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Asymptotic behavior of the nonlinear Schrödinger equation on exterior domain

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We consider the following nonlinear Schrödinger equation on exterior domain.

$$(1) \qquad \begin{cases} iu_t + \Delta_g u + ia(x)u - |u|^{p-1}u = 0 & (x,t) \in \Omega \times (0,+\infty), \\ u\big|_{\Gamma} = 0 & t \in (0,+\infty), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where $1 < p < \frac{n+2}{n-2}$, $\Omega \subset \mathbb{R}^n (n \geq 3)$ is an exterior domain and $(\mathbb{R}^n, g)$ is a complete Riemannian manifold. We establish Morawetz estimates for the system (1) without dissipation ($a(x) \equiv 0$ in (1)) and meanwhile prove exponential stability of the system (1) with a dissipation effective on a neighborhood of the infinity.

It is worth mentioning that our results are different from the existing studies. First, Morawetz estimates for the system (1) are directly derived from the metric g and are independent on the assumption of an (asymptotically) Euclidean metric. In addition, we not only prove exponential stability of the system (1) with non-uniform energy decay rate, which is dependent on the initial data, but also prove exponential stability of the system (1) with uniform energy decay rate. The main methods are the development of Morawetz multipliers in non (asymptotically) Euclidean spaces and compactness-uniqueness arguments.

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1. Introduction

1.1. Notations

Let O be the origin of $\mathbb{R}^n$ $(n \geq 3)$ and

$$r(x) = |x|, \quad x \in \mathbb{R}^n \tag{1.1}$$

be the standard distance function of $\mathbb{R}^n$. Moreover, let $\langle \cdot, \cdot \rangle$, div , ∇, Δ and $I_n = (\delta_{i,j})_{n\times n}$ be the standard inner product of $\mathbb{R}^n$, the standard divergence operator of $\mathbb{R}^n$, the standard gradient operator of $\mathbb{R}^n$, the standard Laplace operator of $\mathbb{R}^n$ and the unit matrix,respectively.

Suppose that $(\mathbb{R}^n, g)$ is a smooth complete Riemannian manifold with

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j, \quad x \in \mathbb{R}^n. \tag{1.2}$$

Let

$$G(x) = (g_{ij}(x))_{n\times n}, \quad x \in \mathbb{R}^n. \tag{1.3}$$

Denote

$$\langle X, Y \rangle_g = \langle G(x)X, Y \rangle, \quad |X|_g^2 = \langle X, X \rangle_g, \quad X, Y \in \mathbb{R}^n_x, \ x \in \mathbb{R}^n. \tag{1.4}$$

Let D be the Levi-Civita connection of the metric g and H be a vector field, then the covariant differential DH of the vector field H is a tensor field of rank 2 as follow:

$$DH(X,Y)(x) = \langle D_Y H, X\rangle_g(x) \quad X,Y \in \mathbb{R}^n_x,\ x \in \mathbb{R}^n. \tag{1.5}$$

Let $S(r)$ be the sphere in $\mathbb{R}^n$ with radius r. Then

$$\left\langle X, \frac{\partial}{\partial r}\right\rangle = 0, \quad \text{for} \quad X \in S(r)_x,\ x \in \mathbb{R}^n \backslash O. \tag{1.6}$$

Finally, we set div_g, ∇_g and Δ_g as the divergence operator of $(\mathbb{R}^n, g)$, the gradient operator of $(\mathbb{R}^n, g)$ and the Laplace−Beltrami operator of $(\mathbb{R}^n, g)$, respectively.

1.2. Nonlinear Schrödinger equation

Let $\Omega \subset \mathbb{R}^n$ be an exterior domain with smooth compact boundary Γ and let $\nu(x)$ be the unit normal vector outside Ω in $(\mathbb{R}^n, g)$ for $x \in \Gamma$. Assume that the origin $O \notin \overline{\Omega}$. Denote

$$d_1 = \inf_{x\in\Gamma} |x| \quad \text{and} \quad d_2 = \sup_{x\in\Gamma} |x|. \tag{1.7}$$

Then $d_2 \geq d_1 > 0$. For any constant $h > d_2$, we define

$$\Omega(h) = \{x | x \in \Omega,\ |x| \leq h\}. \tag{1.8}$$

We consider the following system:

$$\begin{cases} iu_t + \Delta_g u + ia(x)u - |u|^{p-1}u = 0 & (x,t) \in \Omega \times (0, +\infty), \\ u\big|_\Gamma = 0 & t \in (0, +\infty), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1.9}$$

where

$$1 < p < \frac{n+2}{n-2}, \tag{1.10}$$

and $a(x) \in C^2(\overline{\Omega})$ is a nonnegative real function satisfying

$$\sup_{x\in\overline{\Omega}} \left(a(x) + \left|\nabla_g a(x)\right|_g + \left|\Delta_g a(x)\right| \right) < +\infty. \tag{1.11}$$

Define the energy of the system (1.9) by

$$E(t) = \frac{1}{2}\int_{\Omega} \left(|u|^2 + |\nabla_g u|_g^2\right) dx_g + \frac{1}{p+1}\int_{\Omega} |u|^{p+1} dx_g, \tag{1.12}$$

where

$$dx_g = \sqrt{\det\left(G(x)\right)}dx, \quad |u|^2 = u\bar{u}, \quad |\nabla_g u|_g^2 = \langle \nabla_g u, \nabla_g \bar{u}\rangle_g. \tag{1.13}$$

For the free Schrödinger equation on a Riemannian manifold, many Strichartz estimates and local energy estimates are given by [6, 8, 9, 11, 12, 23, 30, 44, 49, 53, 54] under the non-trapping assumption and the assumption of an Euclidean metric at infinity. There exists a wealth of literature on such estimates for the wave equation (see [10, 29, 39, 47, 48, 52] and references therein).

For the linear damped Schrödinger equation on Riemannian manifolds, the local energy decay in an exterior domain has been proved in [1–4, 13, 35, 51] and many others under the geometric control condition (see [5, 50]). Under the non-trapping condition on an exterior domain, exponential decay for the global energy has been proved in [7] for the Schrödinger equation with a dissipation effective on a neighborhood of the infinity. For the nonlinear damped Schrödinger equation on compact manifold or Euclidean space, many stability results are given by [14, 15, 17–21, 25] and references therein. Such results are also based on the non-trapping assumption or geometric control condition.

The non-trapping assumption and geometric control condition are very closely related to the geodesic escape. Since the geodesic depends on a nonlinear ODE, they are hard to check. On the other hand, the non-trapping assumption and geometric control condition are not sufficient to derive Morawetz estimates for hyperbolic equations on global space. In comparison to the existing studies, we here take advantage of the metric g to establish Morawetz estimates for the Schrödinger equation.

As is known, the multiplier method is a simple and effective tool to deal with the energy estimate on PDEs. In particular, the celebrated Morawetz multipliers introduced by [45] have been extensively used to study the energy decay of the wave equation with constant coefficients, see [24, 39, 43, 46] and many others. For bounded domains, Yao[56] developed Morawetz multipliers for the wave equation with variable coefficients, which is a powerful tool in the analysis of systems with variable coefficients and has been extended by [16, 57, 59] and many others mentioned in [58]. However, how to establish the Morawetz estimates in non (asymptotically) Euclidean spaces is still an open problem. Therefore, one purpose of this paper is to establish Morawetz estimates on non (asymptotically) Euclidean spaces.

The organization of our paper goes as follows. In Section 2, we will state our main results. Then some multiplier identities and key lemmas for problem (1.9) will be presented in Section 3. We will show Morawetz estimates for the nonlinear Schrödinger equation without dissipation in Section 4. Then proofs of stability of the damped nonlinear Schrödinger equation with non-uniform decay rate will be presented in Section 5. We will prove stability of the damped nonlinear Schrödinger equation with uniform decay rate in Section 6. Finally, the proof for Assumption (U1) and Assumption (U2) hereinafter under stronger geometric condition is given in Appendix.

2. Main results

2.1. Well-posedness

Denote

$$C_1^\infty(\Omega) = \left\{ w \in C^\infty(\overline{\Omega}) \text{ and } \int_\Omega |w|^2 dx_g < +\infty \right\}. \tag{2.1}$$

$$C_2^\infty(\Omega) = \left\{ w \in C^\infty(\overline{\Omega}) \text{ and } \int_\Omega \left(|w|^2 + |\nabla_g w|_g^2\right) dx_g < +\infty \right\}. \tag{2.2}$$

$$C_3^\infty(\Omega) = \left\{ w \in C^\infty(\overline{\Omega}) \text{ and } \int_\Omega \left(|w|^2 + |\nabla_g w|_g^2 + |\Delta_g u|^2\right) dx_g < +\infty \right\}. \tag{2.3}$$

Let $L^2(\Omega)$ be the closure of $C_1^\infty(\Omega)$ with respect to the tolopogy

$$\|w(x)\|_{L^2(\Omega)}^2 = \int_\Omega |w|^2 dx_g, \tag{2.4}$$

$H^1(\Omega)$ be the closure of $C_2^\infty(\Omega)$ with respect to the tolopogy

$$\|w(x)\|_{H^1(\Omega)}^2 = \int_\Omega \left(|w|^2 + |\nabla_g w|_g^2\right) dx_g, \tag{2.5}$$

and $H^2(\Omega)$ be the closure of $C_3^\infty(\Omega)$ with respect to the tolopogy

$$\|w(x)\|_{H^2(\Omega)}^2 = \int_\Omega \left(|w|^2 + |\nabla_g w|_g^2 + |\Delta_g u|^2\right) dx_g. \tag{2.6}$$

Denote

$$H^1_\Gamma(\Omega) = \{w \in H^1(\Omega), \quad w\big|_\Gamma = 0\}. \tag{2.7}$$

It is well-known that the system (1.9) is subcritical and has been studied extensively in the Euclidean geometry for large classes of nonlinearities, see the books [22, 28], and the references therein. On the hyperbolic spaces, well-posedness and scattering of the system (1.9) without dissipation have been proved in [31, 32]. Therefore, throughout the paper, we assume that the following condition holds true.

Assumption (S). The system (1.9) is well-posed such that

$$u \in C\left([0, +\infty),\ H^1_\Gamma(\Omega) \bigcap H^2(\Omega)\right). \tag{2.8}$$

2.2. Morawetz estimates for the nonlinear Schrödinger equation in non (asymptotically) Euclidean spaces

The main geometric conditions for Morawetz estimates of the nonlinear Schrödinger equation in non (asymptotically) Euclidean spaces are given by the following assumption.

