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Nondiscreteness of F-thresholds

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For every integer $g > 1$ and prime $p > 0$, we give an example of a standard graded domain R (where Proj R is a nonsingular projective curve of genus g over an algebraically closed field of characteristic p), such that the set of F-thresholds of the irrelevant maximal ideal of R is not discrete. This answers a question posed by Mustaţă-Takagi-Watanabe ([MTW], 2005).

These examples are based on a certain Frobenius semistability property of a family of vector bundles on X, which was constructed by D. Gieseker using a specific "Galois" representation (analogous to Schottky uniformization for a genus g Riemann surface).

1. Introduction

Let $(R, \mathbf{m})$ be a Noetherian local ring of positive characteristic p. For an ideal I of R, a set of invariants of singularities in positive characteristic, called F-thresholds, were introduced by [MTW] as follows

$$\{F\text{-thresholds of } I\} := \{c^J(I) \mid J \subseteq \mathbf{m} \text{ such that } I \subseteq \mathrm{Rad}(J)\},$$

where $c^J(I) := \lim_{e\to\infty} \max\{r \mid I^r \not\subseteq J^{[p^e]}\}/p^e$ and where $J^{[p^e]}$ is the ideal generated by the set $\{x^{p^e} \mid x \in J\}$ (the existence of this limit was proved in [DsNbP]).

In the same paper Mustaţă-Takagi-Watanabe posed the following (Question 2.11 in [MTW]):

Question. *Given an ideal* $(0) \neq I \subseteq \mathbf{m}$*, could there exist finite accumulation points for the set of* F*-thresholds of* I*?*

It was shown for regular local ring ([MTW] and for regular ring [1] [BMS2]) that the F-thresholds of an ideal coincide with the F-jumping numbers of the generalized test ideals of I (see [HY]). The F-jumping numbers are positive characteristic analogues of the jumping numbers of a multiplier

[1]With natural finiteness conditions imposed.

ideal in characteristic 0. The first F-jumping number (introduced by [TaW]) corresponds to the log canonical threshold of I.

The set of the jumping numbers, for a given ideal, is known to be discrete and rational.

It is known ([MTW], [BMS1], [BMS2], [KLZ]) that in a regular ring [1] the set of F-jumping numbers for an ideal is discrete and they are all rational (in fact, as pointed out in [BMS2], the discreteness of the set of F-jumping numbers implies the rationality statement due to the fact that if I is generated by m elements and λ is an F-jumping number then $\{p^e\lambda\}+m-1$ is also an F-jumping number, for all $e \geq 1$, where $\{p^e\lambda\}$ is the fractional part of $p^e\lambda$).

More recently, it was shown in [HMNb] (Proposition 4.17) that the set $\{c^J(I)\}_J$ is a discrete set of rational points when R is a direct summand of a regular F-finite domain S. Here the authors extend the theory of Bernstein-Sato polynomials to the direct summands of regular rings, while for regular rings the authors in [MTW] relate the Bernstein-Sato polynomials to the F-jumping numbers and the F-thresholds. Now in [HMNb], each $c^J(I)$ is identified with $c^{JS}(IS)$ and hence is an F-jumping number of IS.

In particular, in all of the above cases, the F-thresholds of an ideal have been studied by identifying them with the F-jumping numbers of some ideal in a regular ring where such a set is discrete and consists of rational numbers.

The discreteness of the set of F-jumping numbers is known in some singular cases too *e.g.* when the ring is an F-finite normal $\mathbb{Q}$-Gorenstein domain ([GrS], [BSTZ], [KSSZ], [ST]). However we cannot conclude the same for F-thresholds, as they can be in general different from the F-jumping numbers, as shown by the following Example 2.5 from [TaW], where the ring $R = k[x,y,z]/(xy-z^2)$ with $\mathbf{m} = (x,y,z)$ and the first F-jumping number of $\mathbf{m}$ < the first F-threshold of $\mathbf{m}$.

In this paper we answer the above question of [MTW] affirmatively (see Corollary 1.2).

