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Lagrangian antisurgery

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We describe an operation which modifies a Lagrangian submanifold L in a symplectic manifold (M, ω) such as to produce a new immersed Lagrangian submanifold L', which as a smooth manifold is obtained by surgery along a framed sphere in L. Intuitively, this can be described as collapsing an isotropic disc with boundary on L to a point. The inverse operation generalizes classical Lagrangian surgery. We also describe corresponding immersed Lagrangian cobordisms between L and L'. After removal of their singular locus, we obtain examples of embedded Lagrangian cobordisms with precisely two ends. As an application, we use this construction to produce interesting examples of Lagrangian cobordisms between Clifford and Chekanov tori.

1. Introduction

A fundamental question in symplectic geometry is what manifolds arise as the Lagrangian submanifolds of a given symplectic manifold (M^{2n}, ω). This question has different flavours and levels of difficulty, depending on whether one asks for embedded or immersed Lagrangian submanifolds, and on whether one incorporates constraints such as exactness or monotonicity.

A natural attempt to construct new Lagrangian submanifolds is to modify given ones by some sort of surgery operation. There is one well known construction which resolves the transverse double points of a Lagrangian immersion $\iota : L \to M$ by replacing neighbourhoods of them by copies of $D^1 \times S^{n-1}$. For example, if L is connected, oriented and immersed with a unique double point, then the resulting Lagrangian L' is embedded and diffeomorphic to the connected sum $L\#(S^1 \times S^{n-1})$, provided that the surgery can be performed compatibly with the orientation. This operation, which we will refer to as *Lagrangian 0-surgery*, is due to Lalonde–Sikorav [11] for $n = 2$ and to Polterovich [14] for general n.

Terminology and notation

In all of the following, "Lagrangian submanifolds" will generally be allowed to be immersed with transverse double points. We will usually not make a notational distinction between abstract smooth manifolds L and their immersed images in M; that is, whenever we have a Lagrangian immersion $\iota : L \to M$, we will slightly abuse notation and denote its image $\iota(L) \subset M$ also by L.

1.1. Surgery of smooth manifolds

On the level of abstract smooth manifolds (i.e., not taking into account the Lagrangian embedding), the passage from L to L' by Lagrangian 0-surgery replaces an embedded copy of $S^0 \times D^n$ by a copy of $D^1 \times S^{n-1}$. This is a special case of the following more general operation originally due to Milnor [13]: Whenever a smooth n-dimensional manifold L contains an embedding $\varphi : S^k \times D^{n-k} \to L$, one can cut out $\varphi(S^k \times D^{n-k})$ and replace it by a copy of $D^{k+1} \times S^{n-k-1}$, such as to obtain a new manifold

$$L' = (L \smallsetminus \varphi(S^k \times D^{n-k})) \cup_{\varphi(S^k \times S^{n-k-1})} (D^{k+1} \times S^{n-k-1}).$$

This works because $\partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1})$. We say that the manifold L', which inherits a smooth structure from L in a canonical way, is obtained from L by *k-surgery* (a.k.a. *surgery of index* $k + 1$).

Surgery theory is closely connected to cobordism theory. The manifold L' resulting from k-surgery on a manifold L is cobordant to L via a cobordism

$$V = ([0, 1] \times L) \cup_{\{1\} \times \varphi(S^k \times D^{n-k})} D^{k+1} \times D^{n-k},$$

i.e., a cobordism that arises from the cylinder $[0,1] \times L$ by attaching a $(k+1)$-handle $D^{k+1} \times D^{n-k}$ along $\{1\} \times \varphi(S^k \times D^{n-k})$. This cobordism is called the *trace* of the corresponding surgery.

1.2. Lagrangian antisurgery

Let now $L \subset M$ be a Lagrangian submanifold containing an embedded copy of $S^k \times D^{n-k}$. It is natural to ask if the manifold L' obtained by k-surgery on L can again be embedded or immersed into M as a Lagrangian submanifold. The answer to a sufficiently strong version of this question is certainly negative: For example, a closed orientable manifold L that can be embedded in $\mathbb{C}^n$ must have Euler characteristic $\chi(L) = 0$. However, k-surgery changes the Euler characteristic according to $\chi(L') = \chi(L) + (-1)^{k+1} + (-1)^{n-k-1}$, and hence does not preserve its vanishing if n is even. So in this case no result of a single k-surgery on L admits a Lagrangian embedding into $\mathbb{C}^n$.

In this paper we will describe a construction which implements k-surgery for Lagrangian submanifolds under certain conditions. Let $L \subset M$ be a Lagrangian submanifold containing an embedding $\varphi : S^k \times D^{n-k} \to L$ together with an *isotropic surgery disc* D, that is, an embedded isotropic $(k+1)$-disc $D \subset M$ intersecting L cleanly along $S = \varphi(S^k \times \{0\})$ and otherwise disjoint from L (this terminology is borrowed from [6]).

Theorem 1.1. *The manifold L' obtained by k-surgery on L with respect to the embedding $\varphi : S^k \times D^{n-k} \to L$ admits a Lagrangian immersion $L' \to M$ whose image agrees with L outside of an arbitrarily small neighbourhood of D, and such that in this neighbourhood it has exactly one transverse double point. Moreover, there exists an immersed Lagrangian cobordism $V : L' \rightsquigarrow L$ given by a Lagrangian immersion of the trace of the k-surgery into $T^*\mathbb{R} \times M$, whose singular locus is a 1-dimensional family of double points along the end corresponding to L'.*

The construction of L' and $V : L' \rightsquigarrow L$, and hence the proof of Theorem 1.1, is the content of Section 2. We refer to the operation that passes from L to L' as *Lagrangian k-antisurgery.* The idea behind the terminology is that the operation *creates* a double point, in contrast to Lagrangian 0-surgery, which *resolves* a double point. To give a quick and intuitive description, one could say that Lagrangian k-antisurgery modifies a Lagrangian L by collapsing an isotropic $(k+1)$-disc with boundary on L to a point.