Assumption (A). Assume that

$$a(x) \equiv 0, \quad \text{in } \Omega, \tag{2.9}$$

$$G(x)\frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^n, \tag{2.10}$$

$$\left\langle \left((1 - \alpha(x))\, G(x) + \frac{r}{2}\frac{\partial G(x)}{\partial r} \right) X, X \right\rangle \geq 0 \ \text{ for } X \in S(r)_x,\ x \in \Omega, \tag{2.11}$$

$$\det\left(G(x)\right) = c_0 r^d, \quad x \in \Omega, \tag{2.12}$$

where $c_0 > 0, d$ are constants and $\alpha(x)$ is a continuous nonnegative function defined on $\mathbb{R}^n$.

Remark 2.1. Let $(r, \theta)=(r, \theta_1, \theta_2, \ldots, \theta_{n-1})$ be the polar coordinates of $x \in \mathbb{R}^n$ in the Euclidean metric. From (2.10), we have

$$g = dr^2 + \sum_{i,j=1}^{n-1} \gamma_{ij}(r, \theta) d\theta_i d\theta_j, \quad x \in \mathbb{R}^n, \tag{2.13}$$

which implies $r(x) = |x|$ is the geodesic distance function of $(\mathbb{R}^n, g)$ from x to the origin O.

Remark 2.2. Let Assumption (**A**) hold true. It follows from relations (4.6) and (4.7) hereinafter that

$$\frac{(n+d/2-1)}{r} = \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det (G(x))}}{\partial r} = \Delta_g r = trD^2 r$$
$$\geq (n-1)\frac{\alpha(x)}{r} \geq 0, \quad x \in \Omega. \tag{2.14}$$

Then

$$d \geq 2(1-n). \tag{2.15}$$

Example 2.1. *Let $d_1 = d_2$ and $G(x)$ satisfy*

$$G(x) = \frac{x \otimes x}{|x|^2} + f(r)\left(I_n - \frac{x \otimes x}{|x|^2}\right), \quad x \in \mathbb{R}^n, \tag{2.16}$$

where $f(r)$ is a smooth function defined on $[0, +\infty)$ such that

$$f(r) = r^m, \quad |x| \geq d_1 \quad \text{and} \quad f(r) = 1, \quad |x| < \frac{d_1}{2}. \tag{2.17}$$

Therefore,

$$G(x)\frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^n, \tag{2.18}$$

$$\left\langle \left(\frac{1}{2}\frac{\partial G(x)}{\partial r}\right) X, X \right\rangle = \frac{m}{2r}|X|_g^2 \quad \text{for } X \in S(r)_x, \ |x| \geq d_1, \tag{2.19}$$

$$det\,(G(x)) = r^{m(n-1)} \quad \text{for } |x| \geq d_1. \tag{2.20}$$

Let

$$\alpha(x) = 1 - \frac{m}{2}, \quad d = m(n-1). \tag{2.21}$$

Then, (2.10),(2.11) and (2.12) hold true.

Theorem 2.1. *Let Assumption* (**A**) *hold true. Assume that*

$$\frac{\partial r}{\partial \nu} \leq 0, \quad x \in \Gamma. \tag{2.22}$$

Then there exists a positive constant C such that for $d = 2(3-n)$,

$$\int_0^T \int_\Omega \frac{|u|^{p+1}}{r} dx_g dt + \int_0^T \int_\Omega \frac{\alpha(x)}{r}(|\nabla_g u|_g^2 - |u_r|^2) dx_g dt \leq CE(0), \tag{2.23}$$

and for $d > 2(3-n)$,

$$\int_0^T \int_\Omega \left(\frac{|u|^2}{r^3} + \frac{|u|^{p+1}}{r} \right) dx_g dt$$
$$+ \int_0^T \int_\Omega \frac{\alpha(x)}{r} (|\nabla_g u|_g^2 - |u_r|^2) dx_g dt \le CE(0). \tag{2.24}$$

2.3. Stability of the damped nonlinear Schrödinger equation with non-uniform energy decay rate

The main geometric conditions for stability of the damped nonlinear Schrödinger equation with non-uniform energy decay rate are given by the following assumption.

Assumption (B). There exist constants $R_0 > d_2, 0 < \delta \le 1$ such that

$$\left\langle \left((1-\delta)G(x) + \frac{r}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle \ge 0 \quad \text{for } X \in \mathbb{R}_x^n,\ x \in \Omega(R_0), \tag{2.25}$$

and $a(x)$ satisfies

$$a(x) \ge a_0 > 0, \quad x \in (\Omega \backslash \Omega(R_0 - \varepsilon_0)) \bigcup \Gamma(\varepsilon_1), \tag{2.26}$$

for some $0 < 2\varepsilon_1 < \varepsilon_0 < R_0 - d_2$, where

$$\Gamma(\varepsilon) = \bigcup_{x \in \Gamma} \{ y \in \Omega \, \big| \, |y - x| < \varepsilon \}, \tag{2.27}$$

and for any $\epsilon > 0$, there exists C_ϵ such that

$$\Big| \Delta_g a(x) \Big| \le C_\epsilon a(x) + \epsilon, \quad x \in \Omega. \tag{2.28}$$

To prove the stability of the system (1.9), the following assumptions are also considered.

Assumption (U1). Let $\widehat{\Omega} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and ω be an open subset of $\widehat{\Omega}$ such that

$$\omega \supset \bigcup_{x \in \partial\widehat{\Omega}} \{ y \in \widehat{\Omega} \, \big| \, |y - x| < \xi \}, \tag{2.29}$$

for some $\xi > 0$. Assume that ω satisfies geometric control condition:

(GCC) There exists constant $T_0 > 0$ such that for any $x \in \widehat{\Omega}$ and any unit-speed geodesic $\gamma(t)$ of $(\mathbb{R}^n, g)$ starting at x, there exists $t < T_0$ such that $\gamma(t) \subset \omega$.

Then there exists $T_1 \geq 0$ such that for any $T > T_1$, the only solution u in $C([0,T], H^1(\widehat{\Omega}))$ to the system

$$\begin{cases} iu_t + \Delta_g u = 0 & (x,t) \in \widehat{\Omega} \times (0,T), \\ u = 0 & (x,t) \in \omega \times (0,T), \end{cases} \tag{2.30}$$

is the trivial one $u \equiv 0$.

Assumption (U2). Let $\widehat{\Omega} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and ω be an open subset of $\widehat{\Omega}$ such that

$$\omega \supset \bigcup_{x \in \partial\widehat{\Omega}} \{y \in \widehat{\Omega} \,\big|\, |y - x| < \xi\}, \tag{2.31}$$

for some $\xi > 0$. Assume that ω satisfies geometric control condition:

(GCC) There exists constant $T_0 > 0$ such that for any $x \in \widehat{\Omega}$ and any unit-speed geodesic $\gamma(t)$ of $(\mathbb{R}^n, g)$ starting at x, there exists $t < T_0$ such that $\gamma(t) \in \omega$.

Then there exists $T_1 \geq 0$ such that for any $T > T_1$, the only solution u in $C([0,T], H^1(\widehat{\Omega}))$ to the system

$$\begin{cases} iu_t + \Delta_g u - |u|^{p-1}u = 0 & (x,t) \in \widehat{\Omega} \times (0,T), \\ u = 0 & (x,t) \in \omega \times (0,T), \end{cases} \tag{2.32}$$

is the trivial one $u \equiv 0$.

Remark 2.3. If $T_1 = 0$, which implies T can be arbitrary small in (2.30) and (2.32), Assumption (**U1**) and Assumption (**U2**) are called as unique continuation condition. On Euclidean space, unique continuation condition for linear(or nonlinear) Schrödinger equation has been proved by [26, 27, 33, 34, 36, 55] and the references therein. On Riemannian manifold, under the assumption that unique continuation condition for linear Schrödinger equation holds true, unique continuation condition for the nonlinear Schrödinger equation was proved by [40] in dimension 3 and [25] in dimension 2 .

By the equivalent relation between the controllability and the observability estimate [42], Assumption (**U1**) follows from Theorem 4.4 in [41]. However, a detailed proof of Theorem 4.4 in [41] is not provided.

Under a stronger geometric condition than (GCC), we can prove Assumption (**U1**) and Assumption (**U2**) directly by multiplier methods. See Proposition (A.1) and Proposition (A.2) in the Appendix.

Theorem 2.2. *Let Assumption* (**B**)*, Assumption* (**U1**) *and Assumption* (**U2**) *hold true. Assume that* $\|u_0\|_{L^2(\Omega)} \le E_0$*. Then there exist positive constants* C_1 *and* C_2*, which are dependent on* E_0*, such that*

$$E(t) \le C_1 e^{-C_2 t} E(0), \quad \forall t > 0. \tag{2.33}$$

2.4. Stability of the damped nonlinear Schrödinger equation with uniform energy decay rate

The main geometric conditions for stability of the damped nonlinear Schrödinger equation with uniform energy decay rate are given by the following assumption.

Assumption (C). There exist constants $R_0 > d_2, 0 < \delta \le 1$ such that

$$G(x)\frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad |x| \le R_0 \quad \text{and} \quad \det(G(x)) = c_0 r^d, \quad x \in \Omega(R_0), \tag{2.34}$$

$$\left\langle \left((1-\delta)G(x) + \frac{r}{2}\frac{\partial G(x)}{\partial r} \right) X, X \right\rangle \ge 0 \quad \text{for } X \in \mathbb{R}^n_x,\ x \in \Omega(R_0), \tag{2.35}$$

where $c_0 > 0, d$ are constants and $a(x)$ satisfies

$$a(x) \ge a_0 > 0, \quad x \in \Omega \backslash \Omega(R_0 - \varepsilon_0), \tag{2.36}$$

for some $0 < \varepsilon_0 < R_0 - d_2$ and for any $\epsilon > 0$, there exists C_ϵ such that

$$\left| \Delta_g a(x) \right| \le C_\epsilon a(x) + \epsilon, \quad x \in \Omega. \tag{2.37}$$

Remark 2.4. Let Assumption (**C**) hold true. It follows from the relations(4.6) and (4.7) hereinafter that

$$\begin{aligned} \frac{(n+d/2-1)}{r} &= \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r} = \Delta_g r = tr D^2 r \\ &\ge (n-1)\frac{\delta}{r}, \quad x \in \Omega(R_0). \end{aligned} \tag{2.38}$$

Then

$$d \ge 2(n-1)(\delta - 1). \tag{2.39}$$

Theorem 2.3. *Let Assumption* (**C**) *hold true. Assume that*

$$\frac{\partial r}{\partial \nu} \le 0, \quad x \in \Gamma. \tag{2.40}$$

Then there exist positive constants C_1, C_2 *such that*

$$E(t) \le C_1 e^{-C_2 t} E(0), \quad \forall t > 0. \tag{2.41}$$

3. Multiplier identities and key lemmas

We need to establish several multiplier identities, which are useful for our problem.

