Math. Res. Lett.
Volume 27, Number 6, 1697–1706, 2020

Holomorphic families of Fatou-Bieberbach domains and applications to Oka manifolds

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We construct holomorphically varying families of Fatou-Bieberbach domains with given centres in the complement of any compact polynomially convex subset K of $\mathbb{C}^n$ for $n > 1$. This provides a simple proof of the recent result of Y. Kusakabe to the effect that the complement $\mathbb{C}^n \setminus K$ of any polynomially convex subset K of $\mathbb{C}^n$ is an Oka manifold. The analogous result is obtained with $\mathbb{C}^n$ replaced by any Stein manifold with the density property.

1. Introduction

A *Fatou-Bieberbach domain* in $\mathbb{C}^n$ is a proper subdomain $\Omega \subsetneq \mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$. No such domains exists for $n = 1$, but they are plentiful for any $n > 1$; see the survey of this topic in [3, Chapter 4]. In particular, the basin of attraction of an attracting fixed point of a holomorphic automorphism of $\mathbb{C}^n$ (or in fact of any complex manifold) is biholomorphic to $\mathbb{C}^n$, cf. [11] and [3, Theorem 4.3.2]. Furthermore, for any compact polynomially convex set $K \subset \mathbb{C}^n$ $(n > 1)$ and point $p \in \mathbb{C}^n \setminus K$ there is a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n$ such that $p \in \Omega$ and $K \cap \Omega = \varnothing$; this is a special case of [5, Proposition 9] where the same result is shown with p replaced by any compact convex set.

In this note we prove the following more general result in this direction.

Theorem 1.1. *Let K be a compact polynomially convex set in $\mathbb{C}^n$ for some $n > 1$, L be a compact polynomially convex set in $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and $f : U \to \mathbb{C}^n$ be a holomorphic map on an open neighbourhood $U \subset \mathbb{C}^N$ of L such that $f(z) \in \mathbb{C}^n \setminus K$ for all $z \in L$. Then there are an open neighbourhood $V \subset U$ of L and a holomorphic map $F : V \times \mathbb{C}^n \to \mathbb{C}^n$ such that for every $z \in V$ we have that $F(z, 0) = f(z)$ and the map $F(z, \cdot\,) : \mathbb{C}^n \to \mathbb{C}^n \setminus K$ is injective. Hence, $\Omega_z := \{F(z, \zeta) : \zeta \in \mathbb{C}^n\}$ is a Fatou-Bieberbach domain in $\mathbb{C}^n \setminus K$ for each $z \in V$.*

A proof of this result based solely on Andersén-Lempert theory is given in Section 2; it also applies if $\mathbb{C}^N$ is replaced by an arbitrary Stein manifold, and also to variable fibres $K_z \subset \mathbb{C}^n$, $z \in L$, with polynomially convex graph (see Remark 2.2). For a convex parameter space $L \subset \mathbb{C}^N$ we prove the analogous result with $\mathbb{C}^n$ replaced by an arbitrary Stein manifold having the density property; see Theorem 3.1.

These two theorems immediately imply the following recent and very interesting result of Yuta Kusakabe.

Theorem 1.2. (Kusakabe, [9, Theorem 1.2 and Corollary 1.3].) *For any compact holomorphically convex subset K in a Stein manifold Y with the density property the complement $Y \setminus K$ is an Oka manifold. In particular, the complement $\mathbb{C}^n \setminus K$ of any compact polynomially convex set K in $\mathbb{C}^n$ for $n > 1$ is an Oka manifold.*

This is the first result in the literature which gives a large class of Oka domains in $\mathbb{C}^n$ for any $n > 1$, and it provides an affirmative answer to a long-standing problem. As noted in [9, Corollary 1.4], it follows from Theorem 1.2 and [4, Theorem 1.1] that for any compact polynomially convex set K in $\mathbb{C}^n$ ($n > 1$), the complement $\mathbb{C}^n \setminus K$ (like any n-dimensional Oka manifold) is the image of a strongly dominating holomorphic map $\mathbb{C}^n \to \mathbb{C}^n \setminus K$.

