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# Generalizations of theorems of Nishino and Hartogs by the $L^2$ method

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*To the memory of Professor Akira Takeuchi*

Three different generalizations will be given for Nishino's rigidity theorem asserting the triviality of Stein families of $\mathbb{C}$ over the polydisc, in connection to generalizations of Hartogs's theorem on the analyticity criterion for continuous functions.

## Introduction

This is a continuation of [Oh-5], where an $L^2$ extension theorem and Hartogs's analyticity criterion in [H] were applied to give an alternate proof of the rigidity theorem of Nishino [Ni] asserting that a Stein manifold fibered over $\mathbb{D}^m$ ($\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$) is biholomorphically equivalent to $\mathbb{D}^m \times \mathbb{C}$ if the fibers are equivalent to $\mathbb{C}$. In [Oh-5] the assertion was stated for $m = 1$ but the proof works for any $m$. See also [Y-1][1] and [Ch][2]. After a remarkable generalization of Nishino's rigidity theorem was given in [Y-2] which replaces $\mathbb{C}$ by any Riemann surface except for the disc and the once-punctured disc, the result has been expected to be strengthened in two ways, by weakening the Steinness assumption and by raising the dimension of the fibers. The purpose of the present article is to generalize Nishino's theorem in each of these two directions, eventually culminating in three generalizations of the theorem. At first we shall replace the Steinness assumption by the existence of a complete Kähler metric on the total space of the family, following the ideas of Grauert [G-1] and Andreotti-Vesentini [A-V]. More precisely we shall prove the following.

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[1] A potential theoretic proof was given.

[2] In [Ch] the Steinness was replaced by the weaker assupmtion of "disc convexity".

**Theorem 0.1.** *Let $M$ be a complex manifold of dimension $m+1$ which admits a complete Kähler metric and a holomorphic map $\pi$ onto $\mathbb{D}^m$ without critical points such that $\pi^{-1}(t) \cong \mathbb{C}$ for all $t \in \mathbb{D}^m$. Then $M$ is biholomorphically equivalent to $\mathbb{D}^m \times \mathbb{C}$.*

Note that neither the $L^2$ extension theorem in [Oh-T] nor Hartogs's theorem is available since $M$ is not assumed to have a Zariski dense Stein open subset of a complex manifold. Accordingly, instead of the $L^2$ extension theorem we shall apply an $L^2$ vanishing theorem in [Oh-1] (see also [Dm] and [Oh-3]) by choosing suitable plurisubharmonic weight functions. As a substitute of Hartogs's theorem we shall apply the following.

**Lemma 0.1.** *Let $f$ be a $\mathbb{C}$-valued continuous function on $\mathbb{D}^m$. Then $f$ is holomorphic if and only if the domain*

$$\mathbb{D}^m \times \mathbb{C} \setminus \{(t, f(t)); t \in \mathbb{D}^m\}$$

*admits a complete Kähler metric.*

The necessity part of Lemma 0.1 is contained in Grauert's observation in [G-1] that Stein manifolds admit complete Kähler metrics. The sufficiency was first proved in [Oh-2] under a restrictive assumption that $f$ is of class $C^1$, by applying the $L^2$ vanishing theorem to extend a holomorphic function from a hyperplane section with a growth control. The proof for the above stronger assertion is more straightforward (see §2).

Since Theorem 0.1 seems to deserve some extension, we shall apply the same method to prove a fibration theorem which describes a condition for a complex manifold to be fibered over a manifold in such a way that the generic fibers are $\mathbb{C}$ (see Theorem 4.2).

It is not known whether or not every Stein family of $\mathbb{C}^n$ for $n \geq 2$ is locally the product. Here we shall be contented with a weaker rigidity result for the family of $\mathbb{C}^n$ paired with the divisor $\{z \in \mathbb{C}^n; z_1 \cdots z_n = 0\}$ (see Theorem 4.3). Rather unexpectedly, this enables us to generalize Nishino's theorem in the following way, too.

**Theorem 0.2.** *Let $M$ be a complex manifold of dimension $m+n$ with a holomorphic submersion onto $\mathbb{D}^m$ whose fibers are once-puctured $\mathbb{CP}^n$, say $\mathbb{CP}^n_*$. Then $M \cong \mathbb{CP}^n_* \times \mathbb{D}^m$ if and only if $M$ is $n$-convex.*

**Corollary 0.1.** *Let $f : \mathbb{D}^m \to \mathbb{C}^n$ be a continuous function. Then $f$ is holomorphic if and only if $(\mathbb{D}^m \times \mathbb{C}^n) \setminus \{(t, f(t)); t \in \mathbb{D}^m\}$ is $n$-convex.*

Recall that a complex manifold $M$ is said to be ***n-convex*** if it admits a $C^2$ exhaustion function $\varphi$ whose Levi form (or complex Hessian) $\partial\bar{\partial}\varphi$ has strictly less than $n$ nonpositive eigenvalues outside some compact subset say $K \subset M$ (cf. [A-G]). $M$ is called ***n-complete*** if $\varphi$ is chosen in such a way that $K = \varnothing$. Theorem 0.2 generalizes Nishino's rigidity theorem because a complex manifold is known to be Stein if and only if it is 1-complete (cf. [G-3]).

After the author finished writing the proof of Corollary 0.1, he was informed by N. Shcherbina that T. Pawlaschyk [P, Theorem 4.7.9] had obtained it in a completely different way. (See also [P-S].) His proof is even simpler in the sense that it does not use the $L^2$ method, but the proof given here might be of independent interest.

