Math. Res. Lett.
Volume 30, Number 2, 319–340, 2023

Metrics of constant negative scalar-Weyl curvature

Giovanni Catino

Extending Aubin's construction of metrics with constant negative scalar curvature, we prove that every n-dimensional closed manifold admits a Riemannian metric with constant negative scalar-Weyl curvature, that is $R + t|W|, t \in \mathbb{R}$. In particular, there are no topological obstructions for metrics with ε-pinched Weyl curvature and negative scalar curvature.

1. Introduction

A natural problem in Riemannian geometry is to understand the relation between curvature and topology of the underlying manifold. Given a smooth n-dimensional manifold M, $n \geq 3$, the curvature tensor of a Riemannian metric g on M can be decomposed in its Weyl, Ricci and scalar curvature part, that is

$$Riem_g = W_g + \frac{1}{n-2} Ric_g \owedge g - \frac{R_g}{2(n-1)(n-2)} g \owedge g,$$

where $\owedge$ is the Kulkarni-Nomizu product. It is common knowledge that *weak positive* curvature conditions, such as positive scalar curvature R_g [8, 17], or *strong negative* ones, such as negative sectional curvature, are in general obstructed. On the other hand, Aubin in [1, 2] showed that, on every smooth n-dimensional closed (compact with empty boundary) manifold, there exists a smooth Riemannian metric with constant negative scalar curvature, $R_g \equiv -1$. This result was extended to the complete, non-compact, case by Bland and Kalka in [3]. In particular, there are no topological obstructions for negative scalar curvature metrics. Actually, a much stronger result is known: Lohkamp in [15] proved that every smooth n-dimensional complete manifold admits a complete smooth Riemannian metric with (strictly) negative Ricci curvature, $Ric_g < 0$ (the three-dimensional case was considered in [4, 7]).

By virtue of the Riemann components, in dimension $n \geq 4$, it is natural to ask if there are unobstructed curvature conditions which involve the Weyl curvature. To the best of our knowledge, the first result in this direction was proved by Aubin [2], who constructed a metric with nowhere vanishing Weyl curvature on every closed n-dimensional manifold. As a consequence, in [6] the authors proved the existence of a canonical metric (weak harmonic Weyl) whose Weyl tensor satisfies a second order Euler-Lagrange PDE, on every given closed four-manifold.

In [9], Gursky studied a variant of the Yamabe problem related to a modified scalar curvature given by

$$R_g + t|W_g|_g, \quad t \in \mathbb{R},$$

where $|W_g|_g$ denotes the norm of the Weyl curvature of g. We will refer to this quantity as the *scalar-Weyl curvature* (see Section 2). Constant scalar-Weyl curvature metrics naturally arise as critical points in the conformal class of the modified Einstein-Hilbert functional

$$g \longmapsto \mathrm{Vol}_g(M)^{-\frac{n-2}{2}} \int_M (R_g + t|W_g|_g)\, dV_g.$$

It is clear that positive scalar-Weyl curvature metrics are obstructed, at least for $t \leq 0$, and naturally we may ask what we can say concerning the negative regime. In this paper we prove the following existence result:

Theorem 1.1. *On every smooth n-dimensional closed manifold M, for every $t \in \mathbb{R}$, there exists a smooth Riemannian metric $g = g_t$ with*

$$R_g + t|W_g|_g \equiv -1 \quad \text{on } M.$$

In particular, there are no topological obstructions for negative scalar-Weyl curvature metrics.

Remark 1.2. In dimension four, Theorem 1.1 was proved also by Seshadri in [18]. We observe that his proof cannot be trivially generalized to higher dimension, since it is based on the existence of a hyperbolic metric on a knot complement of $\mathbb{S}^3$.

It is well known that there are obstructions for the existence of metrics with zero Weyl curvature. On the other hand, choosing $t = 1/\sqrt{\varepsilon}$, $\varepsilon > 0$, in Theorem 1.1 we obtain the following existence result for metrics with ε-pinched Weyl curvature and negative scalar curvature:

Corollary 1.3. *On every smooth n-dimensional closed manifold, for every $\varepsilon > 0$, there exists a smooth Riemannian metric $g = g_\varepsilon$ with*

$$R_g < 0 \quad \textit{and} \quad |W_g|_g^2 < \varepsilon R_g^2 \quad \textit{on } M.$$

The interesting notion of *isotropic curvature* was introduced by Micallef and Moore in [16]: (M, g) has positive (or negative) isotropic curvature if and only if the curvature tensor of g satisfies

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0 \quad (\text{or } < 0)$$

for all orthonormal 4-frames $\{e_1, e_2, e_3, e_4\}$. Using minimal surfaces, the author of [16] proved that any closed simply connected manifold with positive isotropic curvature is homeomorphic to the sphere $\mathbb{S}^n$. As already observed in [18, Theorem 1.1], in dimension four, metrics with negative scalar-Weyl curvature for $t \geq 6$ have negative isotropic curvature. In particular, Theorem 1.1 implies the following:

Corollary 1.4 (Seshadri [18]). *On every smooth four-dimensional orientable closed manifold there exists a smooth Riemannian metric with negative isotropic curvature.*

We finally note that, in dimension $n > 4$, a characterization of negative isotropic curvature was given in [13] in terms of an inequality involving the Weyl tensor and the $(n-4)$-curvature, which coincides with the scalar curvature if $n = 4$. It would be interesting to extend Corollary 1.4 to $n > 4$, by following this path.

2. The scalar-Weyl curvature

In this section we briefly recall the variational and conformal aspects of the scalar-Weyl curvature, first studied by Gursky in [9]. Let (M, g) be a n-dimensional closed (compact with empty boundary) Riemannian manifold. First we recall that the conformal Laplacian is the operator

$$\mathcal{L}_g := -\frac{4(n-1)}{n-2}\Delta_g + R_g,$$

which has the following well known conformal covariance property: if $\widetilde{g} = u^{4/(n-2)}g$, then

$$\mathcal{L}_{\widetilde{g}}\phi = u^{-\frac{n+2}{n-2}}\mathcal{L}_g(\phi u), \quad \forall \phi \in C^2(M).$$

Moreover, the scalar curvature of the conformally related metric $\widetilde{g}$ is given by

$$R_{\widetilde{g}} = u^{-\frac{n+2}{n-2}} \mathcal{L}_g u.$$

Therefore, the operator $\mathcal{L}$ plays a prominent role in the resolution of the Yamabe variational problem. Given $t \in \mathbb{R}$, we define the scalar-Weyl curvature

$$F_g := R_g + t|W_g|_g \tag{2.1}$$

and the associated modified conformal Laplacian

$$\mathcal{L}_g^t := -\frac{4(n-1)}{n-2}\Delta_g + F_g,$$

where $|W_g|_g$ denotes the norm of the Weyl curvature of g. The key observation in [9] is that the couples $(F_g, \mathcal{L}_g^t)$ and $(R_g, \mathcal{L}_g)$ share the same conformal properties. In fact, if $\widetilde{g} = u^{4/(n-2)}g$, then

$$\mathcal{L}_{\widetilde{g}}^t \phi = u^{-\frac{n+2}{n-2}} \mathcal{L}_g^t(\phi u), \quad \forall \phi \in C^2(M), \quad \text{and} \quad F_{\widetilde{g}} = u^{-\frac{n+2}{n-2}} \mathcal{L}_g^t u. \tag{2.2}$$

In particular, a spectral argument shows the following [9, Proposition 3.2]:

Lemma 2.1. *Let (M, g) be a n-dimensional closed Riemannian manifold. Then, there exists a $C^{2,\alpha}$ metric $\widetilde{g} \in [g]$ with either $F_{\widetilde{g}} > 0$, $F_{\widetilde{g}} < 0$, or $F_{\widetilde{g}} \equiv 0$. Moreover, these three possibilities are mutually exclusive.*