Recall that in [TrW], the number $c^I(\mathbf{m})$ was realized as the maximal supporting point of the continuous function, namely the HK density function of (R, I).

Moreover, in the case of dimension two, we had shown there that the F-thresholds of the maximal ideal at graded ideals can be expressed in terms of the Harder-Narasimhan slopes of the associated syzygy bundles. As a result, we had deduced that the set $\{c^I(\mathbf{m}) \mid I \ \text{ is graded}\}_I$ consists of rational points.

Here, we apply this new view point to show that the set of F-thresholds of an ideal can have accumulation points for a cone over a nonsingular projective curve of any genus $g > 1$ over any characteristic $p > 0$. More precisely we prove the following

Theorem 1.1. *Given a prime p and an integer $g > 1$, there is a two-dimensional standard graded normal $\mathbb{Q}$-Gorenstein domain $(R, \mathbf{m})$ (a cone over a nonsingular curve of genus g) over an algebraically closed field of char $p > 0$, and a sequence of $\mathbf{m}$-primary graded ideals $\{I_m\}_{m\geq 0}$, such that the F-threshold of $\mathbf{m}$ at I_m is given by*

$$c^{I_m}(\mathbf{m}) = \frac{3}{2} + \frac{(g-1)}{p^{m+m_0}d}, \quad \textit{for} \quad m \geq 0,$$

where $d = e_0(R, \mathbf{m})$ and $m_0 \geq 0$ is an integer such that $p^{m_0} < g$.

Moreover, each I_m is generated by three homogeneous elements of degree 1 in R. Further, if $p \neq 2$ then we can choose R such that the Gorenstein index of R is coprime to p.

In particular we have the following

Corollary 1.2. *Given a prime p and an integer $g > 1$, there exists a two dimensional standard graded normal $\mathbb{Q}$-Gorenstein domain R with the graded maximal ideal $\mathbf{m}$ such that the set of F-thresholds of $\mathbf{m}$ has accumulation points, where Proj $R = X$ is a nonsigular projective curve of genus g over a field of char p.*

Moreover there is a strictly decreasing sequence consisting of F-thresholds of $\mathbf{m}$; thus, the F-thresholds of an ideal need not satisfy the descending chain condition (unlike in the case of regular rings).

For the proof of Theorem 1.1, we crucially use the following construction by D. Gieseker in [G]. For a given p and $g > 1$, there exists a family X of stable curves of genus g over $\operatorname{Spec} k[[t]]$ (k is an algebraically closed field of char p) with smooth generic fiber, and a closed fiber with particular singularities. By taking a specific representation of G (analogus to the representation arising from a Schottky uniformization for a compact Riemann surface of genus g), where G is the group of covering transformations of Y_0 (and where Y_0 is the universal cover over the special fiber X_0 of X), Gieseker constructed a rank 2 vector bundle F_1 on the generic fiber X_K ($K = k((t))$) with an explicit Harder-Narasimhan filtration. Moreover the bundle F_1, associated to the representation of G, comes equipped with a sequence $\{F_k\}_{k\geq 1}$ of bundles such that $F^*F_{k+1} = F_k$.

From this sequence we construct a set of vector bundles with the similar properties such that, in addition, the new set is a *bounded family* of bundles on the curve X_K. By choosing $\mathcal{L}=$ the power of the canonical bundle of the curve, we ensure that the coordinate ring (corresponding to the embedding of the curve by $\mathcal{L}$) is $\mathbb{Q}$-Gorenstein.

In Section 2 we recall the required basic theory of Harder-Narasimhan filtrations of vector bundles on curves, and also results from [TrW]. In Section 3 we prove Theorem 1.1.

2. Preliminaries

We recall a few generalities about Harder-Narasimhan filtration of vector bundles on curves.

Definition/Notations. Let X be a nonsingular curve over an algebraically closed field k of char $= p > 0$ then for a vector bundle V on X, we denote

$$\deg\ V = \deg(\wedge^{\mathrm{rank} V} V), \quad \text{and} \quad \mu(V) = \deg\ V/\mathrm{rank}\ V.$$

A bundle V is *semistable* if for every subbundle $W \subseteq V$, we have $\mu(W) \leq \mu(V)$.