The local model for the immersed Lagrangian $(k+1)$-handle which enables us to build the cobordism V, as well as the idea of implanting it along an isotropic disc, is inspired by a construction of Dimitroglou-Rizell appearing in [6], which implements k-surgery for Legendrian submanifolds and builds corresponding Lagrangian cobordisms (in a different sense of the word, see Section 2.4).

1.3. Lagrangian cobordisms

The notion of Lagrangian cobordism appearing in Theorem 1.1 is that of Biran–Cornea [3], adapted to the immersed setting in an obvious way: Two ordered collections $(\iota_i : L_i \to M)_{i=1}^r$, $(\iota'_j : L'_j \to M)_{j=1}^s$ of immersed Lagrangian submanifolds of M are called *Lagrangian cobordant* if there exists a smooth cobordism $(V; \coprod_i L_i, \coprod_j L'_j)$ together with a Lagrangian immersion $V \to [0,1] \times \mathbb{R} \times M \subset T^*\mathbb{R} \times M$ such that for some small $\delta > 0$, we have

$$\begin{aligned} V|_{[0,\delta)\times\mathbb{R}} &= \coprod_{i=1}^r [0,\delta) \times \{i\} \times L_i \quad \text{and} \\ V|_{(1-\delta,1]\times\mathbb{R}} &= \coprod_{j=1}^s (1-\delta,1] \times \{j\} \times L'_j. \end{aligned}$$

Here we use the notation $V|_U := V \cap (U \times M)$ to denote the part of V that lies over some subset $U \subset T^*\mathbb{R}$, and we identify $T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ in the standard way. The Lagrangian submanifold $V \subset T^*\mathbb{R} \times M$ is called an *immersed Lagrangian cobordism* with negative ends $(L_i)_{i=1}^r$ and positive ends $(L_j)_{j=1}^r$, and this relationship is denoted by $V : (L'_1, \dots, L'_s) \rightsquigarrow (L_1, \dots, L_r)$. In this article we will only deal with the case $r = s = 1$, i.e., with Lagrangian cobordisms

$$V : L' \rightsquigarrow L$$

with a single positive and a single negative end.

Lagrangian cobordisms have recently attracted a lot of interest due to the fact that, provided certain monotonicity assumptions hold, they preserve Floer theoretic invariants and encode information about the Fukaya category, see [3, 4] and also the recent [12]. So far, there have been essentially two known constructions of Lagrangian cobordisms, which are based on Hamiltonian isotopy resp. Lagrangian 0-surgery. Extending the toolkit for building new ones, such as those provided by Theorem 1.1 and Theorem 1.2 below, was one of the motivations for the present paper.

1.4. Desingularization

The newly created double point p of the Lagrangian L' resulting from Lagrangian antisurgery can be resolved by Lagrangian 0-surgery. Provided that L is embedded, this yields a Lagrangian $L^\natural$ which is also embedded and diffeomorphic to $L'\#P^n$ or to $L'\#Q^n$, where $P^n = S^1 \times S^{n-1}$, and Q^n is the mapping torus of an orientation-reversing involution of S^{n-1}.

There are in fact two families of such resolutions which correspond to the two ways ordering the sheets meeting at $p \in L'$ (as is always the case for Lagrangian 0-surgery). We will show that there exists one such family such that for all $L^\natural$ in that family which are of sufficiently small size (in the sense of Definition 3.1), one can in fact extend the resolution such as to simultaneously remove the singular locus of the immersed Lagrangian cobordism $V : L' \rightsquigarrow L$ produced by Theorem 1.1:

Theorem 1.2. *There exists a choice of ordering of the sheets meeting at the double point $p \in L'$ such that for all $L^\natural$ in the corresponding family of resolutions of p which are of sufficiently small size, there exists an* embedded *Lagrangian cobordism $V^\natural : L^\natural \rightsquigarrow L$ which coincides with the immersed cobordism $V : L' \rightsquigarrow L$ outside of a small neighbourhood of the singular locus of V. As a smooth manifold, $V^\natural$ is diffeomorphic to the manifold obtained from $[0,1] \times L$ by consecutively attaching a $(k+1)$-handle and a 1-handle.*

The construction which constitutes the proof of Theorem 1.2 will be given in Section 4. As stated in Theorem (1.1), the singular locus of V looks like a line of double points (see Proposition 4.1 for a precise description of the singularity of the correponding model the passage from V to $V^\natural$ replaces a neighbourhood of it by a Lagrangian 1-handle.

One should note at this point that our construction of antisurgery cobordisms cannot be replaced by simply appealing to the h-principle [7] satisfied by immersed Lagrangian cobordisms: Indeed, this would not provide the amount of information about the singular locus that we need in order to control the topology of the result of desingularizing the immersed cobordism by a version of Lagrangian surgery.

1.5. Reversing the construction

Lagrangian antisurgery constructs from a Lagrangian $L \subset M$ an new Lagrangian L' with one (additional) double point. Changing perspectives, we can view L as the result of resolving a double point of L' by an operation which is an $(n-k-1)$-surgery on the level of smooth manifolds, and which we therefore refer to as *Lagrangian* $(n-k-1)$*-surgery*. We will discuss this point of view in Section 3.