In analogy with the Yamabe problem, Gursky defined the functional

$$\widehat{Y}(u) := \frac{\int_M u\, \mathcal{L}_g^t u\, dV_g}{\left(\int_M u^{2n/(n-2)}\, dV_g\right)^{(n-2)/2}}$$

and the conformal invariant

$$\widehat{Y}(M, [g]) := \inf_{u \in H^1(M)} \widehat{Y}(u).$$

Using (2.2), it is easy to see that the functional $u \mapsto \widehat{Y}(u)$ is equivalent to the modified Einstein-Hilbert functional

$$\widetilde{g} = u^{4/(n-2)} g \longmapsto \frac{\int_M F_{\widetilde{g}}\, dV_{\widetilde{g}}}{\mathrm{Vol}_{\widetilde{g}}(M)^{(n-2)/2}}.$$

Following a classical subcritical regularization argument, Gursky showed that, if $\widehat{Y}(M, [g]) \leq 0$, then the variational problem of finding a conformal metric $\widetilde{g} \in [g]$ with constant scalar-Weyl curvature F can be solved. The proof (in dimension four) can be found in [9, Proposition 3.5] and it can be trivially generalized to dimension $n \geq 4$. In particular, we have the following sufficient condition to the existence of constant negative scalar-Weyl curvature:

Lemma 2.2. *Let (M, g) be a n-dimensional closed Riemannian manifold. If there exists a metric $g' \in [g]$ such that*

$$\int_M F_{g'}\, dV_{g'} < 0,$$

then, there exists a (unique) $C^{2,\alpha}$ metric $\widetilde{g} \in [g]$ such that $F_{\widetilde{g}} \equiv -1$.

To conclude this section, we observe that the full modified Yamabe problem related to the scalar-Weyl curvature and more generally modified scalar curvatures was treated in [12]. Moreover, these techniques introduced by Gursky, have been used in various contexts, especially in the four-dimensional case. For instance we want to highlight [10, 11, 14, 18].

3. Aubin's metric deformation: two integral inequalities

In this section we first recall the variational formulas for some geometric quantities under the deformation of the metric of the type

$$g' = g + df \otimes df, \quad f \in C^\infty(M).$$

In [1, 2] Aubin, with a clever coupling of this deformation with a conformal one, proved local and global existence results of metrics satisfying special curvature conditions. The proof of the first three formulas can be found in [2]. The variation of the Weyl tensor can be found in [5, Chapter 2].

Lemma 3.1. *Let (M, g) be a n-dimensional Riemannian manifold and consider the variation of the metric g, in a given local coordinate system, defined by*

$$g'_{ij} := g_{ij} + f_i f_j, \quad f \in C^\infty(M).$$

Then we have

$$\begin{aligned} dV_{g'} &= w^{1/2} dV_g, \\ (g')^{ij} &= g^{ij} - \frac{f^i f^j}{w}, \\ R' &= R - \frac{2}{w} R_{ij} f^i f^j + \frac{1}{w}\Big[(\Delta f)^2 - f_{it} f^{it}\Big] \\ &\quad - \frac{2}{w^2}\big[(\Delta f) f^i f^j f_{ij} - f^i f_{ij} f^{jp} f_p\big], \\ W'_{ijkt} &= W_{ijkt} + E_g(f)_{ijkt}, \end{aligned}$$

with $w := 1 + |\nabla f|^2$ *and*

$$\begin{aligned} &E_g(f)_{ijkt} := \frac{1}{w}(f_{ik} f_{jt} - f_{it} f_{jk}) \\ &+ \frac{1}{n-2}(R_{ik} f_j f_t - R_{it} f_j f_k + R_{jt} f_i f_k - R_{jk} f_i f_t) \\ &+ \frac{R}{(n-1)(n-2)}(g_{ik} f_j f_t - g_{it} f_j f_k + g_{jt} f_i f_k - g_{jk} f_i f_t) \\ &+ \frac{f^p f^q}{w(n-2)}[R_{ipkq}(g_{jt} + f_j f_t) - R_{iptq}(g_{jk} + f_j f_k) \\ &\qquad + R_{jptq}(g_{ik} + f_i f_k) - R_{jpkq}(g_{it} + f_i f_t)] \\ &- \frac{2 R_{pq} f^p f^q}{w(n-1)(n-2)}[g_{ik} g_{jt} - g_{it} g_{jk} + g_{ik} f_j f_t - g_{it} f_j f_k + g_{jt} f_i f_k - g_{jk} f_i f_t] \\ &- \frac{1}{w(n-2)}\Big\{\big[(\Delta f) f_{ik} - f_{ip} f^p_k\big](g_{jt} + f_j f_t) - [(\Delta f) f_{it} - f_{ip} f^p_t](g_{jk} + f_j f_k)\Big\} \\ &- \frac{1}{w(n-2)}\Big\{[(\Delta f) f_{jt} - f_{jp} f^p_t](g_{ik} + f_i f_k) - \big[(\Delta f) f_{jk} - f_{jp} f^p_k\big](g_{it} + f_i f_t)\Big\} \\ &+ \frac{1}{w(n-1)(n-2)}\Big[(\Delta f)^2 - \big|\nabla^2 f\big|^2\Big] \\ &\quad \times (g_{ik} g_{jt} - g_{it} g_{jk} + g_{ik} f_j f_t - g_{it} f_j f_k + g_{jt} f_i f_k - g_{jk} f_i f_t) \\ &+ \frac{f^p f^q}{w^2(n-2)}[(f_{ik} f_{pq} - f_{ip} f_{kq})(g_{jt} + f_j f_t) - (f_{it} f_{pq} - f_{ip} f_{tq})(g_{jk} + f_j f_k)] \\ &+ \frac{f^p f^q}{w^2(n-2)}[(f_{jt} f_{pq} - f_{jp} f_{tq})(g_{ik} + f_i f_k) - (f_{jk} f_{pq} - f_{jp} f_{kq})(g_{it} + f_i f_t)] \\ &- \frac{2}{w^2(n-1)(n-2)}[(\Delta f) f^p f^q f_{pq} - f^p f_{pq} f^{qr} f_r](g_{ik} g_{jt} - g_{it} g_{jk}) \\ &- \frac{2}{w^2(n-1)(n-2)}[(\Delta f) f^p f^q f_{pq} - f^p f_{pq} f^{qr} f_r] \\ &\quad \times (g_{ik} f_j f_t - g_{it} f_j f_k + g_{jt} f_i f_k - g_{jk} f_i f_t). \end{aligned}$$

Moreover,

$$R' = R - \frac{R_{ij} f^i f^j}{w} + \nabla^i \left(\frac{\Delta f f_i - f_{ij} f^j}{w} \right)$$

and thus

$$\int_M R' \, dV_g = \int_M R \, dV_g - \int_M \frac{R_{ij} f^i f^j}{1 + |\nabla f|^2} \, dV_g.$$

We will denote by $[g]$ the conformal class of the metric g. Using a conformal deformation, we can show the following first integral sufficient condition for the existence of a constant negative scalar-Weyl curvature:

Lemma 3.2. *Let M be a n-dimensional closed manifold. If there exists a positive smooth function $u \in C^\infty(M)$ such that for a Riemannian metric g on M it holds*

$$\int_M F_g \, u^2 \, dV_g + \frac{4(n-1)}{n-2} \int_M |\nabla u|^2 \, dV_g < 0,$$

then there exists a (unique) $C^{2,\alpha}$ metric $\widetilde{g} \in [g]$ such that $F_{\widetilde{g}} \equiv -1$.

