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Construction of the moduli space of Higgs bundles using analytic methods

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It is a folklore theorem that the Kuranishi slice method can be used to construct the moduli space of semistable Higgs bundles on a closed Riemann surface as a complex space. The purpose of this paper is to provide a proof in detail. We also give a direct proof that the moduli space is locally modeled on an affine GIT quotient of a quadratic cone by a complex reductive group.

1. Introduction

Let X be a closed Riemann surface with genus $g \geq 2$. Introduced by Hitchin in the seminal paper [16], a Higgs bundle on X is a pair $(\mathscr{E}, \Phi)$ consisting of a holomorphic bundle $\mathscr{E} \to X$ and a holomorphic section $\Phi \in H^0(\operatorname{End}\mathscr{E} \otimes \mathscr{K}_X)$, where $\mathscr{K}_X$ is the canonical bundle of X. To obtain a nice moduli space, we recall that a Higgs bundle $(\mathscr{E}, \Phi)$ is stable if $\mu(\mathscr{F}) < \mu(\mathscr{E})$ for every Φ-invariant holomorphic subbundle $0 \subsetneq \mathscr{F} \subsetneq \mathscr{E}$, where $\mu(\mathscr{F})$ is the slope of $\mathscr{F}$. The semistability is defined by replacing $\mu(\mathscr{F}) < \mu(\mathscr{E})$ by $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$.

Finally, $(\mathscr{E}, \Phi)$ is polystable if it is a direct sum of stable Higgs bundles with the same slope. In [16], Hitchin used the Kuranishi slice method to construct the moduli space of stable Higgs bundles first as a smooth manifold and then as a hyperKähler manifold. Such a method was first introduced by Kuranishi in [23] and has been used in several papers to construct moduli spaces in different contexts (for example, see [3, 4, 21, 24] and [20, Chapter 7]). On the other hand, the moduli space of semistable Higgs bundles was constructed by Nitsure in [27] where X is a smooth projective curve and by Simpson in [32] where X is a smooth projective variety. They both used Geometric Invariant Theory (GIT for short), and the method is entirely algebro-geometric. As a consequence, the resulting moduli space is a quasi-projective variety.

It is a folklore theorem that the Kuranishi slice method can be used to construct the moduli space of semistable Higgs bundles as a complex space (for example, see [5, 37]). The purpose of this paper is to provide a proof in detail. More precisely, the problem is stated as follows. Fix a smooth Hermitian vector bundle $E \to X$ and let $\mathfrak{g}_E \to X$ be the bundle of skew-Hermitian endomorphisms of E. For convenience, we assume that the degree of E is zero. This condition is not essential. By the Newlander-Nirenberg theorem, a holomorphic structure on E (described by holomorphic transition functions) is equivalent to an integrable Dolbeault operator $\overline{\partial}_E$. Since $\dim_{\mathbb{C}} X = 1$, the integrability condition is vacuous. Therefore, via the Chern correspondence, the space of holomorphic structures on E can be identified with the space $\mathscr{A}$ of unitary connections on E, which is an infinite-dimensional affine space modeled on $\Omega^1(\mathfrak{g}_E)$. Let $\mathscr{C} = \mathscr{A} \times \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$. Then, the configuration space of Higgs bundles (with a fixed underlying smooth bundle E) is defined as

$$\mathscr{B} = \{(A, \Phi) \in \mathscr{C} \colon \overline{\partial}_A \Phi = 0\} \tag{1.1}$$

(see [37] for more details). Since the complex gauge group $\mathscr{G}^{\mathbb{C}} = \mathrm{Aut}(E)$ naturally acts on the space of holomorphic structures of E, it acts on $\mathscr{A}$ and hence also on $\mathscr{C}$. Then, two Higgs bundles are isomorphic if and only if they are in the same $\mathscr{G}^{\mathbb{C}}$-orbit. Let $\mathscr{B}^{ss}$, $\mathscr{B}^{s}$ and $\mathscr{B}^{ps}$ be the subspaces of $\mathscr{B}$ consisting of semistable, stable and polystable Higgs bundles, respectively. They are $\mathscr{G}^{\mathbb{C}}$-invariant. The moduli space of semistable Higgs bundles is defined as the quotient $\mathscr{M} = \mathscr{B}^{ps}/\mathscr{G}^{\mathbb{C}}$ equipped with the C^∞-topology. Our main result is the following.

Theorem A. *The moduli space $\mathscr{M}$ is a normal complex space.*

More can be said about the local structure of $\mathscr{M}$. To state the theorem, we need some preparation. Recall that the space $\mathscr{C}$ has a natural L^2-metric g and a compatible complex structure I given by multiplication by $\sqrt{-1}$ (see [16, §6]). Let $\mathscr{G}$ be the subgroup of $\mathscr{G}^{\mathbb{C}}$ consisting of unitary gauge transformations. Then, the $\mathscr{G}$-action on $\mathscr{C}$ is Hamiltonian with respect to the Kähler form $\Omega_I = g(I\cdot, \cdot)$. Hitchin's equation can be interpreted as a moment map

$$\mu(A, \Phi) = F_A + [\Phi, \Phi^*]. \tag{1.2}$$

Then, the Hitchin-Kobayashi correspondence (see [16, 29]) states that a Higgs bundle is polystable if and only if its $\mathscr{G}^{\mathbb{C}}$-orbit intersects $\mu^{-1}(0)$. Moreover, the inclusion $\mu^{-1}(0) \cap \mathscr{B} \hookrightarrow \mathscr{B}^{ps}$ induces a homeomorphism

$$(\mu^{-1}(0) \cap \mathscr{B})/\mathscr{G} \xrightarrow{\sim} \mathscr{B}^{ps}/\mathscr{G}^{\mathbb{C}} \tag{1.3}$$

whose inverse is induced by the retraction $r\colon \mathscr{B}^{ss} \to \mu^{-1}(0)$ defined by the Yang-Mills-Higgs flow (see [38]). Finally, we recall the deformation complex for a Higgs bundle (A, Φ):

$$C_{\mu_{\mathbb{C}}}\colon \qquad \Omega^0(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}), \tag{1.4}$$

where $D'' = \overline{\partial}_A + \Phi$. It is an elliptic complex. Let K be the $\mathscr{G}$-stabilizer at (A, Φ). Since the $\mathscr{G}$-action is proper, K is a compact Lie group. Moreover, its complexification $K^{\mathbb{C}}$ is precisely the $\mathscr{G}^{\mathbb{C}}$-stabilizer at (A, Φ) (see Section 3) and acts on $\mathbf{H}^1$ linearly. Then, the local structure of $\mathscr{M}$ is described as follows.

Theorem B. *Let $[A, \Phi] \in \mathscr{M}$ be a point such that $\mu(A, \Phi) = 0$ and $\mathbf{H}^1$ its deformation space, the harmonic space $\mathbf{H}^1(C_{\mu_{\mathbb{C}}})$ defined in $C_{\mu_{\mathbb{C}}}$. Then, the following hold:*

1) *$\mathbf{H}^1$ is a complex symplectic vector space.*
2) *The $K^{\mathbb{C}}$-action on $\mathbf{H}^1$ is complex Hamiltonian with a complex moment map given by*

$$\nu_{0,\mathbb{C}}(x) = \frac{1}{2} H[x, x], \tag{1.5}$$

where H is the harmonic projection defined in $C_{\mu_{\mathbb{C}}}$.

3) *Around* $[A, \Phi]$, *the moduli space* $\mathscr{M}$ *is locally biholomorphic to an open neighborhood of* $[0]$ *in the complex symplectic quotient* $\nu_{0,\mathbb{C}}^{-1}(0) /\!/ K^{\mathbb{C}}$, *which is an affine GIT quotient.*

There are two reasons why this result is not surprising. In [32, §10], Simpson proved that the differential graded Lie algebra $C_{\mu_{\mathbb{C}}}$ is *formal*. As a consequence, the moduli space is locally biholomorphic to a GIT quotient of a quadratic cone in $H^1(C_{\mu_{\mathbb{C}}})$ by a complex reductive group. Another reason is the following. Recall that $\mathscr{C}$ is more than just a Kähler manifold. It has a hyperKähler structure (see [16, §6]) and admits a complex moment map $\mu_{\mathbb{C}}(A, \Phi) = \overline{\partial}_A \Phi$ for the $\mathscr{G}^{\mathbb{C}}$-action. Hence, the moduli space $\mathscr{M}$ is homeomorphic, by the Hitchin-Kobayashi correspondence, to a singular hyperKähler quotient. Then, Theorem B is an infinite-dimensional generalization of Theorem 1.4(iv) in Mayrand [25] to Higgs bundles. We will extend all other statements in Theorem 1.4 to $\mathscr{M}$ in a forthcoming paper.

The major step in the proof of Theorem A and B is to construct a Kuranishi local model for $\mathscr{M}$ at every Higgs bundle (A, Φ) that satisfies Hitchin's equation. This is done in Section 3. Here, a Kuranishi local model is the analytic GIT quotient (developed by Heinzner and Loose in [15]) of a Kuranishi space in $\mathbf{H}^1$ by the $\mathscr{G}^{\mathbb{C}}$-stabilizer at (A, Φ), and is homeomorphic to an open neighborhood of (A, Φ) in $\mathscr{M}$. After that, we will show that the transition functions associated with Kuranishi local models are holomorphic so that $\mathscr{M}$ is a complex space. This is done in Section 4. To prove Theorem B, we adapt Huebschmann's argument in [17, Corollary 2.20] which is further based on Arms-Marsden-Moncrief [1]. This is done in Section 5.

The techniques in the construction of Kuranishi local models mainly come from [34], [9] and [19]. Let K be the $\mathscr{G}$-stabilizer at (A, Φ) with $\mu(A, \Phi) = 0$ so that $K^{\mathbb{C}}$ is the $\mathscr{G}^{\mathbb{C}}$-stabilizer. We will construct a K-equivariant perturbed Kuranishi map Θ (following Székelyhidi's argument in [34, Proposition 7]) that is defined on a Kuranishi space in $\mathbf{H}^1$ and takes values in $\mathscr{B}^{ss}$ such that the pullback moment map $\Theta^*\mu$ is a moment map for the K-action on $\mathbf{H}^1$ with respect to the pullback symplectic form $\Theta^*\Omega_I$. Then, roughly speaking, a $K^{\mathbb{C}}$-orbit is closed in $\mathbf{H}^1$ if and only if it contains a zero of the pullback moment map $\Theta^*\mu$. The precise statement is given in Theorem 3.6 (cf. [9, Theorem 2.9], [19, Proposition 3.8], [6, Proposition 2.4] and [36, Proposition 3.3.2]). Since the perturbed Kuranishi map Θ is no longer holomorphic, $\Theta^*\Omega_I$ is not a Kähler form on $\mathbf{H}^1$, which causes some trouble. To remedy this problem, in the proof of Theorem 3.6, the Yang-Mills-Higgs flow will be used to detect polystable orbits in $\mathscr{B}^{ss}$. Since Kuranishi spaces are locally complete, every Yang-Mills-Higgs flow near (A, Φ)

induces a "reduced flow" in $\mathbf{H}^1$ that stays in a single $K^{\mathbb{C}}$-orbit and converges to a zero of Θ. Therefore, if a $K^{\mathbb{C}}$-orbit is closed, it contains a zero of Θ. Hence, Θ maps polystable $K^{\mathbb{C}}$-orbits in $\mathbf{H}^1$ to polystable orbits in $\mathscr{B}^{ss}$ so that Θ induces a map from a Kuranishi local model to $\mathscr{M}$. The rest of the proof is to show that this map is an open embedding.

After the construction of the moduli space $\mathscr{M}$, it is natural to compare the analytic and the algebraic moduli spaces. More precisely, let us also use $\mathscr{M}_{an}$ to mean the quotient $\mathscr{B}^{ps}/\mathscr{G}^{\mathbb{C}}$ and $\mathscr{M}_{alg}$ the moduli space of semistable Higgs bundles of rank r and degree 0 in the category of schemes, where r is the rank of E. By construction, $\mathscr{M}_{alg}$ parametrizes S-equivalence classes of Higgs bundles. Let us recall the definition of S-equivalence. Every semistable Higgs bundle $(\mathscr{E}, \Phi)$ admits a filtration, called the Seshadri filtration, whose successive quotients are stable, all with slope $\mu(E)$. Let $\mathrm{Gr}(\mathscr{E}, \Phi)$ be the graded object associated with the Seshadri filtration of $(\mathscr{E}, \Phi)$. It is uniquely determined by the isomorphism class of $(\mathscr{E}, \Phi)$. Then, two Higgs bundles $(\mathscr{E}_1, \Phi_1)$ and $(\mathscr{E}_2, \Phi_2)$ are S-equivalent if $\mathrm{Gr}(\mathscr{E}_1, \Phi_1)$ and $\mathrm{Gr}(\mathscr{E}_2, \Phi_2)$ are isomorphic as Higgs bundles. As a consequence, there is a natural comparison map $i\colon \mathscr{M}_{an} \to \mathscr{M}_{alg}$ of the underlying sets that sends each $\mathscr{G}^{\mathbb{C}}$-orbit of a point (A, Φ) in $\mathscr{B}^{ps}$ to the S-equivalence class of the Higgs bundle $(\mathscr{E}_A, \Phi)$ defined by (A, Φ). The following result will be proved in Section 6.

Theorem C. *The comparison map $i\colon \mathscr{M}_{an} \to \mathscr{M}_{alg}$ is a biholomorphism.*

The outline of the proof is the following. It is easy to see that i is a bijection. To show that it is continuous, recall that Nitsure constructed a scheme F^{ss} in [27] that parameterizes semistable Higgs bundles on X, and $\mathscr{M}_{alg}$ is a good quotient of F^{ss}. We show that the comparison map i can be locally lifted to a map σ, called a *classifying map*, that is defined locally on $\mathscr{B}^{ss}$ and takes values in F^{ss}. Here, the terminology comes from Sibley and Wentworth's paper [28], and we adapt the proof of Theorem 6.1 in this paper to show that σ is continuous with respect to the C^∞-topology on $\mathscr{B}^{ss}$ and the analytic topology on F^{ss}. Therefore, i is continuous. By the properness of the Hitchin fibration defined on $\mathscr{M}_{an}$, we see that i is proper and hence a homeomorphism. Then, by constructing Kuranishi families of stable Higgs bundles, we show that the restriction $i\colon \mathscr{M}^s_{an} \to \mathscr{M}^s_{alg}$ is a biholomorphism, where $\mathscr{M}^s_{an}$ and $\mathscr{M}^s_{alg}$ are the open subsets of $\mathscr{M}_{an}$ and $\mathscr{M}_{alg}$ consisting of stable Higgs bundles, respectively. By the normality of $\mathscr{M}_{alg}$, the holomorphicity of $i^{-1}|_{\mathscr{M}^s_{alg}}$ can be extended to i^{-1}. Then, we use Theorem B to prove that $\mathscr{M}_{an}$ is normal. The rest of the proof follows from the fact that a holomorphic bijection between normal, reduced and irreducible complex spaces of the same dimension is a biholomorphism.

