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Stability of tautological bundles on symmetric products of curves

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We prove that, if C is a smooth projective curve over the complex numbers, and E is a stable vector bundle on C whose slope does not lie in the interval $[-1, n-1]$, then the associated tautological bundle $E^{[n]}$ on the symmetric product $C^{(n)}$ is again stable. Also, if E is semi-stable and its slope does not lie in $(-1, n-1)$, then $E^{[n]}$ is semi-stable.

Introduction

Given a smooth projective curve C over the complex numbers, there is an interesting series of related higher-dimensional smooth projective varieties, namely the symmetric products $C^{(n)}$. For every vector bundle E on C of rank r, there is a naturally associated vector bundle $E^{[n]}$ of rank rn on the symmetric product $C^{(n)}$, called *tautological* or *secant bundle*. These tautological bundles carry important geometric information. For example, k-very ampleness of line bundles can be expressed in terms of the associated tautological bundles, and these bundles play an important role in the proof of the gonality conjecture of Ein and Lazarsfeld [6]. Tautological bundles on symmetric products of curves have been studied since the 1960s [11, 13, 14], but there are still new results about these bundles discovered nowadays; see, for example, [2, 10, 15].

A natural problem is to decide when a tautological bundle is stable. Here, stability means slope stability with respect to the ample class H_n that is represented by $C^{(n-1)} + x \subset C^{(n)}$ for any $x \in C$; see Subsection 1.3 for details. This problem has been much studied, mainly in the special case that L is a line bundle; see [1–5, 7, 12]. It is easy to see that $E^{[n]}$ can only be stable if E is stable; see Remark 1.5. Hence, the question is under which circumstances the stability of E implies the stability of $E^{[n]}$. For line bundles and $n = 2$, there is a complete answer given by Biswas and Nagaraj [3]. Namely, $L^{[2]}$ is unstable if and only if $L \cong \mathcal{O}_C$, and $L^{[2]}$ is properly semi-stable if and only if $L \cong \mathcal{O}(\pm x)$ for some point $x \in C$. Mistretta [12, Sect. 4], proved that $L^{[n]}$ is stable whenever $\mathsf{deg}(L) > n-1$. Recently, Dan and Pal [5] and Basu and Dan [2] started the treatment of the problem for E of higher rank, considering the case $n=2$. In *loc. cit.* it is shown that $E^{[2]}$ is stable whenever E is stable with $\mathsf{deg}(E) > \mathsf{rank}(E)$, and $E^{[2]}$ is semi-stable whenever E is semi-stable with $\mathsf{deg}(E) \geq \mathsf{rank}(E)$.

In the present paper, we generalise the result of *loc. cit.* to arbitrary n, and complement it by a similar result for vector bundles of negative degree. Concretely, we prove the following

Theorem 0.1. *Let C be a smooth projective curve, and let $E \in \mathsf{VB}(C)$ be a vector bundle. We set $d := \mathsf{deg}(E)$, $r := \mathsf{rank}(E)$, and $\mu := \mu(E) = \frac{d}{r}$.*

- *(i) Let $n \in \mathbb{N}$, and let E be semi-stable with $d \geq (n-1)r$ or, equivalently, $\mu \geq n-1$. Then $E^{[n]}$ is slope semi-stable with respect to H_n.*
- *(ii) Let $n \in \mathbb{N}$, and let E be stable with $d > (n-1)r$ or, equivalently, $\mu > n-1$. Then $E^{[n]}$ is slope stable with respect to H_n.*
- *(iii) Let E be semi-stable with $d \leq -r$ or, equivalently, $\mu \leq -1$. Then $E^{[n]}$ is slope semi-stable with respect to H_n for every $n \in \mathbb{N}$.*
- *(iv) Let E be stable with $d < -r$ or, equivalently, $\mu < -1$. Then $E^{[n]}$ is slope stable with respect to H_n for every $n \in \mathbb{N}$.*

The slope of a tautological bundle is given by the formula $\mu(E^{[n]}) = \frac{d-(n-1)r}{rn} = \frac{\mu-n+1}{n}$; see Subsection 1.7. Hence, we can reformulate our result as follows:

If the slope $\mu(E^{[n]})$ of a tautological bundle lies outside of the interval $[-1,0]$*, the tautological bundle inherits the properties stability and semi-stability from E. If $\mu(E^{[n]})$ lies on the boundary of this interval, $E^{[n]}$ still inherits semi-stability from E.*

The key to our proof is a short exact sequence relating the tautological bundles $E^{[n-1]}$ and $E^{[n]}$; see Proposition 1.4. This exact sequence allows us to prove Theorem 0.1, by a direct argument if $\mathsf{deg}\, E > 0$, and by induction if $\mathsf{deg}\, E < 0$.

The paper is organised as follows. In Subsections 1.1 and 1.2, we recall the definitions of slope stability and of tautological bundles on the symmetric product of a curve. In Subsection 1.3, we introduce some important divisors on $C^{(n)}$ and C^n, and compute their intersection numbers. In Subsection 1.4 we show that stability of $E^{[n]}$ can be tested by computing the slopes of $\mathfrak{S}_n$-equivariant subsheaves of $\pi_n^*E^{[n]}$, where $\pi_n\colon C^n \to C^{(n)}$ is the $\mathfrak{S}_n$-quotient morphism. Then, in Subsection 1.5, we explain how slopes of

$\mathfrak{S}_n$-equivariant sheaves on C^n can be computed by restriction to appropriate subvarieties. In the next Subsection 1.6, we discuss the key short exact sequence relating $\pi_n^* E^{[n]}$ and $\pi_{n-1}^* E^{[n-1]}$. In Subsection 1.7, we compute the slope of tautological bundles and their pull-backs along the quotient morphisms, and remark that (semi-)stability of $E^{[n]}$ implies (semi-)stability of E. In Section 2, we carry out the proof of Theorem 0.1. Halfway through the proof, we have to separate the cases of negative and positive degree d. These two cases are treated in Subsections 2.2 and 2.3, respectively. In the final Section 3, we observe that Theorem 0.1 is already optimal in the sense that the numerical conditions on the slopes cannot be weakened.

