the composition map equips A_G with the structure of a Hopf dg-algebra over $\mathcal{O}_{G^\circ}$.

Suppose that G is a group dg-scheme acting on a dg-scheme X. Then the category of G-equivariant sheaves on X is defined to be the homotopy limit

$$\mathrm{QCoh}(X)^G = \underset{[n]\in\boldsymbol{\Delta}}{\underleftarrow{\mathrm{holim}}}\, \mathrm{QCoh}(X_n)$$

taken in the category $\mathrm{dgCat}(k)$ of dg-categories over k equipped with Tabuada's model structure (see [12]), and where $X.$ is the classifying space of the action groupoid $G \times X \rightrightarrows X$. Similarly, define

$$\big(\mathcal{D}\mathrm{QCoh}(X)\big)^G = \underset{[n]\in\boldsymbol{\Delta}}{\underleftarrow{\mathrm{holim}}}\, \mathcal{D}\mathrm{QCoh}(X_n).$$

Suppose that N and H are group dg-schemes such that N acts on H by means of automorphisms. Then we can form the external semidirect product $N \rtimes H$ whose underlying dg-scheme is $N \times H$, and with group structure defined by the usual formula. We also call a group dg-scheme G a semidirect product of group dg-subschemes N and H if it is isomorphic to the external semidirect product, and we write $G = N \rtimes H$. This is equivalent to having a short exact sequence of group dg-schemes $1 \to N \to G \to H \to 1$ such that there exists a monomorphism $H \hookrightarrow G$ with $H \hookrightarrow G \to H$ being the identity. Applying H^0, we get that $H^0(A_H) \to H^0(A_G) \to H^0(A_H)$ is also the identity, so $H^0(A_H) \to H^0(A_G)$ is injective, which implies that we also have a short exact sequence $1 \to N^\circ \to G^\circ \to H^\circ \to 1$ of the underlying group schemes. In particular, the underlying group scheme of G is a semidirect product $G^\circ = N^\circ \rtimes H^\circ$.

The same proof as in the classical case shows that we have an isomorphism of dg-schemes $G \cong N \times H$. This yields an isomorphism $A_G \cong A_N \otimes A_H$ of dg-algebras over the underlying field. Therefore, if G is acting on a dg-scheme X, we obtain similarly to Theorem 4.1.3 an isomorphism $A_G^{\otimes n} \otimes A_X \cong A_N^{\otimes n} \otimes A_H^{\otimes n} \otimes A_X$. Applying QCoh and taking homotopy limits, we obtain from Theorems 4.1.1 and 4.1.2 that

4.2.1. Proposition. *If a group dg-scheme $G = N \rtimes H$ acts on an affine dg-scheme X, we have*

$$\mathrm{QCoh}(X)^G \cong \big(\mathrm{QCoh}(X)^N\big)^H,$$

i.e. you can impose G-equivariance by first imposing N-equivariance and then H-equivariance. Similarly, we have for derived categories that

$$\big(\mathcal{D}\mathrm{QCoh}(X)\big)^G \cong \big(\big(\mathcal{D}\mathrm{QCoh}(X)\big)^N\big)^H.$$

We recall from [4] the statement

4.2.2. Theorem. *Suppose that $X_1 \rightrightarrows X_0$ is a groupoid in affine dg-schemes, and consider the associated classifying space $X.$ given by*

$$X_n = X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$

Write $A^n = A_{X_n}$ for the associated cosimplicial system of dg-algebras. Let $A = A^0$ and $C = A^1$, and note that C is a counital coalgebra in A-dgmod-A via the map $\Delta = \partial_1^\# \colon C \to C \otimes_A C$. Then we have a quasi-equivalence of dg-categories

$$\underleftarrow{\mathrm{holim}}_{\mathbf{\Delta}} \mathrm{QCoh}(X.) \cong C\text{-comod}_\infty^{\mathrm{hcu,formal}}(A\text{-dgmod}),$$

where the right-hand side denotes the dg-category of formal, homotopy-counital A_∞-comodules over C in A-dgmod.

4.2.3. Remark. Suppose that we replace $\mathrm{QCoh}(X.)$ by $\mathcal{D}\mathrm{QCoh}(X.)$. We may realize this as $\mathcal{D}\mathrm{QCoh}(X.) = \mathrm{QF}(\mathrm{QCoh}(X.))$. Since the pullbacks are exact and have exact right adjoints, they transform quasi-free objects into quasi-free objects. This gives a direct description of the derived functors.

Therefore, one may repeat the proof of the theorem above to obtain the realization

$$\underleftarrow{\operatorname{holim}}_{\mathbf{\Delta}}\, \mathcal{D}\mathrm{QCoh}(X.) \cong C\text{-comod}_{\infty}^{\mathrm{hcu,formal}}(\mathrm{QF}(\mathcal{A}\text{-dgmod}))$$

for the homotopy limit of the cosimplicial system of derived dg-categories. In particular, derived dg-categories commute with equivariance.