In the case $k = n-1$, this reversed operation is the same as classical Lagrangian 0-surgery, up to Lagrangian isotopy. That is, if L' is the result of an $(n-1)$-antisurgery on L, then L can be obtained back from L' by classical Lagrangian 0-surgery followed by a Lagrangian isotopy, and vice versa. As a consequence, for $k = n-1$ the ends of the desingularized antisurgery cobordisms $V^\natural : L^\natural \rightsquigarrow L$ are both resolutions of the Lagrangian L' by Lagrangian 0-surgery. Recall that Theorem 1.2 only asserts the existence of such a cobordism for $L^\natural$ belonging to *one* of the two families of resolutions $p \in L'$. The next proposition identifies which one it is:

Proposition 1.3 (See Proposition 4.4). *The ends of the desingularized cobordism $V^\natural : L^\natural \rightsquigarrow L$ resulting from $(n-1)$-antisurgery on L belong to* distinct *families of resolutions of $p \in L'$ by Lagrangian 0-surgery.*

1.6. Cobordisms between Clifford and Chekanov tori

As an application of Theorem 1.2, we will construct cobordisms between Clifford tori $T^2_{Cl}(a)$ and Chekanov tori $T^2_{Ch}(A)$ in $\mathbb{R}^4$. These are monotone Lagrangian tori which, in both cases, are specified uniquely up to Hamiltonian isotopy by the areas $a, A > 0$ of any disc of Maslov index 2 with boundary on them.

Theorem 1.4 (See Theorem 5.1). *For every choice of $a < A$ with a/A sufficiently close to 1, there exists a Lagrangian cobordism $T^2_{Cl}(a) \rightsquigarrow T^2_{Ch}(A)$ which as smooth a manifold is obtained from $[0,1] \times T^2$ by successively attaching a 2-handle and a 1-handle.*

To put Theorem 1.4 into context, recall first the classical fact that there does not exist a Hamiltonian isotopy between any two Chekanov and Clifford tori [5, 8]. Since the relation of being Lagrangian cobordant is a generalization of the relation of being Hamiltonian isotopic (as Hamiltonian isotopies give rise to *Lagrangian suspension* cobordisms $L \rightsquigarrow \phi_t(L)$, see [3]), this fact can also be viewed as a restriction on the type of Lagrangian cobordisms that can exist between Clifford and Chekanov tori.

One way of disproving the existence of a Hamiltonian isotopy between $T^2_{Cl}(A)$ and $T^2_{Ch}(A)$ is to note that the numbers of pseudoholomorphic discs of Maslov index 2 through a generic points on these tori are different (this number is 1 for the Chekanov torus, but 2 for the Clifford torus), while the existence of a Hamiltonian isotopy between them would imply that these counts are the same. A similar argument precludes the existence of any *monotone* Lagrangian cobordism between $T^2_{Cl}(A)$ and $T^2_{Ch}(A)$, because such cobordisms also preserve counts of Maslov 2 discs [3, 5]. In particular, the statement of Theorem 1.4 cannot be extended to include the case $a = A$, because the resulting cobordism would be automatically monotone (see Proposition 5.2).

In fact, one can adapt the argument used to prove the latter to obtain the following statement: There does not exist a Lagrangian cobordism $V : T^2_{Cl}(A) \rightsquigarrow T^2_{Ch}(a)$ with $a < A$ (i.e., connecting a Clifford torus to a *smaller* Chekanov torus, in contrast to Theorem 1.4) and with the property that

$$\inf \left\{\omega(\sigma) \mid \sigma \in \pi_2(T^*\mathbb{R} \times \mathbb{R}^4, V),\ \omega(\sigma) > 0\right\} = a. \tag{1}$$

Indeed, if one assumes that such a cobordism exists, one arrives at a contradiction when considering the number of Maslov 2 discs with boundary of the cobordism passing through a generic point and representing the pushforward of the unique class in $H_2(\mathbb{R}^4, T^2_{Ch}(a))$ represented by a Maslov 2 pseudoholomorphic disc: On the one hand, this number would have to be independent of the chosen point, as property (1) ensures that there can be no bubbling and hence that the corresponding moduli space is compact. On the other hand, it would need to be 1 for a point on the Chekanov end, but 0 for a point on the Clifford end.

In contrast to that, there is no obvious obstruction to the existence of a cobordism $T^2_{Cl}(a) \rightsquigarrow T^2_{Ch}(A)$ with property (1), i.e., from a Clifford torus to a *larger* Chekanov torus, as compactness of the moduli space of discs described is not guaranteed in this situation.

To connect these observations with Theorem 1.4, note that a cobordism between Lagrangian 2-tori whose topology is that of a 1-antisurgery cobordism, i.e., as described in the theorem, has property (1) if $A = k \cdot a$ for some $k \in \mathbb{N}$ (this follows e.g. from Proposition 5.2). Therefore, it seems plausible to conjecture that the rôle of Clifford and Chekanov in Theorem 1.4 cannot be swapped, i.e., that a cobordism of this topology connecting a Clifford torus $T^2_{Cl}(A)$ to a *smaller* Chekanov torus $T^2_{Ch}(a)$ does not exist. (Note however that this is just a conjecture, as the existence of such a cobordism for a/A close to 1 is not ruled out by the arguments in the previous paragraph). On the other hand, it is likely that one can extend the statement of Theorem 1.4 to give the existence of an antisurgery cobordism $T^2_{Cl}(a) \rightsquigarrow T^2_{Ch}(A)$

for *any* choice of $0 < a < A$ by deforming the local model for antisurgery suitably (cf. Section 5.2).

While the quest for the "simplest" Lagrangian cobordism connecting given Clifford and Chekanov tori is interesting and subtle, we note that the existence problem for such tori is completely flexible if one does not constrain the topology of the cobordisms one considers: As mentioned above, *immersed* Lagrangian cobordisms are governed by an h-principle, and an immersed cobordism can always be turned into an embedded one by removing all transverse double points by Lagrangian 0-surgery (which are the only singularities after applying a small perturbation).