Conventions

All our varieties are defined over the complex numbers. We denote the set of positive integers by $\mathbb{N}$. Given two varieties X and Y, we write the projections from their product to the factors as $\mathsf{pr}_X = \mathsf{pr}_X^{X\times Y}\colon X\times Y \to X$ and $\mathsf{pr}_Y = \mathsf{pr}_Y^{X\times Y}\colon X\times Y\to X$. We write $\mathsf{VB}(X)$ for the category of vector bundles and $\mathsf{Coh}(X)$ for the category of coherent sheaves on X.

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1. Preliminaries

1.1. The notion of slope stability

Let X be a smooth projective variety. Let us fix an ample class $H \in \mathsf{N}^1(X)$ in the group of divisors modulo numerical equivalence. For a coherent sheaf $A \in \mathsf{Coh}(X)$ with $\mathsf{rank}(A) \geq 1$, we define its *degree* and its *slope* with respect to H by

$$\mathsf{deg}_H(A) := c_1(A)\cdot H^{n-1} := \int_X c_1(A)\cdot H^{n-1}, \quad \mu_H(A) := \frac{\mathsf{deg}_H(A)}{\mathsf{rank}(A)}\,.$$

A vector bundle $E \in \mathsf{VB}(X)$ is called *slope semi-stable* with respect to H if, for every subsheaf $A \subset E$ with $\mathsf{rank}(A) < \mathsf{rank}(E)$, we have $\mu_H(A) \leq \mu_H(E)$.

It is called *slope stable* with respect to H if, for every subsheaf $A \subset E$ with $\mathsf{rank}(A) < \mathsf{rank}(E)$, we have the strict inequality $\mu_H(A) < \mu_H(E)$. Sometimes, we omit the word 'slope' and just speak of semi-stable and stable vector bundles. Note that, if $X = C$ is a curve, the notion of stability and semi-stability is independent of the chosen ample class $H \in \mathsf{N}^1(X)$.

1.2. Symmetric product of a curve and tautological bundles

From now on, let C always be a smooth projective curve, and let $n \in \mathbb{N}$. There is a natural action by the symmetric group $\mathfrak{S}_n$ on the cartesian product C^n by permutation of the factors. The corresponding quotient variety $C^{(n)} := C^n/\mathfrak{S}_n$ is called the *n-th symmetric product* of C. By the Chevalley–Shephard–Todd theorem, the variety $C^{(n)}$ is smooth, and the quotient morphism $\pi_n\colon C^n \to C^{(n)}$ is flat.

The points of $C^{(n)}$ can be identified with the effective degree n divisors on C. Accordingly, we write them as formal sums: $x_1 + \cdots + x_n = \pi_n(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in C$. In fact, the symmetric product is the fine moduli space of effective degree n divisors (or, equivalently, zero-dimensional subschemes of length n) on C, with the universal divisor $\Xi_n \subset C^{(n)} \times C$ given by the image of the closed embedding

$$C^{(n-1)} \times C \hookrightarrow C^{(n)} \times C, \quad (x_1 + \cdots + x_{n-1}, x) \mapsto (x_1 + \cdots + x_{n-1} + x, x)\,.$$

Now, the Fourier–Mukai transform along this universal divisor allows us to construct tautological vector bundles on $C^{(n)}$ from vector bundles on C. Concretely, for $E \in \mathsf{VB}(X)$, the associated *tautological bundle* on $C^{(n)}$ is given by

$$E^{[n]} := \mathsf{pr}^{C^{(n)}\times C}_{C^{(n)}*}(\mathcal{O}_{\Xi_n} \otimes \mathsf{pr}_C^{C^{(n)}\times C*}\, E) \cong a_* b^* E\,,$$

where $a\colon \Xi_n \to C^{(n)}$ and $b\colon \Xi_n \to C$ are the restrictions of the projections $\mathsf{pr}^{C^{(n)}\times C}_{C^{(n)}}$ and $\mathsf{pr}^{C^{(n)}\times C}_{C}$, respectively. Since a is flat and finite of degree n, the coherent sheaf $E^{[n]}$ is a vector bundle with $\mathsf{rank}(E^{[n]}) = n\,\mathsf{rank}(E)$.

1.3. Intersection theory on symmetric and cartesian products of curves

For $i = 1, \ldots, n$, we write $\mathsf{pr}_i\colon C^n \to C$ for the projection to the i-th factor, and $\overline{\mathsf{pr}}_i\colon C^n \to C^{n-1}$ for the projection to the other $n-1$ factors. We set $\widetilde{H}_n = \sum_{i=1}^n [\mathsf{pr}_i^{-1}(x)] \in \mathsf{N}^1(C^n)$ for any point $x \in C$. Indeed, modulo numerical equivalence, the divisor $\mathsf{pr}_i^{-1}(x)$ is independent from the point $x \in C$. Using the Segre embedding, we see that $\widetilde{H}_n$ is ample.

We define $H_n \in \mathsf{N}^1(C^{(n)})$ as the unique class with $\pi^* H_n = \widetilde{H}_n$. One can check easily that H_n is represented by $C^{(n-1)} + x$, the image of the closed embedding $C^{(n-1)} \hookrightarrow C^{(n)}$ with $\alpha \mapsto \alpha + x$, for any $x \in C$. Since $\widetilde{H}_n$ is ample and $\pi_n \colon C^n \to C^{(n)}$ is finite, H_n is ample too. We always consider stability of bundles on $C^{(n)}$ with respect to this ample class.