5. Equivariant sheaves on loop spaces

Let X ba an affine, smooth, and Noetherian scheme over $\mathbb{C}$. Inspired by the Hochschild–Kostant–Rosenberg theorem, we define the **loop space** of the scheme X as the affine dg-scheme $LX = (X, \Omega_X)$, where Ω_X is the non-positively graded algebra of differential forms regarded as a dg-algebra with differential $d = 0$. If $x \in X$ is a point, the **based loop space** at x is the dg-scheme $L_xX = LX \times_X x = (X, \mathrm{Sym}(T_x^* X^\circ[1]))$, the fibre at x of the evaluation map $LX \to X$.

5.0.1. Example. In the case of a group dg-scheme G, we have $LG = L_eG \rtimes G$, and the based loop space is $L_eG = \mathfrak{g}[-1] = (\mathrm{Spec}(\mathbb{C}), \mathrm{Sym}(\mathfrak{g}^*[1]))$.

5.0.2. Theorem. *Let X ba an affine, smooth, and Noetherian scheme over $\mathbb{C}$ acted on by an affine group scheme G. The derived category of (G, Ω_G)-equivariant Ω_X-dg-modules is equivalent to the triangulated subcategory*

$$\langle \mathcal{O}_X \rangle \subset \mathcal{D}\big(\mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{g} \otimes \mathcal{O}_X[-1] \to T_X[-2])\text{-dgmod}\big)^G$$

of the derived category of G-equivariant dg-modules generated by $\mathcal{O}_X$ and closed under small coproducts.

Proof. Because of the decomposition $LG = L_eG \rtimes G$ from Theorem 5.0.1, we recall from Theorem 4.2.1 that

$$\big(\mathcal{D}\mathrm{QCoh}(LX)\big)^{LG} \cong \big(\big(\mathcal{D}\mathrm{QCoh}(LX)\big)^{L_eG}\big)^G.$$

Via Theorem 4.2.3, we obtain

$$\begin{aligned}\big(\mathcal{D}\mathrm{QCoh}(LX)\big)^{LG} &\cong \underleftarrow{\operatorname{holim}}_{[n]\in\mathbf{\Delta}}\, \underleftarrow{\operatorname{holim}}_{[m]\in\mathbf{\Delta}}\, \mathcal{D}(\mathcal{O}_{G^n} \otimes \mathrm{Sym}(\mathfrak{g}^*[1])^{\otimes m} \otimes \Omega_X\text{-dgmod}) \\ &\cong \underleftarrow{\operatorname{holim}}_{[n]\in\mathbf{\Delta}}\big(C_n\text{-comod}_{\infty}^{\mathrm{hcu,formal}}(\mathrm{QF}(R_n\text{-dgmod}))\big)\end{aligned}$$

where $R_n = \mathcal{O}_{G^n} \otimes \Omega_X$ and $C_n = \mathcal{O}_{G^n} \otimes \mathrm{Sym}(\mathfrak{g}^*[1]) \otimes \Omega_X$. The R_n–R_n-bimodule structure on C_n comes from the two face maps

$$\begin{tikzcd} G^n \times L_eG \times LX \arrow[dr, "\partial_0"'] & & G^n \times L_eG \times LX \arrow[dl, "\partial_1"] \\ & G^n \times LX & \end{tikzcd}$$

given by $\partial_0(g,\gamma,\delta) = (g,\delta)$ and $\partial_1(g,\gamma,\delta) = (g,\gamma(\delta))$. Taking comorphisms, the bimodule structure maps are given by

$$\begin{tikzcd} \mathcal{O}_{G^n} \otimes \mathrm{Sym}(\mathfrak{g}^*[1]) \otimes \Omega_X & & \mathcal{O}_{G^n} \otimes \mathrm{Sym}(\mathfrak{g}^*[1]) \otimes \Omega_X \\ & \mathcal{O}_{G^n} \otimes \Omega_X \arrow[ul, "\partial_0^\#"] \arrow[ur, "\partial_1^\#"'] & \end{tikzcd}$$

where $\partial_0^\# = \mathrm{id}_{\mathcal{O}_{G^n}} \otimes 1 \otimes \mathrm{id}_{\Omega_X}$ provides the right module structure, whereas $\partial_1^\# = \mathrm{id}_{\mathcal{O}_{G^n}} \otimes \mathrm{ca}$ provides the left module structure; here,

$$\mathrm{ca}\colon \Omega_X \to \mathrm{Sym}(\mathfrak{g}^*[1]) \otimes \Omega_X$$