Every bundle has the unique *HN (Harder-Narasimhan) filtration*, which is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = V \tag{2.1}$$

such that (1) $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$ and (2) each E_i/E_{i-1} is a semistable bundle.

The HN filtration of V is *strong* HN filtration if, in addition, each E_i/E_{i-1} is strongly semistable, *i.e.*, $F^{n*}(E_i/E_{i-1})$ is semistable for every n^{th}-iterated Frobenius map $F^n : X \longrightarrow X$. It is known (Theorem 2.7 of [L]) that for a vector bundle V there exists $m \geq 0$ such that $F^{m*}V$ has strong HN filtration.

For the vector bundle V with the HN filtration (2.1), we denote

$$\mu_{min}(V) = \mu(V/E_{n-1}) \quad \text{and} \quad a_{min}(V) = \mu_{min}(F^{m*}(V))/p^m,$$

where m is an integer such that $F^{m*}V$ achieves the strong HN filtration.

We recall the following results (Theorem B (1) from [TrW]).

Theorem 2.1 ([TrW]). *Let R be a standard graded two dimensional domain defined over an algebraically closed field. Let $I \subset R$ be a graded ideal of finite colength with a set of s homogeneous generators of degree d. Let $X = Proj\ S$, where S is the normalization of R in its quotient field. Let*

$$(2.2) \qquad 0 \longrightarrow V_0 \longrightarrow M_0 = \oplus^s \mathcal{O}_X(1-d) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

be the canonical sequence of $\mathcal{O}_X$-modules.

Then $c^I(\mathbf{m}) = 1 - a_{min}(V_0)/d$.

3. Nondiscreteness of F-thresholds

We recall a result by Gieseker [G].

Theorem 3.1. (Gieseker) *For each prime $p > 0$ and integer $g > 1$, there is a nonsingular projective curve X of genus g over an algebraically closed field of characterstic p and a semistable vector bundle V of rank 2 and of degree 0 such that F^*V is not semistable.*

Bundles of positive degree with such properties have been previously constructed by J.-P. Serre and H. Tango. But for our result we use the other properties of this bundle, which were proved by Gieseker in the process of proving the above theorem. We elaborate on the relevant results from [G]:

For each $g > 1$ and each algebraically closed field k of char p, there is a family of stable curves X of genus g over Spec $k[[t]]$, such that the special fiber X_0 is a rational curve over k with g nodes and is k-split degenerate, and the generic fiber X_K is smooth and geometrically connected, where K is the quotient field of $k[[t]]$. Now if Y_0 is the universal covering space of the special fiber X_0 and G is the group of the covering transformations of Y_0 over X_0, then (Proposition 2, [G]) any representation ρ of G on K^n gives a rank n bundle F_ρ on X such that the pull back bundle F_1 on the geometric generic fiber $X_{\bar{K}}$ comes with a sequence of bundles $F_1, F_2, F_3, \ldots$ such that $F^*F_{k+1} \simeq F_k$, where F is the absolute Frobenius of $X_{\bar{K}}$. Now, by making a specific choice of a representation ρ (attributed to Mumford by [G]) of the group G on K^2, Gieseker derives (Lemma 4, [G]) a rank 2 bundle F_ρ of degree 0 on X and an exact sequence

$$0 \longrightarrow L \longrightarrow F_\rho \longrightarrow L^{-1} \longrightarrow 0,$$

where $\deg L = g - 1$. Now pull back of L to $X_{\bar{K}}$ gives the HN filtration $0 \subset L \subset F_1$ and also a sequence of bundles F_1, F_2, $F_3, \ldots$ such that $F^*F_{k+1} = F_k$.

By a simple argument it follows (Lemma 5, [G]) that if $g \leq p^{k-1}$ then F_k is semistable. Hence one can choose a (unique) bundle V from the set $\{F_k\}_{k\geq 1}$ such that V is semistable and F^*V is not semistable.

In the following lemma we consider a modified version of such a family $\{F_m\}_m$ of bundles.