Recall that a complex manifold Y is said to be an *Oka manifold* if every holomorphic map from a neigbourhood of a compact (geometrically) convex set L in a Euclidean space $\mathbb{C}^N$ into Y is a uniform limit on L of entire maps $\mathbb{C}^N \to Y$ (see [3, Definition 5.4.1]; this is also called the *convex approximation property* and denoted CAP). By [3, Theorem 5.4.4], holomorphic maps $S \to Y$ from any reduced Stein space S to an Oka manifold Y satisfy all natural Oka-type properties. In his recent paper [8], Kusakabe showed that a complex manifold Y is Oka if (and only if) is satisfies the following condition:

($*$) For any compact convex set $L \subset \mathbb{C}^N$, open set $U \subset \mathbb{C}^N$ containing L, and holomorphic map $f : U \to Y$ there are an open set V with $L \subset V \subset U$ and a holomorphic map $F : V \times \mathbb{C}^m \to Y$ with $F(\cdot\,, 0) = f|_V$ such that

$$\frac{\partial}{\partial \zeta}\Big|_{\zeta=0} F(z, \zeta) : \mathbb{C}^m \to T_{f(z)}Y \quad \text{is surjective for every} \quad z \in V.$$

A map F with these properties is called a *dominating holomorphic spray over* $f|_V$. This is a restricted version of condition Ell_1 introduced by Gromov [6, p. 72] (see also [7]). In [8], Kusakabe used the technique of gluing sprays from [3, Sect. 5.9] to show that this condition implies CAP, so Y is an Oka manifold. Conversely, it has been known before that every Oka manifold satisfies condition Ell_1 (see [3, Corollary 8.8.7]).

Theorem 1.1 provides a very special dominating spray with values in $\mathbb{C}^n \setminus K$ over any given holomorphic map $f : L \to \mathbb{C}^n \setminus K$, thereby proving Theorem 1.2 in the case $Y = \mathbb{C}^n$. In exactly the same way, Theorem 3.1 implies the general case of Theorem 1.2.

Kusakabe also proved in [9, Theorem 4.2] that certain closed noncompact sets in Stein manifolds Y with the density property have Oka complements. He constructed a holomorphically varying family $f(z) \in \Omega_z \subset Y \setminus K$ $(z \in L)$ of nonautonomous basins with uniform bounds (i.e., basins of random sequences of automorphisms of Y which are uniformly attracting at $f(z) \in Y \setminus K$); these are elliptic manifolds as shown by Fornæss and Wold [2], hence Oka. When $Y = \mathbb{C}^n$, the domains Ω_z can be chosen Fatou-Bieberbach domains by using Theorem 1.1 with variable fibres (cf. Remark 2.2). Kusakabe's proof of [9, Theorem 4.2] can also be modified so as to provide a family of Fatou-Bieberbach domains in the general situation under consideration.

2. Proof of Theorem 1.1

We shall use some standard facts concerning polynomial convex sets; we refer the reader to the monograph by E. L. Stout [12]. Firstly, if $K_1, K_2 \subset \mathbb{C}^n$ is a pair of disjoint compact sets such that $K_1 \cup K_2$ is polynomially convex, then for any polynomially convex set $K_1' \subset K_1$ the union $K_1' \cup K_2$ is also polynomially convex. Secondly, every compact polynomially convex set $K \subset \mathbb{C}^n$ is the zero set of a nonnegative plurisubharmonic exhaustion function $\rho : \mathbb{C}^n \to [0, +\infty)$ which is strongly plurisubharmonic on $\mathbb{C}^n \setminus K$. Choosing a sequence $c_1 > c_2 > \cdots > 0$ with $\lim_{i\to\infty} c_i = 0$ and setting $K_i = \{\rho \le c_i\} \supset K$ yields a decreasing sequence of compact polynomially convex sets with K_{i+1} contained in the interior of K_i for every $i \in \mathbb{N}$.

