

The Chow cohomology of affine toric varieties

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We study the Fulton-Macpherson Chow cohomology of affine toric varieties. In particular, we prove that the Chow cohomology vanishes in positive degree. We prove an analogous result for the operational K -theory defined by Anderson and Payne.

1. Introduction

Toric varieties are a rich source of examples for studying delicate questions in both ordinary and equivariant intersection theory. A classical result is that both the Chow and cohomology rings of a smooth projective toric variety are isomorphic to the Stanley-Reisner ring of the associated fan. For singular varieties, however, there is no natural intersection product on algebraic cycles. In this case a natural ring to consider is Fulton and Macpherson's *Chow cohomology ring*.

If X is a scheme then an element c of the Chow cohomology ring $A_{\text{op}}^*(X)$ is a collection of operations on the Chow groups of any scheme X' mapping to X which satisfy appropriate compatibility conditions. If X is smooth then a basic result states that the Chow cohomology ring can be identified with the usual Chow ring. Moreover, a deep result of Fulton and Sturmfels [FS] states that the Chow cohomology ring of a complete toric variety is the ring of Minkowski weights of the associated fan. However, for arbitrary singular schemes, including non-proper toric varieties, little is known about the Chow cohomology ring.

In a situation where a linear algebraic group acts on a scheme there is a corresponding equivariant Chow cohomology ring which was defined in [EG]. In a fundamental paper Payne [Pay] calculated the T -equivariant Chow cohomology ring of an arbitrary toric variety and identified it with the ring of piecewise polynomial functions on the corresponding fan. However, as noted by Katz and Payne [KP], the relationship between equivariant and

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ordinary Chow cohomology is quite subtle and there is no immediate way to use Payne's result to obtain information about the non-equivariant Chow cohomology rings of non-complete toric varieties.

The main result of this paper, Theorem 4.1, is to show that the (non-equivariant) Chow cohomology ring of any affine toric variety vanishes in degree greater than zero. By contrast, the Chow homology groups of an affine toric variety need not vanish and can have both torsion and non-torsion elements.

Although their coordinate rings have prime cycles which are not rationally equivalent to 0, our result shows that the Chow cohomology rings of affine toric varieties behave like that of affine spaces. As observed by Gubeladze [Gub], singular affine toric varieties constitute a class of intuitively contractible varieties naturally generalizing affine spaces. In particular, all vector bundles are trivial so the Grothendieck ring is isomorphic to \mathbb{Z} . In a parallel result we also prove, Theorem 5.1, that the operational K -theory ring defined by Anderson and Payne in [AP] is also isomorphic to \mathbb{Z} . Again, the Grothendieck group of coherent sheaves of an affine toric variety need not be isomorphic to \mathbb{Z} ; see Example 5.7.

Our Chow cohomology result fits into a framework developed in [ES]. There, the first author and Satriano show that for spaces, such as toric varieties, which are good moduli spaces of smooth Artin stacks, elements of the Chow cohomology ring have canonical representatives as *topologically strong* cycles on the corresponding stack.

Based on the results of [ES] the authors of the present paper have a conjectural description of the Chow cohomology ring of an arbitrary toric variety in terms of strong complete intersection cycles on the corresponding canonical stack. For precise statements see [Ric, Conjectures 5.2.8–5.2.9]. As noted in [Ric] our results here imply that the conjectures hold for affine toric varieties.

1.1. Outline of the proof

The proof of Theorem 4.1 uses Kimura's fundamental exact sequence for Chow cohomology [Kim] and the stack-theoretic results proved in [ES].

If $X(\sigma)$ is the affine toric variety defined by a cone σ generated by primitive vectors u_1, \dots, u_k , then we denote by σ^* the fan obtained by star subdivision with respect to the ray generated by $v = u_1 + \dots + u_k$. The map of toric varieties $X(\sigma^*) \rightarrow X(\sigma)$ is a blowup with exceptional divisor $E = V(\rho_v)$. Using Kimura's exact sequence for Chow cohomology, we show that in order to prove that $A_{\text{op}}^k(X(\sigma)) = 0$ for $k > 0$, it suffices to prove the

restriction map $A_{\text{op}}^k(X(\sigma^*)) \rightarrow A_{\text{op}}^k(E)$ is injective. Here $A_{\text{op}}^k(X(\sigma))$ denotes the k -th Chow cohomology group. See Section 3 for a precise definition.

Denote by \mathcal{E} and $\mathcal{X}(\sigma^*)$ the corresponding toric canonical stacks. Using the results of [ES] we can show that $A^k(X(\sigma^*))_{\mathbb{Q}} \rightarrow A^k(E)_{\mathbb{Q}}$ is injective by showing that the corresponding map of Chow rings of smooth stacks $A^k(\mathcal{X}(\sigma^*)) \rightarrow A^k(\mathcal{E})$ is injective. By computing the Chow rings of toric canonical stacks we show directly that the corresponding map is in fact an isomorphism. This shows that $A^k(X(\sigma))_{\mathbb{Q}} = 0$ for $k > 0$. The proof concludes by using results of Hausel and Sturmfels [HS] to show that the Chow cohomology of any affine toric variety is torsion free.

The proof for operational K -theory uses a similar strategy. Here we combine Anderson and Payne's [AP] results with Theorem 5.4 to obtain the desired result.

Since this paper was first submitted, the authors gave a second proof of Theorems 4.1 and 5.1 using different techniques. This appears in [ERi].

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2. Background on toric varieties and canonical stacks

2.1. Basic toric terminology

Following [Ful2] or [CLS], let \mathbb{K} be an algebraically closed field, N and M be rank n dual lattices over \mathbb{Z} (which will be the lattices of 1-parameter subgroups and characters of a torus), and let σ be a strongly convex, rational, and polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. From σ , we can construct the dual cone σ^{\vee} in $M_{\mathbb{R}}$ which satisfies the property that $\sigma^{\vee} \cap M$ is a semigroup. The **affine toric variety** $X(\sigma)$ is defined as: $X(\sigma) := \text{Spec}(k[(\sigma^{\vee} \cap M)])$.

A fan Δ in $N_{\mathbb{R}}$ is a collection of cones satisfying the following two properties: each face of a cone is in Δ , and the intersection of every two cones is a face of each. Each cone in Δ generates an affine toric variety which glue together to define a normal toric variety $X(\Delta)$ with an action of the torus $T_N = X(\{0\})$. Conversely, it is well known that every normal variety containing a dense torus whose action extends to an action on the total variety arises from a fan.

The orbit-cone correspondence associates to each cone σ in a fan Δ a torus orbit $O(\sigma)$ of dimension $n - \dim \sigma$. We denote the closure of $O(\sigma)$ by $V(\sigma)$. The orbit closure is a finite union of orbits corresponding to the faces of σ .

Torus equivariant morphisms of toric varieties $X(\Delta') \rightarrow X(\Delta)$ correspond to morphisms of lattices $\phi: N' \rightarrow N$ which send cones of Δ' into cones of Δ . Various algebro-geometric properties of the morphism can be understood in terms of conditions on the associated map of lattices. In particular, properness is equivalent to the condition that $\phi^{-1}(|\Delta|) = |\Delta'|$. If ϕ is the identity map, then we say Δ' is a **refinement** of Δ . In what follows, we will be interested in the inverse image of a closed subscheme made up of orbit closures, especially in the case that Δ' is a refinement of Δ .