## 1. $L^2$ vanishing theorem and application

For the convenience of the reader, we shall briefly recall two basic things ; an $L^2$ vanishing theorem on complete Kähler manifolds in a primitive form as was stated in [Oh-1] and a method how it is applied to produce holomorphic top forms.

Let $M$ be an $n$-dimensional connected complex manifold equipped with a complete Kähler metric say $g$. We note that $M$ admits a complete Kähler metric of the form $\partial\bar{\partial}\lambda(\varphi)$ for some $\lambda : \mathbb{R} \to \mathbb{R}$ if there exists a proper $C^\infty$ map $\varphi : M \to (-\infty, \infty)$ satisfying $\partial\bar{\partial}\varphi > 0$. Here, by an abuse of notation, $\partial\bar{\partial}\varphi$ stands also for the complex Hessian of $\varphi$ as well as the complex exterior derivatives applied to $\varphi$. We know accordingly that the complement of an analytic set in a Stein manifold admits a complete Kähler metric.

We shall recall below how the $\bar{\partial}$-equation $\bar{\partial}u = v$ can be solved for a given $L^2$ $\bar{\partial}$-closed $(n, 1)$-form $v$ with an $L^2$ solution $u$ that vanishes at a prescribed point $x$ when $M$ admits a $C^\infty$ plurisubharmonic function which is strictly plurisubharmonic at $x$.

Recall that the $L^2$ norm $\|h\|(= \|h\|_g)$ (resp. the weighted $L^2$ norm $\|h\|_\Phi(= \|h\|_{\Phi,g})$) of a measurable $(p, q)$-form $h$ on $M$ is defined as

$$\left(\int_M |h|^2 dV_g\right)^{\frac{1}{2}} \quad \left(\text{resp.} \left(\int_M e^{-\Phi}|h|^2 dV_g\right)^{\frac{1}{2}}\right)$$

where $|h|$ and $dV_g$ respectively stand for the length of $h$ and the volume form with respect to the metric $g$. The space of $L^2$ forms with respect to $\|\cdot\|$ (resp. $\|\cdot\|_\Phi$) will be denoted by $L^{p,q}(M)$(resp. $L^{p,q}_\Phi(M)$). Recall also that

$$\|h\|^2 = \int_M h \wedge \overline{*h} \quad \left(\text{resp. } \|h\|_\Phi^2 = \int_M e^{-\Phi} h \wedge \overline{*h}\right),$$

where $*$ denotes Hodge's star operator, and that $L^{n,0}_\Phi(M)$ does not depend on the choice of the metric $g$. The following was proved in [Oh-1].

**Theorem 1.1.** (cf. [Oh-1, Theorem 1.5 and Corollary 1.6]) *Let $M$ be as above and let $\Phi$ be a strictly plurisubharmonic function of class $C^4$ on $M$. Then, for any $\bar\partial$ closed $(n,1)$-form $f$ on $M$ satisfying*

$$\int_M e^{-\Phi} f \wedge \overline{*_{\partial\bar\partial\Phi} f} < \infty, \tag{1.1}$$

*there exists an $(n,0)$-form $h$ satisfying $\bar\partial h = f$ and*

$$i^{n^2} \int_M e^{-\Phi} h \wedge \overline{h} \le \int_M e^{-\Phi} f \wedge \overline{*_{\partial\bar\partial\Phi} f}.$$

*Here $*_{\partial\bar\partial\Phi}$ denotes Hodge's star operator with respect to $\partial\bar\partial\Phi$.*

Recall that the proof of Theorem 1.1 in [Oh-1] is an application of the Riesz representation theorem or Hahn-Banach's theorem based on the estimate

$$\left|\int_M e^{-\Phi} f \wedge \overline{*_{\partial\bar\partial\Phi+g} u}\right|^2 \le \|\bar\partial^* u\|^2_{\Phi,\partial\bar\partial\Phi+g} \int_M e^{-\Phi} f \wedge \overline{*_{\partial\bar\partial\Phi} f} \tag{1.2}$$

which holds for any $u$ in the domain of the adjoint $\bar\partial^*$ of $\bar\partial$ with respect to the weighted norm $\|\|_{\Phi,\partial\bar\partial\Phi+g}$. The proof of (1.2) for those $u$ with compact support is done by a direct calculation using the Kähler condition on $g$. The completeness of $g$ is needed to extend the estimate to the domain of $\bar\partial^*$. Since $\|f\|_{\Phi,\partial\bar\partial\Phi+g} \le \|f\|_{\Phi,\partial\bar\partial\Phi}$ holds for any $(n,q)$-form $f$, if $\Phi$ is a $C^\infty$ plurisubharmonic function on $M$ and $f$ is a $\bar\partial$-closed $(n,1)$-form on $M$ satisfying

$$\|f\|_{\Phi,\partial\bar\partial\Phi} := \lim_{\epsilon\searrow 0} \|f\|_{\Phi,\partial\bar\partial\Phi+\epsilon g} < \infty,$$

one can find a solution $h$ to $\bar\partial h = f$ satisfying

$$\|h\|_\Phi \le \lim_{\epsilon\searrow 0} \|f\|_{\Phi,\partial\bar\partial\Phi+\epsilon g}.$$