Let f, K and L be as in the theorem. We replace L by a slightly bigger polynomially convex set (still denoted L) contained in U and such that $f(z) \in \mathbb{C}^n \setminus K$ for all $z \in L$. Choose a sequence $K_i \supset K$ as above, with K_1 chosen close enough to K such that $f(z) \in \mathbb{C}^n \setminus K_1$ for every $z \in L$. The compact set $L \times K_i \subset \mathbb{C}^{N+n}$ is polynomially convex for every $i \in \mathbb{N}$. Applying the change of coordinates $\psi(z, \zeta) = (z, \zeta - f(z))$ replaces f by the zero function, and for every $i \in \mathbb{N}$ the set

$$S_i = \psi(L \times K_i) \subset L \times \mathbb{C}^n \subset \mathbb{C}^{N+n} \qquad (1)$$

is polynomially convex and does not intersect $\mathbb{C}^N \times \{0\}^n$. Hence,

$$(L \times \{0\}^n) \cup S_1$$

is polynomially convex. Therefore, there is a small closed ball $B \subset \mathbb{C}^n$ centred at $0 \in \mathbb{C}^n$ such that $(L \times B) \cap S_1 = \varnothing$ and $(L \times B) \cup S_1$ is polynomially convex. Since $S_i \subset S_1$ is polynomially convex, it follows that $(L \times B) \cup S_i$ is polynomially convex for each $i \in \mathbb{N}$.

The following lemma will be used in the inductive construction.

Lemma 2.1. *(Assumptions as above.) Let $B' \subset \mathbb{C}^n$ be a closed ball centred at the origin with $B \subset (B')^\circ$. Then, there are an open neighbourhood $U' \subset U$ of L and a biholomorphic map $\Phi : U' \times \mathbb{C}^n \to U' \times \mathbb{C}^n$ of the form $\Phi(z,\zeta) = (z, \phi(z,\zeta))$ such that*

(a) *Φ approximates the identity map as closely as desired on $L \times B$ and $\Phi(z,0) = (z,0)$ for all $z \in U'$,*

(b) *$\Phi(S_1) \cap (L \times B') = \varnothing$, and*

(c) *the set $\Phi(S_2) \cup (L \times B')$ is polynomially convex.*

Proof. Choose $r > 1$ such that $rB = B'$. Let $\theta_t(z,\zeta) = (z, t\zeta)$ for $z \in \mathbb{C}^N$, $\zeta \in \mathbb{C}^n$, and $t \in \mathbb{C}$. We have that $(L \times B') \cap \theta_r(S_1) = \varnothing$ and

$$(L \times B') \cup \theta_r(S_1) = \theta_r((L \times B) \cup S_1) \text{ is polynomially convex.} \tag{2}$$

Consider the isotopy of biholomorphic maps ϕ_t on a neighbourhood of $(L \times B) \cup S_1$ in $\mathbb{C}^{N+n}$ for $t \in [1,r]$ which equals the identity map on a neighbourhood of $L \times B$ and equals θ_t on a neighbourhood of S_1. Note that $\theta_t(S_1)$ is disjoint from $L \times B$ and the union $(L \times B) \cup \theta_t(S_1)$ is polynomially convex for all $t \in [1,r]$ (since it is contained in $\theta_t((L \times B) \cup S_1) = (L \times tB) \cup \theta_t(S_1)$ which is polynomially convex). Hence, by the parametric Andersén-Lempert theorem (see Kutzschebauch [10] or [3, Theorem 4.12.3]) there is a holomorphic automorphism of $\mathbb{C}^{N+n}$ of the form $\Phi(z,\zeta) = (z, \phi(z,\zeta))$ which approximates the identity map on $L \times B$, it agrees with the identity on $L \times \{0\}^n$, and it approximates θ_r on S_1. Hence, conditions (a) and (b) in the lemma hold. Assuming that the approximations are close enough, we have $\Phi(S_2) \subset \theta_r(S_1)$. Note that $\Phi(S_2)$ is polynomially convex. In view of (2) it follows that $\Phi(S_2) \cup (L \times B')$ is polynomially convex as well which gives condition (c). □

Proof of Theorem 1.1. We apply the push-out method described in [3, Section 4.4]. Using Lemma 2.1 we inductively construct a decreasing sequence of open neighborhoods U_k of L and holomorphic automorphisms $\Phi_k(z,\zeta) = (z, \phi_k(z,\zeta))$ of $U_k \times \mathbb{C}^n$ such that, setting

$$\Phi^k = \Phi_k \circ \Phi_{k-1} \circ \cdots \circ \Phi_1 : U_k \times \mathbb{C}^n \to U_k \times \mathbb{C}^n,$$

the following conditions hold for every $k \in \mathbb{N}$.