Theorem 2.1. [CLS, Lemma 3.3.21 and Proposition 11.1.10] *With notation as above, suppose σ is a cone in Δ with orbit closure $V(\sigma) \subset X(\Delta)$. If $\sigma_1, \dots, \sigma_k$ denote the minimal cones of Δ' such that σ is minimal over $\phi(\sigma_i)$ then the irreducible decomposition of the inverse image of $V(\sigma)$ is:*

$$\varphi^{-1}(V(\sigma)) = \bigcup_{i=1}^k V(\sigma_i).$$

In particular, in the case of a refinement, σ_i can be characterized as the minimal cones intersecting the relative interior of σ .

There is a special refinement which will be important in the remainder of the paper - the star subdivision. Geometrically, this corresponds to the blowup of $X(\Delta)$ along a 0-dimensional subscheme supported at a distinguished point γ_σ . Specifically, given a cone σ , let u_1, \dots, u_k be the primitive generators of the rays of σ , and let $v(\sigma) = \sum_{i=1}^k u_i$. The **star subdivision** of σ , see [CLS, Definition 3.3.13], is the fan denoted σ^* whose maximal cones are all of the form $\text{Cone}(\mu, v(\sigma))$ where $\mu \subset \sigma$ is a facet.

As a final remark, recall that a cone is **smooth** if the minimal generators of the rays form part of an integral basis of N , and that a cone is **simplicial** if the minimal generators are linearly independent over \mathbb{R} . A fan is smooth or simplicial if each of its cones are smooth or simplicial respectively. It is well known that the toric variety $X(\Delta)$ is smooth if and only if each cone in Δ is smooth.

2.2. Cox construction of a toric variety

The Cox construction produces any toric variety as the good quotient of an open subscheme of affine space by the action of a diagonalizable group G . Following [CLS, Chapter 5], we briefly recall the salient details of the construction. Given a full-dimensional fan Δ on a lattice N , the total coordinate ring of $X(\Delta)$ is the polynomial ring $k[x_\rho : \rho \in \Delta(1)]$ freely generated by the variables x_ρ . Suppose $n = |\Delta(1)|$ such that $\mathbb{A}^n = \text{Spec}(k[x_\rho])$, then for each cone $\sigma \in \Delta$, define the monomial:

$$x^\sigma = \prod x_{\rho_i} \text{ where } \{\rho_i\} \text{ are the rays not contained in } \sigma.$$

Furthermore, define $B(\Delta) = (x^\sigma)_{\sigma \in \Delta}$ to be the ideal generated by these monomials. The exceptional set of Δ is the following closed subscheme of \mathbb{A}^n :

$$Z(\Delta) = \mathbb{V}(B(\Delta)) \subset \mathbb{A}^n.$$

The closed set $Z(\Delta)$ has an explicit, combinatorial description. Recall that a **primitive collection** of rays is a collection of rays that satisfies the following two properties: the collection is not contained in a single cone of Δ , but every proper subset is contained in a cone of Δ . For each primitive collection of rays $\{\rho_i\}$, we can generate a linear subspace given by $V((x_{\rho_i})_{i \in I})$.

Proposition 2.2. [CLS, Proposition 5.1.6] *The irreducible components of $Z(\Delta)$ are in bijection with the linear subspaces $\mathbb{V}((x_{\rho_i})_{i \in I})$ where I is a primitive collection of rays.*

Given a fan Δ , let Σ_Δ denote the fan generated by cones $\bar{\sigma} = \text{Cone}(e_\rho : \rho \in \sigma(1)) \subset \mathbb{R}^{|\Delta(1)|}$ where σ runs through the cones of Δ .

Proposition 2.3. [CLS, Proposition 5.1.9] (a) $X(\Sigma_\Delta) = \mathbb{A}^n \setminus Z(\Delta)$.
 (b) *The map $e_\rho \mapsto u_\rho$ defines a map of lattices $\mathbb{Z}^{\Sigma(1)} \rightarrow N$ that is compatible with the fans $\Sigma_\Delta \subset \mathbb{R}^{\Delta(1)}$ and $\Delta \subset N_\mathbb{R}$.*

Remark 2.4. Note that if σ is a cone of Δ , then the orbit closure $V(\sigma)$ is the image of the linear subspace $\mathbb{V}(\{x_\rho\}_{\rho \in \sigma(1)})$.

The diagonalizable group G is constructed as follows: suppose $\dim(\Delta) = k$, then the morphism of lattices $\mathbb{Z}^n \rightarrow \mathbb{Z}^k$ defined by sending the canonical basis vectors to the minimal generators of rays of Δ has finite cokernel; hence, the dual map is injective, and we set L to be the cokernel of this

map. The group G is defined to be $\text{Hom}(L, \mathbb{G}_m)$, the group of characters of L , which injects into the torus of \mathbb{A}^n .

Theorem 2.5. *[CLS, Theorem 5.1.11(a)] Under the above notation, the toric map $(\mathbb{A}^n \setminus Z(\Delta)) \rightarrow X(\Delta)$ identifies $X(\Delta)$ as the good quotient $(\mathbb{A}^n \setminus Z(\Delta))/G$.*

2.3. Canonical toric stacks

Given a full-dimensional fan Δ on a lattice N and its associated toric variety $X(\Delta)$, the canonical toric stack is a smooth quotient stack possessing a special relationship with $X(\Delta)$. Following [GS, Section 5], there are two equivalent definitions for the canonical stack. The first is given in terms of the Cox construction, and the second constructs $\mathcal{X}(\Delta)$ as a toric stack arising from a canonical stacky fan. The theory of stacky fans is worked out in [GS] and provides a combinatorial gadget for toric stacks analogous to fans for toric varieties. A stacky fan is a pair $(\Delta, \beta : N \rightarrow L)$ where Δ is a lattice on N and β is a morphism of lattices with finite cokernel, and β induces a surjective morphism of tori $T_N \rightarrow T_L$ with kernel G_β .

Following [GS, Section 3], toric morphisms between toric stacks arising from stacky fans $(\Delta', N' \rightarrow L')$, $(\Delta, N \rightarrow L)$ correspond to morphisms of lattices which are compatible with the stacky fans: specifically, $\phi : N' \rightarrow N$ and $\Phi : L' \rightarrow L$ are morphisms of lattices such that for every cone $\sigma \in \Delta'$, $\phi(\sigma)$ is contained in a cone of Δ and the following diagram commutes:

$$\begin{array}{ccc} N' & \xrightarrow{\phi} & N \\ \downarrow \beta' & & \downarrow \beta \\ L' & \xrightarrow{\Phi} & L \end{array}$$

Definition 2.1. The canonical stack of $X(\Delta)$ will be denoted by $\mathcal{X}(\Delta)$ and is equipped with a toric morphism $\mathcal{X}(\Delta) \rightarrow X(\Delta)$; it can be constructed in either of the following ways:

- 1) The Cox construction of a toric variety produces an open subscheme $U \subset \mathbb{A}^{|\Delta(1)|}$ and a diagonalizable group G such that $X(\Delta) \cong U/G$, [CLS, Theorem 5.1.11]. The canonical stack of $X(\Delta)$ is the quotient stack $[U/G]$ with morphism $[U/G] \rightarrow X(\Delta)$.
- 2) Equivalently, let $(\Sigma_\Delta, \beta : \mathbb{Z}^{|\Delta(1)|} \rightarrow N)$ be the stacky fan where β is defined by sending the canonical basis vectors to the primitive generators

of Δ , and Σ_Δ is the canonical fan defined in the previous section. The toric stack associated to this stacky fan is the canonical stack $\mathcal{X}(\Delta)$.