1.7. Relation to other work

As mentioned before, one important source of inspiration for our construction is the surgery construction for Legendrian submanifolds appearing in [6]. The local model for the immersed Lagrangian handle we use can be traced back to [1, 2], where it appears in a slightly different guise. It seems that the passage from L to $L^\natural$ in the case $n = 2$ and $k = 1$ is identical to an operation described in [15]. The article [12] is an exploration of relations between Lagrangian surgery and Lagrangian cobordisms in a different direction. Cobordisms as described in Theorem 1.4 have recently also been constructed by Jeff Hicks [10] using Lefschetz fibrations.

2. Lagrangian antisurgery

In this section we will explain the construction of immersed Lagrangian $(k+1)$-handles $\Gamma \subset T^*\mathbb{R} \times T^*\mathbb{R}^n$ for $0 \leq k \leq n-1$, which will serve as the local models for the construction of the cobordisms appearing in Theorem 1.1. Theses handles are immersed Lagrangian cobordisms

$$\Gamma : \Lambda' \rightsquigarrow \Lambda$$

diffeomorphic to $D^{k+1} \times D^{n-k}$ and whose ends are Lagrangian submanifolds $\Lambda \approx S^k \times D^{n-k}$ and $\Lambda' \approx D^{k+1} \times S^{n-k-1}$ of $T^*\mathbb{R}^n$. The construction is inspired by a similar one in [6] (see Section 2.4 for the precise relationship).

Throughout, we will use the standard symplectic form on $T^*\mathbb{R} \times T^*\mathbb{R}^n$ given by

$$\omega_{\text{std}} = dx_0 \wedge dy_0 + \sum_{i=1}^{n} dx_i \wedge dy_i,$$

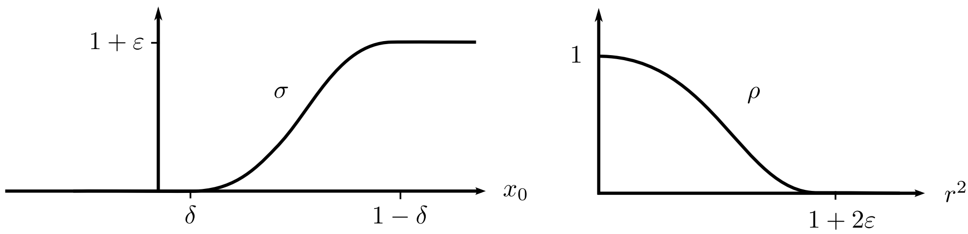

Figure 1: The auxiliary functions $\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ used in the construction of the Lagrangian handles Γ which will serve as the local models for the Lagrangian antisurgery cobordisms.

where x_0, y_0 and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are the usual coordinates on $T^*\mathbb{R}$ resp. $T^*\mathbb{R}^n$.

2.1. Construction of Γ

The handle Γ will be defined as the union of the graphs of exact 1-forms $+dF$ and $-dF$, where $F : \mathfrak{U} \to \mathbb{R}$ is a function defined on a certain subset $\mathfrak{U} \subset \mathbb{R} \times \mathbb{R}^n$.

As a first step in defining $\mathfrak{U}$ and F, consider smooth functions $\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

1) $\sigma(x_0) = 0$ for $x_0 \leq \delta$,
2) $\sigma(x_0) = 1 + \varepsilon$ for $x_0 \geq 1 - \delta$,
3) $\sigma'(x_0) > 0$ for $\delta < x_0 < 1 - \delta$,

and

1) $\rho(r^2) = 1$ for r^2 close to 0,
2) $\rho(r^2) = 0$ for $r^2 \geq 1 + 2\varepsilon$,
3) $-1/(1+\varepsilon) < \rho'(r^2) \leq 0$ for all $r^2 \in \mathbb{R}_{\geq 0}$

for certain small constants $\varepsilon, \delta > 0$. Denote by $r^2, s^2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the functions given by $r^2(\mathbf{x}) = x_1^2 + \cdots + x_{k+1}^2$ and $s^2(\mathbf{x}) = x_{k+2}^2 + \cdots + x_n^2$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then define a function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$f(x_0, \mathbf{x}) = r^2 + \sigma(x_0)\rho(r^2) - s^2 - 1 \tag{2}$$

for $(x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$, where $r^2 \equiv r^2(\mathbf{x})$ and $s^2 \equiv s^2(\mathbf{x})$.

Consider now the set $\mathfrak{U} = \{(x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n \mid f(x_0, \mathbf{x}) \geq 0\}$ and define $F : \mathfrak{U} \to \mathbb{R}$ by

$$F(x_0, \mathbf{x}) = f(x_0, \mathbf{x})^{3/2}. \tag{3}$$

The restriction of F to $\operatorname{int}(\mathfrak{U})$ is smooth, with differential given by

$$dF = \frac{3}{2} f(x_0, \mathbf{x})^{1/2} \big(\sigma'(x_0)\rho(r^2)dx_0 + (1 + \sigma(x_0)\rho'(r^2))dr^2 - ds^2\big). \tag{4}$$