(i) Φ_k approximates the identity map as closely as desired on $L \times kB$ and $\Phi_k(z,0) = (z,0)$ for all $z \in U_k$.

(ii) $\Phi^k(S_k) \cap (L \times (k+1)B) = \varnothing$. (Here, S_k is given by (1).)

(iii) The set $\Phi^k(S_{k+1}) \cup (L \times (k+1)B)$ is polynomially convex.

Indeed, Lemma 2.1 furnishes the first map Φ_1 with $B' = 2B$ and the sets $S_2 \subset S_1$; every subsequent step is of the same form by just increasing the indices. Assuming that the approximations are close enough, [3, Proposition 4.4.1 and Corollary 4.4.2] show that the limit $\Phi = \lim_{k\to\infty} \Phi^k$ exists uniformly on compacts on the domain

$$\begin{aligned} \Omega &= \big\{(z,\zeta) \in L \times \mathbb{C}^n : \Phi^k(z,\zeta) \text{ is a bounded sequence}\big\} \\ &= \bigcup_{k=1}^{\infty} (\Phi^k)^{-1}(L \times kB), \end{aligned}$$

and for every $z \in L$, $\Phi(z,\cdot\,)$ maps the fibre $\Omega_z = \{\zeta \in \mathbb{C}^n : (z,\zeta) \in \Omega\}$ biholomorphically onto $\mathbb{C}^n$. By condition (ii) the set $S = \psi(L \times K)$ does not intersect Ω (it has been pushed to infinity by the sequence Φ^k). Hence, the inverse map $\Phi^{-1}(z,\zeta) = (z, \varphi(z,\zeta))$ provides a holomorphic family of Fatou-Bieberbach maps $\varphi(z,\cdot\,) : \mathbb{C}^n \to \mathbb{C}^n$ ($z \in L$) such that $\varphi(z,0) = 0$ and its image does not intersect the set $K - f(z)$. The map $F(z,\zeta) = \varphi(z,\zeta) + f(z)$ for $z \in L$ and $\zeta \in \mathbb{C}^n$ satisfies the conclusion of the theorem. □

Remark 2.2. The above proof also applies in the case when the product $L \times K$ is replaced by a compact polynomially convex set $\widetilde{K} \subset \mathbb{C}^{N+n}$ projecting onto L whose fibres K_z ($z \in L$) depend on z. The conclusion remains the same, that is, given a holomorphic map $f : L \to \mathbb{C}^n$ with $f(z) \in \mathbb{C}^n \setminus K_z$ for all $z \in L$, there is a holomorphically variable family of Fatou-Bieberbach domains $f(z) \in \Omega_z \subset \mathbb{C}^n \setminus K_z$ for all $z \in L$.

3. Fatou-Bieberbach domains in Stein manifolds with the density property

In this section we give a version of Theorem 1.1 with $\mathbb{C}^n$ replaced by an arbitrary Stein manifold with the density property. (See Varolin [14] or [3, Definition 4.10.1] for this notion.) Every such manifold has dimension > 1. The following result is similar to Theorem 1.1, but we impose the extra condition that the set L is geometrically convex.

Theorem 3.1. *Let X be a Stein manifold with the density property, K be a compact holomorphically convex set in X, L be a compact convex set in $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and $f : U \to X$ be a holomorphic map on an open neighbourhood $U \subset \mathbb{C}^N$ of L such that $f(z) \in X \setminus K$ for all $z \in L$. Then there are a neighbourhood $V \subset U$ of L and a holomorphic map $F : V \times \mathbb{C}^n \to X$ with $n = \dim X$ such that for every $z \in V$ we have that $F(z, 0) = f(z)$ and the map $F(z, \cdot\,) : \mathbb{C}^n \to X \setminus K$ is injective.*

Hence, $\Omega_z := \{F(z, \zeta) : \zeta \in \mathbb{C}^n\} \subset X \setminus K$ is a Fatou-Bieberbach domain of the first kind (i.e., biholomorphic to $\mathbb{C}^n$) for each $z \in V$.