Furthermore, the toric morphism $\mathcal{X}(\Delta) \rightarrow X(\Delta)$ is easily constructed from the following morphism of stacky fans:

$$\begin{array}{ccc} \mathbb{Z}^{|\Delta(1)|} & \longrightarrow & N \\ \downarrow & & \parallel \\ N & \xlongequal{\quad} & N \end{array}$$

Recall [Alp, Definition 4.1], that a morphism from an Artin stack to an algebraic space is a **good moduli space morphism** if it is cohomologically affine and Stein. In the context of quotient stacks, this is a natural generalization of good quotients. As a consequence of [GS, Theorem 6.3], the following holds:

Theorem 2.6. [GS, Example 6.23] *The canonical stack morphism $\mathcal{X}(\Delta) \rightarrow X(\Delta)$ is a good moduli space morphism.*

Example 2.7. Let σ denote the cone generated by $\{(1, 0, 1), (0, -1, 1), (-1, 0, 1), (0, 1, 1)\}$ in \mathbb{R}^3 , then $X(\sigma)$ is a 3-dimensional, affine, singular toric variety with four rays. Let $\beta : \mathbb{Z}^4 \rightarrow \mathbb{Z}^3$ be defined by sending the canonical basis vectors to the primitive generators of σ ; note, the preimage of σ under β is the cone defining \mathbb{A}^4 . Furthermore, the cokernel of the dual of β is $\mathbb{Z}^4 \rightarrow \mathbb{Z}^3$ defined as the matrix $[1, -1, 1, -1]$; hence, if we apply $D(-) := \text{Hom}(-, \mathbb{G}_m)$ to this cokernel, we see that $G_\beta = \mathbb{G}_m$ and defines an action of \mathbb{A}^4 with weights $(1, -1, 1, -1)$. Thus, the canonical stack of $X(\sigma)$ is $\mathcal{X}(\sigma) = [\mathbb{A}^4/\mathbb{G}_m]$.

3. Chow cohomology of algebraic schemes and stacks

Let X be a scheme (or more generally an algebraic space) defined over a field. Following [Ful1], let $A_k(X)$ denote the group of dimension k cycle classes modulo rational equivalence. If X is equidimensional, let $A^k(X)$ denote the group of codimension k cycle classes modulo rational equivalence. In addition if X is smooth, then the intersection product on $A_k(X)$ as constructed in [Ful1, Chapter 6.1] makes $A^*(X)$ into a commutative, graded ring. For general schemes, without any assumptions on non-singularity, [Ful1, Chapter 17] constructs a graded ring $A_{\text{op}}^*(X) := \bigoplus_{k \geq 0} A_{\text{op}}^k(X)$ defined as follows:

an element $c \in A_{\text{op}}^k(X)$ is a collection of homomorphisms of groups:

$$c_g^{(k)} : A_p X' \rightarrow A_{p-k} X'$$

for every morphism $g : X' \rightarrow X$ which are compatible with respect to proper pushforward, and flat and l.c.i. pullbacks (see [Ful1, Definitions 17.1 and 17.3]). If $\alpha \in A_p X'$ then following [Ful1, Notation p.320] we denote $c^{(k)}(\alpha)$ by $c \cap \alpha$. The product is given by composition and turns $A_{\text{op}}^*(X)$ into a unital, associative, graded ring called the **Chow cohomology ring** of X . Moreover, if X has a resolution of singularities (e.g. if X is a toric variety) then $A_{\text{op}}^*(X)$ is known to be commutative. If X is smooth, then [Ful1, Corollary 17.4] proves that the Poincaré-Duality map $A_{\text{op}}^k(X) \rightarrow A^k(X) = A_{n-k}(X)$, $c \mapsto c \cap [X]$, is an isomorphism of rings where the intersection product agrees with the product given by composition.

Remark 3.1. In [Ful1, Chapter 17] the Chow cohomology ring is denoted $A^*(X)$ without the inclusion of the subscript op , regardless if X is smooth or not. We will deviate from this convention to emphasize the distinction between operational classes in the smooth and non-smooth cases.

For smooth quotient stacks $\mathcal{X} = [Z/G]$, the Chow ring $A^*(\mathcal{X})$ is identified with the G -equivariant Chow ring $A_G^*(Z)$ by [EG]. Furthermore, if $\pi : \mathcal{X} \rightarrow X$ is any morphism between a stack and an algebraic space¹, [ES, Proposition 2.10] proves that there is a pullback map $\pi^* : A_{\text{op}}^*(X) \rightarrow A^*(\mathcal{X})$, $c \mapsto c \cap [\mathcal{X}]$. In this paper we make essential use of the following result.

Theorem 3.2. [ES, Theorem 1.1] *If \mathcal{X} is a smooth, connected, properly stable Artin stack with good moduli space $\pi : \mathcal{X} \rightarrow X$, then the pullback π^* is injective after tensoring with \mathbb{Q} .*

Since $\mathcal{X}(\Delta) \rightarrow X(\Delta)$ is a good moduli space morphism, we conclude that $A_{\text{op}}^*(X(\Delta))_{\mathbb{Q}}$ injects into $A^*(\mathcal{X}(\Delta))_{\mathbb{Q}}$. If Δ is simplicial, then $\mathcal{X}(\Delta)$ is Deligne-Mumford and the inclusion is an isomorphism.

3.1. Chow cohomology of toric varieties

For toric varieties, the generators and relations of the Chow groups $A_k(X(\Delta))$ can be explicitly computed. By [Ful2, Chapter 5.1], $A_k(X(\Delta))$ is generated

¹Because the good moduli space of a quotient stack need not be a scheme, [ES] considers maps from stacks to algebraic spaces. Since the moduli space of a toric stack is always a toric variety, this generality is not strictly needed here.

by the classes $[V(\sigma)]$ where σ has codimension k . Furthermore, the group of relations between these generators is generated by all relations of the form:

$$[\text{div}(x^u)] = \sum_{\sigma} \langle u, n_{\sigma, \tau} \rangle [V(\sigma)],$$

where τ runs over all cones of codimension $k + 1$ which are a face of σ , u runs over a generating set of $M(\tau)$, and $n_{\sigma, \tau}$ is a lattice point of σ whose image generates the one-dimensional lattice N_{σ}/N_{τ} . In the case that $X(\Delta)$ is smooth, the above description can be used to compute $A_{\text{op}}^*(X(\Delta))$ since $A_{\text{op}}^k(X(\Delta)) = A_{n-k}(X(\Delta))$. However when $X(\Delta)$ is not smooth, there are only a few known techniques to compute $A_{\text{op}}^*(X(\Delta))$. In particular, for complete toric varieties, the Chow cohomology ring can also be derived from the above description of Chow groups by the following theorem:

Theorem 3.3. *[FS, Proposition 1.4, Theorem 2.1] If X is a complete toric variety, then $A_{\text{op}}^k(X) \cong \text{Hom}(A_k(X), \mathbb{Z})$, and the Chow cohomology is also isomorphic to the ring of Minkowski weights on Δ .*

However, for non-complete toric varieties this isomorphism fails as the following example demonstrates.