Proof. We consider the conformal metric $g'_{ij} = u^{4/(n-2)} g$. By (2.2) we have

$$F_{g'} = R_{g'} + t|W_{g'}|_{g'} = u^{-4/(n-2)} \left(R_g + t|W_g|_g - \frac{4(n-1)}{n-2} \frac{\Delta u}{u} \right).$$

Therefore, since $dV_{g'} = u^{2n/(n-2)} dV_g$, using the assumption we obtain

$$\int_M F_{g'} \, dV_{g'} = \int_M F_g \, u^2 \, dV_g + \frac{4(n-1)}{n-2} \int_M |\nabla u|^2 \, dV_g < 0.$$

The conclusion follows now by Lemma 2.2. □

Using Aubin's deformations, we prove the following second integral sufficient condition for the existence of a constant negative scalar-Weyl curvature:

Lemma 3.3. *Let M be a n-dimensional closed manifold. Suppose that there exists a smooth function $\varphi \in C^\infty(M)$ such that for a Riemannian metric g*

on M and some $t>0$ it holds

$$\int_M (R_g + t|W_g|_\varphi)\, dV_g + t\int_M |E_g(\varphi)|_\varphi\, dV_g - \int_M \frac{R_{ij}\varphi^i\varphi^j}{1+|\nabla\varphi|^2}\, dV_g + \frac{n-1}{n-2}\int_M \left[\frac{\varphi_{ip}\varphi^p\varphi_{iq}\varphi^q}{(1+|\nabla\varphi|^2)^2} - \frac{|\varphi_{ij}\varphi^i\varphi^j|^2}{(1+|\nabla\varphi|^2)^3}\right] dV_g < 0,$$

where $|\cdot|_\varphi$ denotes the norm with respect of $g + d\varphi\otimes d\varphi$ and $E_g(\varphi)$ is defined as in Lemma 3.1. Then, there exists a (unique) $C^{2,\alpha}$ metric $\widetilde{g}\in [g+d\varphi\otimes d\varphi]$ such that $F_{\widetilde{g}}\equiv -1$.

Proof. Let $\varphi\in C^\infty(M)$. Applying Lemma 3.2 to the metric $g' = g + d\varphi\otimes d\varphi$ with

$$u := \left(1+|\nabla\varphi|^2\right)^{-1/4},$$

we know that there exists a conformal metric $g''\in[g']$ with $F_{g''}\equiv -1$, if

$$\int_M \frac{F_{g'}}{\left(1+|\nabla\varphi|^2\right)^{1/2}}\, dV_{g'} + \frac{4(n-1)}{n-2}\int_M \left|\nabla\left(1+|\nabla\varphi|^2\right)^{-1/4}\right|_\varphi^2 dV_{g'} < 0.$$

From Lemma 3.1 we obtain the equivalent inequality

$$\begin{aligned}\int_M F_{g'}\, dV_g &+ \frac{4(n-1)}{n-2} \\ &\times \int_M \partial_i\left(1+|\nabla\varphi|^2\right)^{-1/4}\partial_j\left(1+|\nabla\varphi|^2\right)^{-1/4}\left(g^{ij} - \frac{\varphi_i\varphi_j}{1+|\nabla\varphi|^2}\right) dV_{g'} \\ &= \int_M F_{g'}\, dV_g + \frac{n-1}{n-2}\int_M \left[\frac{\varphi_{ip}\varphi^p\varphi_{iq}\varphi^q}{(1+|\nabla\varphi|^2)^2} - \frac{|\varphi_{ij}\varphi^i\varphi^j|^2}{(1+|\nabla\varphi|^2)^3}\right] dV_g < 0.\end{aligned}$$

Using again Lemma 3.1, we get

$$\begin{aligned}\int_M F_{g'}\, dV_g &= \int_M (R_{g'} + t|W_{g'}|_\varphi)\, dV_g \\ &= \int_M (R_g + t|W_{g'}|_\varphi)\, dV_g - \int_M \frac{R_{ij}\varphi^i\varphi^j}{1+|\nabla\varphi|^2}\, dV_g.\end{aligned}$$

Using that

$$|W_{g'}|_\varphi \le |W_g|_\varphi + |E_g(\varphi)|_\varphi$$

where $E_g(\varphi)$ is defined as in Lemma 3.1, we conclude the proof of this lemma. □

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The strategy of the proof takes strong inspiration from the works of Aubin in [1, 2].

Step 1.

From [1, 2] we know that, on a closed n-dimensional manifold, there exists a Riemannian metric g' with constant scalar curvature -1. In particular, if $t \leq 0$, $F_{g'} < 0$. By Lemma 2.2, there exists a metric $\widetilde{g} \in [g']$ such that $F_{\widetilde{g}} \equiv -1$. Therefore, from now on we focus on the case

$$t > 0.$$

First of all, we can choose a Riemannian metric g with

$$F_g = R_g + t|W_g|_g \geq 0 \quad \text{on } M,$$

otherwise Theorem 1.1 would immediately follow from Lemma 2.1 and Lemma 2.2. Consider a positive smooth function $\psi \in C^\infty(M)$ and a positive constant $k > 0$, and define

$$g' := \psi g, \quad g'' := g' + d(k\psi) \otimes d(k\psi).$$

If we fix $t > 0$ and apply Lemma 3.3 to the metric g' with $\varphi = k\psi$, we obtain that if

$$\begin{aligned}\Phi_M := &\int_M (R_{g'} + t|W_{g'}|_{k\psi})\, dV_{g'} + t\int_M |E_{g'}(k\psi)|_{k\psi}\, dV_{g'} \\ &- \int_M \frac{R'_{ij}\nabla^i_{g'}\psi\nabla^j_{g'}\psi}{1/k^2 + |\nabla_{g'}\psi|^2_{g'}}\, dV_{g'} \\ &+ \frac{n-1}{n-2}\int_M \left[\frac{\nabla^{g'}_{ip}\psi\nabla^p_{g'}\psi\nabla^{g'}_{iq}\psi\nabla^q_{g'}\psi}{(1/k^2 + |\nabla_{g'}\psi|^2_{g'})^2} - \frac{|\nabla^{g'}_{ij}\psi\nabla^i_{g'}\psi\nabla^j_{g'}\psi|^2}{(1/k^2 + |\nabla_{g'}\psi|^2_{g'})^3}\right] dV_{g'} < 0,\end{aligned}$$

then there exists a (unique) $C^{2,\alpha}$ metric $\widetilde{g} \in [g'']$ such that $F_{\widetilde{g}} \equiv -1$. Therefore, to prove Theorem 1.1, it is sufficient to show that $\Phi_M < 0$ for some positive smooth function ψ and positive constant k (concerning the regularity of the metric, see the end of the proof). Let

$$f := \psi^{(n-2)/2}.$$

With respect to the metric g, by standard formulas for conformal transformations (see [5, Chapter 5]), we have

$$\begin{aligned}
R_{g'} &= \frac{1}{\psi}\left(R_g - \frac{2(n-1)}{n-2}\frac{\Delta f}{f} + \frac{n-1}{n-2}\frac{|\nabla f|^2}{f^2}\right),\\
R'_{ij} &= R_{ij} - \frac{f_{ij}}{f} + \frac{n-1}{n-2}\frac{f_i f_j}{f^2} - \frac{1}{n-2}\frac{\Delta f}{f} g_{ij},\\
W'_{ijkt} &= \frac{1}{\psi} W_{ijkt},\\
dV_{g'} &= \psi^{n/2}\, dV_g = f\psi\, dV_g,\\
\nabla^{g'}_{ij}\psi &= \psi_{ij} - \frac{1}{\psi}\left(\psi_i\psi_j - \frac{1}{2}|\nabla\psi|^2 g_{ij}\right).
\end{aligned} \tag{4.1}$$