After this paper was complete, we became aware of Buchdahl and Schumacher's paper [7]. Note that Theorem 3.6 is similar to [7, Theorem 3], which applies to holomorphic vector bundles over a compact Kähler manifold. However, our approach is different. In this paper, the Yang-Mills-Higgs flow plays a major role. Since $\dim_{\mathbb{C}} X = 1$, the necessary analytic inputs are from Wilkin [38]. By contrast, the Yang-Mills flow is not involved in Buchdahl and Schumacher's argument. It is expected that Buchdahl and Schumacher's argument can be adapted to the case of Higgs bundles and used to provide another proof of Theorem A, possibly without the assumption that $\dim_{\mathbb{C}} X = 1$.

Finally, we remark that we only work with reduced complex spaces in this paper. The reason is that the analytic GIT developed by Heinzner and Loose in [15] only applies to reduced complex spaces.

2. Deformation complexes

In this section, after reviewing the deformation complex for Higgs bundles, we introduce another useful Fredholm complex that will be used later. Let $(A, \Phi) \in \mathscr{B}$ such that $\mu(A, \Phi) = 0$. Then, consider the deformation complex

$$C_{\mu_{\mathbb{C}}}: \qquad \Omega^0(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}), \tag{2.1}$$

where $D'' = \overline{\partial}_A + \Phi$. Recall that $C_{\mu_{\mathbb{C}}}$ is obtained by linearizing the equation $\overline{\partial}_A \Phi = 0$ and the $\mathscr{G}^{\mathbb{C}}$-action.

Proposition 2.1 ([30, §1] and [32, §10]). *$C_{\mu_{\mathbb{C}}}$ is an elliptic complex and a differential graded Lie algebra. Moreover, the Kähler identities,*

$$(D'')^* = -i[*, D'], \qquad (D')^* = +i[*, D''], \tag{2.2}$$

hold, where $D' = \partial_A + \Phi^*$ and $*$ is the Hodge star.

There is another useful sequence

$$C_{\mu}: \qquad \Omega^0(\mathfrak{g}_E) \xrightarrow{d_1} \ker D'' \xrightarrow{d_2} \Omega^2(\mathfrak{g}_E), \tag{2.3}$$

where d_2 is the derivative of μ from (1.2) at (A, Φ), and $d_1(u) = (d_A u, [\Phi, u])$. The operator d_2, viewed as a map $\Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \to \Omega^2(\mathfrak{g}_E)$, has a surjective symbol. Hence, $d_2 d_2^* \colon \Omega^2(\mathfrak{g}_E) \to \Omega^2(\mathfrak{g}_E)$ is a self-adjoint elliptic operator. As a consequence, the Hodge decomposition

$$\Omega^2(\mathfrak{g}_E^{\mathbb{C}}) = \operatorname{im} d_2 d_2^* \oplus \ker d_2 d_2^*, \tag{2.4}$$

holds. Moreover, since $d_2 (D'')^* = 0$ and

$$\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) = \ker D'' \oplus \operatorname{im}(D'')^*, \tag{2.5}$$

we have

$$d_2(\ker D'') = d_2(\Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})) \tag{2.6}$$

(In this paper, we routinely identify $\Omega^1(\mathfrak{g}_E)$ with $\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$ using the map $\alpha \mapsto \alpha''$, where α'' is the $(0,1)$-component of α). As a consequence, the natural map $\ker d_2^* \to H^2(C_\mu)$ is an isomorphism. We denote $\ker d_2^*$ by $\mathbf{H}^2(C_\mu)$. Finally, we note that $H^1(C_\mu)$ is equal to the first cohomology of the following elliptic complex that is used by Hitchin in [16, p. 85]

$$C_{Hit}\colon \quad \Omega^0(\mathfrak{g}_E) \xrightarrow{d_1} \Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{d_2 \oplus D''} \Omega^2(\mathfrak{g}_E) \oplus \Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}). \tag{2.7}$$

In fact, by direct computation, the identification $\Omega^1(\mathfrak{g}_E) \xrightarrow{\sim} \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$ induces an isomorphism $\mathbf{H}^1(C_{Hit}) \xrightarrow{\sim} \mathbf{H}^1(C_{\mu_{\mathbb{C}}})$. Therefore, in the rest of the paper, if no confusion can appear, we will simply use $\mathbf{H}^1$ to mean the harmonic space $\mathbf{H}^1(C_{\mu_{\mathbb{C}}})$. In summary, we have obtained

Proposition 2.2. *The sequence C_μ is a Fredholm complex with Hodge decomposition*

$$\Omega^2(\mathfrak{g}_E) = \mathbf{H}^2(C_\mu) \oplus \operatorname{im} d_2. \tag{2.8}$$

Lastly, note that the natural non-degenerate pairing $\Omega^0(\mathfrak{g}_E) \times \Omega^2(\mathfrak{g}_E) \to \mathbb{R}$ restricts to a non-degenerate pairing $\mathbf{H}^0(C_\mu) \times \mathbf{H}^2(C_\mu) \to \mathbb{R}$ so that $\mathbf{H}^2(C_\mu)$ can be identified with the dual space $\mathbf{H}^0(C_\mu)^*$ of $\mathbf{H}^0(C_\mu)$.

3. Kuranishi local models

3.1. Kuranishi maps

A crucial ingredient in the Kuranishi slice method is the Kuranishi maps. They relate polystable orbits in $\mathbf{H}^1$ and polystable orbits in $\mathscr{B}$. Moreover, they eventually induce local charts for the moduli space. To construct Kuranishi maps, we need to use the implicit function theorem, and it is a standard practice to work with the Sobolev completions of relevant spaces. In this paper, we will use Y_k to mean the completion of the space Y with respect to the Sobolev L^2_k-norm. For example, $\Omega^*(\mathfrak{g}_E)_k$ means the completion of $\Omega^*(\mathfrak{g}_E)$ with respect to the L^2_k-norm. Otherwise, we generally use C^∞-topology. Fix $k > 1$.

Now, we describe the Kuranishi maps. Let $(A, \Phi) \in \mathscr{B}$ with $\mu(A, \Phi) = 0$. Recall that $\mathscr{G}^{\mathbb{C}}_{k+1}$ and $\mathscr{G}_{k+1}$ are Hilbert Lie groups and act smoothly on the Hilbert affine manifold $\mathscr{C}_k$. Moreover, the $\mathscr{G}_{k+1}$-action on $\mathscr{C}_k$ is proper (see [11, Section 4.4]). Therefore, if K is the $\mathscr{G}_{k+1}$-stabilizer at (A, Φ), then K is a compact Lie group with Lie algebra $\mathbf{H}^0(C_\mu)$. The following result relates the $\mathscr{G}^{\mathbb{C}}_{k+1}$-stabilizer to the $\mathscr{G}_{k+1}$-stabilizer at (A, Φ).

Proposition 3.1. *The $\mathscr{G}^{\mathbb{C}}_{k+1}$-stabilizer at (A, Φ) is the complexification of K and acts on $\mathbf{H}^1$.*

Proof. This follows from [33, Proposition 1.6]. The rest follows from direct computation. □

If $\mathbf{H}^2(C_{\mu_{\mathbb{C}}}) = 0$, then the implicit function theorem implies that $\mathscr{B}_k$ is locally a complex manifold around (A, Φ). In general, following Lyapunov-Schmidt reduction, we consider

$$\widetilde{\mathscr{B}}_k = [(1 - H)\mu_{\mathbb{C}}]^{-1}(0) \subset \mathscr{C}_k, \tag{3.1}$$

where H is the harmonic projection defined in the elliptic complex $C_{\mu_{\mathbb{C}}}$. By construction, the derivative of $(1 - H)\mu_{\mathbb{C}}$ at (A, Φ) is surjective. Hence, $\widetilde{\mathscr{B}}_k$ is locally a complex manifold around (A, Φ). To parameterize $\widetilde{\mathscr{B}}_k$, consider the map

$$\begin{gathered} F\colon \Omega^{0,1}(\mathfrak{g}^{\mathbb{C}}_E)_k \oplus \Omega^{1,0}(\mathfrak{g}^{\mathbb{C}}_E)_k \to \Omega^{0,1}(\mathfrak{g}^{\mathbb{C}}_E)_k \oplus \Omega^{1,0}(\mathfrak{g}^{\mathbb{C}}_E)_k, \\ F(\alpha, \eta) = (\alpha, \eta) + (D'')^* G[\alpha'', \eta], \end{gathered} \tag{3.2}$$

where α'' is the $(0,1)$-part of α. It has the following properties.

Lemma 3.2.

1) *F is $K^{\mathbb{C}}$-equivariant.*
2) *F is a local biholomorphism around* 0.
3) $D''F(\alpha,\eta) = (1-H)\mu_{\mathbb{C}}(A+\alpha,\Phi+\eta)$.
4) $(D'')^*F(\alpha,\eta) = (D'')^*(\alpha,\eta)$.

Proof. (1) follows from the fact that the $K^{\mathbb{C}}$-action commutes with $(D'')^*$ and G. Since the derivative of F at 0 is the identity map, the inverse function theorem implies (2). Since $(D'')^*(D'')^* = 0$, (4) follows. To prove (3), we compute

$$
\begin{aligned}
&(1-H)\mu_{\mathbb{C}}(A+\alpha,\Phi+\eta)\\
&= D''(D'')^*G(D''(\alpha,\eta)+[\alpha'',\eta])\\
&= D''((\alpha,\eta)-H(\alpha,\eta)-D''(D'')^*G(\alpha,\eta)+(D'')^*G[\alpha'',\eta])\\
&= D''((\alpha,\eta)+(D'')^*G[\alpha,\eta])\\
&= D''F(\alpha,\eta).
\end{aligned}
\tag{3.3}
$$

□

As a consequence, F induces a well-defined map,

$$F\colon \widetilde{\mathscr{B}}_k \cap [(A,\Phi)+\ker(D'')^*] \to \ker D'' \cap \ker(D'')^* = \mathbf{H}^1. \tag{3.4}$$

Since $\widetilde{\mathscr{B}}_k$ and $(A,\Phi)+\ker(D'')^*$ intersect transversely at (A,Φ), their intersection is locally a complex manifold around (A,Φ). Hence, there are an open ball $U \subset \mathbf{H}^1$ in the L^2-norm around 0 and an open neighborhood $\widetilde{U}$ of (A,Φ) in $\widetilde{\mathscr{B}}_k \cap [(A,\Phi)+\ker(D'')^*]$ such that $F\colon \widetilde{U} \to U$ is a biholomorphism. The *Kuranishi map* θ is defined as its inverse viewed as a map $\theta\colon U \hookrightarrow \mathscr{C}_k$, and the *Kuranishi space* is defined as $Z := \theta^{-1}(\mathscr{B} \cap \widetilde{U})$. More concretely, by the construction of $\widetilde{\mathscr{B}}_k$,

$$Z := \{x \in U \colon H[\theta(x),\theta(x)] = 0\}. \tag{3.5}$$

Here, (A,Φ) serves as the origin in the affine manifold $\mathscr{C}_k$. Clearly, Z is a closed complex subspace of U. Moreover, since $\mathscr{B}_k^{ss}$ is open in $\mathscr{B}_k$ (see [38, Theorem 4.1]), by shrinking U and hence Z if necessary, we may assume that $\theta(Z) \subset \mathscr{B}_k^{ss}$.

The next result shows that the Kuranishi space Z is locally complete.

Proposition 3.3. *The map*

$$T\colon \mathbf{H}^0(C_\mu)^\perp_{k+1} \times \mathbf{H}^2(C_\mu)^\perp_{k+1} \times [((A,\Phi)+\ker(D'')^*)\cap \mathscr{B}_k^{ss}] \to \mathscr{B}_k^{ss}, \quad T(u,\beta,B,\Psi) = (B,\Psi)\cdot \exp(-i*\beta)\exp(u), \tag{3.6}$$

is a local homeomorphism around $(0,0,A,\Phi)$*. As a consequence, there exists an open neighborhood* W *of* (A,Φ) *in* $\mathscr{B}_k^{ss}$ *such that the* $\mathscr{G}_{k+1}^{\mathbb{C}}$*-orbit of every* $(B,\Psi)\in W$ *intersects the image* $\theta(Z)$*.*

Proof. Consider the map

$$T\colon \mathbf{H}^0(C_\mu)^\perp_{k+1} \times \mathbf{H}^2(C_\mu)^\perp_{k+1} \times ((A,\Phi)+\ker(D'')^*) \to \mathscr{C}_k, \quad T(u,\beta,B,\Psi) = (B,\Psi)\cdot \exp(-i*\beta)\exp(u), \tag{3.7}$$

where $\mathbf{H}^0(C_\mu)^\perp$ and $\mathbf{H}^2(C_\mu)^\perp$ are the L^2-orthogonal complements of $\mathbf{H}^0(C_\mu)$ and $\mathbf{H}^2(C_\mu)$ in $\Omega^0(\mathfrak{g}_E)$ and $\Omega^2(\mathfrak{g}_E)$, respectively. Its derivative at $(0,0,A,\Phi)$ is given by

$$\begin{aligned} d_{(0,0,A,\Phi)}T(u,\beta,x) &= \frac{d}{dt}\Big|_{t=0} T(tu,t\beta,(A,\Phi)+tx) \\ &= \frac{d}{dt}\Big|_{t=0} (A,\Phi)+tx)\cdot\exp(-i*t\beta)\cdot\exp(tu) \\ &= x + \frac{d}{dt}\Big|_{t=0}(A,\Phi)\cdot\exp(-i*t\beta) \\ &\quad + \frac{d}{dt}\Big|_{t=0}(A,\Phi)\cdot\exp(tu) \\ &= x + D''(-i*\beta) + D''u \\ &= D''(u-i*\beta)+x. \end{aligned} \tag{3.8}$$

Note that

$$\mathbf{H}^0(C_\mu)^\perp \oplus i*\mathbf{H}^2(C_\mu)^\perp = \mathbf{H}^0(C_\mu)^\perp \oplus i\mathbf{H}^0(C_\mu)^\perp = \mathbf{H}^0(C_{\mu_\mathbb{C}})^\perp. \tag{3.9}$$

Since

$$\Omega^{0,1}(\mathfrak{g}_E^\mathbb{C})_k \oplus \Omega^{1,0}(\mathfrak{g}_E^\mathbb{C})_k = \ker(D'')^* \oplus \operatorname{im} D'', \tag{3.10}$$

we conclude that $d_{(0,A,\Phi)}T$ is an isomorphism. Hence, the inverse function theorem implies that there are open neighborhoods $N_1\times N_2\times V$ of

$(0, 0, A, \Phi)$ and W of (A, Φ) such that $T\colon N_1 \times N_2 \times V \to W$ is a diffeomorphism. Since $\mathscr{B}_k^{ss}$ is $\mathscr{G}_{k+1}^{\mathbb{C}}$-invariant, we conclude that

$$T\colon N_1 \times N_2 \times (V \cap \mathscr{B}_k^{ss}) \to W \cap \mathscr{B}_k^{ss} \tag{3.11}$$

is a homeomorphism. Finally, if $U \subset \mathbf{H}^1$ is sufficiently small, then θ is a homeomorphism from Z to $V \cap \mathscr{B}_k^{ss}$. □

Moreover, θ maps $K^{\mathbb{C}}$-orbits to $\mathscr{G}^{\mathbb{C}}$-orbits in the following way.