Example 3.4. Let σ denote the cone generated by $\{(1, 0, 1), (0, -1, 1), (-1, 0, 1), (0, 1, 1)\}$ in \mathbb{R}^3 as seen above. One can compute that $A_2(X(\sigma)) = \mathbb{Z}/2 \oplus \mathbb{Z}$ which does admit non-trivial morphisms $A_2(X(\sigma)) \rightarrow \mathbb{Z}$. However, by Theorem 4.1, $A_{\text{op}}^2(X(\sigma)) = 0$.

3.1.1. Computing Chow cohomology: Kimura’s exact sequence.

There is a second technique to compute the Chow cohomology of a toric variety using Kimura’s exact sequence. This sequence injects $A_{\text{op}}^*(X(\Delta))$ into the Chow cohomology of any envelope of $X(\Delta)$. Recall, a morphism $X' \rightarrow X$ is an envelope if it is proper and for every closed subvariety $V \subset X$, there exists a closed subvariety $V' \subset X'$ mapping onto V such that the restriction $V' \rightarrow V$ is birational.

Proposition 3.5. *[Kim, Theorem 2.3] Let $X' \rightarrow X$ be an envelope which is an isomorphism outside of $S \subset X$ and $E \subset X'$, then*

$$(1) \quad 0 \rightarrow A_{\text{op}}^*(X) \rightarrow A_{\text{op}}^*(X') \oplus A_{\text{op}}^*(S) \rightarrow A_{\text{op}}^*(E)$$

is exact.

If $X' \rightarrow X$ is proper and surjective, then the above sequence is exact after tensoring with \mathbb{Q} . Payne proved [Pay, Lemma 1] that any proper birational, toric morphism is an envelope. In particular, if Δ' is any refinement of Δ , then $A_{\text{op}}^*(X(\Delta))$ injects into $A_{\text{op}}^*(X(\Delta'))$ [Kim, Lemma 2.1]. We make essential use of this fact in our proof of Theorem 4.1.

3.2. Chow rings of canonical toric stacks

The Chow ring of $\mathcal{X}(\Delta)$ admits a simple presentation using equivariant Chow groups. Following [CE, Section 2], let $n = |\Delta(1)|$, $V = \mathbb{A}^n$, $U = V \setminus Z(\Delta)$, G the diagonalizable group such that $\mathcal{X}(\Delta) = [U/G]$ and $X(G)$ the character group of G . The excision exact sequence ([Ful1, Prop. 1.8]) for equivariant Chow groups implies $A^*(\mathcal{X}(\Delta))$ is a quotient of $A_G^*(V) = A_G^*(pt)$; the latter $A_G^*(pt)$ will be denoted by A_G^* . By [EG], $A_G^* \cong \mathbb{Z}[X(G)]$ where the latter is the symmetric algebra of the character group of G . By construction of $\mathcal{X}(\Delta)$, the action of G on V is faithful; hence, G injects into \mathbb{G}_m^n and induces a surjection of character groups $X(\mathbb{G}_m^n) \rightarrow X(G)$. This surjection further induces a surjection of equivariant Chow rings $A_{\mathbb{G}_m^n}^* \rightarrow A_G^*$ leading us to the following definition:

Definition 3.1. The **linear equivalence ideal** of Δ is defined to be:

$$L(\Delta) := \ker(A_{\mathbb{G}_m^n}^* \rightarrow A_G^*).$$

Equivalently, it is the linear ideal generated by the relations among the image of $X(\mathbb{G}_m^n)$ in $X(G)$. Hence, $A_G^* = A_{\mathbb{G}_m^n}^*/L(\Delta)$.

Suppose L_1, \dots, L_k are the irreducible components of $Z(\Delta)$, then by [CE, Propositions 2.1 and 2.2], $A_G^*(U)$ is the quotient of $A_{\mathbb{G}_m^n}^*$ modulo the linear equivalence ideal and the ideal generated by the equivariant fundamental classes of L_i .

Definition 3.2. The **Stanley-Reisner ideal** of Δ is the ideal $\mathcal{Z}(\Delta)$ generated by the equivariant fundamental classes of the irreducible components of $Z(\Delta)$.

Thus, by [CE, Propositions 2.1 and 2.2], the Chow ring of $\mathcal{X}(\Delta)$ has the following presentation:

Theorem 3.6. *Under the above notation,*

$$A^*(\mathcal{X}(\Delta)) \cong A_G^*/\mathcal{Z}(\Delta) \cong A_{\mathbb{G}_m^n}^*/(L(\Delta), \mathcal{Z}(\Delta)).$$

Each of the terms in the above presentation can be explicitly computed. If e_1, \dots, e_n denotes the canonical basis of characters for $X(\mathbb{G}_m^n)$, and t_1, \dots, t_n denote their first Chern classes, then $A_{\mathbb{G}_m^n}^* = \mathbb{Z}[t_1, \dots, t_n]$. For the linear equivalence ideal, let v_1, \dots, v_k be a basis for $\ker(X(\mathbb{G}_m^n) \rightarrow X(G))$, then $L(\Delta)$ is the ideal generated by corresponding first Chern classes of v_1, \dots, v_k : if $v_i = \sum_j a_{ij}e_j$, then $\sum_j a_{ij}t_j \in L(\Delta)$. For the Stanley-Reisner ideal, suppose $L_i = V(x_{i_1}, \dots, x_{i_l})$ where x_1, \dots, x_n are coordinates for V , then the equivariant fundamental class of L_i is $t_{i_1} \cdots t_{i_l}$.

Remark 3.7. Since we know the irreducible components of $Z(\Delta)$ correspond to the primitive collections of Δ , $\mathcal{Z}(\Delta)$ is equivalently generated by the monomials corresponding to primitive collections. Explicitly, if $\{\rho_{i_1}, \dots, \rho_{i_k}\}$ is a primitive collection corresponding to an irreducible component of $Z(\Delta)$, then the monomial $t_{i_1} \cdots t_{i_k}$ is the equivariant fundamental class of this component.

Example 3.8. Let σ be the cone considered in the previous examples. Since $X(\sigma)$ is affine, $Z(\sigma) = 0$. Furthermore, $L(\sigma)$ is generated by the relations $(s_1 - s_3, s_2 - s_4, s_1 + \cdots + s_4)$. Hence, $A^*(\mathcal{X}(\sigma))$ is a polynomial ring in one variable.