Moreover, since

$$g'' = g' + d(k\psi)\otimes d(k\psi) = \psi\left[g + d(2k\sqrt{\psi})\otimes d(2k\sqrt{\psi})\right] =: \psi\overline{g},$$

from the conformal invariance of the Weyl curvature and Lemma 3.1, we obtain

$$\begin{aligned}
W'_{ijkt} + E_{g'}(k\psi)_{ijkt} = W''_{ijkt} &= \frac{1}{\psi}\overline{W}_{ijkt}\\
&= \frac{1}{\psi}\left[W_{ijkt} + E_g(2k\sqrt{\psi})_{ijkt}\right]\\
&= W'_{ijkt} + \frac{1}{\psi}E_g(2k\sqrt{\psi})_{ijkt}.
\end{aligned}$$

Therefore, the "error term" of Weyl tensor under Aubin's deformation of the metric satisfies the following *conformal invariance*:

$$E_{g'}(k\psi) = \frac{1}{\psi}E_g(2k\sqrt{\psi}). \tag{4.2}$$

In particular, we have the relations

$$|W_{g'}|_{k\psi} = |W_{g'}|_{g'+d(k\psi)\otimes d(k\psi)} = \frac{1}{\psi}|W_{g'}|_{\overline{g}} = \frac{1}{\psi^2}|W_g|_{\overline{g}}$$

and

$$|E_{g'}(k\psi)|_{k\psi} = \frac{1}{\psi}|E_{g'}(k\psi)|_{\overline{g}} = \frac{1}{\psi^2}|E_g(2k\sqrt{\psi})|_{\overline{g}}.$$

Following the computation in [2], putting all together we obtain

$$\begin{aligned}
\Phi_M = &\int_M \left(R_g + \frac{t}{\psi}|W_g|_{\overline{g}} - \frac{R_{ij}\psi_i\psi_j}{\psi/k^2 + |\nabla\psi|^2} \right) f\, dV_g \\
&+ t\int_M \frac{f}{\psi}|E_g(2k\sqrt{\psi})|_{\overline{g}}\, dV_g \\
&+ \int_M \frac{f_{ij}\psi^i\psi^j}{\psi/k^2 + |\nabla\psi|^2}\, dV_g + \frac{n-1}{n-2}\int_M \frac{|\nabla f|^2}{f}\, dV_g \\
&- \frac{n-1}{n-2}\int_M \frac{|f_i\psi^i|^2}{f(\psi/k^2 + |\nabla\psi|^2)}\, dV_g \\
&+ \frac{1}{n-2}\int_M \frac{\Delta f|\nabla\psi|^2}{\psi/k^2 + |\nabla\psi|^2}\, dV_g \\
&+ \frac{n-1}{n-2}\int_M \left[\frac{\psi_{ip}\psi^p\psi_{iq}\psi^q}{(\psi/k^2 + |\nabla\psi|^2)^2} - \frac{|\psi_{ij}\psi^i\psi^j|^2}{(\psi/k^2 + |\nabla\psi|^2)^3} \right] f\, dV_g \\
&+ \frac{1}{k^2}\frac{n-1}{n-2}\int_M \frac{\frac{1}{4}|\nabla\psi|^6 - |\nabla\psi|^2(\psi_{ij}\psi^i\psi^j)\psi}{(\psi/k^2 + |\nabla\psi|^2)^3} f\, dV_g.
\end{aligned}$$

Moreover, since

$$\int_M \frac{|\nabla f|^2}{f}\, dV_g - \int_M \frac{|f_i\psi^i|^2}{f(\psi/k^2 + |\nabla\psi|^2)}\, dV_g = \frac{1}{k^2}\frac{n-2}{2}\int_M \frac{f_i\psi^i}{\psi/k^2 + |\nabla\psi|^2}\, dV_g,$$

$$\int_M \frac{\Delta f|\nabla\psi|^2}{\psi/k^2 + |\nabla\psi|^2}\, dV_g = -\frac{1}{k^2}\int_M \frac{\psi\Delta f}{\psi/k^2 + |\nabla\psi|^2}\, dV_g,$$

we finally get

$$\begin{aligned}
\Phi_M = &\int_M \left(R_g + \frac{t}{\psi}|W_g|_{\overline{g}} - \frac{R_{ij}\psi_i\psi_j}{\psi/k^2 + |\nabla\psi|^2} \right) f\, dV_g \\
&+ t\int_M \frac{f}{\psi}|E_g(2k\sqrt{\psi})|_{\overline{g}}\, dV_g + \int_M \frac{f_{ij}\psi^i\psi^j}{\psi/k^2 + |\nabla\psi|^2}\, dV_g \\
&+ \frac{1}{k^2}\frac{n-1}{2}\int_M \frac{f_i\psi^i}{\psi/k^2 + |\nabla\psi|^2}\, dV_g - \frac{1}{k^2}\int_M \frac{\psi\Delta f}{\psi/k^2 + |\nabla\psi|^2}\, dV_g \\
&+ \frac{n-1}{n-2}\int_M \left[\frac{\psi_{ip}\psi^p\psi_{iq}\psi^q}{(\psi/k^2 + |\nabla\psi|^2)^2} - \frac{|\psi_{ij}\psi^i\psi^j|^2}{(\psi/k^2 + |\nabla\psi|^2)^3} \right] f\, dV_g \\
&+ \frac{1}{k^2}\frac{n-1}{n-2}\int_M \frac{\frac{1}{4}|\nabla\psi|^6 - |\nabla\psi|^2(\psi_{ij}\psi^i\psi^j)\psi}{(\psi/k^2 + |\nabla\psi|^2)^3} f\, dV_g.
\end{aligned} \tag{4.3}$$

Step 2.

Let $y = y(x)$ be a fixed smooth real function such that

$$\begin{cases} y(-x) = y(x) & \forall\, x \in \mathbb{R} \\ y(x) = 1 & \forall\, |x| \geq 1 \\ y(x) \geq \delta > 0 & \forall\, x \in \mathbb{R} \\ y'(x) > 0 & \forall\, 0 < x < 1 \\ y'(x) \geq 1 & \forall\, (1/4)^{1/(n-1)} \leq x \leq (3/4)^{1/(n-1)}. \end{cases}$$

Let $p \in M$ and consider a local, normal, geodesic polar coordinate system around p: $\rho, \phi_1, \cdots, \phi_{n-1}$. We have $g_{\rho\rho} = 1$, $g_{\rho i} = 0$, $g_{ij} = \delta_{ij} + \rho^2 a_{ij}$, $g^{\rho\rho} = 1$ (from now on, the indices $i = 1, \ldots, n-1$ correspond to the coordinate ϕ_i). The coefficients a_{ij} are of order 1. In particular, we have that the Christoffel symbols of the metric g satisfy

$$\Gamma^{\rho}_{\rho\rho} = 0, \quad \Gamma^{\rho}_{\rho i} = 0, \quad \Gamma^{\rho}_{ij} = -\frac{\rho}{2}\left(a_{ij} + \rho \partial_\rho a_{ij}\right). \tag{4.4}$$

Let $B_r = B_r(p)$ be the geodesic ball centered at p of radius $0 < r < r_0$, with r_0 such that $B_r \subset M$. For $p' \in B_r$, we choose

$$f(p') := y\left(\frac{\rho}{r}\right), \quad \rho = \mathrm{dist}_g(p', p).$$

In particular, from (4.4), we have

$$f_\rho(p') = \frac{1}{r} y'\left(\frac{\rho}{r}\right), \quad f_i(p') = 0, \tag{4.5}$$

$$\begin{aligned} f_{\rho\rho}(p') &= \frac{1}{r^2} y''\left(\frac{\rho}{r}\right), \quad f_{\rho i}(p') = 0, \\ f_{ij}(p') &= \frac{\rho}{2r}\left(a_{ij} + \rho\partial_\rho a_{ij}\right) y'\left(\frac{\rho}{r}\right). \end{aligned} \tag{4.6}$$

From now on, to simplify the expressions, we will omit arguments in the functions: it will be clear that if f, f_ρ, etc. are computed at $p' \in B_r$, then y, y', y'' will be computed at ρ/r with $\rho = \mathrm{dist}_g(p', p)$. Moreover, we will denote by $C = C(n, \delta, t, p) > 0$ some universal positive constant independent of r and k.