Lemma 3.1. *Let* $\widehat{\Omega} \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary. Suppose that* $u(x,t)$ *solves the following equation:*

$$iu_t + \Delta_g u + ia(x)u - |u|^{p-1}u = 0 \qquad (x,t) \in \widehat{\Omega} \times (0, +\infty). \tag{3.1}$$

Let $\mathcal{H}$ *be a* C^1*vector field defined on* $\overline{\widehat{\Omega}}$*. Then*

$$\begin{aligned}
&\int_0^T \int_{\partial\widehat{\Omega}} \mathrm{Re}\left(\frac{\partial u}{\partial \hat{\nu}} \mathcal{H}(\bar{u})\right) d\Gamma_g dt \\
&\quad + \frac{1}{2}\int_0^T \int_{\partial\widehat{\Omega}} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right) \langle \mathcal{H}, \hat{\nu}\rangle_g d\Gamma_g dt \\
&= \frac{1}{2}\int_{\widehat{\Omega}} \mathrm{Im}\,\left(u\mathcal{H}(\bar{u})\right) dx_g \Big|_0^T + \int_0^T \int_{\widehat{\Omega}} \mathrm{Re}\, D\mathcal{H}(\nabla_g \bar{u}, \nabla_g u) dx_g dt \\
&\quad + \int_0^T \int_{\widehat{\Omega}} \mathrm{Im}\,\left(a(x) u \mathcal{H}(\bar{u})\right) dx_g dt \\
&\quad + \frac{1}{2}\int_0^T \int_{\widehat{\Omega}} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right) \mathrm{div}\,_g \mathcal{H} dx_g dt,
\end{aligned} \tag{3.2}$$

where $\hat{\nu}(x)$ *is the unit normal vector outside* $\widehat{\Omega}$ *in* $(\mathbb{R}^n, g)$ *for* $x \in \partial\widehat{\Omega}$ *and* $d\Gamma_g$ *denotes the volume element of* $(\Gamma, \hat{g})$*, where* $\hat{g}$ *is induced by the metric* g*.*

Moreover, assume that the real function $P \in C^2(\overline{\widehat{\Omega}})$. *Then*

$$
\begin{aligned}
&\int_0^T \int_{\widehat{\Omega}} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - |u|^{p+1} \right) P dx_g dt \\
&= \frac{1}{2}\int_0^T \int_{\partial\widehat{\Omega}} |u|^2 \frac{\partial P}{\partial \hat{\nu}} d\Gamma_g dt - \frac{1}{2}\int_0^T \int_{\widehat{\Omega}} |u|^2 (\Delta_g P) dx_g dt \\
&\quad - \int_0^T \int_{\partial\widehat{\Omega}} \mathrm{Re}\,(P\bar{u}\frac{\partial u}{\partial \hat{\nu}}) d\Gamma_g dt. \qquad (3.3)
\end{aligned}
$$

Proof. Firstly, we multiply (3.1) by $\mathcal{H}(\bar{u})$ and integrate over $\widehat{\Omega} \times (0, T)$. We deduce that

$$
\begin{aligned}
\mathrm{Re}\ (iu_t\mathcal{H}(\bar{u})) &= -\mathrm{Im}\ (u_t\mathcal{H}(\bar{u})) \\
&= -\frac{1}{2}\mathrm{Im}\ (u_t\mathcal{H}(\bar{u}) - \bar{u}_t\mathcal{H}(u)) \\
&= -\frac{1}{2}\mathrm{Im}\ ((u\mathcal{H}(\bar{u}))_t - \mathcal{H}(u\bar{u}_t)) \\
&= -\frac{1}{2}\mathrm{Im}\ (u\mathcal{H}(\bar{u}))_t + \frac{1}{2}\mathrm{Im}\,\mathcal{H}(u\bar{u}_t) \\
&= -\frac{1}{2}\mathrm{Im}\ (u\mathcal{H}(\bar{u}))_t + \frac{1}{2}\mathrm{Im}\ \mathrm{div}_g(u\bar{u}_t\mathcal{H}) - \frac{1}{2}\mathrm{Im}\,(u\bar{u}_t\,\mathrm{div}_g\mathcal{H}), \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Re}\ (\mathcal{H}(\bar{u})\Delta_g u)) &= \mathrm{Re}\ (\,\mathrm{div}_g\mathcal{H}(\bar{u})\nabla_g u - \nabla_g u\langle\mathcal{H}, \nabla_g\bar{u}\rangle_g) \\
&= \mathrm{Re}\ \mathrm{div}_g\mathcal{H}(\bar{u})\nabla_g u - \mathrm{Re}\,\nabla_g u\langle\mathcal{H}, \nabla_g\bar{u}\rangle_g \\
&= \mathrm{Re}\ \mathrm{div}_g\mathcal{H}(\bar{u})\nabla_g u - \mathrm{Re}\,D\mathcal{H}(\nabla_g\bar{u}, \nabla_g u) - \mathrm{Re}\,D^2\bar{u}(\mathcal{H}, \nabla_g u) \\
&= \mathrm{Re}\ \mathrm{div}_g\mathcal{H}(\bar{u})\nabla_g u - \mathrm{Re}\,D\mathcal{H}(\nabla_g\bar{u}, \nabla_g u) - \mathrm{Re}\,D^2\bar{u}(\nabla_g u, \mathcal{H}) \\
&= \mathrm{Re}\ \mathrm{div}_g\mathcal{H}(\bar{u})\nabla_g u - \mathrm{Re}\,D\mathcal{H}(\nabla_g\bar{u}, \nabla_g u) - \frac{1}{2}\mathcal{H}(|\nabla_g u|_g^2) \\
&= \mathrm{Re}\ \mathrm{div}_g\mathcal{H}(\bar{u})\nabla_g u - \mathrm{Re}\,D\mathcal{H}(\nabla_g\bar{u}, \nabla_g u) \\
&\quad - \frac{1}{2}\,\mathrm{div}_g(|\nabla_g u|_g^2\mathcal{H}) + \frac{1}{2}|\nabla_g u|_g^2\,\mathrm{div}_g\mathcal{H}, \qquad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Re}\ \left(ia(x)u - |u|^{p-1}u\right)\mathcal{H}(\overline{u}) &= -\mathrm{Im}\ (a(x)u\mathcal{H}(\overline{u})) \\
&\quad - \frac{1}{p+1}\,\mathrm{div}_g\left(|u|^{p+1}\mathcal{H}\right) + \frac{|u|^{p+1}}{p+1}\,\mathrm{div}_g\mathcal{H}. \qquad (3.6)
\end{aligned}
$$

The equality (3.2) follows from Green's formula.

In addition, by multiplying (3.1) by $P\bar{u}$ and integrating over $\widehat{\Omega}\times(0,T)$, we obtain

$$\text{Re}\,(iPu_t\bar{u}) = -\text{Im}\,(Pu_t\bar{u}) = \text{Im}\,(Pu\bar{u}_t)\,, \tag{3.7}$$

$$\begin{aligned}\text{Re}\,(P\bar{u}\Delta_g u) &= \text{Re}\,(\,\text{div}\,_g P\bar{u}\nabla_g u - \nabla_g u(P\bar{u}))\\ &= \text{Re div}\,_g P\bar{u}\nabla_g u - P|\nabla_g u|_g^2 - \frac{1}{2}\nabla_g P(|u|^2)\\ &= \text{Re div}\,_g P\bar{u}\nabla_g u - P|\nabla_g u|_g^2\\ &\quad -\frac{1}{2}\,\text{div}\,_g|u|^2\nabla_g P + \frac{1}{2}|u|^2\Delta_g P,\end{aligned} \tag{3.8}$$

and

$$\text{Re}\,\big(ia(x)u - |u|^{p-1}u\big)\,P\overline{u} = \text{Re}\,\big(ia(x)P|u|^2\big) - P|u|^{p+1} = -P|u|^{p+1}. \tag{3.9}$$

The equality (3.3) follows from Green's formula. □

The following lemma will be utilized frequently in our subsequent proof.

Lemma 3.2. *Let $x_0\in\mathbb{R}^n$ be a fixed point. Let $H(x) = x - x_0$, then*

$$DH(X,X) = \left\langle\left(G(x) + \frac{\widehat{r}(x)}{2}\frac{\partial G(x)}{\partial\widehat{r}}\right)X,X\right\rangle, \tag{3.10}$$

for $X\in\mathbb{R}^n_x,\ x\in\mathbb{R}^n$,

where $\widehat{r}(x) = |x-x_0|$.

Proof. Let $x\in\mathbb{R}^n, X = \sum_{i=1}^n X_i\frac{\partial}{\partial x_i}\in\mathbb{R}^n_x$. Note that

$$H(x) = \sum_{i=1}^n (x_i - x_{0,i})\frac{\partial}{\partial x_i}. \tag{3.11}$$

Then, we deduce that

$$\begin{aligned}DH(X,X) &= \sum_{i,j,k=1}^n\left\langle D_{\frac{\partial}{\partial x_i}}\left((x_k - x_{0,k})\frac{\partial}{\partial x_k}\right),\frac{\partial}{\partial x_j}\right\rangle_g X_iX_j\\ &= \sum_{i,j=1}^n g_{ij}X_iX_j + \sum_{i,j,k=1}^n (x_k - x_{0,k})\left\langle D_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_j}\right\rangle_g X_iX_j\end{aligned}$$

$$= |X|_g^2 + \sum_{i,j,k=1}^{n} (x_k - x_{0,k}) \left\langle D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g X_i X_j$$

$$= |X|_g^2 + \sum_{i,j,k=1}^{n} \frac{(x_k - x_{0,k})}{2} \frac{\partial g_{ij}}{\partial x_k} X_i X_j$$

$$= \left\langle \left(G(x) + \frac{\widehat{r}(x)}{2} \frac{\partial G(x)}{\partial \widehat{r}} \right) X, X \right\rangle. \tag{3.12}$$

□

The following lemmas show the relationship between the metric g and geometric control condition.