Proposition 3.4 (cf. [8, Lemma 6.1]). *If U is sufficiently small, then the following hold:*

1) *If $x_1, x_2 \in U$ are such that $x_1 = x_2 g$ for some $g \in K^{\mathbb{C}}$, then $\theta(x_1) = \theta(x_2)g$. Hence, if $x_1 \in Z$, then $x_2 \in Z$.*
2) *Conversely, if $d_x\theta(v) = u^{\#}_{\theta(x)}$ for some $u \in \Omega^0(\mathfrak{g}_E^{\mathbb{C}})_{k+1}$, then $u \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$, and $v = u^{\#}_x$, where $u^{\#}$ is the infinitesimal action of u.*

Proof. Since U is an open ball around 0, it is orbit-convex by [33, Lemma 1.14]. Hence, the holomorphicity of θ and [33, Proposition 1.4] imply that $\theta(x_1) = \theta(x_2)g$. Since $\mathscr{B}_k^{ss}$ is $\mathscr{G}_{k+1}^{\mathbb{C}}$-invariant, if $\theta(x_1) \in \mathscr{B}_k^{ss}$, then $\theta(x_2) \in \mathscr{B}_k^{ss}$ so that $x_2 \in Z$. To prove (2), we claim that $u \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$. Then, the claim implies that

(3.12)
$$v = d_{\theta(x)}F(d_x\theta(v)) = d_{\theta(x)}F(u^{\#}_{\theta(x)}) = \frac{d}{dt}\bigg|_{t=0} F(\theta(x)e^{tu}) = \frac{d}{dt}\bigg|_{t=0} xe^{tu} = u^{\#}_x.$$

To prove the claim, write $u = u' + u''$ for some $u' \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$ and $u'' \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})^{\perp}_{k+1}$. Since θ takes values in $(A, \Phi) + \ker(D'')^*$, $(u'')^{\#}_{\theta(x)} \in \ker(D'')^*$. In the proof of Proposition 3.3, we see that the map

$$T\colon \mathbf{H}^0(C_\mu)^{\perp}_{k+1} \times \mathbf{H}^2(C_\mu)^{\perp}_{k+1} \times ((A, \Phi) + \ker(D'')^*) \to \mathscr{C}_k \tag{3.13}$$

is a local diffeomorphism around $(0, 0, A, \Phi)$. Hence, there are open neighborhoods $N_1 \times N_2 \times V$ of $(0, 0, A, \Phi)$ and W of (A, Φ) such that $T\colon N_1 \times N_2 \times V \to W$ is a diffeomorphism. If U is sufficiently small, $\theta\colon Z \to V \cap \mathscr{B}_k^{ss}$ is a homeomorphism. Therefore, the derivative $d_{(0,0,\theta(x))}T$ of T is injective.

Note that

$$\mathbf{H}^0(C_\mu)^\perp \oplus i * \mathbf{H}^2(C_\mu)^\perp = \mathbf{H}^0(C_{\mu_\mathbb{C}})^\perp. \tag{3.14}$$

Then, we see that

$$d_{(0,0,\theta(x))}T(u'',0) = D''_{\theta(x)}u'' = d_{(0,0,\theta(x))}T(0,(u'')^{\#}_{\theta(x)}) \tag{3.15}$$

so that $u'' = 0$. □

3.2. Perturbed Kuranishi maps

The Hitchin-Kobayashi correspondence characterizes polystable orbits in $\mathscr{B}^{ss}$ via the moment map μ. Since θ should eventually induce a local chart for the moduli space, we should be able to relate the polystable orbits in $\mathbf{H}^1$ with respect to the complex reductive group $K^\mathbb{C}$ to the polystable orbits in $\mathscr{B}$. Therefore, we would like to pullback the moment map μ to $U \subset \mathbf{H}^1$ by θ and then use the pullback moment map $\theta^*\mu$ to characterize polystable orbits in U. However, $\theta^*\mu$ takes values in $\Omega^2(\mathfrak{g}_E)_{k-1}$ instead of $\mathbf{H}^2(C_\mu) \cong \mathbf{H}^0(C_\mu)^*$. To fix this issue, we will perturb the Kuranishi map along $\mathscr{G}^\mathbb{C}$-orbits in the following way.

Lemma 3.5. *If $U \subset \mathbf{H}^1$ is sufficiently small, then there is a unique smooth function β defined on U and taking values in an open neighborhood of 0 in $\mathbf{H}^2(C_\mu)^\perp_{k+1}$ such that the perturbed Kuranishi map $\Theta := \theta e^{-i*\beta}$ is smooth and K-equivariant, and $\nu := \Theta^*\mu$ takes values in $\mathbf{H}^2(C_\mu)$ and hence is a moment map for the K-action on U with respect to the symplectic form $\Theta^*\Omega_I$. Moreover, the derivative of Θ at 0 is the inclusion map.*

Before giving the proof, we remark that the perturbed Kuranshi map Θ is no longer holomorphic and hence the form $\Theta^*\Omega_I$ is no longer Kähler.

Proof. We follow the proof of [34, Proposition 7]. Consider the map

$$\begin{gathered} L\colon U \times \mathbf{H}^2(C_\mu)^\perp_{k+1} \to \mathbf{H}^2(C_\mu)^\perp_{k-1}, \\ L(x,\beta) = (1-H)\mu(\theta(x)e^{-i*\beta}), \end{gathered} \tag{3.16}$$

where H is the harmonic projection defined in C_μ. Then, the derivative of L at $(0,0)$ along the direction $(0,\beta)$ is given by

$$d_{(0,0)}L(0,\beta) = (1-H)d_2(-Id_1 * \beta) = d_2 d_2^* \beta, \tag{3.17}$$

where the second equality follows from the formula $d_2^* = -Id_1*$. Since

$$d_2 d_2^* \colon \mathbf{H}^2(C_\mu)_{k+1}^\perp \to \mathbf{H}^2(C_\mu)_{k-1}^\perp \tag{3.18}$$

is an isomorphism, the implicit function theorem guarantees the existence of the desired function β. Since L is K-equivariant, the uniqueness of β implies that Θ is also K-equivariant. A direct computation shows that $d_0\Theta$ is the inclusion map. □

Although we cannot prove a local slice theorem for the $\mathscr{G}^{\mathbb{C}}$-action, the following is a substitute that relates the polystability of Higgs bundles to that of points in $\mathbf{H}^1$ with respect to the $K^{\mathbb{C}}$-action.

Theorem 3.6. *If U is sufficiently small, then the induced map*

$$U \times_K \mathscr{G}_{k+1} \to \mathscr{C}_k, \qquad [x, g] \mapsto \Theta(x)g, \tag{3.19}$$

is injective. Moreover, there is an open ball $B \subset U$ around 0 in the L^2-norm such that the following are equivalent for every $x \in B \cap Z$:

1) *$xK^{\mathbb{C}}$ is closed in $\mathbf{H}^1$.*
2) *$xK^{\mathbb{C}} \cap \nu^{-1}(0) \neq \emptyset$.*

Proof. The derivative of the induced map at $[0, 1]$ is given by

$$\mathbf{H}^1 \oplus \mathbf{H}^0(C_\mu)_{k+1}^\perp \to \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})_k \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})_k, \qquad (x, u) \mapsto x + D''u. \tag{3.20}$$

Since it is injective, we see that the induced map is locally injective around $[0, 1]$. Then, we assume to the contrary that such U does not exist. Therefore, there are sequences $[x_n, g_n]$ and $[x_n', g_n']$ such that

1) x_n, x_n' converge to 0 in $\mathbf{H}^1$.
2) $\Theta(x_n)g_n = \Theta(x_n')g_n'$.
3) $[x_n, g_n] \neq [x_n', g_n']$ for all n.

Since the $\mathscr{G}_{k+1}$-action is proper, by passing to a subsequence, we may assume that $g_n' g_n^{-1}$ converges to some $g \in \mathscr{G}_{k+1}$. Letting $n \to \infty$, we see that $\Theta(0) = \Theta(0)g$ so that $g \in K$. Now, on the one hand, $[x_n', g_n' g_n^{-1}] \neq [x_n, 1]$ for any n. On the other hand, both $[x_n', g_n' g_n^{-1}]$ and $[x_n, 1]$ converge to $[0, 1]$ so that they are equal when $n \gg 0$, since the induced map is locally injective around $[0, 1]$. This is a contradiction.

Now, we prove the second part of the proposition. By Proposition 3.3, there are open neighborhoods $N_1 \times N_2 \times V$ of $(0, 0, A, \Phi)$ and W of (A, Φ) such that $T\colon N_1 \times N_2 \times V \to W$ is a homeomorphism. Here, V and W are open subsets in $\mathscr{B}_k^{ss}$. If U is sufficiently small, $\theta\colon Z \to V$ is a homeomorphism so that Proposition 3.4 holds. Let O be an open neighborhood of 0 in $\mathbf{H}^2(C_\mu)_{k+1}^{\perp}$ such that the smooth function $\beta\colon U \to O$ and hence $\Theta := \theta e^{-i*\beta}$ are defined. By shrinking N_2 if necessary, we may assume that $N_2 \subset O$. Then, by [38, Proposition 3.7], there is an open neighborhood $W' \subset W$ of (A, Φ) in $\mathscr{B}_k^{ss}$ such that the Yang-Mills-Higgs flow starting at any Higgs bundle inside W' stays and converges in W. Moreover, we may assume that $T(N_1' \times N_2' \times V') = W'$ for some open neighborhood $N_1' \times N_2' \times V' \subset N_1 \times N_2 \times V$ of $(0, 0, A, \Phi)$ such that $\theta\colon Z \cap B \to V'$ for some open ball $B \subset U$ around 0.

Now, suppose $x \in B \cap Z$ is such that $xK^{\mathbb{C}}$ is closed in $\mathbf{H}^1$. Let (B_t, Ψ_t) be the gradient flow starting at $\theta(x)$. By the previous setup, $\theta(x) \in V' \subset W'$ so that (B_t, Ψ_t) stays in W. Therefore, we may write $(B_t, \Psi_t) = \theta(x_t)e^{-i*\beta_t}e^{u_t}$ for some $x_t \in Z$ and $(u_t, \beta_t) \in N_1 \times N_2$. We claim that x_t stays in the $K^{\mathbb{C}}$-orbit of x. Since the gradient of $\|\mu\|^2$ is tangent to $\mathscr{G}_{k+1}^{\mathbb{C}}$-orbits, we may write $d_x\theta(\dot{x}_t) = (u_t)^{\#}_{\theta(x_t)}$ for some $u_t \in \Omega^0(\mathfrak{g}_E^{\mathbb{C}})_{k+1}$ that depends on t smoothly. Here, $u_t^{\#}$ is the infinitesimal action of u_t. Then, Proposition 3.4 implies that $u_t \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$ and $\dot{x}_t = (u_t)^{\#}_{x_t}$. On the other hand, the ordinary differential equation in $K^{\mathbb{C}}$,

$$g_t^{-1}\dot{g}_t = u_t, \qquad g_0 = 1, \tag{3.21}$$

has a unique solution $g_t \in K^{\mathbb{C}}$. By the uniqueness, we see that $x_t = xg_t$. Therefore, the claim follows. Then, the fact that T is a homeomorphism implies that both x_t, β_t and u_t converge. Therefore, letting $t \to \infty$, we have $\theta(x_\infty)e^{-i*\beta_\infty}e^{u_\infty} = (B_\infty, \Psi_\infty)$ and $\mu(B_\infty, \Psi_\infty) = 0$. Since $e^{u_\infty} \in \mathscr{G}_{k+1}$, $\theta(x_\infty)e^{-i*\beta_\infty} \in \mu^{-1}(0)$. Since $N_2 \subset O$, the uniqueness of β in Lemma 3.5 implies that $\beta(x_\infty) = \beta_\infty$. Hence,

$$\Theta(x_\infty) = \theta(x_\infty)e^{-i*\beta_\infty} \in \mu^{-1}(0). \tag{3.22}$$

Finally, since $xK^{\mathbb{C}}$ is closed in $\mathbf{H}^1$, we see that $x_\infty \in xK^{\mathbb{C}}$. Again, by the previous setup, $x_\infty \in Z \subset U$.

Conversely, suppose $xK^{\mathbb{C}}$ is not closed in $\mathbf{H}^1$. Note that the complex structure I (the one given by multiplication by $\sqrt{-1}$) on $\mathscr{C}$ restricts to $\mathbf{H}^1$. Since the K-action on $\mathbf{H}^1$ is linear, I-holomorphic and preserves the L^2-metric, it admits a standard moment map ν_0 such that $\nu_0(0) = 0$. Since

$(\operatorname{grad} \| \cdot \|_{L^2}^2, \operatorname{grad} \|\nu_0\|^2)_{L^2} = 8\|\nu_0\|^2$ (see [33, Example 2.3]), the gradient flow of $\|\nu_0\|^2$ starting at x stays in B and converges to some $y \in B \cap Z$ such that $\nu_0(y) = 0$. By the Kempf-Ness theorem, $yK^{\mathbb{C}}$ is closed in $\mathbf{H}^1$. Of course, $y \in \overline{xK^{\mathbb{C}}} \setminus xK^{\mathbb{C}}$. Hence, by the previous paragraph, we can find $y_\infty \in yK^{\mathbb{C}} \cap U$ such that $\mu(\Theta(y_\infty)) = 0$. Hence, we have

$$\Theta(y_\infty) \sim_{\mathscr{G}_{k+1}^{\mathbb{C}}} \Theta(y) \in \overline{\Theta(x)\mathscr{G}_{k+1}^{\mathbb{C}}}, \tag{3.23}$$

where $\sim_{\mathscr{G}_{k+1}^{\mathbb{C}}}$ is the equivalence relation generated by the $\mathscr{G}_{k+1}^{\mathbb{C}}$-action. Now, since $xK^{\mathbb{C}}$ contains a zero of ν in U, we may assume that $\mu(\Theta(x)) = 0$. Then, the following Lemma 3.7 implies that $\Theta(y_\infty) \sim_{\mathscr{G}_{k+1}^{\mathbb{C}}} \Theta(x)$ so that $\Theta(y_\infty) \sim_{\mathscr{G}_{k+1}} \Theta(x)$ by the Hitchin-Kobayashi correspondence. Then, the injectivity of $[x, g] \mapsto \Theta(x)g$ implies that $y_\infty \sim_K x$. This is a contradiction. $\square$

The following result is nothing but the fact that the closure of the $\mathscr{G}_{k+1}^{\mathbb{C}}$-orbit of a semistable Higgs bundle contains a unique polystable orbit. Since we cannot find a proof in the literature, we provide one here:

Lemma 3.7. *Let (B, Ψ) be a semistable Higgs bundle. If $(B_i, \Psi_i) \in \overline{(B, \Psi)\mathscr{G}_{k+1}^{\mathbb{C}}}$ $(i = 1, 2)$ are polystable Higgs bundles, then $(B_1, \Psi_1) \sim_{\mathscr{G}_{k+1}^{\mathbb{C}}} (B_2, \Psi_2)$.*

Proof. We may assume that $\mu(B_i, \Psi_i) = 0$ for $i = 1, 2$. Let $r\colon \mathscr{B}_k^{ss} \to \mu^{-1}(0)$ be the retraction (see [38, Theorem 1.1]) given by the Yang-Mills-Higgs flow. Suppose there are sequences $(B_i^j, \Psi_i^j) \in (B, \Psi)\mathscr{G}_{k+1}^{\mathbb{C}}$ such that $(B_i^j, \Psi_i^j) \xrightarrow{j\to\infty} (B_i, \Psi_i)$. By the openness of $\mathscr{B}_k^{ss}$, each (B_i^j, Ψ_i^j) is semistable if $j \gg 0$. By the continuity of r, we have

$$r(B_i^j, \Psi_i^j) \xrightarrow{j\to\infty} r(B_i, \Psi_i) = (B_i, \Psi_i). \tag{3.24}$$

By [38, Theorem 1.4], we see that each $r(B_i^j, \Psi_i^j)$ is the graded object of the Seshadri filtration of (B_i^j, Ψ_i^j). Since graded objects are determined by $\mathscr{G}_{k+1}^{\mathbb{C}}$-orbits, we conclude that

$$r(B_1^j, \Psi_1^j) \sim_{\mathscr{G}_{k+1}^{\mathbb{C}}} \operatorname{Gr}(B, \Psi) \sim_{\mathscr{G}_{k+1}^{\mathbb{C}}} r(B_2^l, \Psi_2^l) \tag{3.25}$$

for each j, l so that $r(B_1^j, \Psi_1^j) \sim_{\mathscr{G}_{k+1}} r(B_2^l, \Psi_2^l)$. Since the $\mathscr{G}_{k+1}$-action is proper, $\mathscr{G}_{k+1}$-orbits are closed. Letting $j \to \infty$, we see that $(B_1, \Psi_1) \in r(B_2^l, \Psi_2^l)\mathscr{G}_{k+1}$. Now, letting $l \to \infty$, we see that $(B_1, \Psi_1) \sim_{\mathscr{G}_{k+1}} (B_2, \Psi_2)$. $\square$

3.3. Open embeddings into the moduli space

Let $\mathscr{Z} := Z \cap B$ which is a closed complex subspace of B. Note that $\mathscr{Z}$ is K-invariant but not $K^{\mathbb{C}}$-invariant. To fix this issue, recall that every open ball around 0 (in the L^2-norm) in $\mathbf{H}^1$ is K-invariant and orbit-convex (see [33, Definition 1.2 and Lemma 1.14]). By [13, §3.3, Proposition], $\mathscr{Z}K^{\mathbb{C}}$ is a closed complex subspace of $BK^{\mathbb{C}}$, and $\mathscr{Z}$ is open in $\mathscr{Z}K^{\mathbb{C}}$. Recall the standard moment map $\nu_0 \colon \mathbf{H}^1 \to \mathbf{H}^2(C_\mu)$ used in the proof of Theorem 3.6. This is the moment map for the K-action on $\mathbf{H}^1$ with respect to the L^2-metric and the restricted complex structure I. Then, by the analytic GIT developed in [15] or [14, §0], there is a categorical quotient $\pi\colon \mathscr{Z}K^{\mathbb{C}} \to \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}$ in the category of reduced complex spaces such that every fiber of π contains a unique closed $K^{\mathbb{C}}$-orbit, and the inclusion $\nu_0^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}} \hookrightarrow \mathscr{Z}K^{\mathbb{C}}$ induces a homeomorphism

$$(\nu_0^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}})/K \xrightarrow{\sim} \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}. \tag{3.26}$$

Moreover, as a topological space, $\mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}$ is the quotient space defined by the equivalence relation that $x \sim y$ if and only if $\overline{xK^{\mathbb{C}}} \cap \overline{yK^{\mathbb{C}}} \neq \emptyset$.

A corollary of Theorem 3.6 is that $\mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}$ can be realized as a singular symplectic quotient with respect to ν instead of ν_0.

Corollary 3.8. *The inclusion $j\colon \nu^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}} \hookrightarrow \mathscr{Z}K^{\mathbb{C}}$ induces a homeomorphism*

$$\bar{j}\colon (\nu^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}})/K \xrightarrow{\sim} \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}. \tag{3.27}$$

As a consequence, the perturbed Kuranishi map Θ induces well-defined continuous maps $\overline{\Theta}$ and φ in the following commutative diagram

$$\begin{array}{ccc} \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}} & \xrightarrow{\quad\varphi\quad} & \mathscr{B}_k^{ps}/\mathscr{G}_{k+1}^{\mathbb{C}} \\ \sim\uparrow & & \sim\uparrow \\ (\nu^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}})/K & \xrightarrow{\overline{\Theta}} & (\mu^{-1}(0) \cap \mathscr{B}_k)/\mathscr{G}_{k+1} \end{array} \tag{3.28}$$

More explicitly, φ is given by the formula

$$\varphi[x] = [r\theta(x)], \qquad x \in \mathscr{Z}, \tag{3.29}$$

where $r\colon \mathscr{B}_k^{ss} \to \mu^{-1}(0)$ is the retraction defined by the Yang-Mills-Higgs flow.

Proof. Clearly, $\overline{\Theta}$ is a well-defined continuous map. To define φ, it suffices to show that $\overline{j}$ is a homeomorphism. Therefore, we show that it has a continuous inverse and follow the notations and the setup in the proof of Theorem 3.6. Let $\pi\colon \mathscr{Z}K^{\mathbb{C}} \to \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}$ be the quotient map. If $xg \in \mathscr{Z}K^{\mathbb{C}}$ with $x \in \mathscr{Z}$, by using the gradient flow of $\|\nu_0\|^2$, we see that there is a closed $K^{\mathbb{C}}$-orbit $\widetilde{x}K^{\mathbb{C}} \subset \overline{xK^{\mathbb{C}}}$ with $\widetilde{x} \in \mathscr{Z}$. Then, Theorem 3.6 implies that there exists

$$x_\infty \in \nu^{-1}(0) \cap \widetilde{x}K^{\mathbb{C}} \subset \nu^{-1}(0) \cap \overline{xK^{\mathbb{C}}}. \tag{3.30}$$

Therefore, if $\pi(xg) = \pi(yh)$, then $\pi(x_\infty) = \pi(y_\infty)$ so that

$$\overline{\Theta(x_\infty)\mathscr{G}^{\mathbb{C}}_{k+1}} \cap \overline{\Theta(y_\infty)\mathscr{G}^{\mathbb{C}}_{k+1}} \neq \emptyset. \tag{3.31}$$

If we can show that $x_\infty \sim_K y_\infty$, then the map

$$\overline{j}^{-1}\colon \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}} \to (\nu^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}})/K, \qquad [xg] \mapsto [x_\infty], \tag{3.32}$$

is well-defined. Now, $x_\infty \sim_K y_\infty$ follows from the following Lemma.

Lemma 3.9. *If (A_i, Φ_i) $(i = 1, 2)$ are Higgs bundles such that $\mu(A_i, \Phi_i) = 0$ and $\overline{(A_1, \Phi_1)\mathscr{G}^{\mathbb{C}}_{k+1}} \cap \overline{(A_2, \Phi_2)\mathscr{G}^{\mathbb{C}}_{k+1}} \neq \emptyset$, then $(A_1, \Phi_1) \sim_{\mathscr{G}_{k+1}} (A_2, \Phi_2)$.*

Proof. Let (B, Ψ) be a Higgs bundle in the intersection of the closures. Hence, there is a sequence $(A^j_i, \Phi^j_i) \in (A_i, \Phi_i)\mathscr{G}^{\mathbb{C}}_{k+1}$ converging to (B, Ψ). The continuity of r implies that $r(A^j_i, \Phi^j_i) \xrightarrow{j\to\infty} r(B, \Psi)$. By [38, Theorem 1.4],

$$r(A^j_i, \Phi^j_i) \sim_{\mathscr{G}^{\mathbb{C}}_{k+1}} Gr(A_i, \Phi_i) = (A_i, \Phi_i) \tag{3.33}$$

so that $r(A^j_i, \Phi^j_i) \sim_{\mathscr{G}_{k+1}} (A_i, \Phi_i)$. Hence, there is a sequence of $g^j_i \in \mathscr{G}$ such that $(A_i, \Phi_i)g^j_i \xrightarrow{j\to\infty} r(B, \Psi)$. Since the $\mathscr{G}_{k+1}$-action is proper, by passing to a subsequence, we may assume that $g^j_i \xrightarrow{j\to\infty} g_i$ for some $g_i \in \mathscr{G}_{k+1}$. Hence, $(A_i, \Phi_i)g_i = r(B, \Psi)$. □

Continuing with the proof of Corollary 3.8, we show that $\overline{j}^{-1}$ is continuous. Recall that x_∞ is determined by the equation $\theta(x_\infty)e^{-i*\beta_\infty}e^{u_\infty} = r(\theta(\widetilde{x}))$. By the continuity of r, T^{-1} and θ^{-1}, we see that the map $\mathscr{Z} \ni \widetilde{x} \mapsto x_\infty$ is continuous. Moreover, $\mathscr{Z} \ni x \mapsto \widetilde{x}$ is also continuous, which is a general property of the gradient flow of $\|\nu_0\|^2$. Since $\mathscr{Z}$ is open in $\mathscr{Z}K^{\mathbb{C}}$, we conclude that $\overline{j}^{-1}$ is continuous.

It remains to show that $\overline{j}^{-1}$ is indeed the inverse of $\overline{j}$. If $xg \in \nu^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}}$ with $x \in \mathscr{Z}$, then $xK^{\mathbb{C}}$ is closed in $\mathbf{H}^1$ (Theorem 3.6). Since $\overline{j}^{-1}$ is well-defined, we see that

$$(xg)_\infty \sim_K x_\infty \sim_{K^{\mathbb{C}}} \widetilde{x} \sim_{K^{\mathbb{C}}} x \sim_{K^{\mathbb{C}}} xg. \tag{3.34}$$

Then, $\nu((xg)_\infty) = \nu(xg) = 0$ implies that $(xg)_\infty \sim_K xg$. Conversely, if $xg \in \mathscr{Z}K^{\mathbb{C}}$ with $x \in \mathscr{Z}$, then $x_\infty \in \overline{xK^{\mathbb{C}}}$ so that $\pi(xg) = \pi(x_\infty)$.

Finally, to obtain a formula for φ, note that

$$\Theta(x_\infty) \in \overline{\Theta(x)\mathscr{G}^{\mathbb{C}}_{k+1}} = \overline{\theta(x)\mathscr{G}^{\mathbb{C}}_{k+1}}. \tag{3.35}$$

Moreover, $r(\theta(x)) \in \overline{\theta(x)\mathscr{G}^{\mathbb{C}}_{k+1}}$. Hence, by Lemma 3.7, $\Theta(x_\infty) \sim_{\mathscr{G}^{\mathbb{C}}_{k+1}} r(\theta(x))$. □

The next result shows that $\mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}$ is a local model for the quotient $\mathscr{M}_k = \mathscr{B}^{ps}_k / \mathscr{G}^{\mathbb{C}}_{k+1}$. Strictly speaking, $\mathscr{M}_k$ is not the moduli space $\mathscr{M}$. That said, there is a natural map $\mathscr{M} \to \mathscr{M}_k$. Note that [2, Lemma 14.8] and the elliptic regularity for $\overline{\partial}_A$ with $A \in \mathscr{A}$ imply that every point in $\mathscr{M}_k$ has a C^∞ representative. As a consequence, the natural map $\mathscr{M} \to \mathscr{M}_k$ is surjective. Its injectivity follows from [2, Lemma 14.9]. Later, as a consequence of Theorem 3.10, we will show that $\mathscr{M} \to \mathscr{M}_k$ is a homeomorphism, which justifies our use of Sobolev completions.

Theorem 3.10. *If B is sufficiently small, $\varphi\colon \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}} \to \mathscr{M}_k$ is an open embedding.*

Proof. We will follow the notations and the setup in the proof of Theorem 3.6. Since $\overline{\Theta}$ is injective, φ is injective. Let $\Pi\colon \mathscr{B}^{ps}_k \to \mathscr{M}_k$ be the quotient map, and consider the open set $O = \Pi(W' \cap \mathscr{B}^{ps}_k)$. If $(B, \Psi) \in W' \cap \mathscr{B}^{ps}_k$, then $(B, \Psi) = \theta(x)e^{-i*\beta}e^u$ for some $x \in \mathscr{Z}$. We claim that $\varphi[x] = [B, \Psi]$. By the construction of φ in the proof of Corollary 3.8, we see that $\varphi[x] = [\Theta(x_\infty)]$ for some $x_\infty \in \nu^{-1}(0) \cap \mathscr{Z}K^{\mathbb{C}} \cap \overline{xK^{\mathbb{C}}}$ so that

$$\Theta(x_\infty) \in \overline{\theta(x)\mathscr{G}^{\mathbb{C}}_{k+1}} = \overline{(B, \Psi)\mathscr{G}^{\mathbb{C}}_{k+1}}. \tag{3.36}$$

By Lemma 3.7, we have $\Theta(x_\infty) \sim_{\mathscr{G}^{\mathbb{C}}_{k+1}} (B, \Psi)$. As a consequence, the open set O is contained in the image of φ. Hence, we obtain a bijective continuous map $\varphi\colon \widetilde{O} \to O$, where $\widetilde{O} = \varphi^{-1}(O)$.

To show that $\varphi|_{\widetilde{O}}$ is a homeomorphism, we will show that its inverse is continuous. From the previous paragraph, we see that its inverse should be $[B,\Psi]\mapsto[x]$. The continuity follows from the continuity of θ^{-1} and T^{-1}. Therefore, it remains to prove that it is well-defined. If $(B',\Psi')\in W'\cap\mathscr{B}_k^{ps}$ lies in the $\mathscr{G}_{k+1}^{\mathbb{C}}$-orbit of (B,Ψ), then

$$\Theta(x_\infty)\sim_{\mathscr{G}_{k+1}^{\mathbb{C}}}(B,\Psi)\sim_{\mathscr{G}_{k+1}^{\mathbb{C}}}(B',\Psi')\sim_{\mathscr{G}_{k+1}^{\mathbb{C}}}\Theta(x'_\infty) \tag{3.37}$$

so that

$$\overline{xK^{\mathbb{C}}}\ni x_\infty\sim_K x'_\infty\in\overline{x'K^{\mathbb{C}}}. \tag{3.38}$$

Hence, $\overline{xK^{\mathbb{C}}}\cap\overline{x'K^{\mathbb{C}}}\neq\emptyset$.