Another important divisor on C^n is the *big diagonal* $\delta_n = \sum_{1\le i<j\le n} \Delta_{ij}$ where

$$\Delta_{ij} = \{(x_1, \dots, x_n) \mid x_i = x_j\} \subset C^n \,. \tag{1}$$

Note that, in the Chow group modulo numerical equivalence, we have

$$(\widetilde{H}_n)^{n-1} = (n-1)! \sum_{i=1}^{n} \overline{\mathsf{pr}}_i^{-1}(y) \tag{2}$$

for any point $y = (x_1, \dots, x_{n-1}) \in C^{n-1}$. Note that

$$\overline{\mathsf{pr}}_i^{-1}(y) = x_1 \times \cdots \times x_{i-1} \times C \times x_{i+1} \times x_{n-1}.$$

From this, we can easily compute the following intersection numbers

$$(\widetilde{H}_n)^n = n!, \quad \delta_n \cdot (\widetilde{H}_n)^{n-1} = n!(n-1)\,. \tag{3}$$

1.4. Stability under pull-back along quotient morphism

Let G be a finite group acting on a smooth projective variety X. A *G-equivariant sheaf* on X is a coherent sheaf B together with a *G-linearisation*, that means a family of isomorphisms $\{\lambda_g \colon B \xrightarrow{\sim} g^* B\}_{g\in G}$ such that for every pair $g, h \in G$ the following diagram commutes:

$$B \xrightarrow{\lambda_g} g^*B \xrightarrow{g^*\lambda_h} g^*h^*B \xrightarrow{\cong} (hg)^*B\,, \qquad B \xrightarrow{\lambda_{hg}} (hg)^*B.$$

Let $\pi\colon X \to Y := X/G$ be the quotient morphism. Then, for every $g \in G$, we have $\pi \circ g = \pi$, which yields a canonical isomorphism of functors $\mu_g : \pi^* \xrightarrow{\sim} g^*\pi^*$. This gives, for every $F \in \mathsf{Coh}(Y)$, a G-linearisation $\{\mu_g : \pi^*F \xrightarrow{g} {}^* \pi^*F\}_{g\in G}$ of π^*F. We call this the *canonical G-linearisation of the pull-back* π^*F. By a *$\mathfrak{S}_n$-equivariant subsheaf* of π^*F, we mean a subsheaf $A \subset \pi^*F$ which is preserved by the canonical G-linearisation of the pull-back: $\mu_g(A) = g^*A$ as subsheaves of $g^*\pi^*F$.

Lemma 1.1. *Let a finite group G act on a smooth projective variety X such that $Y = X/G$ is again smooth and $\pi\colon X \to Y$ is flat. Let $H \in \mathsf{N}^1(Y)$ be an ample class and $F \in \mathsf{VB}(Y)$.*

(i) *If $\mu_{\pi^*H}(A) \le \mu_{\pi^*H}(\pi_n^* F)$ holds for all $\mathfrak{S}_n$-equivariant subsheaves A of π^*F with $\operatorname{rank} A < \operatorname{rank} F$, then F is slope semi-stable with respect to H.*

(ii) *If $\mu_{\pi^*H}(A) < \mu_{\pi^*H}(\pi_n^* F)$ holds for all $\mathfrak{S}_n$-equivariant subsheaves A of π^*F with $\operatorname{rank} A < \operatorname{rank} F$, then F is slope stable with respect to H.*

Proof. For every $B \in \mathsf{Coh}(Y)$ with $\operatorname{rank} B > 0$, we have

$$\mu_{\pi^*H}(\pi^* B) = |G| \cdot \mu_H(B);$$

see [8, Lem. 3.2.1]. Hence, for F to be semi-stable, it is sufficient to have $\mu_{\pi^*H}(\pi^*B) \le \mu_{\pi^*H}(\pi^*F)$ for every subsheaf $B \subset F$ with $\operatorname{rank} B < \operatorname{rank} F$. The assertion of part (i) follows from the fact that π^*B is a G-equivariant subsheaf of π^*F whenever B is a subsheaf of F. The proof of part (ii) is completely analogous. ☐

See [12, Sect. 4.2] for a similar criterion for slope stability of sheaves on quotients.

1.5. Some technical lemmas concerning restriction of sheaves

Lemma 1.2. *Let $F_1, \dots, F_k \in \mathsf{Coh}(C^n)$ be finite collection of coherent sheaves on C^n, and let $m \le n$. Then there exist points $x_{m+1}, \dots, x_n \in C$ such that, if*

$$\iota\colon C^m \hookrightarrow C^n, \quad (t_1, \dots, t_m) \mapsto (t_1, \dots, t_m, x_{m+1}, \dots, x_n)$$

denotes the closed embedding with image $C^m \times x_{m+1} \times \cdots \times x_n$, the ranks of the F_i do not change under pull-back along ι and all the higher pull-backs of the F_i along ι vanish:

$$\forall\, i = 1, \dots, k : \quad \operatorname{rank}(\iota^* F_i) = \operatorname{rank}(F_i), \quad L^j\iota^*(F_i) = 0 \quad \text{for } j \neq 0.$$

Proof. We consider the projection

$$f\colon C^n \to C^{n-m}, \quad (x_1, \dots, x_n) \mapsto (x_{m+1}, \dots, x_n)$$

to the last $n-m$ components. By generic flatness, we get for every $i = 1,\dots,k$ an open dense subset $U_i \subset C^{n-m}$ such that $F_{i|f^{-1}(U_i)}$ is flat over U_i. Now, choosing $(x_{m+1},\dots,x_n) \in \bigcap_{i=1}^k U_i$ gives the result. □

Lemma 1.3. *Let A be an $\mathfrak{S}_n$-equivariant sheaf on C^n.*

(i) Let $x_2,\dots,x_n \in C$ be points such that $L^j\iota^*(A) = 0$ for all $j \neq 0$ where $\iota\colon C \hookrightarrow C^n$ is given by $\iota(t) = (t, x_2,\dots,x_n)$. Then

$$\mathsf{deg}_{\widetilde{H}_n}(A) = n!\,\mathsf{deg}(\iota^* A)\,.$$

(ii) Let $x \in C$ be a point such that $L^j\iota^*(A) = 0$ for all $j \neq 0$ where $\iota : C^{n-1} \hookrightarrow C^n$ is given by $\iota(t_1,\dots,t_{n-1}) = (t_1,\dots,t_{n-1},x)$. Then

$$\mathsf{deg}_{\widetilde{H}_n}(A) = n\,\mathsf{deg}_{\widetilde{H}_{n-1}}(\iota^* A)\,.$$