Since $0 \le \rho < r$, we have

$$f_\rho = \frac{y'}{r}, \quad f_i = 0, \quad f_{\rho\rho} = \frac{y''}{r^2}, \quad f_{\rho i} = 0, \quad |f_{ij}| \le Crf_\rho \le Cy' \le C.$$

Thus, using that $\psi = f^{2/(n-2)}$ and $0 < \delta \le f \le 1$, we get

$$\begin{aligned} &C^{-1}\frac{y'}{r} \le \psi_\rho \le C\frac{y'}{r}, \quad \psi_i = 0, \quad |\psi_{\rho\rho}| \le \frac{C}{r^2}, \\ &\psi_{\rho i} = 0, \quad |\psi_{ij}| \le Cr\psi_\rho \le Cy' \le C. \end{aligned} \tag{4.7}$$

In particular

$$C^{-1}\frac{(y')^2}{r^2} \le |\nabla\psi|^2 = \psi_\rho^2 \le C\frac{(y')^2}{r^2}.$$

Step 3.

From now on, we consider indices $a, b = \rho, 1, \dots, n-1$, while $i, j = 1, \dots, n-1$. We will estimate the terms in (4.3) not involving the Weyl curvature, restricted to the ball B_r.

We have

$$\begin{aligned} -\frac{R_{ab}\psi^a\psi^b}{\psi/k^2 + |\nabla\psi|^2} &= -\frac{R_{\rho\rho}\psi_\rho^2}{\psi/k^2 + \psi_\rho^2} = -R_{\rho\rho} - \frac{1}{k^2}\frac{\psi R_{\rho\rho}}{\psi/k^2 + \psi_\rho^2} \\ &\le -R_{\rho\rho} + \frac{1}{k^2}\frac{C_1 r^2}{r^2/k^2 + C_2(y')^2} \end{aligned}$$

and thus

$$-\int_{B_r} \frac{R_{ab}\psi_a\psi_b}{\psi/k^2 + |\nabla\psi|^2} f\, dV_g \le C|B_r| + \frac{1}{k^2}\Theta \tag{4.8}$$

where $|B_r|$ denotes the volume of B_r and $\Theta = \Theta(p, 1/k, r) > 0$ will denote a continuous function in $1/k$ and r, for $0 < r < r_0$ and $0 \le 1/k < 1$.

Also

$$\begin{aligned} \frac{f_{ab}\psi^a\psi^b}{\psi/k^2 + |\nabla\psi|^2} &= \frac{f_{\rho\rho}\psi_\rho^2}{\psi/k^2 + \psi_\rho^2} = f_{\rho\rho} - \frac{1}{k^2}\frac{\psi f_{\rho\rho}}{\psi/k^2 + \psi_\rho^2} \\ &\le \frac{y''}{r^2} + \frac{1}{k^2}\frac{C_1}{r^2/k^2 + C_2(y')^2} \end{aligned}$$

and integrating over B_r, we get

$$\int_{B_r} \frac{f_{ab}\psi^a\psi^b}{\psi/k^2 + |\nabla\psi|^2}\, dV_g \le \frac{1}{r^2}\int_{B_r} y''\, dV_g + \frac{1}{k^2}\Theta. \tag{4.9}$$

We have

$$\frac{f_a\psi^a}{\psi/k^2 + |\nabla\psi|^2} \le C\frac{\psi_\rho^2}{\psi/k^2 + \psi_\rho^2} \le C, \quad -\frac{\psi\Delta f}{\psi/k^2 + |\nabla\psi|^2} \le \frac{C_1}{r^2/k^2 + C_2(y')^2}$$

and therefore

$$\frac{1}{k^2}\frac{n-1}{2}\int_{B_r} \frac{f_a\psi^a}{\psi/k^2 + |\nabla\psi|^2}\, dV_g - \frac{1}{k^2}\int_{B_r} \frac{\psi\Delta f}{\psi/k^2 + |\nabla\psi|^2}\, dV_g \le \frac{1}{k^2}\Theta. \tag{4.10}$$

Moreover

$$\begin{aligned}
&\frac{\psi_{ab}\psi^b\psi_{ac}\psi^c}{(\psi/k^2 + |\nabla\psi|^2)^2} - \frac{|\psi_{ab}\psi^a\psi^b|^2}{(\psi/k^2 + |\nabla\psi|^2)^3} \\
&\quad = \frac{\psi_{\rho\rho}^2\psi_\rho^2}{(\psi/k^2 + \psi_\rho^2)^2} - \frac{\psi_{\rho\rho}^2\psi_\rho^4}{(\psi/k^2 + \psi_\rho^2)^3} = \frac{1}{k^2}\frac{\psi\psi_{\rho\rho}^2\psi_\rho^2}{(\psi/k^2 + \psi_\rho^2)^3} \\
&\quad \le \frac{1}{k^2}\frac{C_1}{(r^2/k^2 + C_2(y')^2)^3}
\end{aligned}$$

and thus

$$\frac{n-1}{n-2}\int_{B_r} \left[\frac{\psi_{ab}\psi^b\psi_{ac}\psi^c}{(\psi/k^2 + |\nabla\psi|^2)^2} - \frac{|\psi_{ab}\psi^a\psi^b|^2}{(\psi/k^2 + |\nabla\psi|^2)^3}\right] f\, dV_g \le \frac{1}{k^2}\Theta. \tag{4.11}$$

Finally, reasoning as before, one has

$$\frac{1}{k^2}\frac{n-1}{n-2}\int_{B_r} \frac{\frac{1}{4}|\nabla\psi|^6 - |\nabla\psi|^2(\psi_{ab}\psi^a\psi^b)\psi}{(\psi/k^2 + |\nabla\psi|^2)^3} f\, dV_g \le \frac{1}{k^2}\Theta. \tag{4.12}$$

Therefore, since

$$\int_{B_r} R_g f\, dV_g \le C|B_r|,$$

using (4.8),(4.9),(4.10) and (4.11) in (4.3), we obtain that

$$\begin{aligned}
\Phi_{B_r} \le\, & t\int_{B_r} \frac{f}{\psi}\left(|W_g|_{\overline{g}} + |E_g(2k\sqrt{\psi})|_{\overline{g}}\right) dV_g + C|B_r| \\
& + \frac{1}{r^2}\int_{B_r} y''\, dV_g + \frac{1}{k^2}\Theta,
\end{aligned} \tag{4.13}$$

where Φ_{B_r} denotes the quantity defined in (4.3) restricted to B_r. Note that this intermediate estimate, when $t = 0$, coincides with the one of Aubin in [2].

Step 4.