Before starting the lemma we recall some standard facts.

Remark 3.2. A line bundle $\mathcal{L}$ on a nonsingular curve X of genus g with $\deg\ \mathcal{L} \geq 2g+1$ is very ample, *i.e.*, there is a closed embedding $i : X \longrightarrow \mathbb{P}^n_k$ such that $\mathcal{L} = \mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^n_k}(1)$, for some $n > 0$, (Corollary 5.6, Chap II, [H]). This implies that the section ring $R(X, \mathcal{L}) := \oplus_{m\geq 0} H^0(X, \mathcal{L}^{\otimes m})$ is a quotient of $k[X_0, \ldots, X_n]$ and hence is generated by degree 1 elements.

If $\mathcal{L}$ is an ample line bundle on a nonsingular projective variety X then the section ring $R(X, \mathcal{L})$ is $\mathbb{Q}$-Gorenstein if $\mathcal{L}^{\otimes m} \simeq \omega_X^{\otimes n}$, for some $m, n \in \mathbb{Z} \setminus \{0\}$, where ω_X is the canonical divisor of X. Moreover the least n with this property is called the Gorenstein index of R.

In particular (as $\deg \omega_X^{\otimes m} = (2g-2)m$) the ring $R(X, \mathcal{L})$ is a standard graded normal $\mathbb{Q}$-Gorenstein ring of Gorenstein index m, for any nonsingular curve X of genus $g > 1$ and $\mathcal{L} = \omega_X^{\otimes m}$, provided $m \geq 3$.

Lemma 3.3. *Given an integer $g > 1$ and a prime p, there is a nonsingular curve X of genus g over an algebraically closed field of characteristc p and a family of bundles $\{E_m\}_{m\geq 0}$ such that*

1) rank $E_m = 2$ *and det*$(E_m) = \mathcal{O}_X$*, for* $m \geq 0$ *and*
2) *for each E_m, the number $m \geq 0$ is the least integer such that the bundle $F^{m*}E_m$ is not semistable. Moreover the HN filration (hence the strong HN filtration) of $F^{m*}E_m$ is*

$$0 \subset L_m \subset F^{m*}E_m, \quad \textit{where} \quad \deg(L_m) = (g-1)/p^{m_0},$$

for some $m_0 \geq 0$ where $p^{m_0} < g$.

3) *There exists a very ample line bundle $\mathcal{L}$ on X, such that for every $m \geq 0$, the bundle $E_m \otimes \mathcal{L}$ is generated by its global sections.*

In particular $\{E_m \otimes \mathcal{L}\}_{m\geq 0}$ is a bounded family.

Proof. The results in [G] (see the above discussion) give the following: for given $g > 1$ and p, there is a nonsingular curve X of genus g over an algebraically closed field of char p and a family of bundles $\{F_m\}_{m\geq 1}$ such that

1) F_m is of rank 2 and of *degree* 0, for $m \geq 1$ and
2) $F^*F_{m+1} = F_m$, and F_m is semistable if $g \leq p^{m-1}$,
3) F_1 has the HN filtration $L \subset F_1$, where $\deg L = g - 1$ and $\deg F_1 = 0$.

Hence, there is a unique $m_0 \geq 0$ such that $F_{m_0+2} \in \{F_k\}_{k\geq 1}$ is semistable and $F^*F_{m_0+2} = F_{m_0+1}$ is not semistable. Since $\mathrm{Pic}^0(X)$ (the set of degree 0 line bundles on X) is an abelian variety, (Application 2, page 59 in [Mu1]) the map

$$n_X : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X), \quad \text{given by} \quad \mathcal{L} \mapsto \mathcal{L}^{\otimes n} \quad \text{is surjective.}$$

Therefore, for each m, we can choose $\mathcal{L}_m \in \mathrm{Pic}^0(X)$ such that $\det(F_m) = \mathcal{L}_m^{\otimes 2}$ (recall that $\det(F_m) \in \mathrm{Pic}^0(X)$ as $\deg(\det(F_m)) = \deg(F_m) = 0$).