Proof. In the set-up of part (i), we have $\iota(C) = \overline{\mathsf{pr}}_1^{-1}(y)$ with $y = (x_2,\dots, x_n) \in C^{n-1}$. By projection formula, we have

$$[\overline{\mathsf{pr}}_1^{-1}(y)]\cdot \mathsf{c}_1(A) = \iota^*\,\mathsf{c}_1(A) = \mathsf{c}_1(\iota^* A) = \mathsf{deg}(\iota^* A),$$

where the equality $\iota^*\,\mathsf{c}_1(A) = \mathsf{c}_1(\iota^* A)$ is due to the vanishing of the higher derived pull-backs. By the $\mathfrak{S}_n$-equivariance of A, we get $[\overline{\mathsf{pr}}_i^{-1}(y)]\cdot\mathsf{c}_1(A) = [\overline{\mathsf{pr}}_1^{-1}(y)]\cdot\mathsf{c}_1(A) = \mathsf{deg}(\iota^* A)$ for every $i = 1,\dots,n$. Combining this with (2) gives

$$\begin{aligned}\mathsf{deg}_{\widetilde{H}_n}(A) = (\widetilde{H}_n)^{n-1}\cdot\mathsf{c}_1(A) &= (n-1)!\left(\sum_{i=1}^n [\overline{\mathsf{pr}}_i^{-1}(y)]\right)\cdot\mathsf{c}_1(A)\\ &= (n-1)!n[\overline{\mathsf{pr}}_1^{-1}(y)]\cdot\mathsf{c}_1(A)\\ &= n!\,\mathsf{deg}(\iota^* A)\,.\end{aligned}$$

The proof of part (ii) is very similar. □

1.6. Pull-back of tautological bundles along the $\mathfrak{S}_n$-quotient

For $i = 1,\dots,n$ we consider the divisor $\delta_n(i) := \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \Delta_{ij}$ on C^n; compare (1).

Proposition 1.4. *For every $i = 1, \dots, n$, there is a short exact sequence*

$$0 \to \mathsf{pr}_i^* E(-\delta_n(i)) \to \pi_n^* E^{[n]} \to \overline{\mathsf{pr}}_i^* \pi_{n-1}^* E^{[n-1]} \to 0\,. \tag{4}$$

The subsheaves $\mathrm{U}_n(E,i) := \mathsf{im}\big(\mathsf{pr}_i^* E(-\delta_n(i)) \to \pi_n^* E^{[n]}\big)$ *of* $\pi_n^* E^{[n]}$ *defined by these sequences have pairwise trivial intersections:*

$$\mathrm{U}_n(E,i) \cap \mathrm{U}_n(E,j) \quad \text{for } i \neq j. \tag{5}$$

Furthermore, these subsheaves get permuted by the natural $\mathfrak{S}_n$*-linearisation of the pull-back* $\pi_n^* E^{[n]}$*: If* $\sigma(i) = j$*, we have the equality* $\mu_\sigma(\mathrm{U}_n(E,i)) = \sigma^* \mathrm{U}_n(E,j)$ *of subsheaves of* $\sigma^* \pi_n^* E^{[n]}$*.*

Proof. The pull-back of the universal divisor $\Xi_n \subset C^n \times C$ along the flat morphism $\pi_n \times \mathsf{id}_C \colon C^n \times C \to C^{(n)} \times C$ is given by $(\pi_n \times \mathsf{id}_C)^* \Xi_n = D_n := \sum_{k=1}^n \Gamma_{\mathsf{pr}_k}$. By flat base change along the diagram

$$\begin{array}{ccccc} C^n \times C & \xrightarrow{\pi_n \times \mathsf{id}_C} & C^{(n)} \times C & \xrightarrow{\mathsf{pr}_C} & C \\ \downarrow{\scriptstyle \mathsf{pr}_{C^n}} & & \downarrow{\scriptstyle \mathsf{pr}_{C^{(n)}}} & & \\ C^n & \xrightarrow{\pi_n} & C^{(n)}\,, & & \end{array}$$

we get, setting $q_n := \mathsf{pr}_C^{C^n \times C} = \mathsf{pr}_C^{C^{(n)} \times C} \circ (\pi_n \times \mathsf{id}_C) \colon C^n \times C \to C$, the following isomorphism

$$\pi_n^* E^{[n]} \cong \mathsf{pr}_{C^n *}(\mathcal{O}_{D_n} \otimes q_n^* E)\,. \tag{6}$$

Now, let us fix some $i \in [n]$. We note that $\sum_{k \neq i} \Gamma_{\mathsf{pr}_k} = (\overline{\mathsf{pr}}_i \times \mathsf{id}_C)^* D_{n-1}$, which gives $D_n = \Gamma_{\mathsf{pr}_i} + (\overline{\mathsf{pr}}_1 \times \mathsf{id}_C)^* D_{n-1}$. Hence, we get a short exact sequence

$$0 \to \mathcal{O}_{\Gamma_{\mathsf{pr}_i}}\left(-\sum_{k \neq i} [\Gamma_{\mathsf{pr}_k} \cap \Gamma_{\mathsf{pr}_i}]\right) \to \mathcal{O}_{D_n} \to (\overline{\mathsf{pr}}_i \times \mathsf{id}_C)^* \mathcal{O}_{D_{n-1}} \to 0 \tag{7}$$

of coherent sheaves on $C^n \times C$. All the sheaves of this sequence are finitely supported over C^n. Hence, combining (6) and (7), gives the short exact sequence