Example 3.9. Suppose we star subdivide σ to produce the fan σ^* . The Chow cohomology of the canonical stack of σ^* has the following presentation: the Stanley-Reisner ideal is the ideal (s_1s_3, s_2s_4) since $\{\rho_1, \rho_3\}$ and $\{\rho_2, \rho_4\}$ are the primitive collections of σ^* . Furthermore, the linear equivalence ideal is defined by the equations $(s_1, s_3, s_2 - s_4, s_1 + \cdots + s_5)$. Hence, $A^*(\mathcal{X}(\sigma^*)) = \mathbb{Z}[s_1, s_2]/(s_1^2, s_2^2)$.

4. Chow cohomology of affine toric varieties

In this section, we will establish our main result, the vanishing of the Chow cohomology of an affine toric variety.

Theorem 4.1. *For any affine toric variety $X(\sigma)$,*

$$A_{\text{op}}^k(X(\sigma)) = 0 \text{ for } k > 0.$$

In particular, $A_{\text{op}}^(X(\sigma)) = \mathbb{Z}$.*

To slightly simplify the notation we will assume that the cone is full dimensional. If σ is not full dimensional then $X(\sigma) = T \times X(\bar{\sigma})$ where T

is a torus and $\bar{\sigma}$ is a full-dimensional cone and our full-dimensional proof readily adapts to this case.

The proof will be divided into the following subsections.

4.1. Kimura’s exact sequence

Let σ^* denote the star subdivision of σ with respect to $v = v(\sigma)$, ρ_v the corresponding ray in σ^* , and $\phi : X(\sigma^*) \rightarrow X(\sigma)$ the associated toric morphism. Note, this morphism is an isomorphism outside of $S := V(\sigma)$ and $E := \phi^{-1}(S)$.

Lemma 4.2. *Under the above notation, $E = V(\rho_v)$.*

Proof. By construction, ρ_v is contained in the interior of σ , and if τ is a cone of σ^* non-trivially intersecting the interior of σ , then τ necessarily contains ρ_v . Hence, ρ_v is the unique minimal cone of σ^* intersecting the interior of σ . The Lemma then follows from the description of Theorem 2.1. \square

Furthermore, S is 0-dimensional and irreducible, hence, $A^0(S) = \mathbb{Z}$ and $A^k(S) = 0$ for $k > 0$. Therefore, if we apply Kimura’s exact sequence to ϕ , we obtain the following exact sequence:

$$0 \rightarrow A_{\text{op}}^k(X(\sigma)) \rightarrow A_{\text{op}}^k(X(\sigma^*)) \rightarrow A_{\text{op}}^k(V(\rho_v)) \text{ for } k > 0.$$

Thus, to show $A_{\text{op}}^k(X(\sigma)) = 0$ for $k > 0$, it suffices to show $A_{\text{op}}^k(X(\sigma^*)) \rightarrow A_{\text{op}}^k(V(\rho_v))$ is injective. To accomplish this, in Sections 4.2 and 4.3 we show that it suffices to consider the Chow rings of the associated canonical stacks. In Section 4.4 we will show that these Chow rings are in fact isomorphic.

4.2. A commutative diagram of canonical stacks

Let $\mathcal{X}(\sigma^*) \rightarrow X(\sigma^*)$ and $\mathcal{E} \rightarrow E$ denote the canonical stacks of $X(\sigma^*)$ and E respectively, and let $E \rightarrow X(\sigma^*)$ denote the inclusion. The goal of this subsection is to show that we can construct a closed embedding $\mathcal{E} \rightarrow \mathcal{X}(\sigma^*)$ such that the following diagram of canonical stacks and moduli spaces is commutative:

$$(2) \quad \begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{X}(\sigma^*) \\ \downarrow & & \downarrow \\ E & \longrightarrow & X(\sigma^*) \end{array}$$

Let Σ_{σ^*} denote the fan of $\mathbb{A}^{|\sigma^*(1)|} \setminus Z(\sigma^*)$. It is equipped with a morphism $X(\Sigma_{\sigma^*}) \rightarrow X(\sigma^*)$ as in the canonical stack construction. If $n = |\sigma(1)|$, let x_1, \dots, x_{n+1} be coordinates on $\mathbb{A}^{|\sigma^*(1)|}$ such that x_{n+1} corresponds to the ray ρ_v . By Remark 2.4 we have a commutative diagram of G -quotients

$$(3) \quad \begin{array}{ccc} (\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)) & \longrightarrow & X(\Sigma_{\sigma^*}) \\ \downarrow & & \downarrow \\ E & \longrightarrow & X(\sigma^*) \end{array}$$

where G is the group acting on $X(\Sigma_{\sigma^*})$ such that $\mathcal{X}(\sigma^*) = [X(\Sigma_{\sigma^*})/G]$. We will show that $[(\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)/G]$ is the canonical stack of the toric variety E .

We begin with the following basic result on toric varieties:

Lemma 4.3. *A collection of rays of σ^* is primitive if and only if the corresponding collection in $\text{Star}(\rho_v)$ is primitive.*

Proof. Let C be a primitive collection of σ^* and let C' be the associated collection in $\text{Star}(\rho_v)$. If $D' \subset C'$ is a proper subset and $D \subset C$ denotes the corresponding subset, then since D is contained in some cone, it follows that D' will be contained in the image of that cone. Similarly, if $C' \subset \bar{\tau}(1)$, then $C \subset \tau$ would contradict C being primitive. Thus, if C is primitive, then C' is primitive; the converse holds by a similar argument. \square

The existence of the commutative diagram (2) now follows from the following lemma.

Lemma 4.4. *Let \mathcal{E} denote the canonical stack over E . Then \mathcal{E} is isomorphic to the quotient stack $[(\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)/G]$.*

Proof. By definition, E is the toric variety with fan $\text{Star}(\rho_v)$ in the quotient lattice $N(\rho_v)$. Since $\text{Star}(\rho_v)$ is spanned by the n rays $\bar{\rho}_1, \dots, \bar{\rho}_v$ in $N(\rho_v)$ we see that the canonical stack is the quotient of a

$$[(\mathbb{A}^n \setminus Z(\text{Star}(\rho_v)))/G'] \text{ where } G' = \ker(T_n \rightarrow T_{N(\rho_v)})$$

and T_n is the n -dimensional torus $\text{Hom}(\mathbb{Z}^{|\text{Star}(\rho_v)(1)|}, \mathbb{G}_m)$. By Lemma 4.3, a collection of rays in $\text{Star}(\rho_v)$ is primitive if and only if the corresponding rays in σ^* are primitive. Thus if we identify $\mathbb{V}(x_{n+1}) = \mathbb{A}^n$ then the open sets $\mathbb{A}^n \setminus Z(\text{Star}(\rho_v))$ and $\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)$ are equal. We claim that $G = G'$

and the action of G' on $\mathbb{A}^n \setminus Z(\text{Star}(\rho_v))$ coincides with the action of G on $\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)$ under this isomorphism. Indeed, this follows from the commutativity of the following diagram of character groups:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & X(G') \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^{n+1} & \longrightarrow & X(G) \longrightarrow 0
 \end{array}$$

The vertical maps are the duals of the canonical quotient maps for the primitive generators of ρ_v and e_{n+1} respectively. □

Remark 4.5. Note that unless σ^* is a resolution of singularities, diagram (3) is not cartesian. The following example shows that even if σ^* is a simplicialization the diagram fails to be Cartesian.