In many situations Theorem 1.1 is applied in the following way: Let $M$ and $\Phi$ be as above, let $x \in M$ be any point and let $z = (z_1, \dots, z_n)$ be a local coordinate around $x$ which maps a neighborhood $U$ of $x$ onto $\mathbb{D}^n$, where $\mathbb{D} = \{\zeta \in \mathbb{C}; |\zeta| < 1\}$. Let $\chi : M \to [0,1]$ be a $C^\infty$ function satisfying $\operatorname{supp}\chi \subset U$ and $\chi \equiv 1$ on a neighborhood of $x$, let $\alpha$ be a compactly supported $C^\infty$ $(n,0)$-form on $M$ satisfying $\alpha = \chi dz_1 \wedge \cdots \wedge dz_n$ on $U$ and $\alpha = 0$ outside $U$, and let $\Psi$ be a $C^\infty$ function on $M \setminus \{x\}$ satisfying $\operatorname{supp}\Psi \subset U$ and $\Psi = 2n\chi \log \|z\|$ on $U \setminus \{x\}$, where $\|z\|^2 = \sum_{j=1}^n |z_j|^2$. Clearly $\Psi + m\Phi$ is strictly plurisubharmonic on $M \setminus \{x\}$ for sufficiently large $m$. Then, since $M \setminus \{x\}$ admits a complete Kähler metric, $g + \epsilon\partial\bar{\partial}(\chi \log(-\log \|z\|))$ for sufficiently small $\epsilon > 0$ for instance, by Theorem 1.1 one can find for such $m$ an $(n,0)$-form $u$ on $M \setminus \{x\}$ such that $\bar{\partial} u = \bar{\partial}\alpha$ and

$$i^{n^2} \int_M e^{-\Psi - m\Phi} u \wedge \overline{u} < \infty. \tag{1.3}$$

Then, by (1.3), $\alpha - u$ extends to a holomorphic $n$-form on $M$ say $\tilde{\alpha}$ such that $\tilde{\alpha}(x) \neq 0$. Similarly, for any two distinct points $x, y \in M$, one can find a holomorphic $n$-form $\beta$ on $M$ satisfying $\beta(x) = 0$ and $\beta(y) \neq 0$.

If $\Phi$ is a $C^\infty$ plurisubharmonic function on $M$ and $f$ is a $\bar{\partial}$-closed $(n,1)$-form on $M$ satisfying

$$\lim_{\epsilon \searrow 0} \|f\|_{\Phi, \partial\bar{\partial}\Phi + \epsilon g} < \infty,$$

one can find a solution $h$ to $\bar{\partial} h = f$ satisfying

$$\|h\|_\Phi \leq \lim_{\epsilon \searrow 0} \|f\|_{\Phi, \partial\bar{\partial}\Phi + \epsilon g}.$$

(See [Oh-3, Theorem 2.8] for a more general statement.) Therefore, the above argument works to show the following.

**Theorem 1.2.** *Let $M$ be a complete Kähler manifold of dimension $n$ and let $\varphi$ be a $C^\infty$ plurisubharmonic function on $M \setminus \{x\}$ such that $e^{-\varphi}$ is non-integrable on any neighborhood of $x$ and $\varphi$ is strictly plurisubharmonic on $U \setminus \{x\}$ for some neighborhood $U \ni x$. Then there exists a holomorphic $n$-form $h$ on $M$ such that $h(x) \neq 0$ and*

$$\left| \int_{M \setminus V} e^{-\varphi} h \wedge \overline{h} \right| < \infty$$

*for any neighborhood $V \ni x$.*

The idea that only the local strict positivity of $\partial\bar{\partial}\varphi$ suffices for the existence theorem comes from [Hm], although it is not explicitly stated there.

(See also [Sk] and [Dm].) Of course the global semipositivity of $\partial\bar{\partial}\varphi$ and the existence of a complete Kähler metric are both indispensable here.

## 2. Proof of Lemma 0.1

Let us restate the lemma to fix the notation.

**Lemma 0.1.** Let $\Gamma$ be the graph of a continuous function $z = f(t)$ from $\mathbb{D}^m$ to $\mathbb{C}$. Then $f$ is holomorphic if $(\mathbb{D}^m \times \mathbb{C}) \setminus \Gamma$ has a complete Kähler metric.

*Proof.* Since the problem is local on $\Gamma$, we may assume in advance that $\Gamma \subset \mathbb{D}^m \times \mathbb{D}$. Let $\gamma : W \to (\mathbb{D}^m \times \mathbb{D}) \setminus \Gamma$ be the double covering asociated to the unique index-two subgroup of $\pi_1((\mathbb{D}^m \times \mathbb{D}) \setminus \Gamma) \cong \mathbb{Z}$. For the construction of the covering spaces associated to the subgroups of the fundamental group, see [Ah, Chap. 9] for instance. Then $W$ is a complex manifold admitting a complete Kähler metric pulled up from downstairs. Let $\sigma \in \mathrm{Aut}(W)$ be the covering transformation without fixed points. Then, similarly as in section one, one has a square integrable holomorphic $(m+1)$-form $h \neq 0$ on $W$ satisfying $h(x) = 0$ and $h(y) \neq 0$ for a pair of points $x$ and $y$ satisfying $\sigma(x) = y$. Then, by putting $\hat{h} = h - \sigma^* h$ one has a nonzero holomorphic $(m+1)$-form $\hat{h}$ satisfying $\sigma^*\hat{h} = -\hat{h}$. Then we put

$$\rho = \frac{\hat{h}}{\gamma^*(dt_1 \wedge \cdots \wedge dt_m \wedge dz)}.$$

Clearly $\sigma^*\rho = -\rho$. Note that $\rho^2$ is holomorphic on $(\mathbb{D}^m \times \mathbb{D}) \setminus \Gamma$. Since the denominator and the numerator of $\rho$ are square integrable, by Fubini's theorem for almost every $t \in \mathbb{D}^m$ $\rho^2$ extends to a meromorphic function, say $\rho_t^2$ on $\{t\} \times \mathbb{D}$, whose order of zero or pole at the point $(t, f(t))$ is an odd integer. On the other hand, one knows already by Hartogs's theorem that $\Gamma$ is analytic if $\rho^2$ is not extendable across $\Gamma$. Hence, we may assume that $\rho^2$ is holomorphic on $\mathbb{D}^{m+1}$. In this case $\Gamma$ must be analytic because it is contained in the divisor of $\rho^2$. $\square$