$$(8)\qquad 0 \to \mathsf{pr}_{C^n*}\left(\mathcal{O}_{\Gamma_{\mathsf{pr}_i}}(-\sum_{k\neq i}[\Gamma_{\mathsf{pr}_k}\cap\Gamma_{\mathsf{pr}_i}])\otimes q_n^*E\right) \to \pi_n^*E^{[n]} \to \mathsf{pr}_{C^n*}\big((\overline{\mathsf{pr}}_i\times\mathsf{id}_C)^*\mathcal{O}_{D_{n-1}}\otimes q_n^*E\big) \to 0$$

of coherent sheaves on C^n, which will turn out to be isomorphic to the asserted sequence (4). By flat base change along the diagram

$$\begin{array}{ccccc} C^n\times C & \xrightarrow{\overline{\mathsf{pr}}_i\times\mathsf{id}_C} & C^{n-1}\times C & \xrightarrow{q_{n-1}} & C \\ \downarrow{\scriptstyle \mathsf{pr}_{C^n}} & & \downarrow{\scriptstyle \mathsf{pr}_{C^{n-1}}} & & \\ C^n & \xrightarrow{\overline{\mathsf{pr}}_1} & C^{n-1}, & & \end{array}\qquad (q_n = q_{n-1}\circ(\overline{\mathsf{pr}}_i\times\mathsf{id}_C))$$

we see that $\mathsf{pr}_{C^n*}\big((\overline{\mathsf{pr}}_i\times\mathsf{id}_C)^*\mathcal{O}_{D_{n-1}}\otimes q_n^*E\big)\cong\overline{\mathsf{pr}}_i^*\pi_{n-1}^*E^{[n-1]}$. To bring the first term of (8) into the correct form, we consider the isomorphism

$$t\colon C^n \xrightarrow{\cong} \Gamma_{\mathsf{pr}_i}\subset C^n\times C,\quad (x_1,\dots,x_n)\mapsto(x_1,\dots,x_n;x_i)\,.$$

Because of $\Gamma_{\mathsf{pr}_k}\cap\Gamma_{\mathsf{pr}_i}=\{(x_1,\dots,x_n;x)\in C^n\times C\mid x_k=x_i=x\}$, we see that

$$t^*\left(\sum_{k\neq i}[\Gamma_{\mathsf{pr}_k}\cap\Gamma_{\mathsf{pr}_i}]\right)=\delta_n(i)\,.$$

From this, it follows that $\mathsf{pr}_{C^n*}\big(\mathcal{O}_{\Gamma_{\mathsf{pr}_i}}(-\sum_{k\neq i}[\Gamma_{\mathsf{pr}_i}\cap\Gamma_{\mathsf{pr}_i}])\cong\mathsf{pr}_i^*E(-\delta_n(i))$, which shows that the sheaves in (8) are isomorphic to those in (4).

The fact that, for $i\neq j$, the subsheaves $\mathsf{pr}_i^*E(-\delta_n(i))$ and $\mathsf{pr}_j^*E(-\delta_n(j))$ of $\pi_n^*E^{[n]}$ intersect trivially follows from the fact that $\mathcal{O}_{\Gamma_{\mathsf{pr}_i}}(-\sum_{k\neq i}[\Gamma_{\mathsf{pr}_k}\cap\Gamma_{\mathsf{pr}_i}])$ and $\mathcal{O}_{\Gamma_{\mathsf{pr}_j}}(-\sum_{k\neq j}[\Gamma_{\mathsf{pr}_k}\cap\Gamma_{\mathsf{pr}_j}])$ intersect trivially as subsheaves of $\mathcal{O}_{D_n}$.

The final statement of the proposition follows from the fact that, for $\sigma\in\mathfrak{S}_n$ with $\sigma(i)=j$, we have the equality

$$\nu_\sigma\mathcal{O}_{\Gamma_{\mathsf{pr}_i}}\left(-\sum_{k\neq i}[\Gamma_{\mathsf{pr}_k}\cap\Gamma_{\mathsf{pr}_i}]\right)=\sigma^*\mathcal{O}_{\Gamma_{\mathsf{pr}_j}}\left(-\sum_{k\neq i}[\Gamma_{\mathsf{pr}_k}\cap\Gamma_{\mathsf{pr}_i}]\right)$$

of subsheaves of $\sigma^*\mathcal{O}_{D_n}$, where ν is the natural $\mathfrak{S}_n$-linearisation of the pull-back $\mathcal{O}_{D_n}=(\pi_n^*\times\mathsf{id}_C)^*\mathcal{O}_{\Xi_n}$. □

1.7. Degree and slope of tautological bundles

There are well-known formulae for the Chern classes of tautological bundles; see [11, Sect. 3]. In particular, we have

$$\mathsf{c}_1(\pi_n^* E^{[n]}) = d\widetilde{H}_n - r\delta_n \,. \tag{9}$$

Alternatively, this formula can easily be deduced inductively using the short exact sequence of Proposition 1.4. For doing this, note that $\widetilde{H}_n = \mathsf{pr}_i^*[x] + \overline{\mathsf{pr}}_i^* \widetilde{H}_{n-1}$ and $\delta_n = \delta_n(i) + \overline{\mathsf{pr}}_i^* \delta_{n-1}$ for every $i \in [n]$. Combining (9) with (3), we get $\mathsf{deg}_{\widetilde{H}_n}(\pi_n^* E^{[n]}) = n!\big(d - (n-1)r\big)$ and

$$\mu_{\widetilde{H}_n}(\pi_n^* E^{[n]}) = \frac{(n-1)!(d-(n-1)r)}{r} = (n-1)!(\mu - n + 1)\,. \tag{10}$$

Since π_n is finite of degree $n!$, we also get

$$\mu_{H_n}(E^{[n]}) = \frac{\mu_{\widetilde{H}_n}(\pi_n^* E^{[n]})}{n!} = \frac{(d-(n-1)r)}{nr} = \frac{\mu - n + 1}{n} \tag{11}$$