Lemma 3.3. *Let $\widehat{\Omega} \subset \mathbb{R}^n$ be a bounded domain and $x_0 \in \mathbb{R}^n$ be a fixed point. Assume that there exists $\delta > 0$ such that*

$$\left\langle \left((1-\delta) G(x) + \frac{\widehat{r}(x)}{2} \frac{\partial G(x)}{\partial \widehat{r}} \right) X, X \right\rangle \geq 0, \quad \text{for } X \in \mathbb{R}^n_x, \ x \in \overline{\widehat{\Omega}}, \tag{3.13}$$

where $\widehat{r}(x) = |x - x_0|$. Then, for any $x \in \widehat{\Omega}$ and any unit-speed geodesic $\gamma(t)$ starting at x, if

$$\gamma(t) \in \widehat{\Omega}, \quad 0 \leq t \leq t_0, \tag{3.14}$$

then

$$t_0 \leq \frac{2}{\delta} \sup \left\{ |x - x_0|_g(x) \;\middle|\; x \in \overline{\widehat{\Omega}} \right\}. \tag{3.15}$$

Proof. Let $H(x) = x - x_0$. It follows from (3.10) that

$$DH(X, X) \geq \delta |X|_g^2 \quad \text{for all } X \in \mathbb{R}^n_x, \ x \in \overline{\widehat{\Omega}}. \tag{3.16}$$

Note that

$$|\gamma'(t)|_g = 1, \quad D_{\gamma'(t)} \gamma'(t) = 0. \tag{3.17}$$

Then

$$\langle H, \gamma'(t) \rangle_g \Big|_0^{t_0} = \int_0^{t_0} \gamma'(t) \langle H, \gamma'(t) \rangle_g dt$$

$$= \int_0^{t_0} DH(\gamma'(t), \gamma'(t)) dt \geq \delta t_0. \tag{3.18}$$

Hence

$$t_0 \leq \frac{2}{\delta} \sup \left\{ |H|_g(x) \;\middle|\; x \in \overline{\widehat{\Omega}} \right\}. \tag{3.19}$$

□

Lemma 3.4. *Assume that*

$$G(x)\frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^n, \tag{3.20}$$

$$\frac{\partial G(x)}{\partial r} = -\frac{2}{r_2} G(x) \left(I_n - \frac{x \otimes x}{|x|^2} \right), \quad |x| = r_2, \tag{3.21}$$

where r_2 is a positive constant. Then, for any $x \in S(r_2)$ and any unit-speed geodesic $\gamma(t)$ starting at x with

$$\gamma'(0) \in S(r_2)_x, \tag{3.22}$$

we have

$$\gamma(t) \in S(r_2), \quad \forall t \geq 0. \tag{3.23}$$

Proof. Note that

$$D(rDr) = Dr \otimes Dr + rD^2 r. \tag{3.24}$$

With (3.10), we obtain

$$D^2 r(X, X) = \left\langle \left(\frac{1}{r} G(x) + \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle = 0, \tag{3.25}$$

for $X \in S(r_2)_x, \quad x \in S(r_2)$.

Let $\widehat{g}$ be a Riemannian metric induced by g in $S(r_2)$ and $\widehat{D}$ be the associated Levi-Civita connection.

Let $\widehat{\gamma}(t)$ be a unit-speed geodesic of $(S(r_2), \widehat{g})$ starting at $x \in S(r_2)$, then

$$\left\langle \widehat{\gamma}'(t), \frac{\partial}{\partial r} \right\rangle_g = 0, \quad \widehat{D}_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) = 0, \quad \forall t \geq 0. \tag{3.26}$$

Therefore,

$$\begin{aligned} D_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) &= \widehat{D}_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) + \left\langle D_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t), \frac{\partial}{\partial r} \right\rangle_g \frac{\partial}{\partial r} \\ &= \widehat{D}_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) - D^2 r(\widehat{\gamma}'(t), \widehat{\gamma}'(t)) \frac{\partial}{\partial r} = 0, \end{aligned} \tag{3.27}$$

which implies $\widehat{\gamma}(t)$ is also a geodesic of $(\mathbb{R}^n, g)$. □

4. Proofs of Morawetz estimates in non (asymptotically) Euclidean spaces

Lemma 4.1. *Let $u(x,t)$ solve the system (1.9). Then*

$$\int_\Omega |u|^2 dx_g\Big|_0^T = -2\int_0^T\int_\Omega a(x)|u|^2 dx_g dt, \tag{4.1}$$

$$\begin{aligned}\int_\Omega \left(|\nabla_g u|_g^2 + \frac{2}{p+1}|u|^{p+1}\right) dx_g\Big|_0^T &= -2\int_0^T\int_\Omega a(x)\left(|\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\ &\quad + \int_0^T\int_\Omega |u|^2(\Delta_g a(x)) dx_g dt,\end{aligned} \tag{4.2}$$

for any $T > 0$.

Proof. Multiplying the Schrödinger equation in (1.9) by $2\bar{u}$ and integrating over $\Omega \times (0,T)$, we have

$$\int_\Omega |u|^2 dx_g\Big|_0^T = -2\int_0^T\int_\Omega a(x)|u|^2 dx_g. \tag{4.3}$$

After multiplying the Schrödinger equation in (1.9) by $2\bar{u}_t$ and then integrating over $\Omega \times (0,T)$, we obtain

$$\int_\Omega \left(|\nabla_g u|_g^2 + \frac{2}{p+1}|u|^{p+1}\right) dx_g\Big|_0^T = -2\int_0^T\int_\Omega \operatorname{Im}(a(x)u\bar{u}_t) dx_g dt. \tag{4.4}$$

Let $P = a(x)$ and $\widehat{\Omega} = \Omega(a)$ in (3.3). Substituting (3.3) into (4.4), letting $a \to +\infty$, we get

$$\begin{aligned}\int_\Omega \left(|\nabla_g u|_g^2 + \frac{2}{p+1}|u|^{p+1}\right) dx_g\Big|_0^T &= -2\int_0^T\int_\Omega a(x)\left(|\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\ &\quad + \int_0^T\int_\Omega |u|^2(\Delta_g a(x)) dx_g dt.\end{aligned} \tag{4.5}$$

□

Lemma 4.2. *Let Assumption* (**A**) *hold true. Then*

$$D^2r(X,X) \geq \frac{\alpha(x)}{r}|X|_g^2 \quad \text{for all } X \in S(r)_x,\ x \in \Omega, \tag{4.6}$$

$$\Delta_g r = \frac{n+d/2-1}{r} \quad \text{for } x \in \Omega. \tag{4.7}$$

Proof. Note that

$$D(rDr) = Dr \otimes Dr + rD^2r. \tag{4.8}$$

With (3.10), we obtain

$$D^2r(X,X) = \left\langle \left(\frac{1}{r}G(x) + \frac{1}{2}\frac{\partial G(x)}{\partial r}\right) X, X \right\rangle \geq \frac{\alpha(x)}{r}|X|_g^2, \tag{4.9}$$

for $X \in S(r)_x, \quad x \in \Omega$, and

$$\Delta_g r = \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det\left(G(x)\right)}}{\partial r} = \frac{n+d/2-1}{r} \quad \text{for } x \in \Omega. \tag{4.10}$$

□

Proof of Theorem 2.1. Let $\mathcal{H} = \frac{\partial}{\partial r}$ and $\widehat{\Omega} = \Omega(h)$ in (3.2). It follows from (3.2), (4.6) and (4.7), that

$$\begin{aligned}
&\int_0^T \int_{\partial\Omega(h)} \mathrm{Re}\left(\frac{\partial u}{\partial \hat{\nu}}\mathcal{H}(\bar{u})\right) d\Gamma_g dt \\
&+ \frac{1}{2}\int_0^T \int_{\partial\Omega(h)} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right) \langle \mathcal{H}, \hat{\nu}\rangle_g d\Gamma_g dt \\
\geq{}& \frac{1}{2}\int_{\Omega(h)} \mathrm{Im}\,(u\bar{u}_r)\,dx_g\Big|_0^T + \int_0^T \int_{\Omega(h)} \frac{\alpha(x)}{r}(|\nabla_g u|_g^2 - |u_r|^2) dx_g dt \\
&+ \int_0^T \int_{\Omega(h)} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right) \frac{n-1}{2r} dx_g dt \\
={}& \frac{1}{2}\int_{\Omega(h)} \mathrm{Im}\,(u\bar{u}_r)\,dx_g\Big|_0^T + \int_0^T \int_{\Omega(h)} \frac{\alpha(x)}{r}(|\nabla_g u|_g^2 - |u_r|^2) dx_g dt \\
&+ \int_0^T \int_{\Omega(h)} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - |u|^{p+1}\right) \frac{n-1}{2r} dx_g dt \\
&+ \int_0^T \int_{\Omega(h)} \frac{(n-1)(p-1)}{2r(p+1)}|u|^{p+1} dx_g dt.
\end{aligned} \tag{4.11}$$

Let $P = \frac{n-1}{2r}$ and $\widehat{\Omega} = \Omega(h)$ in (3.3). Substituting (3.3) into (4.11), letting $h \to +\infty$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathrm{Im}\,(u\bar{u}_r)\,dx_g\Big|_0^T - \frac{n-1}{4}\int_0^T\int_\Omega |u|^2\Delta_g\left(\frac{1}{r}\right)dx_g dt \\
&+\int_0^T\int_\Omega \frac{(p-1)(n-1)}{2(p+1)r}|u|^{p+1}dx_g dt \\
&+\int_0^T\int_\Omega \frac{\alpha(x)}{r}(|\nabla_g u|_g^2 - |u_r|^2)dx_g dt \\
&\le \Pi_\Gamma, \qquad (4.12)
\end{aligned}$$

where

$$\begin{aligned}
\Pi_\Gamma = &\int_0^T\int_\Gamma \mathrm{Re}\left(\frac{\partial u}{\partial \nu}\mathcal{H}(\bar{u})\right)d\Gamma_g dt \\
&+\frac{1}{2}\int_0^T\int_\Gamma \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right)\langle \mathcal{H},\nu\rangle_g d\Gamma_g dt \\
&-\frac{1}{2}\int_0^T\int_\Gamma |u|^2\frac{\partial P}{\partial \nu}d\Gamma_g dt + \int_0^T\int_\Gamma \mathrm{Re}\left(P\bar{u}\frac{\partial u}{\partial \nu}\right)d\Gamma_g dt. \qquad (4.13)
\end{aligned}$$