**Remark 2.1.** Shcherbina [Sh] proved that $f$ is holomorphic if $\Gamma$ is pluripolar, i.e. if $\Gamma \subset \varphi^{-1}(-\infty)$ holds for some plurisubharmonic function $\varphi \not\equiv -\infty$ on $\mathbb{D}^m \times \mathbb{C}$. Note that Lemma 0.1 implies that $\Gamma$ is a complex submanifold if $\Gamma = \Phi^{-1}(-\infty)$ for some plurisubharmonic function $\Phi$ on $\mathbb{D}^m \times \mathbb{C}$ such that $\Phi$ is $C^\infty$ on $(\mathbb{D}^m \times \mathbb{C}) \setminus \Gamma$. In fact, for any point $x \in \Gamma$ there exists a neighborhood $U \ni x$ with a complete Kähler metric say $g_U$ such that $\Phi < 0$ on

$U$. Then $U \setminus \Gamma$ admits a complete Kähler metric $g_U - \partial\bar{\partial}\log(-\Phi)$. However, there seems to be no direct way to obtain such $\Phi$ from the given $\varphi$ as above.

## 3. Proof of Theorem 0.1

Let the notation be as in Theorem 0.1. First we briefly recall a connection between Theorem 0.1 and the $L^2$ theory. In [Oh-5] it turned out that Nishino's rigidity theorem is a direct consequence of an $L^2$ extension theorem in [Oh-T][3]. The idea is to regard an open immersion from the total space of the family $\pi : M \to \mathbb{D}^m$ to $\mathbb{D}^m \times \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, as a collection of the reciprocals of injective meromorphic functions from $\pi^{-1}(t)$ to $\hat{\mathbb{C}}$, say $f_t$, and identify the collection of fiberwise exterior derivatives of $f_t$ with a relative canonical form which has a pole of order 2 along an analytic section over $\mathbb{D}^m$. An open embedding of $M$ into $\mathbb{D}^m \times \hat{\mathbb{C}}$ is given by pairing $t$ with a fiberwise primitive of an extension of $dz/z^2$ from $\pi^{-1}(0)$ with inhomogeneous coordinate $z$, which is obtained by applying [Oh-T]. The point is the equivalence

$$f \in \mathbb{C} \cdot \frac{dz}{z^2} \iff f \in L^{1,0}_{b\log^+ \frac{1}{|z|}}(\mathbb{C} \setminus \{0\}) \cap \mathrm{Ker}\bar{\partial} \quad (2 < b \leq 4). \tag{3.1}$$

Here $\log^+ \frac{1}{|z|} := \max\{\log \frac{1}{|z|}, 0\}$ and $L^{1,0}_{b\log^+ \frac{1}{|z|}}(\mathbb{C} \setminus \{0\})$ denotes, as was mentioned in §1, the space of measurable $(1,0)$ forms $u$ on $\mathbb{C} \setminus \{0\}$ such that

$$\sqrt{-1}\int_{\mathbb{C}\setminus\{0\}} e^{-b\log^+ \frac{1}{|z|}} u \wedge \overline{u} < \infty.$$

Note that the fiberwise primitive is recovered as an integral along the paths in the fibers of $\pi$ starting from the points in the image of a holomorphic section. Note that the analyticity of the complement of the image is assured by [H] (or by Lemma 0.1). In view of the argument presented at the end of §1, it is clear that Theorem 1.1 is also applicable to produce such a relative canonical form. Actually the following argument is slightly more delicate because we need to have $m\Phi \leq b\log^+ \frac{1}{|z|}$ $(2 < b \leq 4)$ near $z = 0$.

*Proof of Theorem 0.1.* Let $\pi : M \to \mathbb{D}^m$ be as in the assumption and let $p \in \pi^{-1}(0)$. We choose a neighborhood $U \ni 0$ and a holomorphic map $s : U \to M$ such that $s(0) = p$ and $\pi \circ s = id$. Let $(t, z)$ be a local coordinate around $p$ such that $z \circ s = 0$ and $\pi|_{|z|\leq 1}$ is proper.

[3] For the $L^2$ extension theorem see also [Oh-6].

**Sublemma.** *In the above situation, for any* $\epsilon > 0$ *there exist a neighborhood* $V \ni 0$, *a point* $q \neq p$ *in* $\pi^{-1}(0)$ *and a plurisubharmonic function* $\varphi$ *on* $\pi^{-1}(V) \setminus s(V)$ *such that* $\varphi + (2+\epsilon)\log|z|$ *extends to a bounded function on a neighborhood of* $p$, $\varphi$ *is* $C^\infty$ *on* $\pi^{-1}(V) \setminus (s(V) \cup \{q\})$ *and strictly plurisubharmonic on* $W \setminus \{q\}$ *for some neighborhood* $W \ni q$, $e^{-\varphi}$ *is nonintegrable on any neighborhood of* $q$ *and that* $\varphi$ *is bounded outside* $\{|z| < 1\}$.

*Proof.* Let $\chi : [0, \infty) \to \mathbb{R}$ be a $C^\infty$ function satisfying $\mathrm{supp}\chi \subset [0, \frac{1}{2}]$ and $\mathrm{supp}(\chi - 1) \cap [0, \frac{1}{e}] = \varnothing$.