Remark 1.5. For an arbitrary, not necessarily locally free, coherent sheaf $A \in \mathsf{Coh}(C)$, we can still define an associated tautological sheaf on $C^{(n)}$ by $A^{[n]} := a_* b^*(A)$; compare subsection 1.2. Since a is finite and b is flat, the functor $a_* b^* \colon \mathsf{Coh}(C) \to \mathsf{Coh}(C^{(n)})$ is exact; compare [9, Thm. 1.1]. In particular, if $0 \to E_1 \to E_0 \to A \to 0$ is a locally free resolution of A, then $0 \to E_1^{[n]} \to E_0^{[n]} \to A^{[n]} \to 0$ is a locally free resolution of $A^{[n]}$. It follows that formula (11) extends to a formula for slopes of tautological sheaves of positive rank, namely $\mu_{H_n}(E^{[n]}) = \frac{\mu(A)-n+1}{n}$. It follows that, if E is a vector bundle on C and $A \subset E$ is a destabilising sheaf, then $A^{[n]} \subset E^{[n]}$ is again destabilising. In other words, (semi-)stability of $E^{[n]}$ implies (semi-)stability of E.

2. Proof of the main result

2.1. General part of the proof

Let $E \in \mathsf{VB}(C)$ satisfy the assumptions of one of the four parts (i), (ii), (iii), (iv) of Theorem 0.1. By Lemma 1.1, in order to prove stability or semi-stability of $E^{[n]}$, we need to compare the slopes of A and $\pi_n^* E^{[n]}$ for $A \subset \pi_n^* E^{[n]}$ a $\mathfrak{S}_n$-invariant subsheaf with $s := \mathsf{rank}\, A < nr = \mathsf{rank}\, E^{[n]}$.

For $i = 1, \dots, n$, we set $A'(i) := A \cap \mathrm{U}_n(E, i)$ as an intersection of subsheaves of $\pi_n^* E^{[n]}$; compare Proposition 1.4. We write the corresponding quotient as $A''(i) = A/A'(i)$. We also set $A' = A'(1)$ and $A'' = A''(1)$, and get a commutative diagram with exact columns and rows

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{pr}_1^* E(-\delta_n(1)) & \longrightarrow & \pi_n^* E^{[n]} & \longrightarrow & \overline{\mathsf{pr}}_1^* \pi_{n-1}^* E^{[n-1]} & \longrightarrow & 0
\end{array}
\tag{12}
$$

where the bottom row is the short exact sequence form Proposition 1.4. We set $s' = \mathsf{rank}\, A'$ and $s'' = \mathsf{rank}\, A''$ which gives $s = s' + s''$. By the last statement of Proposition 1.4 together with the $\mathfrak{S}_n$-equivariance of the subsheaf $A \subset \pi_n^* E^{[n]}$, we have $A'(i) \cong \sigma^* A'$ for any $\sigma \in \mathfrak{S}_n$ with $\sigma(i) = 1$. In particular, $\mathsf{rank}\, A'(j) = s'$ for every $j = 1, \dots, n$. By (5), we have $\bigoplus_{j=1}^n A'(j) \subset A$. Hence, we get the following inequalities of the ranks

$$
s \geq ns', \quad s'' \geq (n-1)s', \quad ns'' \geq (n-1)s\,. \tag{13}
$$

Now, we divide the proof that $\mu_{\widetilde{H}_n}(A) \leq \mu_{\widetilde{H}_n}(\pi_n^* E^{[n]})$ (or, for the proof of parts (ii) and (iv) of Theorem 0.1, that we have have a strict inequality) into the two cases of positive and negative $d = \mathsf{deg}\, E$, treated in the following two subsections.

2.2. Proof of the main theorem for bundles of positive degree

In this subsection, we prove parts (i) and (ii) of Theorem 0.1. Let $E \in \mathsf{VB}(C)$ be a semi-stable bundle with $d \geq (n-1)r$, equivalently $\mu \geq (n-1)$. By Lemma 1.2 and Lemma 1.3, there are points $x_2, \dots, x_n \in C$ such that, for $\iota\colon C \hookrightarrow C^n$ with $\iota(t) = (t, x_2, \dots, x_n)$ the closed embedding with image $C \times x_2 \times \cdots \times x_n$, we have

$$
\mathsf{deg}_{\widetilde{H}_n}(A) = n!\, \mathsf{deg}(\iota^* A)\,, \tag{14}
$$

the rank of objects of diagram (12) remain unchanged after pull-back by ι, and the rows and columns of the diagram (12) remain exact after pull-back by ι. Since $C \times x_2 \times \cdots \times x_n$ is a section of $\mathsf{pr}_1\colon C^n \to C$ and a fibre of $\overline{\mathsf{pr}}_1\colon C^n \to C^{n-1}$, the restricted diagram takes the form

(15)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \iota^*A' & \longrightarrow & \iota^*A & \longrightarrow & \iota^*A'' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E(-x_2-x_3-\dots-x_n) & \longrightarrow & \iota^*\pi_n^*E^{[n]} & \longrightarrow & \mathcal{O}_C^{\oplus r(n-1)} & \longrightarrow & 0
\end{array}$$

By the semi-stability of $E(-x_2-x_3-\dots-x_n)$ and $\mathcal{O}_C^{\oplus r(n-1)}$, we get

$$\mu(\iota^*A') \le \mu(E(-x_2-x_3-\dots-x_n)) = \mu-n+1 \tag{16}$$

and $\mu(\iota^*A'') \le 0$. Note that equation (16) is essentially were the upper boundary $n-1$ of the critical interval in the statement of Theorem 0.1 is coming from. Combining the two inequalities, we get

$$\begin{aligned}\mathsf{deg}(\iota^*A) &= \mathsf{deg}(\iota^*A') + \mathsf{deg}(\iota^*A'')\\ &= s'\mu(\iota^*A') + s''\mu(\iota^*A'') \le s'(\mu-n+1)\,.\end{aligned} \tag{17}$$

By (14), and by the inequality $ns' \le s$ of (13) combined with the assumption $\mu \ge n-1$,

$$\begin{aligned}\mu_{\widetilde{H}_n}(A) = n!\frac{\mathsf{deg}(\iota^*A)}{s} &\le n!\frac{s'}{s}(\mu-n+1)\\ &\le (n-1)!(\mu-n+1) = \mu(\pi_n^*E^{[n]})\,.\end{aligned} \tag{18}$$

By Lemma 1.1, this shows that $E^{[n]}$ is semi-stable.