Since $u|_\Gamma = 0$, we obtain $\nabla_{\Gamma_g}\bar{u}\Big|_\Gamma = 0$, that is,

$$\nabla_g \bar{u} = \frac{\partial \bar{u}}{\partial \nu}\nu \quad \text{for } x \in \Gamma. \qquad (4.14)$$

Similarly, we have

$$\mathcal{H}(\bar{u}) = \langle \mathcal{H}, \nabla_g \bar{u}\rangle_g = \frac{\partial \bar{u}}{\partial \nu}\langle \mathcal{H},\nu\rangle_g = \frac{\partial \bar{u}}{\partial \nu}\frac{\partial r}{\partial \nu} \quad \text{for } x \in \Gamma. \qquad (4.15)$$

Using the formulas (4.14) and (4.15) in the formula (4.13) on the portion Γ, with (2.22), we obtain

$$\Pi_\Gamma = \frac{1}{2}\int_0^T\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2\frac{\partial r}{\partial \nu}d\Gamma_g dt \le 0. \qquad (4.16)$$

Substituting (4.16) into (4.12), we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega \text{Im } (u\bar{u}_r)\, dx_g\Big|_0^T - \frac{n-1}{4}\int_0^T\int_\Omega |u|^2\Delta_g\left(\frac{1}{r}\right)dx_g dt\\&+\int_0^T\int_\Omega \frac{(p-1)(n-1)}{2(p+1)r}|u|^{p+1}dx_g dt\\&+\int_0^T\int_\Omega \frac{\alpha(x)}{r}(|\nabla_g u|_g^2-|u_r|^2)dx_g dt\\&\le 0. \end{aligned}\tag{4.17}$$

Note that

$$\Delta_g\left(\frac{1}{r}\right) = -\frac{n+d/2-3}{r}. \tag{4.18}$$

With (4.17), we obtain for $d = 2(3-n)$,

$$\begin{aligned}&\frac{1}{2}\int_\Omega \text{Im } (u\bar{u}_r)\, dx_g\Big|_0^T + \int_0^T\int_\Omega \frac{(p-1)(n-1)}{2(p+1)r}|u|^{p+1}dx_g dt\\&+\int_0^T\int_\Omega \frac{\alpha(x)}{r}(|\nabla_g u|_g^2-|u_r|^2)dx_g dt\\&\le 0, \end{aligned}\tag{4.19}$$

and for $d > 2(3-n)$,

$$\begin{aligned}&\frac{1}{2}\int_\Omega \text{Im } (u\bar{u}_r)\, dx_g\Big|_0^T + \int_0^T\int_\Omega \frac{(n-1)(n+d/2-3)}{4r^3}u^2 dx dt\\&+\int_0^T\int_\Omega \frac{(p-1)(n-1)}{2(p+1)r}|u|^{p+1}dx_g dt\\&+\int_0^T\int_\Omega \frac{\alpha(x)}{r}(|\nabla_g u|_g^2-|u_r|^2)dx dt\\&\le 0. \end{aligned}\tag{4.20}$$

It follows from (2.9), (4.1) and (4.2) that

$$E(t) = E(0), \quad t > 0. \tag{4.21}$$

The estimates (2.23) and (2.24) follows from (4.19) and (4.20). □

5. Proofs of stability with non-uniform energy decay rate

From Lemma 3.3, the following lemma holds true.

Lemma 5.1. *Let assumption* (**B**) *hold true. Then, there exists* $t_0 > 0$, *for any* $x \in \Omega(R_0)$ *and any unit-speed geodesic* $\gamma(t)$ *starting at* x, *there exists* $t < t_0$ *such that*

$$\gamma(t) \in \partial\Omega(R_0). \tag{5.1}$$

Lemma 5.2. *Let assumption* (**B**) *hold true. Let* $u(x,t)$ *solve the system (1.9). Then*

$$\begin{aligned} E(0) + \int_0^T E(t)dt \leq C \int_0^T \int_\Omega a(x)\left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\ + C \int_0^T \int_{\Omega(R_0-\varepsilon_0)} |u|^2 dx_g dt, \end{aligned} \tag{5.2}$$

for sufficiently large T.

Proof. Let $b(x) \in C^\infty(\mathbb{R}^n)$ be a nonnegative function satisfying

$$b(x) = 1, \quad x \in \Omega(R_0 - \varepsilon_0)\backslash\Gamma(\varepsilon_1) \quad \text{and} \quad b(x) = 0, \quad x \in \mathbb{R}^n\backslash\Omega(R_0). \tag{5.3}$$

Let

$$H(x) = b(x)x, \quad x \in \mathbb{R}^n. \tag{5.4}$$

It follows from (2.25) and (3.10) that

$$DH(X,X) \geq \delta|X|_g^2 \quad \text{for all } X \in \mathbb{R}^n_x,\ x \in \Omega(R_0 - \varepsilon_0)\backslash\Gamma(\varepsilon_1), \tag{5.5}$$

$$\operatorname{div}_g H = trDH \geq n\delta \quad \text{for all } x \in \Omega(R_0 - \varepsilon_0)\backslash\Gamma(\varepsilon_1). \tag{5.6}$$

Let $\mathcal{H} = H$ and $\widehat{\Omega} = \Omega(R_0)$ in (3.2). From (3.2), we have

$$
\begin{aligned}
0 \geq & \frac{1}{2}\int_{\Omega(R_0)} \mathrm{Im}\ (uH(\bar{u}))\, dx_g\Big|_0^T + \delta \int_0^T \int_{\Omega(R_0-\varepsilon_0)\backslash\Gamma(\varepsilon_1)} |\nabla_g u|_g^2 dx_g dt \\
& - C\int_0^T \int_{(\Omega(R_0)\backslash\Omega(R_0-\varepsilon_0))\bigcup\Gamma(\varepsilon_1)} |\nabla_g u|_g^2 dx_g dt \\
& + \int_0^T \int_{\Omega(R_0)} \mathrm{Im}\ (a(x)uH(\bar{u}))\, dx_g dt \\
& + \frac{1}{2}\int_0^T \int_{\Omega(R_0)} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right) \mathrm{div}_{\,g} H dx_g dt \\
= & \frac{1}{2}\int_{\Omega(R_0)} \mathrm{Im}\ (uH(\bar{u}))\, dx_g\Big|_0^T + \delta \int_0^T \int_{\Omega(R_0-\varepsilon_0)\backslash\Gamma(\varepsilon_1)} |\nabla_g u|_g^2 dx_g dt \\
& - C\int_0^T \int_{(\Omega(R_0)\backslash\Omega(R_0-\varepsilon_0))\bigcup\Gamma(\varepsilon_1)} |\nabla_g u|_g^2 dx_g dt \\
& + \int_0^T \int_{\Omega(R_0)} \mathrm{Im}\ (a(x)uH(\bar{u}))\, dx_g dt \\
& + \frac{1}{2}\int_0^T \int_{\Omega(R_0)} \left(\mathrm{Im}\,(u\bar{u}_t) - |\nabla_g u|_g^2 - |u|^{p+1}\right) \mathrm{div}_{\,g} H dx_g dt \\
& + \int_0^T \int_{\Omega(R_0)} \frac{(p-1)\,\mathrm{div}_{\,g} H}{2(p+1)}|u|^{p+1} dx_g dt.
\end{aligned} \tag{5.7}
$$

Let $P = \frac{\mathrm{div}_{\,g} H}{2}$ and $\widehat{\Omega} = \Omega(R_0)$ in (3.3). Substituting (3.3) into (5.7), we obtain

$$
\begin{aligned}
& \frac{1}{2}\int_{\Omega(R_0)} \mathrm{Im}\ (uH(\bar{u}))\, dx_g\Big|_0^T - \frac{1}{4}\int_0^T \int_{\Omega(R_0)} |u|^2 \Delta_g(\,\mathrm{div}_{\,g} H) dx_g dt \\
& + \int_0^T \int_{\Omega(R_0)} \mathrm{Im}\ (a(x)uH(\bar{u}))\, dx_g dt \\
& + \delta \int_0^T \int_{\Omega(R_0-\varepsilon_0)\backslash\Gamma(\varepsilon_1)} |\nabla_g u|_g^2 dx_g dt \\
& + \int_0^T \int_{\Omega(R_0-\varepsilon_0)\backslash\Gamma(\varepsilon_1)} \frac{n\delta(p-1)}{2(p+1)}|u|^{p+1} dx_g dt \\
\leq & C\int_0^T \int_{(\Omega(R_0)\backslash\Omega(R_0-\varepsilon_0))\bigcup\Gamma(\varepsilon_1)} \left(|\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt.
\end{aligned} \tag{5.8}
$$

Therefore

$$\begin{aligned}
&\int_0^T \int_{\Omega(R_0-\varepsilon_0)\setminus\Gamma(\varepsilon_1)} \left(|\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\
&\leq C(E(0)+E(T)) \\
&\quad + C\int_0^T \int_{\Omega(R_0)} a(x)\left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\
&\quad + C\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |u|^2 dx_g dt. \qquad (5.9)
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^T E(t)dt \leq{}& C(E(0)+E(T)) \\
&+ C\int_0^T \int_\Omega a(x)\left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\
&+ C\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |u|^2 dx_g dt. \qquad (5.10)
\end{aligned}$$

With (4.1) and (4.2), we deduce that

$$\begin{aligned}
CE(T) \leq{}& CE(0) + C\int_0^T \int_\Omega a(x)\left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\
&+ \frac{C}{2}\int_0^T \int_\Omega |u|^2 \Big|\Delta_g a(x)\Big| dx_g dt, \qquad (5.11)
\end{aligned}$$

and

$$\begin{aligned}
4CE(0) &= \int_0^{4C} E(t)dt - \int_0^{4C} (E(t)-E(0))dt \\
&\leq \int_0^{4C} E(t)dt + 4C\int_0^{4C} \int_\Omega a(x)\left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\
&\quad + 2C\int_0^T \int_\Omega |u|^2 \Big|\Delta_g a(x)\Big| dx_g dt. \qquad (5.12)
\end{aligned}$$