We put

$$\Phi = \max\left\{(2+\epsilon-\delta-\delta')\chi(|z|)\log\frac{|z-z(q)|}{|z|} + \delta\lambda\left(\log\frac{1}{|z|}\right), \log\|t\|^N\right\}.$$

Here $\lambda$ is a $C^\infty$ convex increasing function on $(-\infty, \infty)$ satisfying $\mathrm{supp}\lambda \subset [0, \infty)$,

$$\lambda'(x) = 1 \ \text{ on } \ [2, \infty)$$

and

$$\lambda''(x) > 0 \ \text{ on } \ (0, 1].$$

Then it is easy to see that, for any choice of positive numbers $\delta$ and $\delta'$ with $\max\{\delta, \delta'\} < \max\{\epsilon, 1\}$, one can find a neighborhood $V \ni 0$ such that, by extending $\Phi$ as 0 outside the neighborhood $\{(t, z); |z| < 1\}$ of $p$, $\Phi + \delta'\lambda(\log\frac{1}{|z|})$ satisfies the requirement if $|z(q)| \ll \delta < \epsilon$ and $N \gg \frac{1}{\epsilon - \delta}$, except for the smoothness outside $\{z = 0\}$ and strict plurisubharmonicity near $q$. Here one uses the fact that the $C^2$-norm of $\log\frac{|z-z(q)|}{|z|}$ on $\mathrm{supp}\partial\chi(|z|)$ tends to 0 as $q$ approaches to $p$. Hence, by Richberg's theorem on the approximation of continuous plurisubharmonic functions by $C^\infty$ ones (cf. [Ri]), one can find, for any $\epsilon' > 0$, a $C^\infty$ strictly plurisubharmonic function $\Phi_{\epsilon'}$ on $\pi^{-1}(V) \cap \{0 < |z| < 1\}$ such that

$$\left|\Phi + \delta'\lambda\left(\log\frac{1}{|z|}\right) + \delta''\|t\|^2 - \Phi_{\epsilon'}\right| < \epsilon'$$

holds on $\pi^{-1}(V) \cap \{0 < |z| < 1\}$ and $\Phi_{\epsilon'}$ extends to $\pi^{-1}(V)$ as a $C^\infty$ function in such a way that the extension coincides with $\delta''\|t\|^2$ outside $\{|z| < 1\}$. Then one may take this extended $\Phi_{\epsilon'}$ as $\varphi$. □

*Continuation of the proof of Theorem 0.1.* Let the situation be as above. By Theorem 1.2, for any $\epsilon > 0$ one has the existence of

$$h \in L^{m+1,0}_{(2+\epsilon)\log^+(1/|z|)}(\pi^{-1}(V) \setminus s(V)) \cap \mathrm{Ker}\bar{\partial}$$

such that $h(q) \neq 0$, where the values of $\log^+ \frac{1}{|z|}$ outside $\{|z| < 1\}$ are defined to be 0, by an abuse of notation. Hence, by choosing $\epsilon < 2$ one can find a holomorphic $(m+1)$-form $h$ on $\pi^{-1}(V) \setminus s(V)$ such that $h/\pi^*(dt_1 \wedge \cdots \wedge dt_m)$ is fiberwise the exterior derivative of a holomorphic function on $\pi^{-1}(V) \setminus s(V)$, say $F$, which is fiberwise univalent. Here we note that, for any $h \in L^{m+1,0}_{(2+\epsilon)\log^+(1/|z|)}(\pi^{-1}(V) \setminus s(V)) \cap \mathrm{Ker}\bar{\partial} \setminus \{0\}$ with $0 < \epsilon < 2$, $h$ does not have any zeros because it would otherwise contradict the $L^2$ condition. (Recall also that $H^{1,0}(\hat{\mathbb{C}}, \mathcal{O}(2)) \cong \mathbb{C} \cdot \frac{dz}{z^2}$ and note that $d(1/z) = -dz/z^2$ concerning the univalence of $F$.) By Riemann's mapping theorem, for instance, the complements of such injective maps consist of single points in a bounded subset of $\mathbb{C}$. Hence the complement of the image of $(t, F)$ is the graph of a continuous function because of its closedness, by virtue of a theorem of Bolzano and Weierstrass. Hence it is a complex submanifold by Lemma 0.1, so that we obtain the desired conclusion. $\square$

**An alternate argument.** Based on the fact that

$$f dz \in \mathbb{C}\frac{dz}{z^2}$$

holds if and only if $f \in \mathcal{O}(\mathbb{C} \setminus \{z = 0\})$ and

$$\int_{\mathbb{C}\setminus\{z=0\}} e^{-3\log^+(1/|z|)}|f(z)|^2 < \infty$$

and that any holomorphic 1-form on $\mathbb{C} \setminus \{0\}$ satisfying

$$i\int_{\mathbb{C}\setminus\{0\}} e^{-5\log^+(1/|z|)} u \wedge \overline{u} < \infty$$

is of the form

$$\left(\frac{a}{z^2} + \frac{b}{z^3}\right) dz, \quad a,\ b \in \mathbb{C},$$

one has also a univalent map from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$ by taking the ratio $v/u$ of an $L^2$ holomorphic 1-form $u$ with respect to the weight $3\log^+(1/|z|)$ and an $L^2$ holomorphic 1-form $v$ with respect to the weight $5\log^+(1/|z|)$. Hence, instead of taking the fiberwise primitive of $h$ in the above proof, one may divide by $h$ some element of $L^{m+1,0}_{5\log^+\frac{1}{|z|}}(\pi^{-1}(V) \setminus s(V)) \cap \mathrm{Ker}\bar{\partial}$.