We now estimate the remaining terms in (4.3) which involve the Weyl curvature. Since

$$\overline{g} = g + d(2k\sqrt{\psi}) \otimes d(2k\sqrt{\psi}),$$

from Lemma 3.1, we have

$$\overline{g}^{\rho\rho} = \frac{1}{1 + 4k^2(\sqrt{\psi})_\rho^2}, \quad \overline{g}^{\rho i} = 0, \quad \overline{g}^{ij} = g^{ij}.$$

Therefore, for any Riemann-type 4-tensor, T, we obtain

$$\begin{aligned} |T_g|_{\overline{g}}^2 = \sum_{i,j,k,t=1}^{n-1} T_{ijkt}^2 + \frac{4}{1 + 4k^2(\sqrt{\psi})_\rho^2} \sum_{i,k,t=1}^{n-1} T_{i\rho kt}^2 \\ + \frac{4}{\left[1 + 4k^2(\sqrt{\psi})_\rho^2\right]^2} \sum_{i,k=1}^{n-1} T_{i\rho k\rho}^2. \end{aligned} \tag{4.14}$$

In particular (this follows immediately from $\overline{g} \geq g$):

$$|W_g|_{\overline{g}} \leq |W_g|_g \quad \text{and} \quad t\int_{B_r} \frac{f}{\psi} |W_g|_{\overline{g}} \, dV_g \leq C|B_r|.$$

From (4.13), we obtain

$$\Phi_{B_r} \leq t\int_{B_r} \frac{f}{\psi} |E_g(2k\sqrt{\psi})|_{\overline{g}} \, dV_g + C|B_r| + \frac{1}{r^2}\int_{B_r} y'' \, dV_g + \frac{1}{k^2}\Theta. \tag{4.15}$$

Concerning the first integral, we have the following key estimate:

Lemma 4.1. *We have*

$$t\int_{B_r} \frac{f}{\psi} |E_g(2k\sqrt{\psi})|_{\overline{g}} \, dV_g \leq C|B_r| + \frac{1}{k^2}\Theta,$$

for some $C = C(n, \delta, t, p) > 0$ *and* $\Theta = \Theta(p, 1/k, r) > 0$ *as above.*

Proof. We set $\eta = 2\sqrt{\psi}$ and $E = E_g(2k\sqrt{\psi}) = E_g(k\eta)$. From (4.7), since $0 < \delta^{2/(n-2)} \leq \psi \leq 1$, we have

$$C^{-1}\frac{y'}{r} \leq \eta_\rho \leq C\frac{y'}{r}, \quad \eta_i = 0, \quad |\eta_{\rho\rho}| \leq \frac{C}{r^2}, \tag{4.16}$$
$$\eta_{\rho i} = 0, \quad |\eta_{ij}| \leq Cr\eta_\rho \leq Cy' \leq C.$$

Firstly, from Lemma 3.1 and (4.16), we get

$$\begin{aligned}
E_{ijkt} &= \frac{k^2}{1+k^2\eta_\rho^2}(\eta_{ik}\eta_{jt} - \eta_{it}\eta_{jk}) \\
&\quad + \frac{k^2\eta_\rho^2}{(1+k^2\eta_\rho^2)(n-2)}(R_{i\rho k\rho}g_{jt} - R_{i\rho t\rho}g_{jk} + R_{j\rho t\rho}g_{ik} - R_{j\rho k\rho}g_{it}) \\
&\quad - \frac{2k^2R_{\rho\rho}\eta_\rho^2}{(1+k^2\eta_\rho^2)(n-1)(n-2)}(g_{ik}g_{jt} - g_{it}g_{jk}) \\
&\quad - \frac{k^2}{(1+k^2\eta_\rho^2)(n-2)}\Big[\big((\Delta\eta)\eta_{ik} - \eta_{ip}\eta_k^p\big)g_{jt} - ((\Delta\eta)\eta_{it} - \eta_{ip}\eta_t^p)g_{jk} \\
&\qquad\qquad + ((\Delta\eta)\eta_{jt} - \eta_{jp}\eta_t^p)g_{ik} - \big((\Delta\eta)\eta_{jk} - \eta_{jp}\eta_k^p\big)g_{it}\Big] \\
&\quad + \frac{k^2}{(1+k^2\eta_\rho^2)(n-1)(n-2)}\Big[(\Delta\eta)^2 - \big|\nabla^2\eta\big|^2\Big](g_{ik}g_{jt} - g_{it}g_{jk}) \\
&\quad + \frac{k^4\eta_\rho^2\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)^2(n-2)}(\eta_{ik}g_{jt} - \eta_{it}g_{jk} + \eta_{jt}g_{ik} - \eta_{jk}g_{it}) \\
&\quad - \frac{2k^4\eta_\rho^2\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)^2(n-1)(n-2)}(\Delta\eta - \eta_{\rho\rho})(g_{ik}g_{jt} - g_{it}g_{jk}).
\end{aligned}$$

Since $\Delta\eta = \eta_{\rho\rho} + \eta_p^p$, we can simplify the expression, obtaining

$$\begin{aligned}
E_{ijkt} &= \frac{k^2}{1+k^2\eta_\rho^2}(\eta_{ik}\eta_{jt} - \eta_{it}\eta_{jk}) \\
&\quad + \frac{k^2\eta_\rho^2}{(1+k^2\eta_\rho^2)(n-2)}(R_{i\rho k\rho}g_{jt} - R_{i\rho t\rho}g_{jk} + R_{j\rho t\rho}g_{ik} - R_{j\rho k\rho}g_{it}) \\
&\quad - \frac{2k^2R_{\rho\rho}\eta_\rho^2}{(1+k^2\eta_\rho^2)(n-1)(n-2)}(g_{ik}g_{jt} - g_{it}g_{jk}) \\
&\quad - \frac{k^2}{(1+k^2\eta_\rho^2)(n-2)}\Big[\big(\eta_p^p\eta_{ik} - \eta_{ip}\eta_k^p\big)g_{jt} - \big(\eta_p^p\eta_{it} - \eta_{ip}\eta_t^p\big)g_{jk} \\
&\qquad\qquad + \big(\eta_p^p\eta_{jt} - \eta_{jp}\eta_t^p\big)g_{ik} - \big(\eta_p^p\eta_{jk} - \eta_{jp}\eta_k^p\big)g_{it}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{k^2}{(1+k^2\eta_\rho^2)(n-1)(n-2)}\big[(\eta_p^p)^2 + 2\eta_{\rho\rho}\eta_p^p - |\eta_{ij}|^2\big](g_{ik}g_{jt} - g_{it}g_{jk}) \\
&- \frac{k^2\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)^2(n-2)}(\eta_{ik}g_{jt} - \eta_{it}g_{jk} + \eta_{jt}g_{ik} - \eta_{jk}g_{it}) \\
&- \frac{2k^4\eta_\rho^2\eta_{\rho\rho}\eta_p^p}{(1+k^2\eta_\rho^2)^2(n-1)(n-2)}(g_{ik}g_{jt} - g_{it}g_{jk}).
\end{aligned}
$$

In particular, we have simplified the fourth block with the sixth one. Coupling the fifth block with the last one, we obtain

$$
\begin{aligned}
E_{ijkt} = &\frac{1}{1/k^2+\eta_\rho^2}(\eta_{ik}\eta_{jt} - \eta_{it}\eta_{jk}) \\
&+ \frac{\eta_\rho^2}{(1/k^2+\eta_\rho^2)(n-2)}(R_{i\rho k\rho}g_{jt} - R_{i\rho t\rho}g_{jk} + R_{j\rho t\rho}g_{ik} - R_{j\rho k\rho}g_{it}) \\
&- \frac{2R_{\rho\rho}\eta_\rho^2}{(1/k^2+\eta_\rho^2)(n-1)(n-2)}(g_{ik}g_{jt} - g_{it}g_{jk}) \\
&- \frac{1}{(1/k^2+\eta_\rho^2)(n-2)}\Big[\big(\eta_p^p\eta_{ik} - \eta_{ip}\eta_k^p\big)g_{jt} - \big(\eta_p^p\eta_{it} - \eta_{ip}\eta_t^p\big)g_{jk} \\
&\qquad + \big(\eta_p^p\eta_{jt} - \eta_{jp}\eta_t^p\big)g_{ik} - \big(\eta_p^p\eta_{jk} - \eta_{jp}\eta_k^p\big)g_{it}\Big] \\
&+ \frac{1}{(1/k^2+\eta_\rho^2)(n-1)(n-2)}\big[(\eta_p^p)^2 - |\eta_{ij}|^2\big](g_{ik}g_{jt} - g_{it}g_{jk}) \\
&- \frac{1}{k^2}\frac{\eta_{\rho\rho}}{(1/k^2+\eta_\rho^2)^2(n-2)}(\eta_{ik}g_{jt} - \eta_{it}g_{jk} + \eta_{jt}g_{ik} - \eta_{jk}g_{it}) \\
&+ \frac{1}{k^2}\frac{2\eta_\rho^2\eta_{\rho\rho}\eta_p^p}{(1/k^2+\eta_\rho^2)^2(n-1)(n-2)}(g_{ik}g_{jt} - g_{it}g_{jk}).
\end{aligned}
$$