Let now E be stable and $\mu > n-1$. Then, by the stability of $E(-x_2-x_3-\dots-x_n)$, the inequality (16) is strict. Accordingly, the inequality in (17) and the first inequality in (18) are strict, except for if $s'=0$. However, for $s'=0$ the second inequality of (18) is strict, due to the assumption $\mu > n-1$. Hence, in any case, we have $\mu_{\widetilde{H}_n}(A) < \mu_{\widetilde{H}_n}(\pi_n^*E^{[n]})$ so that $E^{[n]}$ is stable by Lemma 1.1.

2.3. Proof of the main theorem for bundles of negative degree

In this subsection, we prove part (iii) and (iv) of Theorem 0.1. So, let $E \in \mathsf{VB}(C)$ be a semi-stable bundle with $\mu \le -1$. We argue by induction on n

that $\mu_{\widetilde{H}_n}(A) \leq \mu_{\widetilde{H}_n}(\pi_n^* E)$ for every $\mathfrak{S}_n$-equivariant subsheaf with

$$s := \operatorname{rank} A < nr = \operatorname{rank} E^{[n]}.$$

For $n = 1$, the assertion is trivial as $\pi_1^* E^{[1]} = E$. Let now $n \geq 2$. By Lemma 1.2 and Lemma 1.3, there is an $x \in C$ such that, for $\iota\colon C^{n-1} \hookrightarrow C^n$ with $\iota(t_2, \dots, t_n) = (x, t_2, \dots, t_n)$ the closed embedding with image $x \times C^{n-1}$, we have

$$\deg_{\widetilde{H}_n}(A) = n \deg_{\widetilde{H}_{n-1}}(\iota^* A)\,, \tag{19}$$

the rank of objects of diagram (12) remain unchanged after pull-back by ι, and the rows and columns of the diagram (12) remain exact after pull-back by ι. Noting that $\iota^*(\delta_n(1)) = \widetilde{H}_{n-1}$ and $\overline{\mathsf{pr}}_1 \circ \iota = \mathsf{id}_{C^{n-1}}$, the restricted diagram takes the form

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \iota^* A' & \longrightarrow & \iota^* A & \longrightarrow & \iota^* A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-\widetilde{H}_{n-1})^{\oplus r} & \longrightarrow & \iota^* \pi_n^* E^{[n]} & \longrightarrow & \pi_{n-1}^* E^{[n-1]} & \longrightarrow & 0 \end{array} \tag{20}$$

By the induction hypothesis, together with (10), we get

$$\mu_{\widetilde{H}_{n-1}}(\iota^* A'') \leq \mu_{\widetilde{H}_{n-1}}(\pi_{n-1}^* E^{[n-1]}) = (n-2)!(\mu - n + 2)\,. \tag{21}$$

Furthermore, the inclusion $\iota^* A' \hookrightarrow \mathcal{O}(-\widetilde{H}_{n-1})^{\oplus r}$ combined with (3) gives

$$\mu_{\widetilde{H}_{n-1}}(\iota^* A') \leq \mu_{\widetilde{H}_{n-1}}(\mathcal{O}(-\widetilde{H}_{n-1})^{\oplus r}) = -(n-1)!\,. \tag{22}$$

Combining (21) and (22), and replacing s' by $s - s''$, we get

$$\begin{aligned} \deg_{\widetilde{H}_{n-1}}(\iota^* A) &= s'' \mu_{\widetilde{H}_{n-1}}(\iota^* A'') + s' \mu_{\widetilde{H}_{n-1}}(\iota^* A') \\ &\leq (n-2)!\big(s''(\mu - n + 2) - (s - s'')(n-1)\big) \\ &= (n-2)! s''(\mu + 1) - (n-1)! s\,. \end{aligned}$$

Combining this with the assumption $\mu+1\le 0$ and the inequality $ns''\ge (n-1)s$ from (13) gives

$$n\,\mathsf{deg}_{\widetilde{H}_{n-1}}(\iota^*A)\le (n-2)!ns''(\mu+1)-n!s \le (n-2)!(n-1)s(\mu+1)-n!s = (n-1)!s(\mu-n+1)\,. \tag{23}$$

Together with (19) and (10), we get

$$\mu_{\widetilde{H}_n}(A)=\frac{n\,\mathsf{deg}_{\widetilde{H}_n}(\iota^*A)}{s}\le (n-1)!(\mu-n+1)=\mu_{\widetilde{H}_n}(\pi_n^*E^{[n]})\,. \tag{24}$$

Let now E be stable and $\mu<-1$. Proceeding as before by induction, we see that, if $s''<(n-1)r$, the inequality (21) is strict. Hence, the first inequality of (23) is strict too, which leads to a strict inequality in (24). If $s''=(n-1)r$, we have $ns''>(n-1)s$ since $s<nr$. It follows that, in this case, the second inequality of (23) is strict, which again leads to a strict inequality in (24).

3. The numerical conditions are sharp

In this section, we observe that the numerical conditions in Theorem 0.1 on the slope cannot be weakened. For this, we consider examples of (semi)-stable bundles on C with various values $\mu(E)\in[-1,n-1]$ such that $E^{[n]}$ is unstable.

Let $\ell\ge 0$, $x\in C$, and $L=\mathcal{O}_C(\ell\cdot x)$. Any non-zero section of L induces a non-zero section of $L^{[n]}$; see [11, Corollary of Prop. 1]. Hence, $\mathcal{O}_{X^{(n)}}$ is a subsheaf of $L^{[n]}$. For $0\le\ell<n-1$, we have $\mu(L^{[n]})<0$; see (11). Hence, in this case, the subsheaf $\mathcal{O}_{X^{(n)}}$ is destabilising. For $\ell=n-1$, we have $\mu(L^{[n]})=0$ and $L^{[n]}$ is properly semi-stable.