## 4. Globalizing Nishino's rigidity theorem

By Theorem 0.1, we know that $U \cong \mathbb{D}^m \times \mathbb{C}$ if $M$ is a complete Kähler manifold of dimension $m+1$ and the fibers of $\pi$ are $\mathbb{C}$. On the other hand, it follows immediately from Nishino's rigidity theorem and Oka's principle that a Stein manifold $M$ with a submersion $\pi : M \to N$ onto a Stein manifold $N$ with $H^2(N, \mathbb{Z}) = 0$ is equivalent to $N \times \mathbb{C}$ if $\pi^{-1}(t) \cong \mathbb{C}$ for all $t \in N$. We shall show that the method of proving Theorem 0.1 is available to show another global version of Nishino's theorem.

**Definition 4.1.** A closed complex analytic set $S$ of pure codimension $m$ in a complex manifold $M$ is said to be **plumbed** if $S$ is equipped with a neighborhood $U$ and a surjective holomorphic map $\pi$ from $U$ onto $\mathbb{D}^m$ such that $\pi^{-1}(0) = S$ and the fibers of $\pi$ are closed in $M$.

**Example 4.1.** Every complete intersection in an affine algebraic variety is plumbed.

From now on, we assume that $M$ is a connected $n$-dimensional complex manifold admitting a $C^\infty$ plurisubharmonic exhaustion function say $\varphi$ and a Kähler metric say $g$. Note that $g + \partial\bar{\partial}e^{\varphi}$ is then a complete Kähler metric. For a Hermitian line bundle $(B, b)$ over $M$, we denote by $\Theta_b$ the curvature form of $b$. Following [N], we shall call a manifold admitting a $C^\infty$ plurisubharmonic exhaustion function a ***weakly 1-complete manifold***.

For the convenience of the reader, we recall a generalization of Theorem 1.1 following [Dm] and [Oh-3] according to the notation of §1 with self-explanatory generalizations.

**Theorem 4.1.** *Let $M$ be an $n$-dimensional complex manifold admitting a complete Kähler metric and let $(B, b)$ be a semipositive line bundle over $M$. Then, for any $q \geq 1$ and a measurable $B$-valued $(n, q)$ form $v$ on $M$ such that $\bar{\partial}v = 0$ and $\|v\|_{\Theta_b, b} < \infty$, one can find a measurable $B$-valued $(n, q-1)$-form $u$ on $M$ satisfying $\bar{\partial}u = v$ and $\|u\|^2_{\Theta_b, b} \leq q\|v\|^2_{\Theta_b, b}$.*

Applying Theorem 4.1 for $q = 1$ instead of Theorem 1.1, one can generalize Theorem 0.1 as follows.

**Theorem 4.2.** *Let $M$ be as above with a connected and reduced divisor $S$ and a plumbed submanifold $(T, V, \sigma)$ with $T \cong \mathbb{C}$ such that $S$ intersects with $T$ at one point transversally. Assume moreover that $\#(S \cap \sigma^{-1}(t)) = 1$ for all $t$ and that there exists a semipositive line bundle $(B, b)$ over $M$ with*

*$\Theta_b|_{\sigma^{-1}(t)} \equiv 0$ for all $t$ and that the bundle $[S]$ admits a fiber metric for which the length of a canonical section $s$ of $[S]$, say $|s|$, satisfies* $\sup|s| < \infty$ *and* $-\partial\bar{\partial}\log|s| + \Theta_b > 0$ *on $M \setminus S$. Then $S$ is connected and nonsingular, and there exists a surjective holomorphic map from $M$ to $S$ whose fibers over a Zariski dense open set are $\mathbb{C}$.*

*Proof.* If $n = 2$, then $T$ is a divisor. Let $[T]$ be the line bundle associated to $T$. Then, similarly as in the proof of Theorem 0.1, under the above situation one can produce $[T]^\nu$-valued holomorphic 2-forms $f_0, \dots, f_m$ on $M \setminus S$, for sufficiently large $\nu$, such that $f_k$ are proportional to each other on the fibers of $\sigma$ and the ratio $(f_0 : \cdots : f_m)$ is constant along the fibers, so that it is an extension of an embedding of $S$ into $\mathbb{CP}^m$ via a surjective holomorphic map $\pi : M \to S$. Hence $S$ is nonsingular because it is a holomorphic retract of $M$. Since $M$ is weakly 1-complete, it follows by Maitani-Yamaguchi's theorem (cf. [M-Y] or [B-1, B-2]) that all fibers of $\pi$ have the trivial Bergman kernel since so do they for $\sigma$. Therefore, by the weak 1-completeness of $M$ again, the irreducible fibers of $\pi$ are all $\mathbb{C}$ because so are they for $\sigma$. Hence $M$ surjectively maps onto $S$ with generic fibers equivalent to $\mathbb{C}$, as desired.

If $n \geq 3$, it is easy to see that a similar argument works, since the assumption on the line bundle $B$ implies that it plays the role of $[T]$ as above. □

A similar method based on Theorem 4.1 can be applied to prove the following.