Substituting (5.11) and (5.12) into (5.10), for $T > 4C$, with (2.28), we have

$$E(0) + \int_0^T E(t)dt \leq C \int_0^T \int_\Omega a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\ + C \int_0^T \int_{\Omega(R_0-\varepsilon_0)} |u|^2 dx_g dt. \tag{5.13}$$

The estimate (5.2) holds true. □

Lemma 5.3. *Let assumption* (**B**), *assumption* (**U1**) *and assumption* (**U2**) *hold true and let* T *be sufficiently large. Then for any* $\|u_0\|_{L^2(\Omega)} \leq E_0$, *there exists positive constant* $C(E_0, T)$ *such that*

$$E(0) + \int_0^T E(t)dt \\ \leq C(E_0, T) \int_0^T \int_\Omega a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt. \tag{5.14}$$

Proof. We apply compactness-uniqueness arguments to prove the conclusion. It follows from (5.2) that

$$E(0) + \int_0^T E(t)dt \leq C \int_0^T \int_\Omega a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt \\ + C \int_0^T \int_{\Omega(R_0-\varepsilon_0)} |u|^2 dx_g dt. \tag{5.15}$$

Then, if the estimate (5.14) doesn't hold true, there exist $\{u_k\}_{k=1}^{\infty}$ such that

$$\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |u_k|^2 dx_g dt \\ \geq k \int_0^T \int_\Omega a(x) \left(|u_k|^2 + |\nabla_g u_k|_g^2 + |u_k|^{p+1}\right) dx_g dt. \tag{5.16}$$

Thus,

$$E_k(0) + \int_0^T E_k(t)dt \leq C E_0, \tag{5.17}$$

where

$$E_k(t) = \frac{1}{2} \int_\Omega \left(|u_k|^2 + |\nabla_g u_k|_g^2\right) dx_g + \frac{1}{p+1} \int_\Omega |u_k|^{p+1} dx_g. \tag{5.18}$$

Therefore, there exists $\hat{u}_0$ and a subset of $\{u_k\}_{k=1}^{\infty}$, still denoted by $\{u_k\}_{k=1}^{\infty}$, such that

$$u_k \to \hat{u}_0 \text{ weakly in } L^2([0,T], H_\Gamma^1(\Omega)), \tag{5.19}$$

and

$$u_k \to \hat{u}_0 \text{ strongly in } L^2(\Omega(R_0-\varepsilon_0)) \text{ for arbitrarily fixed } t \in [0,T]. \tag{5.20}$$

Note that

$$\|u_k - \hat{u}\|_{L^2(\Omega(R_0-\varepsilon_0))}^2 \leq \widehat{C}(T)E_0, \quad \forall t \in [0,T], \ \forall 1 \leq k < +\infty. \tag{5.21}$$

Lebesgue's dominated convergence theorem yields

$$u_k \to \hat{u}_0 \text{ strongly in } L^2(\Omega(R_0-\varepsilon_0) \times (0,T)). \tag{5.22}$$

Case a:

$$\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |\hat{u}_0|^2 dx_g dt > 0. \tag{5.23}$$

It follows from (1.11), (4.1), (4.2) and (5.17) that there exists $C(T) > 0$ such that

$$E_k(t) \leq C(T)E_0, \quad \forall 0 \leq t \leq T. \tag{5.24}$$

Denote

$$q = \frac{2n}{(n-2)p}, \quad q^* = \frac{q}{q-1}. \tag{5.25}$$

Since $1 < p < \frac{n+2}{n-2}$, then

$$\frac{2n}{n+2} < q, q^* < \frac{2n}{n-2}. \tag{5.26}$$

Note that

$$\frac{1}{q} + \frac{1}{q^*} = 1, \tag{5.27}$$

then, $L^{q^*}(\Omega(R_0-\varepsilon_0))$ is the dual space of $L^q(\Omega(R_0-\varepsilon_0))$.

Note that

$$H^1_\Gamma\left(\Omega(R_0-\varepsilon_0)\right) \hookrightarrow L^{\frac{2n}{n-2}}\left(\Omega(R_0-\varepsilon_0)\right). \tag{5.28}$$

therefore, it follows from (5.24) that

$$\{|u_k|^{p-1}u_k\} \text{ are bounded in } L^\infty([0,T],L^q(\Omega(R_0-\varepsilon_0))). \tag{5.29}$$

Then

$$\{|u_k|^{p-1}u_k\} \text{ are bounded in } L^q\left(\Omega(R_0-\varepsilon_0)\times(0,T)\right). \tag{5.30}$$

Hence, there exists a subset of $\left\{u_k\right\}_{k=1}^\infty$, still denoted by $\left\{u_k\right\}_{k=1}^\infty$, such that

$$|u_k|^{p-1}u_k \to |\hat{u}_0|^{p-1}\hat{u}_0 \text{ weakly in } L^q\left(\Omega(R_0-\varepsilon_0)\times(0,T)\right). \tag{5.31}$$

It follows from (5.16) that

$$a(x)\hat{u}_0 = 0 \qquad (x,t)\in\Omega\times(0,T). \tag{5.32}$$

Therefore, with (5.19) and (5.31), we obtain

$$\begin{cases} i\hat{u}_{0t} + \Delta_g\hat{u}_0 - |\hat{u}_0|^{p-1}\hat{u}_0 = 0 & (x,t)\in\left(\Omega(R_0-\varepsilon_0)\times(0,T)\right), \\ a(x)\hat{u}_0 = 0 & (x,t)\in\Omega\times(0,T). \end{cases} \tag{5.33}$$

With (5.1) and Assumption **(U2)**, we have

$$\hat{u}_0 \equiv 0, \qquad (x,t)\in\Omega\times(0,T), \tag{5.34}$$

which contradicts (5.23).

Case b:

$$\hat{u}_0 \equiv 0 \text{ on } \Omega(R_0-\varepsilon_0)\times(0,T). \tag{5.35}$$

Denote

$$v_k = u_k\Big/\sqrt{c_k} \quad \text{for } k\geq 1, \tag{5.36}$$

where

$$c_k = \int_0^T\int_{\Omega(R_0-\varepsilon_0)} |u_k|^2 dx_g dt. \tag{5.37}$$

Then v_k satisfies

$$\begin{cases} iv_{kt} + \Delta_g v_k + ia(x)v_k - |u_k|^{p-1}v_k = 0 & (x,t) \in \Omega \times (0,T), \\ v_k\Big|_{\Gamma} = 0 & t \in (0,T), \end{cases} \tag{5.38}$$

and

$$\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |v_k|^2 dx_g dt = 1. \tag{5.39}$$

It follows from (5.16) that

$$1 \geq k \int_0^T \int_\Omega a(x) \left(|v_k|^2 + |\nabla_g v_k|_g^2 + |u_k|^{p-1}|v_k|^2\right) dx_g dt. \tag{5.40}$$

Therefore, it follows from (5.15) that

$$\widehat{E}_k(0) + \int_0^T \widehat{E}_k(t)dt \ \leq 1 + \frac{1}{k} \leq 2, \tag{5.41}$$

where

$$\widehat{E}_k(t) = \int_\Omega \left(|v_k|^2 + |\nabla_g v_k|_g^2 + |u_k|^{p-1}|v_k|^2\right) dx_g. \tag{5.42}$$

Hence, there exists v_0 and a subset of $\{v_k\}_{k=1}^\infty$, still denoted by $\{v_k\}_{k=1}^\infty$, such that

$$v_k \to v_0 \text{ weakly in } L^2([0,T], H^1_\Gamma(\Omega)), \tag{5.43}$$

and

$$v_k \to \hat{v}_0 \text{ strongly in } L^2(\Omega(R_0-\varepsilon_0)) \text{ for arbitrarily fixed } t \in [0,T]. \tag{5.44}$$

Then by Lebesgue's dominated convergence theorem, we obtain

$$v_k \to v_0 \text{ strongly in } L^2(\Omega(R_0-\varepsilon_0) \times (0,T)). \tag{5.45}$$

It follows from (1.11), (4.1) and (4.2) that there exists $C(T) > 0$ such that

$$E_k(t) \leq C(T)E_k(0), \quad \forall 0 \leq t \leq T. \tag{5.46}$$

With (5.36) and (5.41), we obtain

$$\widehat{E}_k(t) \le C(T)\widehat{E}_k(0) \le 2C(T), \quad \forall 0 \le t \le T. \tag{5.47}$$

Let q, q^* be given by (5.25). Note that

$$H^1_\Gamma\left(\Omega(R_0-\varepsilon_0)\right) \hookrightarrow L^{\frac{2n}{n-2}}\left(\Omega(R_0-\varepsilon_0)\right). \tag{5.48}$$

Therefore, it follows from (5.47) that

$$\{|v_k|^{p-1}v_k\} \text{ are bounded in } L^\infty([0,T], L^q(\Omega(R_0-\varepsilon_0))). \tag{5.49}$$

Hence

$$\begin{aligned}&\int_0^T \int_{\Omega(R_0-\varepsilon_0)} \left(|u_k|^{p-1}|v_k|\right)^q dx_g dt \\ &= c_k^{\frac{q(p-1)}{2}} \int_0^T \int_{\Omega(R_0-\varepsilon_0)} |v_k|^{\frac{2n}{n-2}} dx_g dt \\ &\le c_k^{\frac{q(p-1)}{2}} C(T).\end{aligned} \tag{5.50}$$

With (5.35), (5.37) and (5.45), we obtain

$$\lim_{k\to+\infty} \int_0^T \int_{\Omega(R_0-\varepsilon_0)} \left(|u_k|^{p-1}|v_k|\right)^q dx_g dt = 0. \tag{5.51}$$

It follows from (5.40) that

$$a(x)v_0 = 0 \qquad (x,t) \in \Omega \times (0,T). \tag{5.52}$$

Therefore, it follows from (5.38), (5.43) and (5.51) that

$$\begin{cases} iv_{0t} + \Delta_g v_0 = 0 & (x,t) \in \Omega(R_0-\varepsilon_0) \times (0,T), \\ a(x)v_0 = 0 & (x,t) \in \Omega \times (0,T). \end{cases} \tag{5.53}$$

With (5.1) and Assumption **(U1)**, we have

$$v_0 \equiv 0, \qquad (x,t) \in \Omega \times (0,T). \tag{5.54}$$

It follows from (5.39) that

$$\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |v_0|^2 dx_g dt = 1, \tag{5.55}$$

which contradicts (5.54). □

Proof of Theorem 2.2. Let T be sufficiently large. It follows from (4.1) that $\|u\|_{L^2(\Omega)}$ is non increasing. Hence, with (5.14), we obtain

$$\begin{aligned} &E(S)+\int_S^{S+T}E(t)dt \\ &\leq C(E_0,T)\int_S^{S+T}\int_\Omega a(x)\left(|u|^2+|\nabla_g u|_g^2+|u|^{p+1}\right)dx_g dt, \end{aligned} \tag{5.56}$$

for any $S\geq 0$.