Example 4.6. Consider the cone σ with ray generators given by $\{(1, 1, 1), (-1, 0, 1), (1, 0, 1), (0, -1, 1)\}$, and its associated star subdivision σ^* obtained by adding the ray generator $(1, 0, 4)$ and subdividing appropriately. We claim that in this example, \mathcal{E} is not scheme-theoretically saturated with respect to π . Following [ES, Remark 3.4], since $\dim(\mathcal{E}) = \dim(E)$ and $\pi(\mathcal{E}) = E$, it suffices to show \mathcal{E} is not a strong divisor.

With the notation as in this section, the action of \mathbb{G}_m^2 on $\mathbb{A}^5 \setminus V(x_1, x_3) \cup V(x_2, x_4)$ is given by the matrix:

$$\begin{bmatrix} 3 & -2 & 1 & -2 & 0 \\ 2 & -3 & 0 & -3 & 1 \end{bmatrix}$$

If \mathcal{E} is strong, then the ideal (x_5) is generated by an invariant function on each open set $D(x_1x_2)$, $D(x_2x_3)$, $D(x_3x_4)$, and $D(x_1x_4)$. However, on $D(x_1x_2)$ where x_1, x_2 are invertible, the weight $(0, 1)$ of x_5 cannot be expressed as an integral linear combination of the weights $(3, 2)$ and $(-2, -3)$ for x_1, x_2 respectively; thus, \mathcal{E} is not strong. Note that $(0, 1)$ can be expressed rationally in terms of these weights on each affine open set and it is easy to check that (x_5^{15}) is locally generated by an invariant function.

4.3. The injectivity result of [ES]

Since the canonical stack morphisms are good moduli space morphisms, we have that $A_{\text{op}}^*(E)_{\mathbb{Q}} \subset A^*(\mathcal{E})_{\mathbb{Q}}$ and $A_{\text{op}}^*(X(\sigma^*))_{\mathbb{Q}} \subset A^*(\mathcal{X}(\sigma^*))_{\mathbb{Q}}$. Hence, by

combining these injections with the main result of the previous section and Kimura’s exact sequence, the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_{\text{op}}^k(X(\sigma))_{\mathbb{Q}} & \longrightarrow & A_{\text{op}}^k(X(\sigma^*))_{\mathbb{Q}} & \longrightarrow & A_{\text{op}}^k(E)_{\mathbb{Q}} \\
 & & & & \downarrow & & \downarrow \\
 & & & & A^k(\mathcal{X}(\sigma^*))_{\mathbb{Q}} & \longrightarrow & A^k(\mathcal{E})_{\mathbb{Q}}
 \end{array}$$

Since the vertical maps are injections, to show $A_{\text{op}}^k(X(\sigma^*)) \rightarrow A_{\text{op}}^k(E)$ is injective at least over \mathbb{Q} , it suffices to show $A^*(\mathcal{X}(\sigma^*))_{\mathbb{Q}} \rightarrow A^*(\mathcal{E})_{\mathbb{Q}}$ is an injection. In fact, we claim this map is an isomorphism over \mathbb{Z} .

4.4. The pullback of Chow rings of canonical stacks is an isomorphism

Theorem 4.7. *Let $\mathcal{E} \rightarrow \mathcal{X}(\sigma^*)$ be the injective morphism given by Lemma 4.4, then the pullback induces an isomorphism $A^*(\mathcal{E}) \cong A^*(\mathcal{X}(\sigma^*))$.*

Proof. By the previous lemma, $\mathcal{E} \cong [(\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*)) / G] \subset \mathcal{X}(\sigma^*) = [(\mathbb{A}^{n+1} \setminus Z(\sigma^*)) / G]$. Hence, the Chow rings of the canonical stacks are the following equivariant Chow rings:

$$A^*(\mathcal{X}(\sigma^*)) = A_G^*(\mathbb{A}^{n+1} \setminus Z(\sigma^*)), \quad \text{and} \quad A^*(\mathcal{E}) = A_G^*(V(x_{n+1}) \setminus Z(\sigma^*)).$$

Since $Z(\sigma^*)$ does not contain x_{n+1} , we can write $\mathbb{V}(x_{n+1}) \setminus Z(\sigma^*) = \mathbb{A}^n \setminus Z'$ and $\mathbb{A}^{n+1} \setminus Z(\sigma^*) = (\mathbb{A}^n \setminus Z') \times \mathbb{A}^1$ for the obvious exceptional set $Z' \subset \mathbb{A}^n$. In particular, the inclusion of canonical stacks is the quotient by G of the zero section $\mathbb{A}^n \setminus Z' \subset (\mathbb{A}^n \setminus Z') \times \mathbb{A}^1$ of a G -equivariant vector bundle over $\mathbb{A}^n \setminus Z'$. Hence, the pullback in G -equivariant Chow groups along this inclusion is an isomorphism. □

Thus, $A_{\text{op}}^k(X(\sigma))_{\mathbb{Q}} = 0$ for $k > 0$. The final step will be to show $A_{\text{op}}^k(X(\sigma))$ is torsion-free.

4.5. Chow cohomology of semi-projective toric varieties is torsion-free

From [HS], a toric variety $X(\Delta)$ is semi-projective if it has at least one torus fixed point and the natural map $X(\Delta) \rightarrow \text{Spec}(\Gamma(\mathcal{O}_{X(\Delta)}))$ is projective; equivalently, it is semi-projective if Δ has full-dimensional convex support and $X(\Delta)$ is quasi-projective. Hence, for an affine toric variety $X(\sigma)$,

any regular subdivision (for example a star subdivision) Δ of σ produces a semi-projective toric variety $X(\Delta)$ which is projective over $X(\sigma)$. In particular, let σ' be a resolution of singularities of σ obtained by regular subdivisions, then by [Kim, Lemma 2.1], $A_{\text{op}}^*(X(\sigma)) \subset A_{\text{op}}^*(X(\sigma'))$. Hence, it suffices to show $A_{\text{op}}^*(X(\sigma'))$ is torsion-free where $X(\sigma')$ is a smooth semi-projective toric variety.

Remark 4.8. For a smooth projective toric variety, [Ful2, Section 5.2] demonstrates that its Chow ring is torsion-free by producing a Bialynicki-Birula decomposition from a particular ordering of the maximal dimensional cones.

In [HS, Section 2], Hausel and Sturmfels generalize the construction of [Ful2, Section 5.2] to smooth semi-projective toric varieties by utilizing the moment map associated to a 1-parameter subgroup, and in particular, they construct the Bialynicki-Birula decomposition of a smooth semiprojective variety by producing a collection of locally closed subsets $\{U_j\}$ satisfying the following properties:

- 1) Each U_i is a union of orbits, and hence, the closure of U_i is a union of U_j 's.
- 2) $X(\sigma')$ is the disjoint union of the U_i 's.
- 3) Each $U_i \cong \mathbb{A}^{n-k_i}$.