**Theorem 4.3.** *Let $n \geq 2$ and let $\pi$ be a holomorphic submersion from a complex manifold $M$ onto a Stein manifold $N$ such that $H^2(N, \mathbb{Z}) = 0$. Assume that there exists a complete Kähler metric on $M$, $\pi^{-1}(t) \cong \mathbb{C}^n$ for all $t \in N$ and that there exists a proper holomorphic embedding*

$$\sigma : N \times \{z = (z_1, \dots, z_n) \in \mathbb{C}^n; \min\{|z_1|, \dots, |z_n|\} \leq 1\} \hookrightarrow M$$

*satisfying $\pi \circ \sigma(t, z) \equiv t$. Then there exists a biholomorphic map $F$ from $M$ to $N \times \mathbb{C}^n$ commuting with $\pi$ and the projection to $N$ such that the image of $\sigma$ is mapped by $F$ onto $N \times \{z = (z_1, \dots, z_n) \in \mathbb{C}^n; \min\{|z_1|, \dots, |z_n|\} \leq 1\}$.*

*Proof.* We shall exploit the identification of

$$\left\{\left(\frac{a}{z^2} + \frac{b}{z^3}\right) dz; a, b \in \mathbb{C}\right\}$$

with

$$\left\{ f dz; f \in \mathcal{O}(\mathbb{C} \setminus \{0\}) \text{ and } \int_{\mathbb{C}\setminus\{0\}} e^{-5\log^+ |z^{-1}|} |f(z)|^2 < \infty \right\}.$$

Note that

$$f dz_1 \wedge \cdots \wedge dz_n \in \mathbb{C} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^2 \cdots z_n^2}$$

holds if and only if $f \in \mathcal{O}(\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\})$ and

$$\int_{\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}} e^{-3\sum_{k=1}^n \log^+ |z_k|} |f(z)|^2 < \infty.$$

Similarly, any holomorphic $n$-form $u$ on $\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}$ satisfying

$$i^{n^2} \int_{\mathbb{C}^n \setminus \{z_1 \cdots z_n = 0\}} e^{-2\log^+ (1/|z_j|) - 3\sum_{k=1}^n \log^+ (1/|z_k|)} u \wedge \overline{u} < \infty$$

is of the form

$$\left( \frac{a}{\prod_{k=1}^n {z_k}^2} + \frac{b}{z_j \prod_{k=1}^n {z_k}^2} \right) dz_1 \wedge \cdots \wedge dz_n, \quad a,\ b \in \mathbb{C}.$$

We put $X = \{z \in \mathbb{C}^n; z_1 \cdots z_n = 0\}$. Then it is easy to see from the above mentioned $L^2$ interpretation that, similarly as in the proof of Theorem 0.1, one can find an open embedding of $N \setminus \sigma(N \times X)$ into $N \times (\hat{\mathbb{C}})^n$ by taking the ratios of $L^2$ holomorphic top forms with respect to suitable weights. Analyticity of the complement of the image follows similarly, too. $\square$

Note that $F$ is unique up to the composition, in the second factor, of an element of $\mathrm{Aut}(\mathbb{C}^n)$ that fixes $\{z; \min_j |z_j| \le 1\}$.

## 5. The Rigidity of $\mathbb{CP}^n_*$

For the proof of Theorem 0.2, we first recall briefly what Kodaira's theory [K, Main Theorem and its proof] says for analytic families of fiberwise hyperplanes in a given analytic family $\pi : M \to \mathbb{D}^m$ whose fibers are biholomorphically equivalent to $\mathbb{CP}^n_*$.

Let $H$ be any hypreplane in $\pi^{-1}(0)$, i.e. a compact complex analytic subset of $\pi^{-1}(0)$ which is a hyperplane in the one-point compactification $\mathbb{CP}^n$ of $\pi^{-1}(0)$[4]. Then the normal bundle of $H$ in $M$ is equivalent to the

[4] A hyperplane in $\mathbb{CP}^n$ is defined to be a nonsingular divisor whose associated line bundle is of degree one.

direct sum of the trivial bundle of rank $m$ and the hyperplane section bundle, so that there exist a neighborhood $U$ of $0 \in \mathbb{D}^m$, a neighborhood $V$ of $0 \in \mathbb{D}^n$, a proper and smooth analytic family $\varpi : \mathscr{H} \to U \times V$ satisfying $\dim \varpi^{-1}(t, v) = n - 1$ for all $(t, v) \in U \times V$ and a smooth holomorphic map $\sigma : \mathscr{H} \to \pi^{-1}(U)$ such that $\pi \circ \sigma = pr_U \circ \varpi$ and $\sigma(\varpi^{-1}((0,0)) = H$. In [K] this is proved in a somewhat more general situation by an elementary method of constructing power series and proving their convergence. The proof of Theorem 0.2 stated below is to be understood in this context.

*Proof of Theorem 0.2.* Since every automorphism of $\mathbb{CP}^n_*$ extends to $\mathbb{CP}^n$ as an automorphism and every analytic $\mathbb{CP}^n$-bundle over $\mathbb{D}^m$ is analytically trivial by the Grauert-Oka principle (cf. [G-1]), it suffices to prove that every point of $\mathbb{D}^m$ has a neighborhood say $W$ such that $\pi^{-1}(W) \cong W \times \mathbb{CP}^n_*$. Hence we may assume in advance that there exist holomorphic sections say $s_0, \ldots, s_n$ of the fibration $\pi : M \to \mathbb{D}^m$ such that each fiber $\pi^{-1}(t)$ is spanned by $\{s_0(t), \ldots, s_n(t)\}$ and that $\{s_1(t), \ldots, s_n(t)\}$ spans a compact divisor for every $t$. Here a subset $A \subset \pi^{-1}(t)$ is said to be spanned by $B \subset A$ if so is the closure of $A$ in the one-point compactification ($\cong \mathbb{CP}^n$) of $\pi^{-1}(t)$ with respect to the projective linear structure of $\mathbb{CP}^n$. We shall call $A$ the linear span of $B$ by an abuse of langulage. For a fixed $t$, let us denote by $\langle v_1, \ldots, v_p \rangle$ the linear span of $\{v_j; 1 \le j \le p\} \subset \pi^{-1}(t)$. Note that

$$M_t^* := \{\langle v_1, \ldots, v_n \rangle; v_j \in \pi^{-1}(t), \langle v_1, \ldots, v_n \rangle \text{ is compact}\} \cong \mathbb{C}^n.$$