It follows from (4.1) and (4.2) that

$$\int_\Omega |u|^2 dx_g\Big|_S^{S+T} = -2\int_S^{S+T}\int_\Omega a(x)|u|^2 dx_g dt, \tag{5.57}$$

and

$$\begin{aligned} &\int_\Omega\left(|\nabla_g u|_g^2+\frac{2}{p+1}|u|^{p+1}\right)dx_g\Big|_S^{S+T} \\ &= -2\int_S^{S+T}\int_\Omega a(x)\left(|\nabla_g u|_g^2+|u|^{p+1}\right)dx_g dt \\ &\quad + \int_S^{S+T}\int_\Omega |u|^2(\Delta_g a(x))dx_g dt. \end{aligned} \tag{5.58}$$

Therefore, with (5.56), we deduce that

$$\begin{aligned} &E(S)+\int_S^{S+T}E(t)dt \\ &\leq C(E_0,T)(E(S)-E(S+T)) \\ &\quad + C(E_0,T)\int_S^{S+T}\int_\Omega \Big|\Delta_g a(x)\Big||u|^2 dx_g dt \\ &= C(E_0,T)(E(S)-E(S+T)) - M\int_\Omega |u|^2 dx_g\Big|_S^{S+T} \\ &\quad + \int_S^{S+T}\int_\Omega\left(C(E_0,T)\Big|\Delta_g a(x)\Big| - Ma(x)\right)|u|^2 dx_g dt. \end{aligned} \tag{5.59}$$

For sufficiently large M, with (2.28) we have

$$E(S)\leq C(E_0,T)(E(S)-E(S+T)) - M\int_\Omega |u|^2 dx_g\Big|_S^{S+T}. \tag{5.60}$$

Denote

$$\widetilde{E}(t) = E(t) + \frac{M}{C(E_0, T)} \int_\Omega |u|^2 dx_g. \tag{5.61}$$

From (5.60), we obtain

$$\widetilde{E}(S) \leq \widetilde{C}(E_0, T)(\widetilde{E}(S) - \widetilde{E}(S+T)). \tag{5.62}$$

Then

$$\widetilde{E}(S+T) \leq \frac{\widetilde{C}(E_0, T) - 1}{\widetilde{C}(E_0, T)} \widetilde{E}(S). \tag{5.63}$$

It follows from (1.11), (4.1), (4.2) and (5.61) that there exists $\widetilde{C}(T) > 0$ such that

$$\widetilde{E}(S+t) \leq \widetilde{C}(T)\widetilde{E}(S), \quad \forall 0 \leq t \leq T. \tag{5.64}$$

With (5.63), $\widetilde{E}(t)$ is of exponential decay. Hence, there exist

$$C_1(E_0), C_2(E_0) > 0$$

such that

$$E(t) \leq C_1(E_0) e^{-C_2(E_0)t} E(0), \forall t > 0. \tag{5.65}$$

□

6. Proofs of stability with uniform energy decay rate

Lemma 6.1. *Let Assumption* (**C**) *hold true. Assume that*

$$\frac{\partial r}{\partial \nu} \leq 0, \quad x \in \Gamma. \tag{6.1}$$

Let $u(x,t)$ solve the system (1.9). Then

$$E(0) + \int_0^T E(t) dt \leq C \int_0^T \int_\Omega a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt, \tag{6.2}$$

for sufficiently large T.

Proof. Let $b(z)$ be a smooth nonnegative function defined on $[0, +\infty)$ satisfying

$$b(z) = 1, \quad 0 \le z \le R_0 - \varepsilon_0 \quad \text{and} \quad b(z) = 0, \quad z \ge R_0. \tag{6.3}$$

Let

$$H(x) = b(r)x, \quad x \in \mathbb{R}^n. \tag{6.4}$$

It follows from (2.34), (2.35) and (3.10) that

$$DH(X,X) \ge \delta |X|_g^2 \quad \text{for all } X \in \mathbb{R}^n_x,\ x \in \Omega(R_0 - \varepsilon_0), \tag{6.5}$$

$$\begin{aligned} \operatorname{div}_g H &= 1 + r\Delta_g r \\ &= 1 + r\left(\frac{n-1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r}\right) \\ &= n + d/2 \quad \text{for all } X \in \mathbb{R}^n_x,\ x \in \Omega(R_0 - \varepsilon_0). \end{aligned} \tag{6.6}$$

Let $\mathcal{H} = H$ and $\widehat{\Omega} = \Omega(R_0)$ in (3.2). From (3.2), we have

$$\begin{aligned} &\int_0^T \int_\Gamma \operatorname{Re}\left(\frac{\partial u}{\partial \nu}\mathcal{H}(\bar u)\right) d\Gamma_g dt \\ &+ \frac{1}{2}\int_0^T \int_\Gamma \left(\operatorname{Im}(u\bar u_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right)\langle \mathcal{H}, \nu\rangle_g d\Gamma_g dt \\ \ge\ & \frac{1}{2}\int_{\Omega(R_0)} \operatorname{Im}\,(uH(\bar u))\,dx_g\Big|_0^T + \delta\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |\nabla_g u|_g^2 dx_g dt \\ &- C\int_0^T \int_{x\in\Omega(R_0)\backslash\Omega(R_0-\varepsilon_0)} |\nabla_g u|_g^2 dx_g dt \\ &+ \int_0^T \int_{\Omega(R_0)} \operatorname{Im}\,(a(x)uH(\bar u))\,dx_g dt \\ &+ \frac{1}{2}\int_0^T \int_{\Omega(R_0)} \left(\operatorname{Im}(u\bar u_t) - |\nabla_g u|_g^2 - \frac{2}{p+1}|u|^{p+1}\right)\operatorname{div}_g H dx_g dt \\ =\ & \frac{1}{2}\int_{\Omega(R_0)} \operatorname{Im}\,(uH(\bar u))\,dx_g\Big|_0^T + \delta\int_0^T \int_{\Omega(R_0-\varepsilon_0)} |\nabla_g u|_g^2 dx_g dt \\ &- C\int_0^T \int_{x\in\Omega(R_0)\backslash\Omega(R_0-\varepsilon_0)} |\nabla_g u|_g^2 dx_g dt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_{\Omega(R_0)} \mathrm{Im}\,(a(x)uH(\bar u))\,dx_g dt \\
&+\frac{1}{2}\int_0^T\int_{\Omega(R_0)} \left(\mathrm{Im}\,(u\bar u_t)-|\nabla_g u|_g^2-|u|^{p+1}\right)\,\mathrm{div}\,_g H dx_g dt \\
&+\int_0^T\int_{\Omega(R_0)} \frac{(p-1)\,\mathrm{div}\,_g H}{2(p+1)}|u|^{p+1}dx_g dt. \qquad (6.7)
\end{aligned}
$$

Let $P=\frac{\mathrm{div}\,_g H}{2}$ and $\widehat{\Omega}=\Omega(R_0)$ in (3.3). Substituting (3.3) into (6.7), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega(R_0)} \mathrm{Im}\,(uH(\bar u))\,dx_g\Big|_0^T-\frac{1}{4}\int_0^T\int_{\Omega(R_0)}|u|^2\Delta_g(\,\mathrm{div}\,_g H)dx_g dt \\
&+\int_0^T\int_{\Omega(R_0)} \mathrm{Im}\,(a(x)uH(\bar u))\,dx_g dt+\delta\int_0^T\int_{\Omega(R_0-\varepsilon_0)}|\nabla_g u|_g^2 dx_g dt \\
&+\int_0^T\int_{\Omega(R_0-\varepsilon_0)}\frac{(n+d/2)(p-1)}{2(p+1)}|u|^{p+1}dx_g dt \\
&\le C\int_0^T\int_{\Omega(R_0)\backslash\Omega(R_0-\varepsilon_0)}\left(|\nabla_g u|_g^2+|u|^{p+1}\right)dx_g dt+\Pi_\Gamma, \qquad (6.8)
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_\Gamma=&\int_0^T\int_\Gamma \mathrm{Re}\,\left(\frac{\partial u}{\partial\nu}H(\bar u)\right)d\Gamma_g dt \\
&+\frac{1}{2}\int_0^T\int_\Gamma\left(\mathrm{Im}\,(u\bar u_t)-|\nabla_g u|_g^2-\frac{2}{p+1}|u|^{p+1}\right)\langle H,\nu\rangle_g d\Gamma_g dt \\
&-\frac{1}{2}\int_0^T\int_\Gamma |u|^2\frac{\partial P}{\partial\nu}d\Gamma_g dt+\int_0^T\int_\Gamma \mathrm{Re}\,\left(P\bar u\frac{\partial u}{\partial\nu}\right)d\Gamma_g dt. \qquad (6.9)
\end{aligned}
$$

Since $u|_\Gamma=0$, we obtain $\nabla_{\Gamma_g}\bar u\Big|_\Gamma=0$, that is,

$$
\nabla_g\bar u=\frac{\partial\bar u}{\partial\nu}\nu\quad\text{for } x\in\Gamma. \qquad (6.10)
$$

Then, with (6.4), we have

$$
H(\bar u)=\langle H,\nabla_g\bar u\rangle_g=\frac{\partial\bar u}{\partial\nu}\langle H,\nu\rangle_g=r\frac{\partial\bar u}{\partial\nu}\frac{\partial r}{\partial\nu}\quad\text{for } x\in\Gamma. \qquad (6.11)
$$

Using the formulas (6.10) and (6.11) in formula (6.9) on the portion Γ, with (6.1), we obtain

$$\Pi_\Gamma = \frac{1}{2}\int_0^T \int_\Gamma r\Big|\frac{\partial u}{\partial \nu}\Big|^2 \frac{\partial r}{\partial \nu} d\Gamma_g dt \leq 0. \tag{6.12}$$