Finally, we show that if B is sufficiently small, then φ is an open embedding. Write $\pi^{-1}(\widetilde{O})=\mathscr{Z}K^{\mathbb{C}}\cap Q$ for some open set Q in $\mathbf{H}^1$, where $\pi\colon\mathscr{Z}K^{\mathbb{C}}\to\mathscr{Z}K^{\mathbb{C}}/\!\!/K^{\mathbb{C}}$ is the quotient map. Since $0\in Q$, choose some open ball $B'\subset Q\cap B$ around 0. By [33, Lemma 1.14], we know that B and B' are ν_0-convex (see [15, (2.6), Definition]). Hence, by definition of $\mathscr{Z}$, $\mathscr{Z}$ is also ν_0-convex. Hence, by [15, (3.1), Lemma], we see that $\mathscr{Z}K^{\mathbb{C}}\cap B'K^{\mathbb{C}}=(\mathscr{Z}\cap B')K^{\mathbb{C}}$. Then, we claim that $(\mathscr{Z}\cap B')K^{\mathbb{C}}\subset\pi^{-1}(\widetilde{O})$. In fact, if $xg\in(\mathscr{Z}\cap B')K^{\mathbb{C}}$ with $x\in\mathscr{Z}\cap B'$, then $x\in ZK^{\mathbb{C}}\cap Q$. Since $ZK^{\mathbb{C}}\cap Q$ is $K^{\mathbb{C}}$-invariant, $xg\in ZK^{\mathbb{C}}\cap Q$. Finally, we claim that $(\mathscr{Z}\cap B')K^{\mathbb{C}}$ is also π-saturated so that $(\mathscr{Z}\cap B')K^{\mathbb{C}}/\!\!/K^{\mathbb{C}}$ is an open neighborhood of $[0]$ in $\mathscr{Z}K^{\mathbb{C}}/\!\!/K^{\mathbb{C}}$. Therefore, if B is shrunk to B', and $\mathscr{Z}$ is shrunk to $\mathscr{Z}\cap B'$, we see that φ is an open embedding.

Suppose $\pi(xg)=\pi(yh)$ for some $x\in\mathscr{Z}$ and $y\in\mathscr{Z}\cap B'$. We want to show that $xg\in(\mathscr{Z}\cap B')K^{\mathbb{C}}$. By using the gradient flow of $\|\nu_0\|^2$, we can find a closed orbit $y'K^{\mathbb{C}}\subset\overline{yK^{\mathbb{C}}}$ with $y'\in\mathscr{Z}\cap B'$. Since every fiber of π contains a unique closed orbit, $y'K^{\mathbb{C}}\subset\overline{xK^{\mathbb{C}}}$. Since B' is open, $xK^{\mathbb{C}}\cap B'\neq\emptyset$. Hence, $x\in B'K^{\mathbb{C}}\cap\mathscr{Z}K^{\mathbb{C}}=(\mathscr{Z}\cap B')K^{\mathbb{C}}$. □

To show that $\mathscr{M}\to\mathscr{M}_k$ is a homeomorphism, we need the following lemma.

Lemma 3.11. *Elements in $\mathscr{B}_k\cap[(A,\Phi)+\ker(D'')^*]$ are of class C^∞.*

Proof. Suppose $(D'')^*(\alpha'',\eta)=0$ and $(\overline{\partial}_A+\alpha'')(\Phi+\eta)=0$, where α'' is the $(0,1)$-part of α. The second equation is also equivalent to $D''(\alpha'',\eta)+[\alpha'',\eta]=0$. Hence, $\Delta(\alpha,\eta)=-(D'')^*[\alpha'',\eta]$ where $\Delta=D''(D'')^*+(D'')^*D''$ is the Laplacian defined in $C_{\mu_{\mathbb{C}}}$. Since $k>1$, the Sobolev multiplication theorem (see [11, Theorem 4.4.1]) implies that $[\alpha'',\eta]$ is in L_k^2 and hence $(D'')^*[\alpha'',\eta]$ is in L_{k-1}^2. By the elliptic regularity, (α'',η) is hence in L_{k+1}^2. By induction, (α'',η) is in C^∞. □

Lemma 3.12. *The map φ in Corollary 3.8 factors through the natural map $\mathscr{M} \to \mathscr{M}_k$.*

Proof. Recall that the formula for φ is given by $\varphi[x] = [r\theta(x)]$ where $x \in \mathscr{Z}$. By Lemma 3.11, θ restricts to a continuous map $Z \to \mathscr{B}^{ss} \cap ((A, \Phi) + \ker(D'')^*)$. Since $r\colon \mathscr{B}^{ss} \to \mu^{-1}(0)$ is continuous, $\mathscr{Z} \ni x \mapsto [r\theta(x)] \in \mathscr{M}$ is continuous. Finally, [2, Lemma 14.9] and the fact that φ is well-defined imply that φ factors through $\mathscr{M} \to \mathscr{M}_k$. □

Corollary 3.13. *The natural map $\mathscr{M} \to \mathscr{M}_k$ is a homeomorphism. Therefore, the map $\varphi\colon \mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}} \to \mathscr{M}$ is an open embedding.*

Proof. By Lemma 3.12 and Theorem 3.10, $\mathscr{M} \to \mathscr{M}_k$ is locally an open map and hence open. □

4. Gluing local models

For the rest of the paper, we will drop the subscripts that indicate Sobolev completions for notational convenience. By Lemma 3.11, 3.12 and Corollary 3.13, this should not cause any confusion. The main result in this section is the following, which is part of Theorem A. The normality of $\mathscr{M}$ will be proved in Lemma 6.7.

Theorem 4.1. *The moduli space $\mathscr{M}$ is a complex space locally biholomorphic to a Kuranishi local model $\mathscr{Z}K^{\mathbb{C}} /\!\!/ K^{\mathbb{C}}$.*

Let (A_i, Φ_i) $(i = 1, 2)$ be Higgs bundles such that $\mu(A_i, \Phi_i) = 0$. We will use subscript i to denote relevant objects associated with (A_i, Φ_i). Let $\mathscr{Z}_i$ be their Kuranishi spaces and $\mathscr{Z}_i K_i^{\mathbb{C}} /\!\!/ K_i^{\mathbb{C}}$ Kuranishi local models, where K_i is the $\mathscr{G}$-stabilizer of (A_i, Φ_i). Let

$$\varphi_i\colon \mathscr{Z}_i K_i^{\mathbb{C}} /\!\!/ K_i^{\mathbb{C}} \xrightarrow{\sim} O_i \subset \mathscr{M} \tag{4.1}$$

be the map constructed in Theorem 3.10 such that $O_1 \cap O_2 \neq \emptyset$. Hence, the transition function is given by

$$\varphi_2^{-1}\varphi_1\colon \varphi_1^{-1}(O_1 \cap O_2) \to \varphi_2^{-1}(O_1 \cap O_2). \tag{4.2}$$

Our goal is to show that $\varphi_2^{-1}\varphi_1$ is holomorphic so that $\mathscr{M}$ is a complex space. Since holomorphicity is a local condition, the idea is that the transition function $\varphi_2^{-1}\varphi_1$ should be locally induced by a holomorphic $K_1^{\mathbb{C}}$-invariant map from an open set in $\mathscr{Z}_1 K_1^{\mathbb{C}}$ to $\mathscr{Z}_2 K_2^{\mathbb{C}} /\!\!/ K_2^{\mathbb{C}}$. Then, the rest of the argument follows from the universal property of the quotient map $\pi_i\colon \mathscr{Z}_i K_i^{\mathbb{C}} \to \mathscr{Z}_i K_i^{\mathbb{C}} /\!\!/ K_i^{\mathbb{C}}$. Here, the technical difficulty is to find an appropriate open set in $\mathscr{Z}_1 K_1^{\mathbb{C}}$ that is also π_1-saturated. This will be overcome in the following Lemma 4.2.

To proceed, we follow the notations and the setup in the proof of Theorem 3.6. Let $[x] \in \varphi_1^{-1}(O_1 \cap O_2)$. Using the gradient flow of $\|\nu_0\|^2$, we may assume that $x \in \mathscr{Z}_1$ has a closed $K_1^{\mathbb{C}}$-orbit. Hence, $\theta_1(x)$ is polystable (Theorem 3.6), and $\varphi_1[x] = [r\theta_1(x)] = [\theta_1(x)]$. Similarly, there is some $x' \in \mathscr{Z}_2$ with closed $K_2^{\mathbb{C}}$-orbit such that $\varphi_2[x'] = \varphi_1[x]$ so that $\theta_1(x) \sim_{\mathscr{G}^{\mathbb{C}}} \theta_2(x')$. Since $\theta_i\colon \mathscr{Z}_i \to V_i' \subset W_i'$ is a homeomorphism, $\theta_1(x) \in W_1' \cap W_2' h^{-1}$ for some $h \in \mathscr{G}^{\mathbb{C}}$.

Lemma 4.2. *There is an open neighborhood C of x in $\mathscr{Z}_1$ such that*

1) $CK_1^{\mathbb{C}}$ *is π_1-saturated.*
2) $\theta_1(C) \subset W_1' \cap W_2' h^{-1}$.
3) $[x] \in \pi_1(C) \subset \varphi_1^{-1}(O_1 \cap O_2)$.

Proof. Since $T_1\colon N_1' \times V_1' \to W_1'$ and $\theta_1\colon \mathscr{Z}_1 \to V_1'$ are homeomorphisms, there is an open ball Q around x such that

$$\theta_1(\mathscr{Z}_1 \cap Q) \subset W_1' \cap W_2' h^{-1}. \tag{4.3}$$

Since $\mathscr{Z}_1$ is open in $\mathscr{Z}_1 K_1^{\mathbb{C}}$, $(\mathscr{Z}_1 \cap Q)K_1^{\mathbb{C}}$ is open in $\mathscr{Z}_1 K_1^{\mathbb{C}}$. Then, set

$$C = \pi_1^{-1}\pi_1(\nu_1^{-1}(0) \cap (\mathscr{Z}_1 \cap Q)K_1^{\mathbb{C}}) \cap (\mathscr{Z}_1 \cap Q). \tag{4.4}$$

By Corollary 3.8, C is open in $\mathscr{Z}_1$. Clearly, (2) follows and $x \in C$.

To show that $CK_1^{\mathbb{C}}$ is π_1-saturated, let $y \in \mathscr{Z}_1 K_1^{\mathbb{C}}$ be such that $\pi_1(y) = \pi_1(y')$ for some $y' \in C$. By definition of C, $\pi_1(y') = \pi_1(y'')$ for some $y'' \in \nu_1^{-1}(0) \cap (\mathscr{Z}_1 \cap Q)K_1^{\mathbb{C}}$. Since $y'' K_1^{\mathbb{C}}$ is closed, $y'' K_1^{\mathbb{C}} \subset \overline{yK_1^{\mathbb{C}}}$. Since $y'' K_1^{\mathbb{C}} \cap C \neq \emptyset$, and C is open, we conclude that $yK_1^{\mathbb{C}} \cap C \neq \emptyset$. This shows (1). If $y \in C$, then $\pi_1(y) = \pi_1(y'g)$ for some $y'g \in \nu_1^{-1}(0) \cap (\mathscr{Z}_1 \cap Q)K_1^{\mathbb{C}}$ with $y' \in \mathscr{Z}_1 \cap Q$. Therefore, $\varphi_1[y] = [\theta_1(y')]$. By the construction of φ_i in Corollary 3.8 and Theorem 3.10, we see that

$$O_i = \Pi r\theta_i(\mathscr{Z}_i) = \Pi r(V_i') = \Pi r(W_i'), \tag{4.5}$$

where $\Pi\colon \mathscr{B}^{ps} \to \mathscr{M}$ is the quotient map. Since $\theta_1(y') \in W_1' \cap W_2' h^{-1}$ is polystable, it is easy to see that $[\theta_1(y')] \in O_1 \cap O_2$. This proves (3). $\square$

Now, for $y \in C$, $\theta_1(y)h \in W_2'$. Since T_2 is a homeomorphism, there is $g(y) \in \mathscr{G}^{\mathbb{C}}$, as a function of $y \in C$, such that $\theta_1(y)hg(y) \in V_2'$. Hence, we have obtained a map

$$\psi_{21}\colon C \to \mathscr{Z}_2 K_2^{\mathbb{C}} /\!\!/ K_2^{\mathbb{C}}, \qquad \psi_{21}(y) = \pi_2\theta_2^{-1}(\theta_1(y)hg(y)). \tag{4.6}$$

Lemma 4.3.

1) *ψ_{21} is holomorphic.*
2) *If $y, y' \in C$ are in the same $K_1^{\mathbb{C}}$-orbit, then $\psi(y) = \psi(y')$.*

Proof. Explicitly, we have

$$g(y) = \exp(-p_1 T_2^{-1}(\theta_1(y)h)), \tag{4.7}$$

where p_1 is the projection onto the first factor. Since

$$T_2\colon \mathbf{H}^0(C^2_{\mu_{\mathbb{C}}})^{\perp} \times ((A_2, \Phi_2) + \ker D_2''^*) \to \mathscr{C} \tag{4.8}$$

is holomorphic, its inverse, when restricted to appropriate open neighborhoods, is also holomorphic. Moreover, since the Kuranishi map is holomorphic, θ_1 is also holomorphic when the codomain is appropriately extended. Therefore, we conclude that $g\colon C \to \mathscr{G}^{\mathbb{C}}$ is holomorphic. Finally, since the $\mathscr{G}^{\mathbb{C}}$-action is holomorphic, we conclude that ψ_{21} is holomorphic.