Using (4.16), since $|\eta_{ik}\eta_{jt}| \le C\eta_\rho^2$, it is easy to see that the first five blocks are bounded by $C = C(n,\delta,t,p) > 0$ while the last two are controlled by

$$
\frac{1}{k^2}\frac{C_1}{[r^2/k^2 + C_2(y')^2]^2}.
$$

Therefore

$$
|E_{ijkt}| \le C + \frac{1}{k^2}\frac{C_1}{[r^2/k^2 + C_2(y')^2]^2}. \tag{4.17}
$$

Secondly, from Lemma 3.1 and (4.16), we get

$$
E_{i\rho kt} = 0. \tag{4.18}
$$

Lastly, using again Lemma 3.1 and (4.16), we obtain

$$
\begin{aligned}
E_{i\rho k\rho} = {} & \frac{k^2\eta_{ik}\eta_{\rho\rho}}{1+k^2\eta_\rho^2} + \frac{k^2R_{ik}\eta_\rho^2}{n-2} + \frac{k^2Rg_{ik}\eta_\rho^2}{(n-1)(n-2)} + \frac{k^2R_{i\rho k\rho}\eta_\rho^2}{n-2} - \frac{2k^2R_{\rho\rho}g_{ik}\eta_\rho^2}{(n-1)(n-2)} \\
& - \frac{k^2}{n-2}\big[(\Delta\eta)\eta_{ik} - \eta_{ip}\eta_k^p\big] - \frac{k^2g_{ik}\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)(n-2)}(\Delta\eta - \eta_{\rho\rho}) \\
& + \frac{k^2g_{ik}}{(n-1)(n-2)}\Big[(\Delta\eta)^2 - \big|\nabla^2\eta\big|^2\Big] \\
& + \frac{k^4\eta_\rho^2\eta_{ik}\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)(n-2)} - \frac{2k^4g_{ik}\eta_\rho^2\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)(n-1)(n-2)}(\Delta\eta - \eta_{\rho\rho}).
\end{aligned}
$$

Since $\Delta\eta = \eta_{\rho\rho} + \eta_p^p$, we can simplify this expression, obtaining

$$
\begin{aligned}
E_{i\rho k\rho} = {} & \frac{k^2\eta_{ik}\eta_{\rho\rho}}{1+k^2\eta_\rho^2} + \frac{k^2R_{ik}\eta_\rho^2}{n-2} + \frac{k^2Rg_{ik}\eta_\rho^2}{(n-1)(n-2)} + \frac{k^2R_{i\rho k\rho}\eta_\rho^2}{n-2} - \frac{2k^2R_{\rho\rho}g_{ik}\eta_\rho^2}{(n-1)(n-2)} \\
& - \frac{k^2}{n-2}\big[\eta_{\rho\rho}\eta_{ik} + \eta_p^p\eta_{ik} - \eta_{ip}\eta_k^p\big] - \frac{k^2g_{ik}\eta_{\rho\rho}\eta_p^p}{(1+k^2\eta_\rho^2)(n-2)} \\
& + \frac{k^2g_{ik}}{(n-1)(n-2)}\big[(\eta_p^p)^2 + 2\eta_{\rho\rho}\eta_p^p - |\eta_{ij}|^2\big] \\
& + \frac{k^4\eta_\rho^2\eta_{ik}\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)(n-2)} - \frac{2k^4g_{ik}\eta_\rho^2\eta_{\rho\rho}\eta_p^p}{(1+k^2\eta_\rho^2)(n-1)(n-2)} \\
= {} & \frac{k^2\eta_{ik}\eta_{\rho\rho}}{1+k^2\eta_\rho^2} + \frac{k^2R_{ik}\eta_\rho^2}{n-2} + \frac{k^2Rg_{ik}\eta_\rho^2}{(n-1)(n-2)} + \frac{k^2R_{i\rho k\rho}\eta_\rho^2}{n-2} - \frac{2k^2R_{\rho\rho}g_{ik}\eta_\rho^2}{(n-1)(n-2)} \\
& - \frac{k^2}{n-2}\big[\eta_p^p\eta_{ik} - \eta_{ip}\eta_k^p\big] - \frac{k^2g_{ik}\eta_{\rho\rho}\eta_p^p}{(1+k^2\eta_\rho^2)(n-2)} \\
& + \frac{k^2g_{ik}}{(n-1)(n-2)}\big[(\eta_p^p)^2 - |\eta_{ij}|^2\big] + \frac{k^2\eta_{ik}\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)(n-2)} \\
& + \frac{k^2g_{ik}\eta_{\rho\rho}\eta_p^p}{(1+k^2\eta_\rho^2)(n-1)(n-2)}.
\end{aligned}
$$

Rearranging the terms, we get

$$
\begin{aligned}
E_{i\rho k\rho} = {} & \frac{k^2R_{ik}\eta_\rho^2}{n-2} + \frac{k^2Rg_{ik}\eta_\rho^2}{(n-1)(n-2)} + \frac{k^2R_{i\rho k\rho}\eta_\rho^2}{n-2} - \frac{2k^2R_{\rho\rho}g_{ik}\eta_\rho^2}{(n-1)(n-2)} \\
& - \frac{k^2}{n-2}\big[\eta_p^p\eta_{ik} - \eta_{ip}\eta_k^p\big] + \frac{k^2g_{ik}}{(n-1)(n-2)}\big[(\eta_p^p)^2 - |\eta_{ij}|^2\big] \\
& + \frac{n-1}{n-2}\frac{k^2\eta_{ik}\eta_{\rho\rho}}{1+k^2\eta_\rho^2} - \frac{k^2g_{ik}\eta_{\rho\rho}\eta_p^p}{(1+k^2\eta_\rho^2)(n-1)}.
\end{aligned}
$$

Therefore, from (4.16), we deduce

$$|E_{i\rho k\rho}| \leq Ck^2\eta_\rho^2 + \frac{C_1}{r^2/k^2 + C_2(y')^2},$$

and thus

$$\frac{1}{1+k^2\eta_\rho^2}|E_{i\rho k\rho}| \leq C + \frac{1}{k^2}\frac{C_1}{[r^2/k^2 + C_2(y')^2]^2}. \tag{4.19}$$

As a consequence, using (4.14) and (4.17), (4.18), (4.19), we obtain

$$|E_g(2k\sqrt{\psi})|_{\overline{g}} \leq C + \frac{1}{k^2}\frac{C_1}{[r^2/k^2 + C_2(y')^2]^2}$$

which implies

$$t\int_{B_r}\frac{f}{\psi}|E_g(2k\sqrt{\psi})|_{\overline{g}}\,dV_g \leq C|B_r| + \frac{1}{k^2}\Theta,$$

for some $C = C(n,\delta,t,p) > 0$ and $\Theta = \Theta(p,1/k,r) > 0$. □

Step 5.