Substituting (6.12) into (6.8), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega(R_0)} \operatorname{Im}\,(uH(\bar{u}))\,dx_g\Big|_0^T - \frac{1}{4}\int_0^T\int_{\Omega(R_0)} |u|^2\Delta_g(\operatorname{div}_g H)dx_g dt \\
&+\int_0^T\int_{\Omega(R_0)} \operatorname{Im}\,(a(x)uH(\bar{u}))\,dx_g dt \\
&+\delta\int_0^T\int_{\Omega(R_0-\varepsilon_0)} |\nabla_g u|_g^2 dx_g dt \\
&+\int_0^T\int_{\Omega(R_0-\varepsilon_0)} \frac{(n+d/2(p-1)}{(p+1)}|u|^{p+1}dx_g dt \\
&\leq C\int_0^T\int_{\Omega(R_0)\backslash\Omega(R_0-\varepsilon_0)} \left(|\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt.
\end{aligned} \tag{6.13}$$

Therefore,

$$\begin{aligned}
\int_0^T\int_{\Omega(R_0-\varepsilon_0)} \left(|u|^{p+1} + |\nabla_g u|_g^2\right) dx_g dt &\leq C(E(0)+E(T)) \\
&+ C\int_0^T\int_{\Omega(R_0)} a(x)\left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt.
\end{aligned} \tag{6.14}$$

Note that

$$\int_0^T\int_{\Omega(R_0-\varepsilon_0)} |u|^2 dx_g dt \leq C(R_0)\int_0^T\int_{\Omega(R_0-\varepsilon_0)} |\nabla_g u|_g^2 dx_g dt. \tag{6.15}$$

Hence

$$\begin{aligned}
\int_0^T E(t)dt &\leq C(E(0)+E(T)) \\
&+ C\int_0^T\int_\Omega a(x)\left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt.
\end{aligned} \tag{6.16}$$

With (4.1) and (4.2) , we deduce that

$$CE(T) \leq CE(0) + C \int_0^T \int_\Omega a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt$$

$$+ \frac{C}{2} \int_0^T \int_\Omega |u|^2 \Big|\Delta_g a(x)\Big| dx_g dt, \tag{6.17}$$

and

$$4CE(0) = \int_0^{4C} E(t)dt - \int_0^{4C} (E(t) - E(0))dt$$

$$\leq \int_0^{4C} E(t)dt + 4C \int_0^{4C} \int_\Omega a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt$$

$$+ 2C \int_0^T \int_\Omega |u|^2 \Big|\Delta_g a(x)\Big| dx_g dt. \tag{6.18}$$

Substituting (6.17) and (6.18) into (6.16), for $T > 4C$, with (2.37), we have

$$E(0) + \int_0^T E(t)dt \leq C \int_0^T \int_\Omega a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1}\right) dx_g dt. \tag{6.19}$$

The estimate (6.2) holds true. □

Proof of Theorem 2.3. From (4.1), (4.2) and (6.2), we deduce that

$$E(0) + \int_0^T E(t)dt$$

$$\leq C(E(0) - E(T)) + C \int_0^T \int_{\Omega(R_0 - \varepsilon_0)} \Big|\Delta_g a(x)\Big| |u|^2 dx_g dt$$

$$= C(E(0) - E(T)) - M \int_0^T \int_\Omega |u|^2 dx_g dt \Big|_0^T$$

$$+ \int_0^T \int_{\Omega(R_0 - \varepsilon_0)} \left(C\Big|\Delta_g a(x)\Big| - M \int_0^T \int_\Omega a(x) \right) |u|^2 dx_g dt. \tag{6.20}$$

For sufficiently large M, with (2.37) we have

$$E(0) \leq C(E(0) - E(T)) - M \int_0^T \int_\Omega |u|^2 dx_g dt \Big|_0^T. \tag{6.21}$$

Denote

$$\widetilde{E}(t) = E(t) + M \int_0^T \int_\Omega |u|^2 dx_g dt. \tag{6.22}$$

From (6.21), we obtain

$$\widetilde{E}(0) \leq \widetilde{C}(\widetilde{E}(0) - \widetilde{E}(T)). \tag{6.23}$$

Then

$$\widetilde{E}(T) \leq \frac{\widetilde{C} - 1}{\widetilde{C}} \widetilde{E}(0). \tag{6.24}$$

It follows from (1.11), (4.1), (4.2) and (6.22) that there exists $\widetilde{C}(T) > 0$ such that

$$\widetilde{E}(t) \leq \widetilde{C}(T)\widetilde{E}(0), \quad \forall 0 \leq t \leq T. \tag{6.25}$$

With (6.24), $\widetilde{E}(t)$ is exponentially decaying. Hence, there exist $C_1, C_2 > 0$ such that

$$E(t) \leq C_1 e^{-C_2 t} E(0), \forall t > 0. \tag{6.26}$$

□

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Appendix A. Proofs of Assumption (U1) and Assumption (U2) under a strong geometric condition

Let $\widehat{\Omega} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and ω be an open subset of $\widehat{\Omega}$ such that

$$\omega \supset \bigcup_{x \in \partial\widehat{\Omega}} \{y \in \widehat{\Omega} \big| \ |y - x| < \xi\}, \tag{A.1}$$

for some $\xi > 0$.

Assume that the origin $O \notin \overline{\widehat{\Omega}}$ and

$$G(x)\frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^n, \quad \text{and} \quad \det\left(G(x)\right) = c_0 r^d, \quad x \in \widehat{\Omega}, \tag{A.2}$$

$$\left\langle \left((1-\delta)G(x) + \frac{r}{2}\frac{\partial G(x)}{\partial r} \right) X, X \right\rangle \geq 0 \quad \text{for } X \in \mathbb{R}^n_x, \ x \in \widehat{\Omega}, \tag{A.3}$$

where $0 < \delta \leq 1$, $c_0 > 0$ and d are constants.

Remark A.1. It follows from (4.6) and (4.7) that

$$\begin{aligned}\frac{(n+d/2-1)}{r} &= \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r} = \Delta_g r = trD^2 r \\ &\geq (n-1)\frac{\delta}{r}, \quad x \in \widehat{\Omega}. \end{aligned} \tag{A.4}$$

Then

$$d \geq 2(n-1)(\delta - 1). \tag{A.5}$$

Proposition A.1. *There exists $T_1 \geq 0$ such that for any $T > T_1$, the only solution u in $C([0,T], H^1(\widehat{\Omega}))$ to the system*

$$\begin{cases} iu_t + \Delta_g u - |u|^{p-1}u = 0 & (x,t) \in \widehat{\Omega} \times (0,T), \\ u = 0 & (x,t) \in \omega \times (0,T), \end{cases} \tag{A.6}$$

is the trivial one $u \equiv 0$.

Proof. Let $b(x) \in C^\infty(\mathbb{R}^n)$ be a nonnegative function satisfying

$$b(x) = 1, \quad x \in \widehat{\Omega}\backslash\omega \quad \text{and} \quad b(x) = 0, \quad \mathbb{R}^n\backslash\widehat{\Omega}. \tag{A.7}$$

Let

$$H(x) = b(x)x, \quad x \in \mathbb{R}^n. \tag{A.8}$$

It follows from (A.2), (A.3) and (3.10) that

$$DH(X,X) \geq \delta |X|_g^2 \quad \text{for all } X \in \mathbb{R}^n_x,\ x \in \widehat{\Omega}\backslash\omega, \tag{A.9}$$

$$\begin{aligned}\operatorname{div}_g H &= 1 + r\Delta_g r \\ &= 1 + r\left(\frac{n-1}{r} + \frac{\partial \ln\sqrt{\det(G(x))}}{\partial r}\right) \\ &= n + d/2 \quad \text{for all } X \in \mathbb{R}^n_x,\ x \in \widehat{\Omega}\backslash\omega. \end{aligned} \tag{A.10}$$

Let $a(x) = 0$ in (3.1). Let $\mathcal{H} = H$ in (3.2) and $P = \frac{\operatorname{div}_g H}{2}$ in (3.3). Substituting (3.3) into (3.2), we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\widehat{\Omega}} \operatorname{Im}\,(uH(\bar{u}))\,dx_g\Big|_0^T - \frac{1}{4}\int_0^T\int_{\widehat{\Omega}} |u|^2\Delta_g(\operatorname{div}_g H)dx_g dt \\ &+\int_0^T\int_{\widehat{\Omega}} \operatorname{Re}\,(DH(\nabla_g\bar{u},\nabla_g u))\,dx_g dt \\ &+\int_0^T\int_{\widehat{\Omega}} \frac{(p-1)\operatorname{div}_g H}{2(p+1)}|u|^{p+1}dx_g dt \\ &= 0. \end{aligned} \tag{A.11}$$

Then

$$\begin{aligned} &\int_0^T\int_{\widehat{\Omega}\backslash\omega}\left(|\nabla_g u|^2 + \frac{2}{p+1}|u|^{p+1}\right)dx_g dt \\ &\le C\Big|\int_{\widehat{\Omega}} \operatorname{Im}\,(uH(\bar{u}))\,dx_g\Big|_0^T\Big| \\ &\quad + C\int_0^T\int_\omega \left(|\nabla_g u|^2 + |u|^2 + |u|^{p+1}\right)dx_g dt. \end{aligned} \tag{A.12}$$

Hence

$$\int_0^T\int_{\widehat{\Omega}}\left(|\nabla_g u|^2 + \frac{2}{p+1}|u|^{p+1}\right)dx_g dt \le 2CE(0). \tag{A.13}$$

Note that

$$\int_0^T\int_{\widehat{\Omega}} |u|^2 dx_g dt \le C\int_0^T\int_{\widehat{\Omega}} |\nabla_g u|_g^2 dx_g dt. \tag{A.14}$$

Therefore

$$\int_0^T\int_{\widehat{\Omega}}\left(|u|^2 + |\nabla_g u|^2 + \frac{2}{p+1}|u|^{p+1}\right)dx_g dt \le 2CE(0), \tag{A.15}$$

which implies

$$(T-C)E(0) \le 0. \tag{A.16}$$

The assertion (A.6) holds true. □

By a similar proof with Proposition (A.1), the following assertion holds.

Proposition A.2. *There exists* $T_1 \geq 0$ *such that for any* $T > T_1$*, the only solution* u *in* $C([0,T], H^1(\widehat{\Omega}))$ *to the system*

$$\begin{cases} iu_t + \Delta_g u = 0 & (x,t) \in \widehat{\Omega} \times (0,T), \\ u = 0 & (x,t) \in \omega \times (0,T), \end{cases} \tag{A.17}$$

is the trivial one $u \equiv 0$*.*

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