We define $E_m = F_{m+m_0+1} \otimes \mathcal{L}_{m+m_0+1}^{-1}$, for $m \geq 0$.

Then $\det(E_m) = \det(F_{m+m_0+1}) \otimes (\mathcal{L}_{m+m_0+1}^{-1})^{\otimes 2} = \mathcal{O}_X$. This proves Assertion (1).

Note that

$$F^{k*}E_m = F^{k*}F_{m+m_0+1} \otimes (\mathcal{L}_{m+m_0+1}^{-1})^{\otimes p^k} = F_{m-k+m_0+1} \otimes (\mathcal{L}_{m+m_0+1}^{-1})^{\otimes p^k},$$

hence for any $m \geq 0$, the bundles $E_m, F^*E_m, \ldots, F^{m-1*}E_m$ are semistable. Since $F^{m*}E_m = F_{m_0+1} \otimes (\mathcal{L}_{m+m_0+1}^{-1})^{\otimes p^m}$, it has the HN filtration

$$L_m \subset F^{m*}E_m \quad \text{if and only if} \quad F^{m_0*}(L_m \otimes \mathcal{L}_{m+m_0+1}^{\otimes p^m}) \subset F^{m_0*}F_{m_0+1} = F_1$$

is the HN filtration of F_1. Therefore, by the uniqueness of the HN filtration we have $\deg L_m = (g-1)/p^{m_0}$. Moreover $p^{m_0} < g$ as F_{m_0+1} is not semistable. This proves Assertion (2).

Now we fix a very ample line bundle $\mathcal{O}_X(1) = \omega_X^{\otimes l_0}$ on X, where $l_0 \geq 3$ and (this is a standard argument in the literature)

Claim. For $m \geq 1$, the bundle E_m is 2-regular, *i.e.*, $H^1(X, E_m(n-1)) = 0$, for $n \geq 2$.

Proof of the claim. By Serre duality $H^1(X, E_m(n-1)) = \mathrm{Hom}(E_m, \omega_X(1-n))^\vee$. If $E_m \longrightarrow \omega_X(1-n)$ is a nonzero map then the semistability property of the sheaf E_m implies $\mu(E_m) \leq \mu(\omega_X(1-n))$. Therefore $0 \leq (2g-2) + (1-n)\deg \mathcal{O}_X(1) < 0$. This proves the claim. $\square$

Hence (Chapter 14, [Mu2]), for $m \geq 1$, every $E_m(2)$ is generated by its global sections. Moreover, we can choose $n_0 \geq 2$ (Theorem 5.17, [H]) such that $E_0(n_0)$ is generated by its global sections. Hence Assertion (3) follows by taking $\mathcal{L} = \mathcal{O}_X(n_0) = \omega_X^{n_0 l_0}$.

Moroever each $E_m \otimes \mathcal{L}$ has the same Hilbert polynomial with respect to $\mathcal{O}_X(1)$ (as each E_m has the same rank and degree). Therefore the family $\{E_m \otimes \mathcal{L}\}_{m\geq 0}$ is a bounded family. □

Remark 3.4. (a) For fix $l_0 \geq 3$, there exists $\widetilde{l_0}$ such that for every $n_0 \geq \widetilde{l_0}$ we can choose $\mathcal{L} = \omega_X^{\otimes l_0 n_0}$ (for $\mathcal{L}$ as in Lemma 3.3).

(b) Lemma 3.3 implies that, for any prime p and $g > 1$, there is a nonsingular curve X of genus g over an algebraically closed field of characteristic p and a bounded family $\mathbb{F}$ of vector bundles on X, such that if m_V denotes the minimum integer m for which $F^{m*}V$ achieves the strong HN filtration then the set $\{m_V \mid V \in \text{ the bounded family } \mathbb{F}\}$ is unbounded.