In a similar way, we get examples of higher rank and non-integer slope: Whenever $E\in\mathsf{VB}(C)$ has $\mu(E)<n-1$ and $h^0(E)>0$, the structure sheaf $\mathcal{O}_{C^{(n)}}$ is a destabilising subsheaf of $E^{[n]}$. For many curves C and many values of d and r such that $\mu(E)<n-1$, the existence of stable bundles with $h^0(E)>0$ is guaranteed by Brill–Noether theory.

The tautological bundles $L^{[n]}$ associated to $L=\mathcal{O}(-x)$, which have slope $\mu(L^{[n]})=-1$ for every $n\in\mathbb{N}$, can also be shown to be properly semi-stable as follows. We consider the bundle $L^{\boxplus n}:=\bigoplus_{i=1}^n \mathsf{pr}_i^*L$ on C^n equipped with the $\mathfrak{S}_n$-linearisation given by permutation of the direct summands. We have an isomorphism $L^{[n]}\cong\pi_{n*}^{\mathfrak{S}_n}L^{\boxplus n}$, where $\pi_{n*}^{\mathfrak{S}_n}L^{\boxplus n}$ are the invariants of $\pi_{n*}L^{\boxplus n}$

under the $\mathfrak{S}_n$-linearisation. Every morphism $s\colon L \hookrightarrow \mathcal{O}_C$ induces an $\mathfrak{S}_n$-equivariant embedding $L^{\boxtimes n} := \bigotimes_{i=1}^n \mathsf{pr}_i^* L \hookrightarrow L^{\boxplus n}$ with components

$$s^{\boxtimes i-1} \boxtimes \mathsf{id} \boxtimes s^{\boxtimes n-i} \colon L^{\boxtimes n} \to \mathsf{pr}_i^* L = \mathcal{O}_C^{\boxtimes i-1} \boxtimes L \boxtimes \mathcal{O}_C^{\boxtimes n-1} .$$

Since $\pi_{n*}^{\mathfrak{S}_n}$ is exact, we have an inclusion $\pi_{n*}^{\mathfrak{S}_n} L^{\boxtimes n} \hookrightarrow L^{[n]}$. Furthermore, $\pi_n^* \pi_{n*}^{\mathfrak{S}_n} L^{\boxtimes n} \cong L^{\boxtimes n}$. Hence,

$$\mu_{H_n}(\pi_{n*}^{\mathfrak{S}_n} L^{\boxtimes n}) = \frac{\mu_{\widetilde{H}_n}(L^{\boxtimes n})}{n!} = -1 = \mu_{H_n}(L^{[n]}) ,$$

which shows that $L^{[n]}$ is properly semi-stable.

Note, however, that it is still possible that there are stable tautological bundles with slope lying in the interval $[-1, 0]$. At least, there are stable tautological bundles on the boundary of this interval in the case $n = 2$: If L is of degree 1 but not isomorphic to $\mathcal{O}_C(x)$ for any $x \in C$, or of degree -1 but not isomorphic to $\mathcal{O}_C(-x)$ for any $x \in C$, the tautological bundle $L^{[2]}$ is stable (not only semi-stable) of slope -1 or 0; see [3].

References

[1] V. Ancona and G. Ottaviani, *Stability of special instanton bundles on* $\mathbf{P}^{2n+1}$, Trans. Amer. Math. Soc. **341** (1994), no. 2, 677–693.

[2] S. Basu and K. Dan, *Stability of secant bundles on the second symmetric power of curves*, Arch. Math. (Basel) **110** (2018), no. 3, 245–249.

[3] I. Biswas and D. S. Nagaraj, *Stability of secant bundles on second symmetric power of a curve*, in: Commutative Algebra and Algebraic Geometry (CAAG-2010), Vol. 17 of Ramanujan Math. Soc. Lect. Notes Ser., pp. 13–18, Ramanujan Math. Soc., Mysore, (2013).

[4] G. Bohnhorst and H. Spindler, *The stability of certain vector bundles on* $\mathbf{P}^n$, in: Complex Algebraic Varieties (Bayreuth, 1990), Vol. 1507 of Lecture Notes in Math., pp. 39–50, Springer, Berlin, (1992).

[5] K. Dan and S. Pal, *Semistability of certain bundles on second symmetric power of a curve*, J. Geom. Phys. **103** (2016), 37–42.

[6] L. Ein and R. Lazarsfeld, *The gonality conjecture on syzygies of algebraic curves of large degree*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 301–313.

[7] A. El Mazouni, F. Laytimi, and D. S. Nagaraj, *Secant bundles on second symmetric power of a curve*, J. Ramanujan Math. Soc. **26** (2011), no. 2, 181–194.

[8] D. Huybrechts and M. Lehn, The Geometry of Moduli Spaces of Sheaves, Cambridge Mathematical Library, Cambridge University Press, Cambridge, second edition (2010), ISBN 978-0-521-13420-0.

[9] A. Krug and J. V. Rennemo, *Some ways to reconstruct a sheaf from its tautological image on a Hilbert scheme of points*, arXiv:1808.05931, (2018).

[10] A. Marian, D. Oprea, and R. Pandharipande, *The combinatorics of Lehn's conjecture*, J. Math. Soc. Japan **71** (2019), no. 1, 299–308.

[11] A. Mattuck, *Secant bundles on symmetric products*, Amer. J. Math. **87** (1965), 779–797.

[12] E. Mistretta, *Some constructions around stability of vector bundles on projective varieties*, PhD thesis (2006).

[13] R. L. E. Schwarzenberger, *Vector bundles on the projective plane*, Proc. London Math. Soc. (3) **11** (1961), 623–640.

[14] R. L. E. Schwarzenberger, *The secant bundle of a projective variety*, Proc. London Math. Soc. (3) **14** (1964), 369–384.

[15] Z. L. Wang, *Tautological integrals on symmetric products of curves*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 8, 901–910.

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