Note that dF extends to a section of $T^*\mathbb{R} \times T^*\mathbb{R}^n$ defined on all of $\mathfrak{U}$ which vanishes along $\partial\mathfrak{U} = \{f(x_0, \mathbf{x}) = 0\}$; we will denote this extended section by dF as well. The graphs $\Gamma_\pm \subset T^*\mathbb{R} \times T^*\mathbb{R}^n$ of $\pm dF : \mathfrak{U} \to T^*\mathbb{R} \times T^*\mathbb{R}^n$ are Lagrangian submanifolds with boundary, and the tangent spaces along the boundary are given by

$$T\Gamma_\pm|_{\partial\Gamma_\pm} = T(N^*(\partial\mathfrak{U}))|_{\partial\mathfrak{U}},$$

where $N^*(\partial\mathfrak{U})$ denotes the conormal bundle of $\partial\mathfrak{U}$. Hence Γ_+ and Γ_- fit together smoothly along $\partial\mathfrak{U}$, in the sense that their union

$$\Gamma = \Gamma_+ \cup \Gamma_- = \{((x_0, \mathbf{x}), \pm dF(x_0, \mathbf{x})) \mid (x_0, \mathbf{x}) \in \mathfrak{U}\} \tag{5}$$

is a submanifold of $T^*\mathbb{R} \times T^*\mathbb{R}^n$ which is embedded near $\partial\mathfrak{U}$. The singular locus $\Gamma^s \subset \Gamma$ along which Γ is not embedded is the set of points $(x_0, \mathbf{x}) \in \operatorname{int}\mathfrak{U}$ at which dF vanishes, which is given by

$$\Gamma^s = \{((x_0, 0), (0, 0)) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \mid x_0 \geq 1 - \delta\}, \tag{6}$$

see Proposition 4.1; that is, Γ^s is a 1-dimensional family of double points.

Γ is the immersed image of a $(k+1)$-handle $D^{k+1} \times D^{n-k}$, and moreover an immersed Lagrangian cobordism

$$\Gamma : \Lambda' \rightsquigarrow \Lambda$$

in the sense of Section 1.3. To see the latter and to describe the ends, set $\mathfrak{U}_{x_0} = \{\mathbf{x} \in \mathbb{R}^n \mid (x_0, \mathbf{x}) \in \mathfrak{U}\}$ for $x_0 \in \mathbb{R}$ and define $F_{x_0} : \mathfrak{U}_{x_0} \to \mathbb{R}$ to be the function given by $F_{x_0}(\mathbf{x}) = F(x_0, \mathbf{x})$ for $\mathbf{x} \in \mathfrak{U}_{x_0}$. Since F_{x_0} is independent of x_0 if either $x_0 \leq \delta$ or $x_0 \geq 1 - \delta$, it follows that the part of Γ lying over

$(-\infty,\delta]\times\mathbb{R}\cup[1-\delta,\infty)\times\mathbb{R}\subset T^*\mathbb{R}$ is

$$(-\infty,\delta]\times\{0\}\times\Lambda \quad\cup\quad [1-\delta,\infty)\times\{0\}\times\Lambda' \tag{7}$$

with

$$\begin{aligned}\Lambda&=\{(\mathbf{x},\pm dF_0(\mathbf{x}))\in T^*\mathbb{R}^n \mid \mathbf{x}\in\mathfrak{U}_0\},\\ \Lambda'&=\{(\mathbf{x},\pm dF_1(\mathbf{x}))\in T^*\mathbb{R}^n \mid \mathbf{x}\in\mathfrak{U}_1\}.\end{aligned} \tag{8}$$

This shows that Γ is a Lagrangian cobordism (up to modifying the ends in an obvious way).

2.2. Isotropic surgery discs

We will now describe the situation in which it is possible to implant the local model described above, such as to produce from a given Lagrangian L a new immersed Lagrangian L' together with a Lagrangian cobordism $V : L' \rightsquigarrow L$.

The following definition is an adaptation of Definition 4.2 in [6] to our setting.

Definition 2.1. Let $L\subset M$ be a Lagrangian submanifold and let $S\subset L$ be a embedded k-sphere with trivializable normal bundle. An *isotropic surgery disc* for S consists of the following data:

1) An embedded isotropic $(k+1)$-disc $D\subset M$ such that
   - $\partial D = S$,
   - $\operatorname{int} D\cap L=\emptyset$,
   - any vector field $X\subset TD|_S$ which is outward-pointing normal to $S=\partial D$ is nowhere contained in TL.
2) A symplectic subbundle E of $(TD)^\omega$ such that $TD\oplus E=(TD)^\omega$, and a symplectic trivialization $\Psi : D\times\mathbb{C}^{n-k-1}\to E$ such that the Lagrangian subbundle $\Psi(S\times\mathbb{R}^{n-k-1})$ of $E|_S$ is contained in $TL|_S$.

We will usually denote isotropic surgery discs simply by D, omitting the bundle E and its trivialization Ψ from the notation.

An isotropic surgery disc $D\equiv(D,E,\Psi)$ for a sphere $S\subset L$ determines a homotopy class of trivializations of the normal bundle of $S\subset L$ as follows: Let $Y\subset TL|_S$ be a vector field which is normal to $S\subset L$ and such that $\omega(X,Y)>0$ for a vector field $X\subset TD|_S$ which is outward-pointing normal to S (such a vector field Y exists due to the assumption on such

X in Definition 2.1). Then the subbundle $\Psi(S \times \mathbb{R}^{n-k-1}) \oplus \mathbb{R}Y$ of $TL|_S$ is complementary to TS and of rank $n-k$, and thus it spans the normal bundle of $S \subset L$. Since the space of all such vector fields Y is non-empty and contractible, the corresponding trivialization is determined up to homotopy.