To show (2), suppose there are $z, z' \in \mathscr{Z}_2$ such that

$$\begin{aligned} \theta_2(z) &= \theta_1(y)hg(y), \\ \theta_2(z') &= \theta_1(y')hg(y'). \end{aligned} \tag{4.9}$$

We want to show that $\pi_2(z) = \pi_2(z')$. Since y and y' are in the same $K_1^{\mathbb{C}}$-orbit,

$$\theta_2(z) \sim_{\mathscr{G}^{\mathbb{C}}} \theta_1(y) \sim_{\mathscr{G}^{\mathbb{C}}} \theta_1(y') \sim_{\mathscr{G}^{\mathbb{C}}} \theta_2(z') \tag{4.10}$$

so that $r\theta_2(z) \sim_{\mathscr{G}} r\theta_2(z')$. This means that $\varphi_2[z] = \varphi_2[z']$. Since φ_2 is injective, $[z] = [z']$. □

Lemma 4.4. *The transition function $\varphi_2^{-1}\varphi_1$ is holomorphic.*

Proof. By Lemma 4.3, ψ_{21} extends to a $K_1^{\mathbb{C}}$-invariant holomorphic map

$$\psi_{21}\colon CK_1^{\mathbb{C}} \to \mathscr{Z}_2 K_2^{\mathbb{C}} /\!\!/ K_2^{\mathbb{C}}. \tag{4.11}$$

Since $CK_1^{\mathbb{C}}$ is a π_1-saturated open set (Lemma 4.2),

$$\pi_2\colon CK_1^{\mathbb{C}} \to \pi_1(CK_1^{\mathbb{C}}) =: CK_1^{\mathbb{C}} /\!\!/ K_1^{\mathbb{C}} \tag{4.12}$$

is also a categorical quotient. As a consequence, ψ_{21} descends to a holomorphic map

$$\overline{\psi}_{21}\colon CK_1^{\mathbb{C}} /\!\!/ K_1^{\mathbb{C}} \to \mathscr{Z}_2 K_2^{\mathbb{C}} /\!\!/ K_2^{\mathbb{C}}. \tag{4.13}$$

Let $[c] \in CK_1^{\mathbb{C}} /\!\!/ K_1^{\mathbb{C}}$ with $c \in C$ and $z = \theta_2^{-1}(\theta_1(c)hg(c))$. Hence, $\theta_2(z) \sim_{\mathscr{G}^{\mathbb{C}}} \theta_1(c)$. Therefore,

$$\varphi_2\overline{\psi}_{21}[c] = \varphi_2\psi_{21}(c) = \varphi_2\pi_2(z) = \Pi(r\theta_2(z)) = \Pi(r\theta_1(z)) = \varphi_1[c]. \tag{4.14}$$

This shows that the transition function $\varphi_2^{-1}\varphi_1$ coincides with a holomorphic map $\overline{\psi}_{21}$ on an open neighborhood $CK_1^{\mathbb{C}} /\!\!/ K_1^{\mathbb{C}}$ of $[x]$ in $\varphi_1^{-1}(O_1 \cap O_2)$. This completes the proof. □

Proof of Theorem 4.1. By the properness of the $\mathscr{G}$-action, $(\mu^{-1}(0) \cap \mathscr{B})/\mathscr{G}$ is Hausdorff. The Hitchin-Kobayashi correspondence implies that $\mathscr{M}$ is Hausdorff. The Kuranishi local models are constructed in Corollary 3.8 and Theorem 3.10. By Lemma 4.4, the transition functions are holomorphic. □

5. Singularities in Kuranishi spaces

In this section, we will show that Kuranishi spaces have only cone singularities. We will use the same notations as in Section 3. The main result in this section is the following (cf. [17, Theorem 2.24] and [1, Theorem 3]).

Theorem 5.1. *The following diagram commutes:*

$$\begin{array}{ccc} \widetilde{\mathscr{B}} \cap ((A,\Phi) + \ker(D'')^*) & \xrightarrow{\;F\;} & \mathbf{H}^1 \\ \Big\downarrow{\scriptstyle \mu_{\mathbb{C}}} & \swarrow{\scriptstyle \frac{1}{2}H[\cdot,\cdot]} & \\ \mathbf{H}^2(C_{\mu_{\mathbb{C}}}) & & \end{array} \tag{5.1}$$

Proof. By construction of $\widetilde{\mathscr{B}}$, the restriction of $\mu_{\mathbb{C}}$ to $\widetilde{\mathscr{B}}$ is given by

$$\begin{aligned}\mu_{\mathbb{C}}(A+\alpha,\Phi+\eta) &= H\mu_{\mathbb{C}}(A+\alpha,\Phi+\eta)\\ &= \frac{1}{2}H[\alpha'',\eta;\alpha'',\eta] = H[\alpha'',\eta],\end{aligned} \tag{5.2}$$

where $(A+\alpha'',\Phi+\eta)\in\widetilde{\mathscr{B}}$. By definition of the Kuranishi space Z, it suffices to prove

1) $H[(\alpha'',\eta),(D'')^*G[\alpha'',\eta;\alpha'',\eta]]=0$, and
2) $H[(D'')^*G[\alpha'',\eta;\alpha'',\eta],(D'')^*G[\alpha'',\eta;\alpha'',\eta]]=0$

for any $(\alpha'',\eta)\in\ker(D'')^*$. By Kähler identities,

$$H[(\alpha'',\eta),(D'')^*G[\alpha'',\eta;\alpha'',\eta]] = \pm iH[(\alpha'',\eta),D' * G[\alpha'',\eta;\alpha'',\eta]] \tag{5.3}$$

and $(\alpha'',\eta)\in\ker D'$. Since D' is a derivation with respect to $[\cdot,\cdot]$, we see that

$$H[(\alpha'',\eta),D' * G[\alpha'',\eta;\alpha'',\eta]] = \pm HD'[(\alpha'',\eta),*G[\alpha'',\eta;\alpha'',\eta]]=0. \tag{5.4}$$

This proves (1). The same argument shows (2). This completes the proof. □

As a corollary, we obtain a description of singularities in the Kuranishi spaces.

Corollary 5.2. *The Kuranishi space Z is an open neighborhood of* 0 *in the quadratic cone*

$$Q=\{x\in\mathbf{H}^1\colon \frac{1}{2}H[x,x]=0\}. \tag{5.5}$$

Proof. This is clear by definition of Kuranishi spaces and Theorem 5.1. □

It is easy to see that the complex structures on $\mathscr{C}$ restrict to $\mathbf{H}^1$ so that $\mathbf{H}^1$ has a linear hyperKähler structure. In particular, the complex symplectic form $\Omega_{\mathbb{C}}$ on $\mathscr{C}$ restricts to $\mathbf{H}^1$. Hence, there is a standard complex moment map $\nu_{0,\mathbb{C}}\colon \mathbf{H}^1\to\mathbf{H}^2(C_{\mu_{\mathbb{C}}})$ for the $K^{\mathbb{C}}$-action with respect to the linear complex symplectic structure. More precisely, $\nu_{0,\mathbb{C}}$ is defined by

$$\langle\nu_{0,\mathbb{C}}(x),\xi\rangle=\frac{1}{2}\Omega_{\mathbb{C}}(x\cdot\xi,x),\qquad \xi\in\mathbf{H}^0(C_{\mu_{\mathbb{C}}}). \tag{5.6}$$

Since $i\colon\mathbf{H}^1\hookrightarrow\mathscr{C}$ is $K^{\mathbb{C}}$-equivariant, and $\mu_{\mathbb{C}}$ is a complex moment map, $Hi^*\mu_{\mathbb{C}}$ is a complex moment map for the $K^{\mathbb{C}}$-action on $\mathbf{H}^1$, where H is the harmonic projection onto $\mathbf{H}^2(C_{\mu_\mathbb{C}})$. Since $Hi^*\mu_\mathbb{C}(0) = 0$, we see that $Hi^*\mu_\mathbb{C} = \nu_{0,\mathbb{C}}$. On the other hand, $Hi^*\mu_\mathbb{C} = \frac{1}{2}H[\cdot,\cdot]$. Hence, Q is the zero set of the standard complex moment map $\nu_{0,\mathbb{C}}$.

Obviously, $\nu_{0,\mathbb{C}}^{-1}(0)$ is a closed complex subspace of $\mathbf{H}^1$. In fact, it is an affine variety. Therefore, the affine GIT quotient $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ K^\mathbb{C}$ exists such that the inclusion $\nu_0^{-1}(0) \cap \nu_{0,\mathbb{C}}^{-1}(0) \hookrightarrow \nu_{0,\mathbb{C}}^{-1}(0)$ induces a homeomorphism (see [15, (1.4)])

$$(\nu_0^{-1}(0) \cap \nu_{0,\mathbb{C}}^{-1}(0))/K \xrightarrow{\sim} \nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ K^\mathbb{C}. \tag{5.7}$$

Note that $(\nu_0^{-1}(0) \cap \nu_{0,\mathbb{C}}^{-1}(0))/K$ is precisely the hyperKähler quotient with respect to the standard hyperKähler moment maps on $\mathbf{H}^1$.

Theorem 5.3 (=Theorem B). *Let $[A,\Phi] \in \mathscr{M}$ be a point such that $\mu(A,\Phi) = 0$ and $\mathbf{H}^1$ its deformation space, a harmonic space defined in $C_{\mu_\mathbb{C}}$. Then, the following hold:*

1) *$\mathbf{H}^1$ is a complex-symplectic vector space.*
2) *The $\mathscr{G}^\mathbb{C}$-stabilizer $K^\mathbb{C}$ at (A,Φ) is a complex reductive group, acts on $\mathbf{H}^1$ linearly and preserves the complex-symplectic structure on $\mathbf{H}^1$. Moreover, the $K^\mathbb{C}$-action on $\mathbf{H}^1$ admits a canonical complex moment map $\nu_{0,\mathbb{C}}$ such that $\nu_{0,\mathbb{C}}(0) = 0$.*
3) *Around $[A,\Phi]$, the moduli space $\mathscr{M}$ is locally biholomorphic to an open neighborhood of $[0]$ in the complex symplectic quotient $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ K^\mathbb{C}$ which is an affine GIT quotient.*

Proof. It remains to show (3). Since $\mathscr{X}$ is open in Z which is also open in Q, we have $\mathscr{X}K^\mathbb{C}$ is open in Q. Since $\mathscr{X}K^\mathbb{C}$ is saturated with respect to the quotient $Q \to Q /\!\!/ K^\mathbb{C}$, $\mathscr{X}K^\mathbb{C} /\!\!/ K^\mathbb{C}$ is an open neighborhood of $[0]$ in $Q /\!\!/ K^\mathbb{C}$. The rest follows from Theorem 3.10 and 4.1. □

6. Comparison with the algebraic construction

Let $\mathscr{M}_{an}$ be the moduli space $\mathscr{B}^{ps}/\mathscr{G}^\mathbb{C}$ and $\mathscr{M}_{alg}$ the coarse moduli space of the semistable Higgs bundles of rank r and degree 0, where r is the rank of E. By [32, Theorem 4.7, Theorem 11.1], $\mathscr{M}_{alg}$ is a normal irreducible quasi-projective variety. By abusing the notation, we also use $\mathscr{M}_{alg}$ to mean its analytification. Then, there is a natural comparison map

$$i\colon \mathscr{M}_{an} \to \mathscr{M}_{alg}, \qquad [A,\Phi] \mapsto [\mathscr{E}_A, \Phi]_S. \tag{6.1}$$

Here, $(\mathscr{E}_A, \Phi)$ is the Higgs bundle determined by (A, Φ), and $[\mathscr{E}_A, \Phi]_S$ means the S-equivalence class of $(\mathscr{E}_A, \Phi)$. We will prove Theorem C in this section. By [38, Proposition 5.1], we see that i is a bijection of sets.

6.1. Continuity

The first step towards our goal is to show that i is a homeomorphism. To this end, we need some preparations. First, we may assume that the degree of E is sufficiently large. This can be arranged as follows. Fix a holomorphic line bundle $\mathscr{L} = (L, \overline{\partial}_L)$ of degree $d > 0$. Here, L is the underlying smooth line bundle of $\mathscr{L}$, and $\overline{\partial}_L$ is the $\overline{\partial}$-operator defined by the holomorphic structure on $\mathscr{L}$. We may also fix a Hermitian metric on L so that the Chern connection of $\overline{\partial}_L$ is d_L. Then, there is a map

$$\mathscr{B}(E) \to \mathscr{B}(E \otimes L), \qquad (A, \Phi) \mapsto (A \otimes 1 + 1 \otimes d_L, \Phi \otimes 1). \tag{6.2}$$

Here, $\mathscr{B}(E)$ and $\mathscr{B}(E \otimes L)$ are the configuration spaces of Higgs bundles with underlying smooth bundles E and $E \otimes L$, respectively. Since $(\mathscr{E}, \Phi)$ is (semi)stable if and only if $(\mathscr{E} \otimes \mathscr{L}, \Phi)$ is (semi)stable, this map restricts to a map

$$\mathscr{B}(E)^{ps} \to \mathscr{B}(E \otimes L)^{ps} \tag{6.3}$$

and eventually descends to a homeomorphism (in the C^∞-topology)

$$\mathscr{M}_{an} \xrightarrow{\otimes \mathscr{L}} \mathscr{M}_{an}(rd), \tag{6.4}$$

where $\mathscr{M}_{an}(rd) = \mathscr{B}^{ps}(E \otimes L)^{ps} / \operatorname{Aut}(E \otimes L)$, and rd is the degree of $E \otimes L$. On the other hand, there is a homeomorphism (in the analytic topology) $\mathscr{M}_{alg} \to \mathscr{M}_{alg}(rd)$ given by tensoring by $\mathscr{L}$. Here, $\mathscr{M}_{alg}(rd)$ is the moduli space of the semistable Higgs bundles of rank r and degree rd in the category of schemes. Finally, these maps fit into the following commutative diagram

$$\begin{array}{ccc} \mathscr{M}_{an} & \xrightarrow{i} & \mathscr{M}_{alg} \\ \downarrow{\scriptstyle \otimes\mathscr{L}} & & \downarrow{\scriptstyle \otimes\mathscr{L}} \\ \mathscr{M}_{an}(rd) & \xrightarrow{i} & \mathscr{M}_{alg}(rd) \end{array} \tag{6.5}$$

Therefore, the bottom map is a homeomorphism if and only if the top one is a homeomorphism.

Now, let us recall Nitsure's construction of $\mathscr{M}_{alg}$ in [27]. By the previous paragraph, we may assume that the degree d of E is sufficiently large so that if $(\mathscr{E}_A, \Phi)$ is a semistable Higgs bundle defined by $(A, \Phi) \in \mathscr{B}$ then $\mathscr{E}_A$ is generated by global sections and $H^1(X, \mathscr{E}_A) = 0$. Let $p = d + r(1-g)$ and Q be the Quot scheme parameterizing isomorphism classes of quotients $\mathscr{O}_X^p \to \mathscr{E} \to 0$, where $\mathscr{E}$ is a coherent sheaf on X with rank r and degree d, and $\mathscr{O}_X$ is the structure sheaf of X. Let $\mathscr{O}_{X\times Q}^p \to \mathscr{U} \to 0$ be the universal quotient sheaf on $X \times Q$, and $R \subset Q$ be the subset of all $q \in Q$ such that

1) the sheaf $\mathscr{U}_q$ is locally free, and
2) the map $H^0(X, \mathscr{O}_X^p) \to H^0(X, \mathscr{U}_q)$ is an isomorphism.

It is shown that R is open in Q. Moreover, Nitsure constructed a linear scheme F over R such that closed points in F_q correspond to Higgs fields on $\mathscr{U}_q$ for any $q \in Q$. Let F^{ss} denote the subset of F consisting of semistable Higgs bundles $(\mathscr{O}_X^p \to \mathscr{E} \to 0, \Phi)$. It is open in F. Moreover, the group $PGL(p)$ acts on Q, and the action lifts to F. Finally, Nitsure showed that the good quotient of F^{ss} by the group $PGL(p)$ exists and is the moduli space $\mathscr{M}_{alg}$.