We put $M^* = \coprod_{t \in \mathbb{D}^m} M_t^*$ and call the elements of $M_t^*$ the hyperplanes in $\pi^{-1}(t)$. Then $M^*$ is naturally equipped with a structure of a complex manifold whose local coordinates are associated to the local analytic families of hyperplanes parametrized by a set of analytic sections of the normal bundles of hyperplanes. In short, $M^*$ is the component of the Douady space of $M$ containing $\langle s_1(0), \ldots, s_n(0) \rangle$, i.e. the (so called) space of displacements of $\langle s_1(0), \ldots, s_n(0) \rangle$ in $M$. (See [K] for an explicit construction of the local coordinates in more general situations.) For any $1 \le j \le n$, let $A_{j,t}$ be the set of hyperplanes in $\pi^{-1}(t)$ that contain $s_j(t)$, i.e. the collection of $\langle v_1, \ldots, s_j(t), \ldots, v_n \rangle$ such that $(v_1, \ldots, \check{v_j}, \ldots, v_n)$ runs through the product space $\prod_{k \ne j} \langle s_k(t), s_0(t) \rangle$ of the lines $\langle s_k(t), s_0(t) \rangle$ except for those points for which $\langle v_1, \ldots, s_j(t), \ldots, v_n \rangle$ are noncompact. Then it is clear that $A_{j,t} \cong \mathbb{C}^{n-1}$ as a complex submanifold of the dual space of $\mathbb{CP}^n$, so that the disjoint union of $A_{j,t}$ for all $t \in \mathbb{D}^m$ say $A_j$ is naturally equipped with the structure of an analytic family of $\mathbb{C}^{n-1}$ over $\mathbb{D}^m$ which is a divisor in $M^*$. Notice that $\coprod_{t \in \mathbb{D}^m} A_{j,t}$ are analytic because they are hypersurfaces in the

Douady space of $M$ which are defined by the equations corresponding to the constraints of containing the sections $s_j(t)$ $(1 \leq j \leq n)$.

By assumption, there exists a $C^2$ exhaustion function $\varphi$ on $M$ whose Levi form has at least $(m+1)$ positive eigenvalues outside some compact subset of $M$. Then, by defining $\psi : M^* \to \mathbb{R}$ as

$$\psi(x) = \sup\{\varphi(p); p \in x\},$$

it is easily seen that $M^*$ is 1-convex (cf. [N-S][5]). Hence $M^*$ is Stein since it obviously contains no compact complex analytic subsets of positive dimension (cf. [G-3]). Thus, replacing $\mathbb{D}^m$ by a smaller neighborhood of 0 if necessary, we may assume in advance that the map $\pi^* : M^* \to \mathbb{D}^m$ induced by $\pi$ satisfies the assumption of Theorem 4.3.

Therefore, there exists a biholomorphic map $\beta : M^* \to \mathbb{D}^m \times \mathbb{C}^n$ commuting with the projections in such a way that the local coordinates of $M$ locally parametrize the hyperplanes

$$\{(t, H); t \in \mathbb{D}^m \text{ and } H \text{ is a hyperplane in } \mathbb{C}^n\}$$

via $\beta$. Hence, $M$ is identified with an open subset of the Douady space of $\mathbb{D}^m \times \mathbb{CP}^n$. Thus $M$ must be biholomorphically equivalent to $\mathbb{D}^m \times \mathbb{CP}^n_*$.

The converse holds because $\mathbb{D}^m \times \mathbb{CP}^n_*$ is $n$-complete with respect to

$$-\log(1 - \|t\|^2) - \log \sum_{j \neq 0} \left| \frac{z_j}{z_0} \right|^2 .$$

□

*Proof of Corollary 0.1.* Let $f : \mathbb{D}^m \to \mathbb{C}^n$ be a continuous function and let $\Gamma_f = \{(t, f(t)); t \in \mathbb{D}^m\}$. Assume that the domain $(\mathbb{D}^m \times \mathbb{C}^n) \setminus \Gamma_f$ is $n$-complete. Fixing an open embedding $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ and put $M = (\mathbb{D}^m \times \mathbb{CP}^n) \setminus \Gamma_f$. Then there exists a biholomorphic map say $\alpha$ from $M$ to $\mathbb{D}^m \times \mathbb{CP}^n_*$ by Theorem 0.2, which means that $f$ is holomorphic because $\alpha$ is holomorphically extendable across $\Gamma_f$. The converse is obvious. □

## 6. Open questions

Q1. Is every $n$-complete family of a once-punctured compact complex manifold locally trivial? Namely, given a surjective holomorphic submersion

[5] See [Ba] for a more detailed exposition in a more general circumstance. See also [Oh-4] for an alternate method to see the holomorphic convexity of $M^*$.

$\pi : M \to T$ such that $M$ is $n$-complete and $\pi^{-1}(t) \cong N \setminus \{p\}$ for all $t$ for some $n$-dimensional compact complex manifold $N$ and a point $p \in N$, does it follow that every point $t \in T$ has a neighborhood $U$ such that $\pi^{-1}(U) \cong U \times (N \setminus \{p\})$?

Q2. Is every complete Kähler family of $\mathbb{C}^n$ locally trivial?

Q3. Let $\pi : M \to \mathbb{D}$ be a holomorphic submersion such that $M$ is complete Kähler and $\pi^{-1}(t) \cong \mathbb{C}^n$ for all $t \neq 0$. Does it follow that $\pi^{-1}(0) \cong \mathbb{C}^n$?

Q4. Is there any reasonable generalization of the theorems in [Y-1, Y-2], [M-Y] and [B-1, B-2] to $n$-convex families?

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