Example 1. The prototypical example for the situation described in Definition 2.1 is given by the Lagrangian $\Lambda \subset T^*\mathbb{R}^n$ described in (8) and the k-sphere

$$S_0 = \{(\mathbf{x}, \mathbf{y}) \in T^*\mathbb{R}^n \mid x_1^2 + \cdots + x_{k+1}^2 = 1, \\ x_{k+2} = \cdots = x_n = 0, \mathbf{y} = 0\}; \tag{9}$$

the obvious choice of isotropic surgery disc for $S_0 \subset \Lambda$ is

$$D_0 = \{(\mathbf{x}, \mathbf{y}) \in T^*\mathbb{R}^n \mid x_1^2 + \cdots + x_{k+1}^2 \leq 1, \\ x_{k+2} = \cdots = x_n = 0, \mathbf{y} = 0\} \tag{10}$$

together with the symplectic subbundle

$$E_0 = \langle \partial_{x_{k+2}}, \ldots, \partial_{x_n}, \partial_{y_{k+2}}, \ldots, \partial_{y_n} \rangle \tag{11}$$

of $(TD_0)^\omega$ and the identification $\Psi_0 : D_0 \times \mathbb{C}^{n-k-1} \to E_0$ taking $D_0 \times \mathbb{R}^{n-k-1}$ to the subbundle $\langle \partial_{x_{k+2}}, \ldots, \partial_{x_n} \rangle$ and $D_0 \times i\mathbb{R}^{n-k-1}$ to the subbundle $\langle \partial_{y_{k+2}}, \ldots, \partial_{y_n} \rangle$.

Assume that we are in the situation of Definition 2.1, i.e., that we have a Lagrangian L with a sphere $S \subset L$ and a corresponding isotropic surgery disc $D \equiv (D, E, \Psi)$. Let $\phi : D_0 \to D$ be a diffeomorphism; together with the symplectic trivialization $\Psi : D \times \mathbb{C}^{n-k-1} \times E$, this determines an isomorphism of symplectic vector bundles $T^*D_0 \oplus E_0 \cong T^*D \oplus E$ (here we use the notation of Example 1). An application of the isotropic neighbourhood theorem then yields an extension of ϕ to a symplectomorphism

$$\phi : \mathcal{W}_0 \to \mathcal{W}$$

between appropriate Darboux-Weinstein neighbourhoods $\mathcal{W}_0 \supset D_0$ and $\mathcal{W} \supset D$ of the discs in $T^*\mathbb{R}^n$ resp. M, and we may assume that this extension satisfies

$$\phi(\Lambda \cap \mathcal{W}_0) = L \cap \mathcal{W}. \tag{12}$$

To see this, note that the condition that the outward normal vector field to $S \subset D$ is nowhere tangent to L guarantees that one can arrange $D\phi(T\Lambda|_{S_0}) = TL|_S$; after adjusting ϕ by a Hamiltonian isotopy and possibly shrinking the Weinstein neighbourhoods, one obtains (12).

2.3. Implantation of the local model and proof of Theorem 1.1

We now explain how to implant the local model and give the definition of Lagrangian antisurgery and of the corresponding Lagrangian cobordism. This will complete the proof of Theorem 1.1 (up to the description of the singular locus of the cobordism, for which we refer to Proposition 4.1).

As before, we assume that we have a Lagrangian L containing an embedded sphere $S \subset L$ together with an isotropic surgery disc D. To prepare the construction, consider the neighbourhood of $D_0 \subset T^*\mathbb{R}^n$ (see Example 1) given by

$$\mathcal{U}_0 = \left\{ (\mathbf{x}, \mathbf{y}) \in T^*\mathbb{R}^n \mid r^2 < 1 + 2\varepsilon,\ s^2 < 2\varepsilon,\ \|\mathbf{y}\|^2 < 6\sqrt{2\varepsilon}(1 + 4\varepsilon) \right\} \tag{13}$$

and denote by $\mathcal{U}_0^c$ the complement of $\mathcal{U}_0$ in $T^*\mathbb{R}^n$. The following lemma shows that the part of the model cobordism Γ that projects to $\mathcal{U}_0^c \subset T^*\mathbb{R}^n$ lies over $\mathbb{R} \subset T^*\mathbb{R}$ and is "cylindrical":

Lemma 2.1. *We have* $\Gamma \cap (T^*\mathbb{R} \times \mathcal{U}_0^c) = \mathbb{R} \times (\Lambda \cap \mathcal{U}_0^c) = \mathbb{R} \times (\Lambda' \cap \mathcal{U}_0^c)$.

Proof. We first claim that for $((x_0, \mathbf{x}), (y_0, \mathbf{y})) = ((x_0, \mathbf{x}), \pm dF(x_0, \mathbf{x})) \in \Gamma$ with $r^2 < 1 + 2\varepsilon$, we already have $(\mathbf{x}, \mathbf{y}) \in \mathcal{U}_0$. To see that, recall that the set $\mathfrak{U} \subset \mathbb{R} \times \mathbb{R}^n$ over which Γ lives is characterized by $f(x_0, \mathbf{x}) \geq 0$, where $f(x_0, \mathbf{x}) = r^2 + \sigma(x_0)\rho(r^2) - s^2 - 1$. Since $r^2 \mapsto r^2 + \sigma(x_0)\rho(r^2) - 1$ is strictly increasing (this follows from the assumptions that $\sigma(x_0) \leq 1 + \varepsilon$ for all x_0 and $-1/(1+\varepsilon) < \rho'(r^2)$ for all r^2) with value 2ε at $r^2 = 1 + 2\varepsilon$ for every $x_0 \in \mathbb{R}$, it follows that $s^2 < 2\varepsilon$. Moreover, one can read off from the expression (4) for $dF(x_0, \mathbf{x})$ that the bound on $\|\mathbf{y}\|^2$ is satisfied whenever $r^2 < 1 + 2\varepsilon$ and $s^2 < 2\varepsilon$. Let now $((x_0, \mathbf{x}), (y_0, \mathbf{y})) \in \Gamma \cap (T^*\mathbb{R} \times \mathcal{U}_0^c)$. As a consequence of the claim above, we obtain $r^2 \geq 1 + 2\varepsilon$, and hence (4) for $dF(x_0, \mathbf{x})$ simplifies to $dF(x_0, \mathbf{x}) = \frac{3}{2}(r^2 - s^2 - 1)^{1/2}(dr^2 - ds^2)$, as $\rho(r^2) \equiv 0$ for $r^2 \geq 1 + 2\varepsilon$. Since this has vanishing dx_0 component $y_0 = 0$ and is independent of x_0, it follows that $((x_0, \mathbf{x}), (y_0, \mathbf{y})) = ((x_0, \mathbf{x}), (0, \mathbf{y}))$ lies in $\mathbb{R} \times (\Lambda \cap \mathcal{U}_0^c)$ and in $\mathbb{R} \times (\Lambda' \cap \mathcal{U}_0^c)$. Thus $\Gamma \cap (T^*\mathbb{R} \times \mathcal{U}_0^c)$ is contained in both of these sets. The inclusions in the other direction are obvious. $\square$