denotes the coaction map. This coaction map can be made explicit: The action of G on X yields a map

$$(\mathfrak{g} \otimes \mathcal{O}_X) \oplus TX \longrightarrow TX$$

given fibrewise by $d\rho_{e,x}\colon \mathfrak{g} \oplus T_xX \longrightarrow T_xX$, $(v,\xi) \longmapsto \xi + d(\rho(x))_e(v)$, where the map $\rho(x)\colon G \to X$ is given by $g \mapsto gx$. This may be dualized to a map

$$T^*X \longrightarrow (\mathfrak{g}^* \otimes \mathcal{O}_X) \oplus T^*X, \qquad \omega \longmapsto \varphi(\omega) + \omega,$$
$$\text{where } \varphi\colon T^*X \longrightarrow \mathfrak{g}^* \otimes \mathcal{O}_X.$$

Denoting $\mathrm{Sym}(\varphi)$ also by φ, this provides us with the map

$$\mathrm{ca}\colon \mathrm{Sym}_{\mathcal{O}_X}(T^*X[1]) \longrightarrow \mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{g}^* \otimes \mathcal{O}_X[1]) \otimes_{\mathcal{O}_X} \mathrm{Sym}_{\mathcal{O}_X}(T^*X[1])$$

which is our coaction map

$$\mathrm{ca}\colon \Omega_X \longrightarrow \mathrm{Sym}(\mathfrak{g}^*[1]) \otimes \Omega_X, \qquad \omega \longmapsto 1 \otimes \omega + \varphi(\omega) \otimes 1.$$

The coalgebra structure on C_n comes from the composition

$$\partial_1\colon G^n \times L_eG \times LX \times_{G^n \times LX} G^n \times L_eG \times LX \longrightarrow G^n \times L_eG \times LX,$$

taking a pair $((g,\gamma_2,\delta_2),(g,\gamma_1,\delta_1))$ with $\delta_2=\gamma_1(\delta_1)$ to $(g,\gamma_2\gamma_1,\delta_1)$. In other words, the coalgebra map $\Delta=\partial_1^{\#}\colon C_n\to C_n\otimes_{R_n}C_n$ is given by

$$\begin{array}{ccc}
C_n=\mathcal{O}_{G^n}\otimes\mathrm{Sym}(\mathfrak{g}^*[1])\otimes\Omega_X & & x\otimes y\otimes z \\
\downarrow & & \downarrow \\
\mathcal{O}_{G^n}\otimes\mathrm{Sym}(\mathfrak{g}^*[1])\otimes\mathrm{Sym}(\mathfrak{g}^*[1])\otimes\Omega_X & & x\otimes(1\otimes y+y\otimes 1)\otimes z \\
\| & & \downarrow \\
C_n\otimes_{R_n}C_n & & (1\otimes 1\otimes 1)\otimes(x\otimes y\otimes z) \\
 & & +\,(1\otimes y\otimes 1)\otimes(x\otimes 1\otimes z).
\end{array}$$

Now the coalgebra C_n is free of finite rank as a right dg-module over the dg-algebra R_n. Therefore, we may consider the algebra $A_n=\mathrm{Hom}_{\text{mod-}R_n}(C_n,R_n)$ in the category $\mathcal{O}_{G^n}\otimes\Omega_X$-dgmod-$\mathcal{O}_{G^n}\otimes\Omega_X$. By adjunction, we get a quasi-equivalence of dg-categories

$$C_n\text{-comod}_\infty^{\text{hcu,formal}}(\mathrm{QF}(R_n\text{-dgmod}))\cong A_n\text{-mod}_\infty^{\text{hu}}(\mathrm{QF}(R_n\text{-dgmod}))$$

between C_n-comod$_\infty^{\text{hcu,formal}}$ and the category of homotopy-unital A_∞-modules over A_n. The "formal" attribute becomes redundant in this case, as A_∞-modules have no similar convergence condition. We then apply Theorems 2.2.5 and 2.2.6 to obtain that the right-hand side is a presentation of the derived dg-category $\mathcal{D}(R_n\text{-dgmod})$. We sum up the conclusion so far:

5.0.3. Lemma. $(\mathcal{D}\mathrm{QCoh}(LX))^{LG}\cong\underleftarrow{\mathrm{holim}}_{[n]\in\mathbf{\Delta}}\,\mathcal{D}(A_n\text{-dgmod})$.

We claim that the algebra A_n has a a description as a Heisenberg algebra:

5.0.4. Lemma. *The algebra A_n is the Heisenberg algebra*

$$A_n=\mathrm{Heis}_{\mathcal{O}_{G^n\times X}}(M_n[1]\oplus N_n[-1])$$

where

$$M_n=\mathcal{O}_{G^n}\otimes T^*X\qquad\textit{and}\qquad N_n=\mathcal{O}_{G^n\times X}\otimes\mathfrak{g}$$

and the pairing is induced by the map $\varphi\colon M_n\to N_n^*$ from above.