The proof of Theorem 3.1 depends on the following interpolation result for graphs. We denote by dist_X a distance function on X compatible with the manifold topology.

Lemma 3.2. *Let X, K, L, U and f be as above, and let $z_0 \in L$ be arbitrary. Then for any $\epsilon > 0$ there exist a neighbourhood $V \subset U$ of L and a fibred holomorphic automorphism $\phi(z, x) = (z, \varphi(z, x))$ of $V \times X$ such that $\phi(z, f(z)) = (z, f(z_0))$ for all $z \in V$, and $\mathrm{dist}_X(\varphi(z, x), x) < \epsilon$ for all $z \in L$ and $x \in K$.*

Proof. We may assume that $z_0 = 0 \in \mathbb{C}^N$. Let $V_1, \ldots, V_m$ be complete holomorphic vector fields on X such that $V_1(x), \ldots, V_m(x)$ span the tangent space $T_x X$ for all $x \in X$ (such exist by [3, Proposition 5.6.23] since X is Stein and has the density property). Let $\psi_{1,s}, \ldots, \psi_{m,s}$ denote their respective flows, $s \in \mathbb{C}$. Consider the map $\Psi : \mathbb{C}^m \times X \to X$ defined for $s = (s_1, \ldots, s_m) \in \mathbb{C}^m$ and $x \in X$ by

$$\Psi(s_1, \ldots, s_m, x) = \psi_{m,s_m} \circ \cdots \circ \psi_{1,s_1}(x).$$

Note that $\Psi(s, \cdot\,) \in \mathrm{Aut}(X)$ for every $s \in \mathbb{C}^m$. Then the partial differential $\partial_s|_{s=0} \Psi(s, f(0))$ has maximal rank $n = \dim X$, so there exists an n-dimensional linear subspace $\Lambda \subset \mathbb{C}^m$ on which this differential has rank n.

We may assume that $\Lambda = \mathbb{C}^n \times \{0\}^{m-n}$. Write $s = (s', s'')$, with $s' \in \mathbb{C}^n$ and $s'' \in \mathbb{C}^{m-n}$, and set $\widetilde{\Psi}_{s'} := \Psi((s', 0''), \cdot\,) \in \mathrm{Aut}(X)$. It follows that there exists $\delta > 0$ such that the map $s' \mapsto \widetilde{\Psi}_{s'}(f(0))$ is an embedding of the open δ-ball $B_\delta \subset \mathbb{C}^n$ centred at $0 \in \mathbb{C}^n$ onto an open neighbourhood of $f(0) \in X$.

We replace L by a slightly larger convex set $L' \subset U$ with $L \subset (L')^\circ$ without changing the notation. We also choose a compact holomorphically convex set $K' \subset X$ containing K in its interior and such that $f(z) \in X \setminus K'$ for all $z \in L$. Set $f_t(z) = f(t \cdot z)$ for $z \in L$ and $t \in [0,1]$. Consider the isotopy $\phi_t(z,x)$ defined to be the identity near $L \times K'$ and $\phi_t(z, f(z)) = (z, f_{1-t}(z))$, $0 \le t \le 1$ on the graph $Z := \{(z, f(z)) : z \in L\} \subset \mathbb{C}^N \times X$. The image of ϕ_t is the disjoint union of $L \times K'$ and the holomorphic graph of f_{1-t} over L, so it is holomorphically convex in $\mathbb{C}^N \times X$. By using [3, Proposition 3.3.2] (a fibred version of the tubular neighbourhood theorem for Stein manifolds) along with the Oka-Grauert principle we can extend ϕ_t to a fibred isotopy of injective holomorphic maps on an open neighbourhood of Z in $\mathbb{C}^N \times X$. Since X has the density property, given $\eta > 0$ there is a fibred holomorphic automorphism $\tilde{\phi}(z,x) = (z, \tilde{\varphi}(z,x))$ of $L \times X$ such that $\mathrm{dist}_X(\tilde{\varphi}(z, f_1(z)), f(0)) < \eta$ for $z \in L$ and $\mathrm{dist}_X(\tilde{\varphi}(z,x), x) < \epsilon/2$ for $z \in L$ and $x \in K'$ (see [10] and [3, Theorems 4.10.5 and 4.12.3]). Note that $f_1 = f$. If $\eta > 0$ is chosen small enough, there exists for each $z \in L$ a unique point $\lambda(z) \in B_\delta \subset \mathbb{C}^n$ such that $\widetilde{\Psi}_{\lambda(z)}(f(0)) = \tilde{\varphi}(z, f(z))$. The fibred holomorphic automorphism