The neighbourhood $\mathcal{U}_0$ of D_0 can be made arbitrarily small by letting the parameter ε tend to zero. In particular, by choosing the parameter ε sufficiently small, we may assume that the closure $\overline{\mathcal{U}_0}$ is contained in a Weinstein neighbourhood $\mathcal{W}_0$ of D_0 as described in Section 2.2, i.e., such that we have a symplectic identification $\phi : \mathcal{W}_0 \to \mathcal{W}$ with a Weinstein neighbourhood $\mathcal{W}$ of D.

Definition 2.2. Given such choices of ε and ϕ, we define the immersed Lagrangian $L' \subset M$ obtained from L by *Lagrangian k-antisurgery* along the isotropic disc D by

$$L' = (L \cap \mathcal{W}^c) \;\cup\; \phi(\Lambda' \cap \mathcal{W}_0), \tag{14}$$

and its *immersed Lagrangian trace* $V : L' \rightsquigarrow L$ by

$$V = \mathbb{R} \times (L \cap \mathcal{W}^c) \;\cup\; (\mathrm{id} \times \phi)(\Gamma \cap (T^*\mathbb{R} \times \mathcal{W}_0)), \tag{15}$$

using the symplectomorphism $\mathrm{id} \times \phi : T^*\mathbb{R} \times \mathcal{W}_0 \to T^*\mathbb{R} \times \mathcal{W}$.

The fact that the pieces which we glue in fit together as required is a consequence of Lemma 2.1, which implies that $\phi(\Lambda' \cap (\mathcal{W}_0 \setminus \overline{\mathcal{U}_0})) = L \cap (\mathcal{W} \setminus \overline{\mathcal{U}})$ and $(\mathrm{id} \times \phi)(\Gamma \cap (T^*\mathbb{R} \times (\mathcal{W}_0 \setminus \overline{\mathcal{U}_0}))) = \mathbb{R} \times (L \cap (\mathcal{W} \setminus \overline{\mathcal{U}}))$, where $\mathcal{U} = \phi(\mathcal{U}_0)$; hence the pieces of Λ' resp. Γ we glue overlap with corresponding pieces of L resp. $\mathbb{R} \times L$ as required.

The Lagrangian submanifold $L' \subset M$ given by Definition 2.2 is the immersed image of the manifold obtained from L by a k-surgery along S with respect to the trivialization of the normal bundle of $S \subset L$ determined by the surgery disc D. The Lagrangian cobordism $V : L' \rightsquigarrow L$ is the immersed image of the trace corresponding to that surgery.

2.4. Relation to the construction in [6]

The Lagrangian handle Γ constructed in this section is closely related to the Lagrangian handle constructed by Dimitroglou-Rizell in [6, Section 4]. To make the connection precise, consider the function

$$\hat{F} : \mathfrak{U} \to \mathbb{R}, \quad (x_0, \mathbf{x}) \mapsto F(x_0, \mathbf{x}) \cdot x_0,$$

where $\mathfrak{U} \subset \mathbb{R} \times \mathbb{R}^n$ and $F : \mathfrak{U} \to \mathbb{R}$ are as defined in Section 2.1. The handle in [6] is obtained by gluing the graphs $\hat{\Gamma}_{\pm} \subset T^*(\mathbb{R} \times \mathbb{R}^n)$ of $\pm d\hat{F} : \mathfrak{U} \to$

$T^*(\mathbb{R}\times\mathbb{R}^n)$, in analogy with the construction of Γ in 5, such as to obtain

$$\hat{\Gamma}=\hat{\Gamma}_-\cup\hat{\Gamma}_+=\{((x_0,\mathbf{x}),\pm d\hat{F}(x_0,\mathbf{x}))\mid (x_0,\mathbf{x})\in\mathfrak{U}\}.$$

The part of $\hat{\Gamma}$ lying over $\mathfrak{U}\cap\{x_0>0\}$ is then mapped to $\mathbb{R}\times J^1(\mathbb{R}^n)$, the symplectization of the 1-jet space of $\mathbb{R}^n$, using a symplectic identification $T^*(\mathbb{R}_{>0}\times\mathbb{R}^n)\cong\mathbb{R}\times J^1(\mathbb{R}^n)$, which results in a Lagrangian cobordism with ends which are cylindrical over *Legendrian* submanifolds of $J^1(\mathbb{R}^n)$ (see e.g. the introduction of [6] for the relevant definitions). In particular, the result is not a Lagrangian cobordism in the sense of [3] (and neither is $\hat{\Gamma}$), as it does not have the required cylindrical ends described in Section 1.3.

3. Lagrangian antisurgery and surgery

In this section we explain the relationship between Lagrangian $(n-1)$-antisurgery and classical Lagrangian 0-surgery [14]. We start by recalling the construction of Lagrangian 0-surgery as described e.g. in [3, Section 6.1].