Following [28], if U is an open subset of $\mathscr{B}^{ss}$ (in the C^∞-topology), a map $\sigma\colon U \to F^{ss}$ is called a *classifying map* if $\sigma(A, \Phi)$ is a Higgs bundle isomorphic to $(\mathscr{E}_A, \Phi)$.

Lemma 6.1. *Fix $(A_0, \Phi_0) \in \mathscr{B}^{ss}$. There exists an open neighborhood U of (A_0, Φ_0) in $\mathscr{B}^{ss}$ in the C^∞-topology such that a classifying map $\sigma\colon U \to F^{ss}$ exists and is continuous with respect to the analytic topology on F^{ss}.*

Before giving the proof, we first show how it implies the continuity of i.

Corollary 6.2. *The comparison map $i\colon \mathscr{M}_{an} \to \mathscr{M}_{alg}$ is a homeomorphism.*

Proof. Fix $[A_0, \Phi_0] \in \mathscr{M}_{an}$ such that $(A_0, \Phi_0) \in \mathscr{B}^{ps}$. By Lemma 6.1, there exists an open neighborhood U of (A_0, Φ_0) such that a continuous classifying map $\sigma\colon U \to F^{ss}$ exists. Composed with the categorical quotient $F^{ss} \to \mathscr{M}_{alg}$, which is continuous in the analytic topology, we obtain a continuous map $U \to \mathscr{M}_{alg}$. By construction, it descends to the restriction of i to the open set $\pi(U)$, where $\pi\colon \mathscr{B}^{ps} \to \mathscr{M}_{an}$ is the quotient map.

To see that i is a homeomorphism, we show that it is proper. Since $\mathscr{M}_{alg}$ is locally compact in the analytic topology, if i is proper, then it is a closed map and hence a homeomorphism. Let us recall the definitions of Hitchin fibrations in the analytic and algebraic settings. Given a Higgs bundle $(\mathscr{E}, \Phi)$, the coefficient of λ^{n-i} in the characteristic polynomial $\det(\lambda + \Phi)$ is a holomorphic section of $\mathscr{K}_X^i$, where n is the rank of $\mathscr{E}$, $i = 1, \cdots, n$, and $\mathscr{K}_X$ is the canonical bundle on the Riemann surface X. Since these sections are clearly $\mathscr{G}^{\mathbb{C}}$-invariant, we have obtained a well-defined map

$$h_{an} \colon \mathscr{M}_{an} \to \bigoplus_{i=1}^{n} H^0(X, \mathscr{K}_X^i). \tag{6.6}$$

It is known that h_{an} is a proper map (see [16, Theorem 8.1] or [37, Theorem 2.15]). On the other hand, let $(\mathscr{V}, \mathbf{\Phi})$ be the local universal family of semistable Higgs bundles parameterized by the scheme F^{ss}. Therefore, $(\mathscr{V}, \mathbf{\Phi})$ is a pair of a vector bundle $\mathscr{V} \to X \times F^{ss}$ and a section $\mathbf{\Phi} \in H^0(X \times F^{ss}, p_X^* \mathscr{K}_X \otimes \operatorname{End} \mathscr{V})$, where $p_X \colon X \times F^{ss} \to X$ is the projection onto the first factor. Moreover, if $q = (\mathscr{E}, \Phi) \in F^{ss}$ is a semistable Higgs bundle then the restriction $(\mathscr{V}_q, \mathbf{\Phi}_q)$ of $(\mathscr{V}, \mathbf{\Phi})$ to $X \times \{q\}$ is isomorphic to $(\mathscr{E}, \Phi)$. Hence, there is a map $\widetilde{h}_{alg} \colon F^{ss} \to \oplus_{i=1}^n H^0(X, \mathscr{K}_X^i)$ sending a closed point $q \in F^{ss}$ to the coefficients of the characteristic polynomial $\det(\lambda + \mathbf{\Phi}_q)$. Since the Higgs fields of two S-equivalent Higgs bundles have the same characteristic polynomial, $\widetilde{h}_{alg}$ induces a well-defined map $h_{alg} \colon \mathscr{M}_{alg} \to \oplus_{i=1}^n H^0(X, \mathscr{K}_X^i)$ (see [27, §6] for more details). The maps h_{an} and h_{alg} are called Hitchin fibrations. Therefore, if $[A, \Phi] \in \mathscr{M}_{an}$ and $q = (\mathscr{E}_A, \Phi) \in F^{ss}$ is the Higgs bundle determined by (A, Φ), then

$$h_{alg} \circ i[A, \Phi] = h_{alg}([\mathscr{E}_A, \Phi]_S) = \widetilde{h}_{alg}(q). \tag{6.7}$$

By definition, $\widetilde{h}_{alg}(q) \in \oplus_{i=1}^n H^0(X, \mathscr{K}_X^i)$ is the coefficients of the characteristic polynomial $\det(\lambda + \mathbf{\Phi}_q)$. Since $(\mathscr{V}_q, \mathbf{\Phi}_q)$ is isomorphic to $(\mathscr{E}_A, \Phi)$, $\widetilde{h}_{alg}(q) = h_{an}(A, \Phi)$, and we have proved that $h_{alg} \circ i = h_{an}$.

As a consequence, if K is a compact subset in $\mathscr{M}_{alg}$ in the analytic topology, then $i^{-1}(K) \subset h_{an}^{-1} h_{alg}(K)$. Since h_{alg} is continuous, $h_{alg}(K)$ is compact and hence $h_{an}^{-1} h_{alg}(K)$ is compact by the properness of h_{an}. Since $\mathscr{M}_{alg}$ is a separated scheme, $\mathscr{M}_{alg}$ is Hausdorff in the analytic topology. Hence, K is closed and $i^{-1}(K)$ is also closed and contained in a compact set. Therefore, $i^{-1}(K)$ is compact. □

Proof of Lemma 6.1. The proof is essentially taken from that of [28, Theorem 6.1]. We first show that a classifying map σ exists and then prove its continuity. Let $V_0 = \ker \overline{\partial}_{A_0} \subset \Omega^0(E)$. By definition of $\overline{\partial}$-operators, $V_0 = H^0(X, \mathscr{E}_A)$. Since $H^1(\mathscr{E}_A) = 0$, the Riemann-Roch theorem implies that $\dim V_0 = p$. Hence, by choosing a basis for V_0, we may identify V_0 with

$\mathbb{C}^p$. Moreover, since $\mathscr{E}_{A_0}$ is generated by global sections, the evaluation map

$$X \times V_0 \to \mathscr{E}_A, \qquad (x,s) \mapsto s(x), \tag{6.8}$$

realizes $\mathscr{E}_{A_0}$ as a quotient of $V_0 \otimes \mathscr{O}_X \cong \mathscr{O}_X^p$. Let (A,Φ) be another point in $\mathscr{B}^{ss}$, and consider the map defined by the composition

$$\pi_A \colon V_A = \overline{\partial}_A \hookrightarrow \Omega^0(E) \to V_0, \tag{6.9}$$

where $\Omega^0(E) \to V_0$ is given by the harmonic projection defined in the following elliptic complex

$$C(A_0)\colon \Omega^0(E) \xrightarrow{\overline{\partial}_{A_0}} \Omega^{0,1}(E). \tag{6.10}$$

We claim that there exists an open neighborhood U of (A_0,Φ_0) such that π_A is an isomorphism for every $(A,\Phi) \in U$. Write $\pi_A(s) = s + u_s$ for some $u_s \in V_0^{\perp}$ and $\overline{\partial}_A = \overline{\partial}_{A_0} + a$ for some $a \in \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$. Let G_0 be the Green operator in the elliptic complex $C(A_0)$. Since $u_s \in V_0^{\perp}$,

$$u_s = \overline{\partial}^*_{A_0}\overline{\partial}_{A_0} G_0 u_s = \overline{\partial}^*_{A_0} G_0 \overline{\partial}_{A_0} u_s = \overline{\partial}^*_{A_0} G_0(-\overline{\partial}_{A_0} s) = \overline{\partial}^*_{A_0} G_0(as). \tag{6.11}$$

Hence, π_A has a natural extension

$$\widetilde{\pi}_A \colon \Omega^0(E) \to \Omega^0(E), \qquad s \mapsto s + \overline{\partial}^*_{A_0} G_0(as), \tag{6.12}$$

satisfying the following estimate

$$\begin{aligned} \|\overline{\partial}^*_{A_0} G_0(as)\|_{L^2_k} &\le C\|as\|_{L^2_{k-1}} \le C\|as\|_{L^2_k} \\ &\le C\|a\|_{L^2_k}\|s\|_{L^2_k} \le C\|a\|_{C^\infty}\|s\|_{L^2_k}, \end{aligned} \tag{6.13}$$

where we have used the Sobolev multiplication theorem (see [11, Theorem 4.4.1]). Therefore, if $A_1, A_2 \in \mathscr{B}^{ss}$ and $\overline{\partial}_{A_i} = \overline{\partial}_{A_0} + a_i$ for some $a_i \in \Omega^{0,1}(E)$, we have

$$\|(\widetilde{\pi}_{A_2} - \widetilde{\pi}_{A_1})s\|_{L^2_k} = \|\overline{\partial}^*_{A_0} G_0(a_2 - a_1)s\|_{L^2_k} \le C\|a_2 - a_1\|_{C^\infty}\|s\|_{L^2_k}. \tag{6.14}$$

Now if U is sufficiently small, we may assume that

$$\|\overline{\partial}^*_{A_0} G_0(as)\|_{L^2_k} \le (1/2)\|s\|_{L^2_k} \tag{6.15}$$

so that

$$\|\widetilde{\pi}_A s\|_{L^2_k} \ge (1/2)\|s\|_{L^2_k}. \tag{6.16}$$

This shows that $\widetilde{\pi}_A$ is injective. Since $H^1(\mathscr{E}_A) = 0$, $\dim V_A = \dim V_0 = p$, π_A is an isomorphism. Therefore, the map

$$X \times V_0 \xrightarrow{1\times\pi_A^{-1}} X \times V_A \xrightarrow{(x,s)\mapsto s(x)} \mathscr{E}_A \tag{6.17}$$

realizes $\mathscr{E}_A$ as a quotient of $V_0 \otimes \mathscr{O}_X \cong \mathscr{O}_X^p$, since $\mathscr{E}_A$ is generated by global sections. As a consequence, the classifying map

$$\sigma\colon U \to F^{ss}, \qquad (A, \Phi) \mapsto (\mathscr{O}_X^p \to \mathscr{E}_A \to 0, \Phi), \tag{6.18}$$

is well-defined.

Now, we show that σ is continuous. Let $G(p, r)$ be the Grassmannian parameterizing isomorphism classes of quotients $\mathbb{C}^p \to V \to 0$, where V is a vector space of dimension r. Over $G(p, r)$, there is a universal quotient bundle $H \to G(p, r)$. Fix $x \in X$ and choose a basis for the fiber $(\mathscr{K}_X)_x$ of the canonical bundle $\mathscr{K}_X$ over x. Therefore, any Higgs field $\Phi \in H^0(\operatorname{End}\mathscr{E} \otimes \mathscr{K}_X)$ induces an endomorphism $\Phi_x\colon E_x \to E_x \otimes (\mathscr{K}_X)_x \cong E_x$. Then, Nitsure showed in [27] that there is a morphism

$$\tau_x\colon F \to \operatorname{End} H, \qquad (\mathscr{O}_X^p \to \mathscr{E}_A \to 0, \Phi) \mapsto (\mathbb{C}^p \to E_x, \Phi_x\colon E_x \to E_x), \tag{6.19}$$

where $\mathbb{C}^p \to E_x$ is obtained by evaluating the map $\mathscr{O}_X^p \to \mathscr{E}_A$ at x. Moreover, [27, Proposition 5.7] states that there are N points $x_1, \cdots, x_N \in X$ such that $\{\tau_{x_i}\}$ induces an injective and proper morphism (in the category of schemes) $\tau\colon F^{ss} \to W$ for some open subset W of $(\operatorname{End} H)^N$. Therefore, the underlying continuous map of τ is a closed embedding with respect to the analytic topology. Hence, σ is continuous if the composition

$$\sigma_x\colon U \xrightarrow{\sigma} F^{ss} \xrightarrow{\tau_x} \operatorname{End} H \tag{6.20}$$

is continuous for any $x \in X$. More explicitly, σ_x is given by

$$(A, \Phi) \mapsto (V_0 \to E_x \to 0, \Phi_x\colon E_x \to E_x), \tag{6.21}$$

where $V_0 \to E_x$ is defined by

$$V_0 \xrightarrow{\pi_A^{-1}} V_A \xrightarrow{s\mapsto s(x)} E_x. \tag{6.22}$$

Clearly, the map $\Phi \mapsto \Phi_x$ is continuous. It suffices to show that

$$A \mapsto (V_0 \to E_x \to 0) \tag{6.23}$$

is continuous. Fix $s \in V_0$ and $A_1, A_2 \in U$. Write $\overline{\partial}_{A_i} = \overline{\partial}_{A_0} + a_i$ for some $a_i \in \Omega^{0,1}(E)$ $(i = 1, 2)$. Then, the following estimate follows from (6.14), (6.16), and Sobolev embedding $L^2_k \hookrightarrow C^0$,

$$\begin{aligned}
|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s(x)| &\leq \|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s\|_{C^0} \\
&\leq C\|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s\|_{L^2_k} \\
&\leq C\|\widetilde{\pi}_{A_1}^{-1}(s - \widetilde{\pi}_{A_1}\widetilde{\pi}_{A_2}^{-1}s)\|_{L^2_k} \\
&\leq C\|s - \widetilde{\pi}_{A_1}\widetilde{\pi}_{A_2}^{-1}s\|_{L^2_k} \\
&= C\|(\widetilde{\pi}_{A_2} - \widetilde{\pi}_{A_1})\pi_{A_2}^{-1}s\|_{L^2_k} \\
&\leq C\|a_2 - a_1\|_{C^\infty}\|\widetilde{\pi}_{A_2}^{-1}s\|_{L^2_k} \\
&\leq C\|a_2 - a_1\|_{C^\infty}\|s\|_{L^2_k}.
\end{aligned} \tag{6.24}$$

Hence, $A \mapsto (V_0 \to E_x \to 0)$ is continuous. □

6.2. Holomorphicity

We continue to show that the comparison map i is a biholomorphism. Let $\mathscr{M}_{an}^s$ and $\mathscr{M}_{alg}^s$ be the subsets of $\mathscr{M}_{an}$ and $\mathscr{M}_{alg}$ consisting of stable Higgs bundles, respectively. We first show that the restriction $i\colon \mathscr{M}_{an}^s \to \mathscr{M}_{alg}^s$ is a biholomorphism. By [31, Theorem 4.7], $\mathscr{M}_{alg}^s$ is open in $\mathscr{M}_{alg}$. By [31, Corollary 11.7] and [27, Proposition 7.1], we see that $\mathscr{M}_{alg}^s$ is smooth. On the other hand, a polystable Higgs bundle (A, Φ) is stable if and only if its $\mathscr{G}^{\mathbb{C}}$-stabilizer is equal to $\mathbb{C}^*$ or equivalently $\dim \mathbf{H}^0(C_{\mu_{\mathbb{C}}}(A, \Phi)) = 1$. Since $\mathbb{C}^*$ is contained in every $\mathscr{G}^{\mathbb{C}}$-stabilizer, by the upper semicontinuity of dimensions of cohomology (see [20, Chapter VII, (2.37)]), we conclude that $\mathscr{M}_{an}^s$ is open in $\mathscr{M}_{an}$.