$$\phi(z,x) = \left(z, \widetilde{\Psi}^{-1}_{\lambda(z)}(\tilde{\varphi}(z,x))\right), \quad z \in L,\ x \in X$$

then satisfies the lemma provided $\eta > 0$ is chosen small enough. □

We will also need the following basic result which we include lacking a reference. (The existence of a Fatou-Bierberbach domain of the first kind containing a point $p \in X$ was proved by Varolin [13], but this is not sufficient for our purpose.)

Lemma 3.3. *Let X be a Stein manifold with the density property, let $K \subset X$ be a holomorphically convex compact set, and let $p \in X \setminus K$. Then there exists a Fatou-Bieberbach domain $\Omega \subset X \setminus K$ of the first kind such that $p \in \Omega$.*

Proof. Let K' be a holomorphically convex compact set in X containing K in its interior and such that $p \notin K'$. Choose a local holomorphic coordinate $\phi : U_p \to \mathbb{C}^n$ on X such that $p \in U_p \subset X \setminus K'$ and $\phi(p) = 0$. Denote by B_δ the open ball of radius δ centred at the origin in $\mathbb{C}^n$. Let $\delta > 0$ be chosen small enough such that $\overline{B_\delta} \subset \phi(U_p)$ and $\phi^{-1}(\overline{B_\delta}) \cup K'$ is holomorphically convex in X. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be the automorphism $F(z_1, \ldots, z_n) = (\frac{z_1}{2}, \ldots, \frac{z_n}{2})$. Since X has the density property, we can apply [3, Theorem 4.10.5] to approximate the map which equals $\phi^{-1} \circ F \circ \phi$ on a neighbourhood of $\phi^{-1}(\overline{B_\delta})$ and equals the identity map on a neighbourhood of K' to obtain a sequence $G_j \in \mathrm{Aut}(X)$ such that for any $k \in \mathbb{N}$ we have that $G_k \circ \cdots \circ G_1(K) \subset K'$, and setting $F_j = \phi \circ G_j \circ \phi^{-1}$ we have that

$$s \cdot \|z\| \leq \|F_j(z)\| \leq r \cdot \|z\|, \quad j \in \mathbb{N}$$

on B_δ, with $0 < s < \frac{1}{2} < r < 1$ and $r^2 < s < 1$. Now, following [15, proof of Theorem 4] we have that the abstract basin of attraction, or the tail space $\widetilde{\Omega}$ (see [1]) associated to $\{F_j\}_{j\in\mathbb{N}}$, is biholomorphic to $\mathbb{C}^n$, and the basin of attraction Ω of the sequence $\{G_j\}_{j\in\mathbb{N}}$ is biholomorphic to $\widetilde{\Omega}$. □

Proof of Theorem 3.1. This is an immediate consequence of Lemmas 3.2 and 3.3. □

Acknowledgements

The first named author is supported by the research program P1-0291 and grant J1-9104 from ARRS, Republic of Slovenia, and by the Stefan Bergman Prize 2019 from the American Mathematical Society. The second named author is supported by the RCN grant 240569 from Norway.

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Received May 25, 2020
Accepted September 30, 2020