Proof. The underlying complex of A_n is

$$A_n = \mathcal{O}_{G^n} \otimes \mathrm{Sym}(\mathfrak{g}[-1]) \otimes \Omega_X.$$

For an $a \in A_n$, we have, by the comodule structure on C_n, that $a(x \otimes y \otimes z) = a(1 \otimes y \otimes 1) \cdot x \otimes z$ for all elements $x \otimes y \otimes z \in C_n$. Therefore, a is determined by the element $a(1 \otimes y \otimes 1)$ for $y \in \mathrm{Sym}(\mathfrak{g}^*[1])$. By the above coalgebra structure, we have for $a, b \in A_n$ that

$$(a \cdot b)(1 \otimes y \otimes 1) = a(1 \otimes 1 \otimes 1)b(1 \otimes y \otimes 1) + a(1 \otimes y \otimes 1)b(1 \otimes 1 \otimes 1)$$

so $a \cdot b = a \wedge b$, and we recover the free multiplication on $\mathrm{Sym}(\mathfrak{g}[-1])$.

The left R_n-multiplication on A_n is given by the left R_n-multiplication on R_n. The right R_n-multiplication is given by the left R_n-multiplication on C_n. To rewrite this, we let $r \in \mathcal{O}_{G^n} \otimes T^*X[1] \subset R_n$ and $a \in \mathcal{O}_{G^n} \otimes \mathfrak{g}[-1] \otimes 1 \subset A_n$. Then the map $a\colon \mathcal{O}_{G^n} \otimes \mathrm{Sym}(\mathfrak{g}^*[1]) \otimes \Omega_X \to \mathcal{O}_{G^n} \otimes \Omega_X$ kills all elements except those of the form $x \otimes y \otimes z$ for $y \in \mathfrak{g}^*[1] \subset \mathrm{Sym}(\mathfrak{g}^*[1])$. The element $\varphi(r)$ is of this form. It follows that we have

$$\begin{aligned} a \cdot r &= a(\mathrm{ca}(r) \cdot -) = a\big((r + \varphi(r)) \cdot -\big) = (-1)^{|r||-|} a(- \cdot r) + \langle \varphi(r), a\rangle \\ &= (-1)^{|r||-|} a(-) \cdot r + \langle \varphi(r), a\rangle \\ &= (-1)^{|r||-|} (-1)^{|r|(|a|+|-|)} r \cdot a(-) + \langle \varphi(r), a\rangle \\ &= (-1)^{|a||r|} r \cdot a + \langle \varphi(r), a\rangle \end{aligned}$$

(note that $|a| = 1$ and $|r| = -1$, but they have been kept in the equation for clarity). In other words, we obtain the desired Heisenberg algebra structure. □

Applying the Koszul duality statement of Theorem 3.4.4 over the base $S_n = \mathrm{Sym}_{\mathcal{O}_{G^n \times X}}(N[-1])$, we obtain a quasi-equivalence of dg-categories

$$\mathcal{D}(A_n\text{-dgmod}) \cong \mathcal{D}(A_n^!\text{-dgmod})_{\mathrm{tors}}$$

where

$$\begin{aligned} A_n^! &= \mathrm{Sym}_{\mathcal{O}_{G^n \times X}}\big(N_n[-1] \xrightarrow{\varphi^*} M_n^*[-2]\big) \\ &= \mathrm{Sym}_{\mathcal{O}_{G^n \times X}}\big(\mathcal{O}_{G^n} \otimes \mathfrak{g} \otimes \mathcal{O}_X[-1] \xrightarrow{\varphi^*} \mathcal{O}_{G^n} \otimes TX[-2]\big). \end{aligned}$$

Thus

$$
\begin{aligned}
\big(\mathcal{D}\mathrm{QCoh}(LX)\big)^{LG} &\cong \underset{[n]\in\mathbf{\Delta}}{\underleftarrow{\mathrm{holim}}}\,\mathcal{D}(A_n^!\text{-dgmod})_{\mathrm{tors}}\\
&\cong \mathcal{D}\big(\mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{g}\otimes\mathcal{O}_X[-1]\xrightarrow{\varphi^*}TX[-2])\text{-dgmod}\big)^G_{\mathrm{tors}}
\end{aligned}
$$

which is what we wanted to prove. □

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Department of Mathematics, Aarhus University
Ny Munkegade 118, 8000 Aarhus C, Denmark
E-mail address: hippie@math.au.dk

Department of Mathematics, University of Haifa
199 Abba Khoushy Avenue, Mount Carmel, Haifa, 3498838, Israel
E-mail address: sorsted@gmail.com

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