Proposition 6.3. *$\mathscr{M}_{an}^s$ is a smooth submanifold of $\mathscr{M}_{an}$.*

Proof. Fix $(A, \Phi) \in \mathscr{B}^s$ that satisfies Hitchin's equation. Let K be its $\mathscr{G}$-stabilizer so that $K^{\mathbb{C}}$ is its $\mathscr{G}^{\mathbb{C}}$-stabilizer. To show that $\mathscr{M}_{an}^s$ is smooth, we will use Theorem B. It is enough to show that $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ K^{\mathbb{C}} = \mathbf{H}^1$. In fact, since $K^{\mathbb{C}} = \mathbb{C}^*$, $K^{\mathbb{C}}$ acts on $\mathbf{H}^1$ trivially. Moreover, $\nu_{0,\mathbb{C}}(x) = \frac{1}{2}H[x, x]$ is trace-free for every $x \in \mathbf{H}^1$. Since $\mathbf{H}^2(C_{\mu_{\mathbb{C}}}) = \mathbb{C}^*\omega_X$, we conclude that $H[x, x] = 0$ for every $x \in \mathbf{H}^1$, where ω_X is a fixed Kähler form on X. □

Fix $[A, \Phi] \in \mathscr{M}_{an}^s$ such that $(A, \Phi) \in \mathscr{B}^s$ satisfies Hitchin's equation. By Corollary 3.13 and Proposition 6.3, we see that $\varphi\colon \mathscr{Z} \to \mathscr{M}_{an}^s$ is a biholomorphism onto an open neighborhood of $[A, \Phi]$ in $\mathscr{M}_{an}^s$, where $\mathscr{Z}$ is an open neighborhood of 0 in $\mathbf{H}^1$ and φ the map induced by the Kuranishi map $\theta\colon \mathscr{Z} \to \mathscr{B}^s$ (see Section 3). Therefore, to show that $i|_{\mathscr{M}^s_{an}}$ is holomorphic, it is enough to show that $i\varphi\colon \mathscr{Z} \to \mathscr{M}^s_{alg}$ is holomorphic. By the remark after the proof of [31, Corollary 5.6], we see that the analytification of $\mathscr{M}_{alg}$ is the coarse moduli space of semistable Higgs bundles in the category of complex spaces. Therefore, to show that $i\varphi$ is holomorphic, we need to construct a family $(\mathscr{V}, \mathbf{\Phi})$, called the *Kuranishi family* associated with θ, of stable Higgs bundles over $\mathscr{Z}$ such that $(\mathscr{V}_t, \mathbf{\Phi}_t)$ is isomorphic to $(\mathscr{E}_{A_t}, \Phi_t)$ for every $t \in \mathscr{Z}$, where $(A_t, \Phi_t) = \theta(t)$. In general, a family $(\mathscr{V}, \mathbf{\Phi})$ of Higgs bundles over a complex space T is a holomorphic vector bundle $\mathscr{V} \to X \times T$ together with a holomorphic section $\mathbf{\Phi} \in H^0(X \times T, p_X^*\mathscr{K}_X \otimes \operatorname{End}\mathscr{V})$, where $p_X\colon X \times T \to X$ is the projection onto the first factor.

Proposition 6.4. *For any $(A, \Phi) \in \mathscr{B}^s$, let $\theta\colon \mathscr{Z} \to \mathscr{B}^s$ be the Kuranishi map defined by (A, Φ). Then, there exists a Kuranishi family $(\mathscr{V}, \mathbf{\Phi})$ of stable Higgs bundles over $\mathscr{Z}$ such that $(\mathscr{V}_t, \mathbf{\Phi}_t)$ is isomorphic to $(\mathscr{E}_{A_t}, \Phi_t)$ for every $t \in \mathscr{Z}$, where $(A_t, \Phi_t) = \theta(t)$.*

Proof. We adapt the proof of [10, Proposition 2.6]. Let $V = p_X^*E$ be the smooth vector bundle over $X \times \mathscr{Z}$, and $\mathbf{\Phi}(x,t) := \Phi_t(x)$ can be regarded as a smooth section of $p_X^*\Lambda^{1,0}X \otimes \operatorname{End}(U) \subset \Omega^{1,0}(X \times \mathscr{Z}, \operatorname{End} U)$. Then, we need to put a holomorphic structure on V so that $\mathbf{\Phi}$ is a holomorphic section.

Let $\{s_i\}$ be a smooth local frame for E. Then $\{p_X^* s_i\}$ is a smooth local frame for V. Then, we define a $\overline{\partial}$-operator $\overline{\partial}_V\colon \Omega^0(V) \to \Omega^{0,1}(V)$ by the requirement that

$$\overline{\partial}_V(p_X^* s_i) = \overline{\partial}_{A_t} s_i. \tag{6.25}$$

Here, $\overline{\partial}_{A_t} s_i$ is regarded as a local section of $\Lambda^{0,1}(X \times \mathscr{Z}) \otimes V$. It is easy to show that $\overline{\partial}_V$ is independent of the choices of smooth local frames $\{s_i\}$. Therefore, $\overline{\partial}_V$ is a well-defined $\overline{\partial}$-operator on V.

Then, we show that $\overline{\partial}_V$ is integrable so that $\mathscr{V} = (V, \overline{\partial}_V)$ is a holomorphic vector bundle over $X \times \mathscr{Z}$. Write $\overline{\partial}_{A_t} s_i = f_i^j s_j$ for some smooth local function f_i^j on $X \times \mathscr{Z}$. Since θ is holomorphic, each f_i^j is holomorphic in the direction of $\mathscr{Z}$. As a consequence,

$$\overline{\partial}_V^2(p_X^* s_i) = \overline{\partial}_{X \times \mathscr{Z}} f_i^j \wedge s_j + f_i^j \overline{\partial}_{A_t} s_j = \overline{\partial}_X f_i^j \wedge s_j + f_i^j \overline{\partial}_{A_t} s_j, \tag{6.26}$$

where $\overline{\partial}_{X\times\mathscr{Z}}$ and $\overline{\partial}_X$ are usual $\overline{\partial}$-operators on the complex manifolds $X \times \mathscr{Z}$ and X, respectively. On the other hand,

$$0 = \overline{\partial}^2_{A_t} s_i = \overline{\partial}_X f_i^j \wedge s_j + f_i^j \overline{\partial}_{A_t} s_j. \tag{6.27}$$

Then, we show that $\overline{\partial}_V \mathbf{\Phi} = 0$. Write $\Phi_s = \phi^i s_i$ for some smooth local function ϕ^i on $X \times \mathscr{Z}$. Since θ is holomorphic, ϕ^i is holomorphic in the direction of $\mathscr{Z}$. As a consequence,

$$\overline{\partial}_V \mathbf{\Phi} = \overline{\partial}_{X\times\mathscr{Z}} \phi^i \wedge s_i + \phi^i \overline{\partial}_{A_t} s_i = \overline{\partial}_X \phi^i \wedge s_i + \phi^i \overline{\partial}_{A_t} s_i = \overline{\partial}_{A_t} \Phi_t = 0. \tag{6.28}$$

Finally, we need to show that if $(\mathscr{V}_t, \mathbf{\Phi}_t)$ is isomorphic to $(\mathscr{E}_{A_t}, \Phi_t)$ for any $t \in \mathscr{Z}$. If $i_t(x) = (x,t)$ is the holomorphic map $X \to X \times \mathscr{Z}$, then the holomorphic structure on $i_t^*\mathscr{V}$ is given by the pullback $\overline{\partial}$-operator $i_t^*\overline{\partial}_V$. Since

$$[i_t^*(\overline{\partial}_V)](i_t^* p_X^* s) = i_t^*(\overline{\partial}_V s) = \overline{\partial}_{A_t} s \tag{6.29}$$

for any smooth local section s of E, we see that $i_t^*\mathscr{V}$ is isomorphic to $\mathscr{E}_{A_t}$. Moreover, $i_t^*\mathbf{\Phi} = \Phi_t = \Phi$. □

Corollary 6.5. *The comparison map $i\colon \mathscr{M}^s_{an} \to \mathscr{M}^s_{alg}$ is a biholomorphism.*

Proof. Since the analytification of $\mathscr{M}_{alg}$ is the coarse moduli space of semi-stable Higgs bundles in the category of complex spaces, the family $(\mathscr{V}, \mathbf{\Phi})$ constructed in Proposition 6.4 induces a holomorphic map

$$\mathscr{Z} \to \mathscr{M}^s_{alg}, \qquad t \mapsto [\mathscr{V}_t, \mathbf{\Phi}_t]. \tag{6.30}$$

On the other hand, the map $i\varphi\colon \mathscr{Z} \to \mathscr{M}^s_{alg}$ is given by

$$i\varphi(t) = i[A_t, \Phi_t] = [\mathscr{E}_{A_t}, \Phi_t] = [\mathscr{V}_t, \mathbf{\Phi}_t]. \tag{6.31}$$

Hence, $i\varphi$ is holomorphic. Since both $\mathscr{M}^s_{an}$ and $\mathscr{M}^s_{alg}$ are smooth complex manifolds, and i is a holomorphic bijection, i is a biholomorphism. □

Then, we extend the holomorphicity of i^{-1} on $\mathscr{M}^s_{alg}$ to the full moduli space $\mathscr{M}_{alg}$.

Corollary 6.6. *The map $i^{-1}\colon \mathscr{M}_{alg} \to \mathscr{M}_{an}$ is holomorphic.*

Proof. Recall that $\mathscr{M}_{an}$ is assumed to be reduced, and $\mathscr{M}_{alg}$ is reduced. Take a holomorphic $f\colon U \to \mathbb{C}$ where U is an open subset of $\mathscr{M}_{an}$. Then, the pullback $(i^{-1})^*f$ is continuous on the open set $i(U)$ and holomorphic on $i(U) \cap \mathscr{M}^s_{alg}$. By [22], the normality of $\mathscr{M}_{alg}$ implies the normality of its analytification. Since $\mathscr{M}^s_{alg}$ is open in the Zariski topology, $\mathscr{M}_{alg} \setminus \mathscr{M}^s_{alg}$ is a closed analytic subset of $\mathscr{M}_{alg}$ in the analytic topology. Since $(i^{-1})^*f$ is already continuous on $i(U)$, the Riemann extension theorem for normal complex spaces implies that the restriction $(i^{-1})^*f\colon \mathscr{M}^s_{alg} \cap i(U) \to \mathbb{C}$ can be extended to a holomorphic function g on $i(U)$. Since $\mathscr{M}_{alg}$ is irreducible, the open set $\mathscr{M}^s_{alg}$ is dense in the Zariski topology and hence in the analytic topology ([26, §10, Theorem 1]). Since both $(i^{-1})^*f$ and g are continuous and agree on an open dense subset $\mathscr{M}^s_{alg} \cap i(U)$ of $i(U)$, $(i^{-1})^*f = g$. This shows that i^{-1} is holomorphic. □

The final ingredient is the normality of $\mathscr{M}_{an}$.

Lemma 6.7. *$\mathscr{M}_{an}$ is a normal complex space.*

Proof. Let us temporarily use Q to mean $\nu^{-1}_{0,\mathbb{C}}(0)$ viewed as an affine variety in $\mathbf{H}^1$ and Q^{an} to mean the analytification of Q. By Theorem 5.3, it suffices to prove that $Q^{an} /\!\!/ K^{\mathbb{C}}$ is normal at the origin [0]. Here, $Q^{an} /\!\!/ K^{\mathbb{C}}$ is the analytic GIT quotient of Q^{an} by $K^{\mathbb{C}}$. By [15], the analytification of the affine GIT quotient $Q /\!\!/ K^{\mathbb{C}}$ is $Q^{an} /\!\!/ K^{\mathbb{C}}$.

Now, we fix a Higgs bundle (A, Φ) such that $\mu(A, \Phi) = 0$. By choosing a point $x \in X$, the holomorphic bundle $(\mathscr{E}_A, \Phi, x)$ defines a point in the moduli space $\mathbf{R}_{Dol}(X, x, n)$ of the semistable Higgs bundles of rank n and degree 0 and with a frame at x. In [32, Corollary 11.7], it is shown that $\mathbf{R}_{Dol}(X, x, n)$ is normal. Moreover, in the proof of [32, Proposition 10.5], it is shown that the formal completion of Q (regarded as an affine variety in $\mathbf{H}^1$) at 0 is isomorphic to the formal completion of a subscheme Y at $(\mathscr{E}_A, \Phi, x)$. Here, Y is a local slice, provided by Luna's slice theorem (see [18, Theorem 4.2.12]) at $(\mathscr{E}_A, \Phi, x)$ for the $GL_n(\mathbb{C})$ action on $\mathbf{R}_{Dol}(X, x, n)$. Moreover, since $\mathbf{R}_{Dol}(X, x, n)$ is normal at $(\mathscr{E}_A, \Phi, x)$, Y can be taken to be normal at $(\mathscr{E}_A, \Phi, x)$. As a consequence, the formal completion of Q is normal at 0. By [35, Tag 0FIZ], Q is normal at 0. Since taking invariants commutes with localizations and preserves the normality, we conclude that $Q /\!\!/ K^{\mathbb{C}}$ is normal at [0]. Since normality is preserved by the analytification (see [22]), we see that $Q^{an} /\!\!/ K^{\mathbb{C}}$ is normal at [0]. □

The proof of Theorem C rests on the following theorem.

Theorem 6.8 ([12, Theorem, p.166]). *Let $f\colon X \to Y$ be an injective holomorphic map between reduced and pure dimensional complex spaces. Assume that Y is normal and that $\dim X = \dim Y$. Then f is open, and f maps X biholomorphically onto $f(X)$. In particular, the space X is normal.*

Proof of Theorem C. Now the map

$$i^{-1}\colon \mathscr{M}_{alg} \to \mathscr{M}_{an} \tag{6.32}$$

is a holomorphic homeomorphism. To use Theorem 6.8, we verify that $\mathscr{M}_{an}$ is pure dimensional, normal and $\dim \mathscr{M}_{an} = \dim \mathscr{M}_{alg}$. By Lemma 6.7, $\mathscr{M}_{an}$ is normal. Since $\mathscr{M}_{alg}$ is connected in the analytic topology, $\mathscr{M}_{an}$ is connected. Then, the normality and connectedness of $\mathscr{M}_{an}$ implies that $\mathscr{M}_{an}$ is irreducible and hence pure dimensional (see [12, Theorem, p.168]). Finally, by Corollary 6.5, $\dim \mathscr{M}_{an} = \dim \mathscr{M}_{alg}$. □

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