Proof of Theorem 1.1. For given p and g, we select a nonsingular curve X and a family $\{E_m\}_{m\geq 0}$ of bundles and a line bundle $\mathcal{L} = \omega_X^{l_0 n_0}$, where l_0, n_0 as in Remark 3.4 (a). Since E_m is a vector bundle of rank two over a curve, the (globally generated) bundle $E_m \otimes \mathcal{L}$ is generated by 3 global sections (Ex. 8.2, Chap II, [H]). Hence there is a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow M_m \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{O}_X \longrightarrow E_m \otimes \mathcal{L} \longrightarrow 0.$$

Now $M_m = (\det(E_m \otimes \mathcal{L}))^{-1} = (\mathcal{L}^{\otimes 2})^\vee$. Dualizing the above short exact sequence we get

$$(3.1) \qquad 0 \longrightarrow (E_m \otimes \mathcal{L})^\vee \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{O}_X \xrightarrow{\eta} \mathcal{L}^{\otimes 2} \longrightarrow 0.$$

Let

$$R = \oplus_{n\geq 0} R_n = \oplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes 2n}) \quad \textit{and} \quad I_m = h_{m1}R + h_{m2}R + h_{m3}R,$$

where the above map η is given by $(s_1, s_2, s_3) \mapsto h_{m1}s_1 + h_{m2}s_2 + h_{m3}s_3$, for some $h_{mi} \in H^0(X, \mathcal{L}^{\otimes 2})$.

By Remark 3.2, the section ring $R = R(X, \mathcal{L}^{\otimes 2})$ is a normal $\mathbb{Q}$-Gorenstein standard graded domain. Let $\mathbf{m}$ be the graded maximal ideal of R. Note that h_{m1}, h_{m2}, h_{m3} in R_1 and $\deg\ X = e_0(R, \mathbf{m}) = \deg \mathcal{L}^{\otimes 2}$. By Theroem 2.1, we have

$$c^{I_m}(\mathbf{m}) = 1 - a_{min}((E_m \otimes \mathcal{L})^\vee)/\deg(\mathcal{L}^{\otimes 2}).$$

Now, for any $m \geq 0$, the bundle $F^{m-1*}E_m$ is semistable and $0 \subset L_m \subset F^{m*}E_m$ is the strong HN filtration. This implies that $0 \subset L_m \otimes F^{m*}(\mathcal{L}^\vee) \subset$

$F^{m*}((E_m \otimes \mathcal{L})^\vee)$ is the strong HN filtration and $F^{m-1*}((E_m \otimes \mathcal{L})^\vee)$ is semistable.

Hence

$$\begin{aligned} a_{min}((E_m \otimes \mathcal{L})^\vee) &= \mu_{min}(F^{m*}((E_m \otimes \mathcal{L})^\vee))/p^m \\ &= \mu(L_m^{-1} \otimes F^{m*}(\mathcal{L}^\vee))/p^m \\ &= -\deg(\mathcal{L}) - (g-1)/p^{m+m_0}. \end{aligned}$$

Therefore

$$c^{I_m}(\mathbf{m}) = 1 + \frac{1}{2\deg(\mathcal{L})}\left[\deg\ \mathcal{L} + \frac{g-1}{p^{m+m_0}}\right] = \frac{3}{2} + \frac{(g-1)}{dp^{m+m_0}},$$

where $d = e_0(R, \mathbf{m}) = \deg \mathcal{L}^{\otimes 2}$.

By Remark 3.4 (a), given $p \neq 2$, we can choose l_0 and n_0 such that the Gorenstein index $2l_0n_0$ is coprime to p. This proves the theorem. □

Remark 3.5. We recall that when R is a regular local ring, then, apart from the set of F-thresholds (of an ideal) being discrete and rational, there can be no strictly decreasing sequence of F-thresholds of an ideal I (Remark 2.9, [MTW]). This is because in the regular case there is a bijection between the set of F-thresholds of I and the set of test ideals of I, given by $c \mapsto \tau(I^c)$ such that if c_1 and c_2 are two F-threhsolds of I then $c_1 < c_2$ if and only of $\tau(I^{c_2}) \subset \tau(I^{c_1})$.

However the above example in Theorem 1.1 shows that any such "order reversing" bijective correspondence between the set of F-thresholds and a set of ideals of some kind, would not hold.

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