In particular, by [EH, Definition 1.16], this decomposition of $X(\sigma')$ is an affine stratification. By [Tot], the classes of U_i form a basis for $A^*(X(\sigma'))$, and hence, $A^*(X(\sigma'))$ is necessarily torsion-free. Thus, since $A_{\text{op}}^*(X(\sigma)) \subset A^*(X(\sigma'))$, it follows that $A_{\text{op}}^*(X(\sigma))$ is torsion-free, and by combining all of the previous subsections, this concludes the proof of the main theorem. \square

The argument presented in this subsection can be utilized to demonstrate that the Chow cohomology rings of a wide class of toric varieties are torsion-free. As a natural generalization of semi-projectivity, we say that a toric variety $X(\Delta)$ is semi-proper if it has at least one torus fixed point and the natural map $X(\Delta) \rightarrow \text{Spec}(\Gamma(\mathcal{O}_{X(\Delta)}))$ is proper.

Theorem 4.9. *Let $X(\Delta)$ be any semi-proper toric variety, then $A_{\text{op}}^*(X(\Delta))$ is torsion-free.*

Proof. By the toric Chow's lemma [CLS, Theorem 6.1.18], there exists a projective toric variety $X(\Delta')$, and toric morphism $X(\Delta') \rightarrow X(\Delta)$, where Δ' is

a smooth refinement of Δ , such that the map $X(\Delta') \rightarrow X(\sigma)$ factors through $X(\Delta) \rightarrow X(\sigma)$. Hence, by [Kim, Lemma 2.1], $A_{\text{op}}^*(X(\Delta)) \subset A^*(X(\Delta))$ and the latter is torsion-free by the previous arguments in this section. \square

Example 4.10. Let σ denote the cone generated by $\{(1, 0, 1), (0, -1, 1), (-1, 0, 1), (0, 1, 1)\}$ in \mathbb{R}^3 , then σ^* is actually a resolution of singularities of σ . In particular, the vertical maps in the square

$$\begin{CD} A_{\text{op}}^k(X(\sigma^*)) @>>> A_{\text{op}}^k(E) \\ @VVV @VVV \\ A^k(\mathcal{X}(\sigma^*)) @>\simeq>> A^k(\mathcal{E}) \end{CD}$$

are isomorphisms. Furthermore, $E = V(\rho_v) = \mathbb{P}^1 \times \mathbb{P}^1$, and we can explicitly compute $A^*(\mathcal{X}(\sigma^*))$ as:

$$A^*(\mathcal{X}(\sigma^*)) = \mathbb{Z}[s_1, \dots, s_4, s]/(s_1 - s_3, s_2 - s_4, s_1 + \dots + s_4 + s, s_1 s_3, s_2 s_4).$$

Hence, $s_1 = s_3$, $s_2 = s_4$, and $s = -2s_1 - 2s_2$; and $A^*(\mathcal{X}(\sigma^*)) = \mathbb{Z}[s_1, s_2]/(s_1^2, s_2^2)$ is clearly isomorphic to the Chow ring of $\mathbb{P}^1 \times \mathbb{P}^1$ as predicted by Theorem 4.7. This calculation also directly verifies that $A_{\text{op}}^*(X(\sigma))$ is torsion free since it injects into the ring $\mathbb{Z}[s_1, s_2]/(s_1^2, s_2^2)$ which has no \mathbb{Z} -torsion.

5. Operational K -theory

In this part we prove Theorem 5.1. The proof is similar to the proof of Theorem 4.1 so we give an extended sketch. The main technical difficulty is establishing a version of the injectivity result of [ES] for operational K -theory

5.1. Notation and background on operational K -theory

We briefly recall the notation and basic results on operational K -theory defined by Anderson and Payne in the paper [AP].

Following [AP], if X is a scheme (or more generally a stack) we denote by $K_0(X)$ the Grothendieck group of coherent sheaves, and $K^0(X)$ the Grothendieck group of perfect complexes. If X has an ample family of line bundles, then $K^0(X)$ is the same as the naive Grothendieck group of vector bundles.

For any scheme X , Anderson and Payne define the **operational K -theory** $\text{op } K^0(X)$ of X as follows. An element $c \in \text{op } K^0(X)$ is a collection of

operators $c_f: K_0(X') \rightarrow K_0(X')$ indexed by morphisms $X' \xrightarrow{f} X$ compatible with proper pushforward, flat and l.c.i. pullback.

For any scheme X , there is a canonical map $\text{op } K^0(X) \rightarrow K_0(X)$ given by $c \mapsto c_{\text{id}_X}(\mathcal{O}_X)$. If X is smooth, then [AP, Corollary 4.5] states that this map is an isomorphism. They also prove [AP, Proposition 5.3] that the groups $\text{op } K^0(X)$ satisfy descent for Chow envelopes. In particular, the exact sequence (1) also holds for operational K -theory.

Theorem 5.1. *For any affine toric variety $X(\sigma)$, $\text{op } K^0(X(\sigma)) = \mathbb{Z}$.*

5.2. The injectivity theorem for operational K -theory

Lemma 5.2. *If $X' \xrightarrow{f} X$ is an envelope (resp. proper surjective), then the pullback map on operational K -theory $f^*: \text{op } K^0(X) \rightarrow \text{op } K^0(X')$ is injective (resp. rationally injective).*

Proof. The pushforward map $f_*: K_0(X') \rightarrow K_0(X)$ is surjective (resp. rationally surjective) by [Full, Lemma 18.3(6)]. Given this fact, the proof is identical to the proof of [Kim, Lemma 2.1] with operational K -theory replacing Chow cohomology. □

The following is the K -theoretic analogue of [ES, Proposition 2.10].

Proposition 5.3. *If $\mathcal{X} \rightarrow X$ is a good moduli space morphism and X has a resolution of singularities, then there is a pullback $\text{op } K^0(X) \rightarrow K_0(\mathcal{X})$, $c \mapsto c(\mathcal{O}_{\mathcal{X}'})$. In general this map is defined with \mathbb{Q} coefficients.*

Proof. Assuming X admits a resolution of singularities, then there is a birational proper morphism $f: X' \rightarrow X$ with X' a smooth scheme. Let $\mathcal{X}' = X' \times_X \mathcal{X}$ and let $f: \mathcal{X}' \rightarrow \mathcal{X}$ be the morphism of stacks obtained by base change. If $c \in \text{op } K^0(X)$ then since $\text{op } K^0(X') = K_0(X') = K^0(X')$ we can identify f^*c with an element of $K^0(X')$. Since pullbacks in K -theory of vector bundles exist for arbitrary morphisms, we obtain a class $(\pi')^* f^*c \in K^0(\mathcal{X}')$. Using the same name for its image in $K_0(\mathcal{X}')$ under the map $K^0(\mathcal{X}') \rightarrow K_0(\mathcal{X}')$ which takes a vector bundle to the class of the corresponding coherent sheaf, we define π^*c to be $g_*(\pi')^* f^*c \in K_0(\mathcal{X})$. The projection formula implies that this definition is independent of the choice of resolution X' . □

We now obtain the K -theoretic analogue of the injectivity result proved in [ES].