Using Lemma 4.1 in (4.15), we obtain

$$\Phi_{B_r} \leq C|B_r| + \frac{1}{r^2}\int_{B_r} y''\,dV_g + \frac{1}{k^2}\Theta \tag{4.20}$$

for some $C = C(n,\delta,t,p) > 0$ and $\Theta = \Theta(p,1/k,r) > 0$. Since, $y'(1) = 0$, integrating by parts, we obtain

$$\begin{aligned}\frac{1}{r^2}\int_{B_r} y''\,dV_g &= -\frac{1}{r}\int_{B_r} y'\partial_\rho \log\sqrt{\det g_{ij}}\,dV_g - \frac{n-1}{r}\int_{B_r}\frac{y'}{\rho}\,dV_g \\ &\leq \frac{C}{r}|B_r| - \frac{n-1}{r}\int_{B_r}\frac{y'}{\rho}\,dV_g.\end{aligned}$$

Hence, from (4.20), we get

$$\Phi_{B_r} \leq C\left(1+\frac{1}{r}\right)|B_r| - \frac{n-1}{r}\int_{B_r}\frac{y'}{\rho}\,dV_g + \frac{1}{k^2}\Theta.$$

Using that, by assumption, $y'(x) \geq 1$ for all $(1/4)^{1/(n-1)} \leq x \leq (3/4)^{1/(n-1)}$, we obtain

$$\begin{aligned}\Phi_{B_r} &\leq C\left(1+\frac{1}{r}\right)|B_r| - \frac{n-1}{r}|\mathbb{S}^{n-1}| \inf_M \sqrt{\det g_{ij}} \int_{r(\frac{1}{4})^{1/(n-1)}}^{r(\frac{3}{4})^{1/(n-1)}} \rho^{n-2}\, d\rho + \frac{1}{k^2}\Theta \\ &\leq C\left(1+\frac{1}{r}\right)|B_r| - \frac{C_2}{r^2}|B_r| + \frac{1}{k^2}\Theta,\end{aligned}$$

where we used the fact that $|B_r| \sim cr^n$ as $r \to 0$. In particular, there exist a continuous function $\lambda(p) > 0$ and, for $p \in M$ fixed, a continuous function $\Theta_p(r) > 0$ in r, for $0 < r < r_0$, such that

$$\Theta(p, 1/k, t) \leq \Theta_p(r),$$

and

$$\Phi_{B_r} \leq \left[C\left(1+\frac{1}{r}\right) - \frac{\lambda}{r^2}\right]|B_r| + \frac{1}{k^2}\Theta_p(r). \tag{4.21}$$

Since, by assumption, $F_g = R_g + t|W_g|_g \geq 0$, given $\nu > 0$, there exists a positive radius $0 < r_1 < r_0$ such that

$$\frac{\lambda}{r_1^2} - C\left(1+\frac{1}{r_1}\right) - 1 \geq \nu \overline{F}_g, \tag{4.22}$$

where $\overline{F}_g := \left(\int_M F_g\, dV_g\right)/\mathrm{Vol}_g(M)$. Consider h disjoint geodesic balls $B^j_{r_1}(p_j)$ of radius $r = r_1$ centered at $p_j \in M$, $j = 1, \dots, h$; as well as corresponding functions $f^{[j]}$ and $\psi^{[j]}$, as constructed above. Moreover, for ν sufficiently large, we can assume that

$$\sum_{j=1}^{h} |B^j_{r_1}(p_j)| > \frac{1}{\nu}\mathrm{Vol}_g(M).$$

On every ball B^j, we choose

$$k^2 := \max\left\{1, \sup_{j=1,\dots,h} \frac{\Theta_{p_j}(r_1)}{|B^j_{r_1}(p_j)|}\right\}.$$

From (4.21) and (4.22), for all $j = 1, \dots, h$, we get

$$\Phi_{B^j_{r_1}} \leq -\nu\overline{F}_g|B^j_{r_1}(p_j)| - |B^j_{r_1}(p_j)| + \frac{1}{k^2}\Theta_{p_j}(r_1) \leq -\nu\overline{F}_g|B^j_{r_1}(p_j)|.$$

Now we define f (and ψ accordingly) setting $f \equiv f^{[j]}$ inside the ball B^j and $f \equiv 1$ in the complement of the union of all the balls B^j, $j = 1, \ldots, h$. Therefore, for all $j = 1, \ldots, h$, we obtain

$$\begin{aligned}\Phi_M &\leq \int_M F_g \, dV_g - \nu \overline{F}_g \sum_{j=1}^{h} |B^j_{r_1}(p_j)| \\ &< \overline{F}_g \left(\mathrm{Vol}_g(M) - \nu \sum_{j=1}^{h} |B^j_{r_1}(p_j)| \right) \leq 0.\end{aligned}$$

This concludes the proof of Theorem 1.1. To be precise, we note that the proof above gives a $C^{2,\alpha}$ metric with negative constant scalar-Weyl curvature F. The density of smooth metrics in the space of $C^{2,\alpha}$ metrics (with the $C^{2,\alpha}$ norm) will then give us a smooth metric with negative scalar-Weyl curvature. From Lemma 2.2 we obtain a smooth metric with constant negative scalar-Weyl curvature. □

Acknowledgments. *The author is member of the "Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni" (GNSAGA) of the Istituto Nazionale di Alta Matematica (INdAM).*

References

[1] T. Aubin, Sur la courbure scalaire des variétés riemanniennes compactes, C. R. Acad. Sci. Paris **262** (1966) 130–133.

[2] T. Aubin, Métriques riemanniennes et courbure. J. Diff. Geom. **4** (1970), 383–424.

[3] J. Bland, M. Kalka, Negative scalar curvature metrics on non-compact manifolds, Trans. A.M.S., **2** (1989), 433–446.

[4] R. Brooks, A construction of metrics of negative Ricci curvature, J. Diff. Geom. **29** (1989), 85–94.

[5] G. Catino, P. Mastrolia, A Perspective on Canonical Riemannian Metrics, Progress in Mathematics **336** (2020), Birkhäuser Basel.

[6] G. Catino, P. Mastrolia, D. Monticelli, F. Punzo, Four dimensional closed manifolds admit a weak harmonic Weyl metric, Comm. Cont. Math. **25** (2023), no. 8, 2250047.

[7] L.-Z. Gao, S.-T. Yau, The existence of negatively Ricci curved metrics on three mani- folds, Invent. Math. **85** (1986), 637–652.

[8] M. Gromov, H.B Lawson, The classification of simply connected manifolds of positive scalar curvature, Ann. Math. **111** (1980), no. 3, 423–434.

[9] M. Gursky, Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere, Math. Ann. **318** (2000), 417–431.

[10] M. J. Gursky, C. LeBrun, Yamabe invariants and spin^c structures, Geom. Funct. Anal. **8** (1998), 965–977.

[11] M. J. Gursky and C. LeBrun, On Einstein manifolds of positive sectional curvature, Annals Global Anal. Geom. **17** (1999), 315–328.

[12] M. Itoh, The modified Yamabe problem and geometry of modified scalar curvatures, J. Geom. Anal. **15** (2005), 63–81.

[13] M.-L. Labbi, On compact manifolds with positive isotropic curvature, Proc. Amer. Math. Soc. **128** (2000) no. 5, 1467–1474.

[14] C. LeBrun, Ricci curvature, minimal volumes, and Seiberg-Witten theory, Invent. Math. **145** (2001), 279–316.

[15] J. Lohkamp, Metrics of negative Ricci curvature, Ann. of Math. **140** (1994), no. 3, 655–683.

[16] M. J. Micallef and J. D. Moore, Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes, Ann. of Math. **127** (1988), 199–227.

[17] R. Schoen, S.-T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math. **28** (1979), no. 1, 159–183.

[18] H. Seshadri, A note on negative isotropic curvature, Math. Res. Lett. **11** (2004), 365–375.

Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo da Vinci 32, 20133 Milano, Italy
E-mail address: giovanni.catino@polimi.it

Received February 26, 2021
Accepted October 19, 2021