Theorem 5.4. *Let $\pi: \mathcal{X} \rightarrow X$ be a properly stable, good moduli space morphism with \mathcal{X} a smooth Artin stack with generically trivial stabilizer and X a scheme. Then the pullback $\pi^*: \text{op } K^0(X) \rightarrow K_0(\mathcal{X})$ is injective after tensoring with \mathbb{Q} .*

Proof. By [ERy, Corollary 7.2], there is a commutative diagram of stacks and good moduli spaces

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

with \mathcal{X}' a smooth tame stack with smooth coarse space X' such that the map $f: X' \rightarrow X$ is proper and birational. By Lemma 5.2, we know that $f^*: \text{op } K^0(X) \rightarrow \text{op } K^0(X')$ is injective after tensoring with \mathbb{Q} . Thus, it suffices to prove that $\pi'^*: \text{op } K^0(X') \rightarrow K_0(\mathcal{X}')$ is injective after tensoring with \mathbb{Q} . In other words, we are reduced to the case that \mathcal{X} is a smooth tame stack with smooth coarse moduli space X . In this case $\text{op } K^0(X) = K_0(X) = K^0(X)$. By the Riemann-Roch theorem for smooth schemes, the Chern character $\text{ch}: K_0(X)_{\mathbb{Q}} \simeq A^*(X)_{\mathbb{Q}}$ is an isomorphism. The Chern character commutes with the pullback $\pi^*: K_0(X) \rightarrow K_0(\mathcal{X})$, and the pullback map in Chow groups $\pi^*: A^*(X)_{\mathbb{Q}} \rightarrow A^*(\mathcal{X})_{\mathbb{Q}}$ is an isomorphism. Hence, the map π^* is necessarily injective². It follows from functoriality of the Riemann-Roch map that the composition $K_0(X)_{\mathbb{Q}} \rightarrow K_0(\mathcal{X})_{\mathbb{Q}} \rightarrow A^*(\mathcal{X})_{\mathbb{Q}}$ is injective. \square

5.3. K -theory of toric canonical stacks

Like the Chow ring, the K -theory of a canonical toric stack $\mathcal{X}(\Delta)$ admits a simple Stanley-Reisner type presentation.

The **K -theoretic linear equivalence ideal** of Δ is defined to be:

$$KL(\Delta) := \ker (R(\mathbb{G}_m^n) \rightarrow R(G)).$$

If e_1, \dots, e_n is the canonical basis of characters, then

$$R(\mathbb{G}_m^n) = \mathbb{Z}[e_1, e_1^{-1}, \dots, e_n, e_n^{-1}] \text{ and } KL(\Delta)$$

²Note that the Chern character $\text{ch}: K_0(\mathcal{X}) \rightarrow A^*(\mathcal{X})$ is defined because the hypothesis ensures that \mathcal{X} is a quotient stack by [EHKV]. The Chern character is surjective but not injective. See [Edi] for a discussion of the Riemann-Roch theorem on tame stacks.

is generated by expressions of the form $e_1^{a_1} \cdots e_n^{a_n} - 1$ with $a_1 e_1 + \cdots + a_n e_n \in \ker(X(\mathbb{G}_m^n) \rightarrow X(G))$.

Definition 5.1. The *K-theoretic Stanley-Reisner ideal* of Δ is the ideal $\mathcal{KZ}(\Delta)$ generated by the equivariant *K-theoretic fundamental classes* of the irreducible components of $Z(\Delta)$.

If $V(x_{i_1}, \dots, x_{i_k}) \subset \mathbb{A}^n$ is a linear subspace then $[V] = \prod_{j=1}^k (1 - e_{i_j}^{-1})$ in $K_{\mathbb{G}_m}(\mathbb{A}^n) = \mathbb{Z}[e_1, e_1^{-1}, \dots, e_n, e_n^{-1}]$. Using the excision sequence for equivariant *K-theory*, we obtain the analogue of Theorem 3.6:

Theorem 5.5. *With notation as above,*

$$K^0(\mathcal{X}(\Delta)) \cong R(G)/\mathcal{KZ}(\Delta) \cong \mathbb{Z}[e_1, e_1^{-1}, \dots, e_n, e_n^{-1}]/(KL(\Delta) + \mathcal{KZ}(\Delta)).$$

Example 5.6. Once again, let σ be the cone of Example 2.7, and let σ^* be its star subdivision as in Example 3.9. Then $K(\mathcal{X}(\sigma^*))$ is the quotient of the character algebra of \mathbb{G}_m^5 , $\mathbb{Z}[e_1, e_1^{-1}, e_2, e_2^{-1}, e_3, e_3^{-1}, e_4, e_4^{-1}, e, e^{-1}]$, by the *K-theoretic ideal of linear relations* and the *K-theoretic Stanley-Reisner ideal*. In our case, the *K-theoretic ideal of linear relations* is generated by $(e_1 e_3^{-1} - 1, e_2 e_4^{-1} - 1, e_1 e_2 e_3 e_4 e - 1)$, and the *K-theoretic Stanley-Reisner ideal* is generated by $(1 - (e_1 e_3)^{-1}, 1 - (e_2 e_4)^{-1})$. Thus,

$$K_0(\mathcal{X}(\sigma^*)) \simeq \mathbb{Z}[e_1, e_2]/(e_2^2 - 1, e_1^2 - 1) = K_0(\mathbb{P}^1) \otimes K_0(\mathbb{P}^1).$$

5.4. Completion of the proof

The proof now proceeds as the proof of Theorem 4.1. Using the analogue of Kimura’s exact sequence for operational *K-theory* [AP] and Proposition 5.3, we are reduced to showing (with the same notation as in Section 4.2) that the restriction $K^0(\mathcal{E}) \rightarrow K^0(\mathcal{X})$ is an isomorphism. This fact follows from the Stanley-Reisner description of $K(\mathcal{X})$ and the same formal argument used in the proof of Theorem 4.7.

Finally, if X is any smooth variety such that $A^*(X)$ is torsion free, then $K(X)$ is also torsion free. This follows from the fact that the natural map $A_k(X) \rightarrow \text{Gr}_k K^0(X)$, $[V] \mapsto [\mathcal{O}_V]$ is surjective and becomes an isomorphism after tensoring with \mathbb{Q} [Ful1, Examples 15.1.5, 15.2.16]. Hence, the same argument used in the proof of Theorem 4.9 implies that $\text{op } K^0(X(\Delta))$ is torsion free for any semi-proper toric variety. In particular, $\text{op } K^0(X(\sigma))$ is torsion free for an affine toric variety.

Example 5.7. Let σ be the cone of Example 2.7. In Example 3.4, we showed that $\text{rk } A_*(X(\sigma)) \geq 2$. Thus, by the Riemann-Roch theorem for singular schemes $\text{rk } K_0(X(\sigma)) \geq 2$, hence $\text{op } K^0(X(\sigma)) \neq K_0(X(\sigma))$ even after tensoring with \mathbb{Q} .

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