3.1. Classical Lagrangian 0-surgery

Let $\gamma=(a,b):\mathbb{R}\to T^*\mathbb{R}$ be an embedded smooth curve satisfying

$$\begin{aligned}&\gamma(t)=(t,0),\ t\in(-\infty,-\kappa],\\ &a(t)<0<b(t),\ t\in(-\kappa,\kappa),\\ &\gamma(t)=(0,t),\ t\in[\kappa,\infty),\end{aligned}\tag{16}$$

where $\kappa>0$ is a small parameter, see Figure 2. Then consider the embedding

$$h_\gamma:\mathbb{R}\times S^{n-1}\to T^*\mathbb{R}^n,\quad (t,\mathbf{x})\mapsto(a(t)\mathbf{x},b(t)\mathbf{x})\tag{17}$$

where $S^{n-1}=\{\mathbf{x}\in\mathbb{R}^n\mid \|\mathbf{x}\|_2=1\}$ is the unit sphere in $\mathbb{R}^n$. The image of h_γ is an embedded Lagrangian submanifold of $T^*\mathbb{R}^n$ which outside of the ball B^{2n}_κ of radius κ centered at $0\in T^*\mathbb{R}^n$ coincides with $\mathbb{R}^n\times\{0\}\cup\{0\}\times\mathbb{R}^n$. We can also view this Lagrangian as the orbit of γ, viewed now as living in $T^*\mathbb{R}\times\{0\}\subset T^*\mathbb{R}^n$, under the $SO(n)$-action on $T^*\mathbb{R}^n$ given by

$$A(\mathbf{x},\mathbf{y})=(A\mathbf{x},A\mathbf{y})\tag{18}$$

for $A\in SO(n)$.

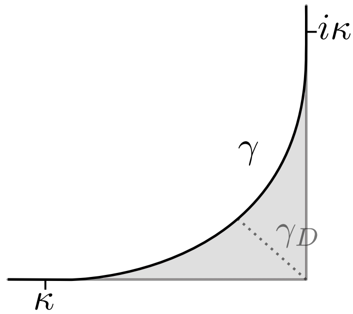

Figure 2: A curve γ of the type used in the definition of Lagrangian 0-surgery.

Given now a Lagrangian $L \subset M$ with a transverse double point $p \in L$, we can implant this local model using a Darboux chart which identifies neighbourhoods of the sheets of L meeting at p with neighbourhoods of 0 in $\mathbb{R}^n \times \{0\}$ resp. in $\{0\} \times \mathbb{R}^n$. The result is a new Lagrangian submanifold

$$L^\sharp \subset M. \tag{19}$$

For a fixed choice of Darboux chart, any two choices of γ subject to the specification in (16) are related by isotopies that are constant outside of compact sets, and these induce Lagrangian isotopies of the corresponding versions of $L^\sharp$. Modifying this specification so that $-\gamma \cup \gamma$ lies in the first and third quadrants of $T^*\mathbb{R}$, instead of the second and fourth ones as in the description above, has the same effect as reversing the order of the sheets in the above sense and leads results in a second family of resolutions which are Lagrangian isotopic to one another. Resolutions which do not belong to the same family are usually not globally Lagrangian or even smoothly isotopic, and sometimes even distinct as smooth manifolds (e.g., orientable in one case, but non-orientable in the other case).

In the case that L is the union of two Lagrangian submanifolds L_- and L_+ that intersect transversely at $p \in L_- \cap L_+$, we denote by

$$L_- \# L_+$$

the result of any 0-surgery resulting from implanting the local model in such a way that locally L_- gets identified with $\mathbb{R}^n \times \{0\}$ while L_+ gets identified with $\{0\} \times \mathbb{R}^n$.[1]

[1]This is the convention used e.g. in [3]. Other papers, such as [14], use the opposite convention.

Within each family of resolutions which are Lagrangian isotopic through such "local" isotopies (i.e., induced by isotopies of 0-surgery models), the obstruction to being Hamiltonian isotopic is the difference between their sizes in the sense of the following definition:

Definition 3.1. Let $L^\natural$ be the resolution a transverse double point $p \in L$ obtained by implanting a local model for Lagrangian 0-surgery as described above. We define the *size* of the resolution to be the symplectic area between the curve γ and the coordinate axes in the local model (the shaded region in Figure 2).

We note that one could give a more flexible definition of Lagrangian 0-surgery including versions with non-positive size (by removing the requirement that γ be contained in a quadrant); however, we only consider surgeries of positive size.

3.2. Lagrangian 0-surgery and $(n-1)$-antisurgery

We now explain the connection between Lagrangian 0-surgery and Lagrangian $(n-1)$-antisurgery, and how to see that the two operations are inverse to one another up to Lagrangian isotopy.

Recall that Lagrangian $(n-1)$-antisurgery replaces an embedded copy of $\Lambda \cong S^{n-1} \times D^1$ in a Lagrangian L by an immersed copy of $\Lambda' \cong D^n \times S^0$, in which the two copies of D^n intersect transversely at a double point p in the resulting Lagrangian L'. This double point can be resolved by Lagrangian 0-surgery. To see that this resolution can be implemented in such a way that the resulting Lagrangian[2] $L^\flat$ is Lagrangian isotopic to L, we first consider the local situation for the case $n = 1$, for which the antisurgery model is depicted in Figure 3. The green arcs are the parts of Λ which, when performing antisurgery, are cut out and replaced by the singular red part such as to produce Λ'. If one applies Lagrangian 0-surgery to Λ' in such a way that the curve γ in Figure 2 gets mapped to the dotted blue arc in Figure 3, then the resulting non-singular Lagrangian $\Lambda^\flat$ is evidently Lagrangian isotopic to the

[2]The notation $L^\flat$ is chosen to distinguish this resolution from $L^\natural$, the one that comes up in Theorem 1.2 (as they belong to distinct families of resolutions, see Proposition 4.4).

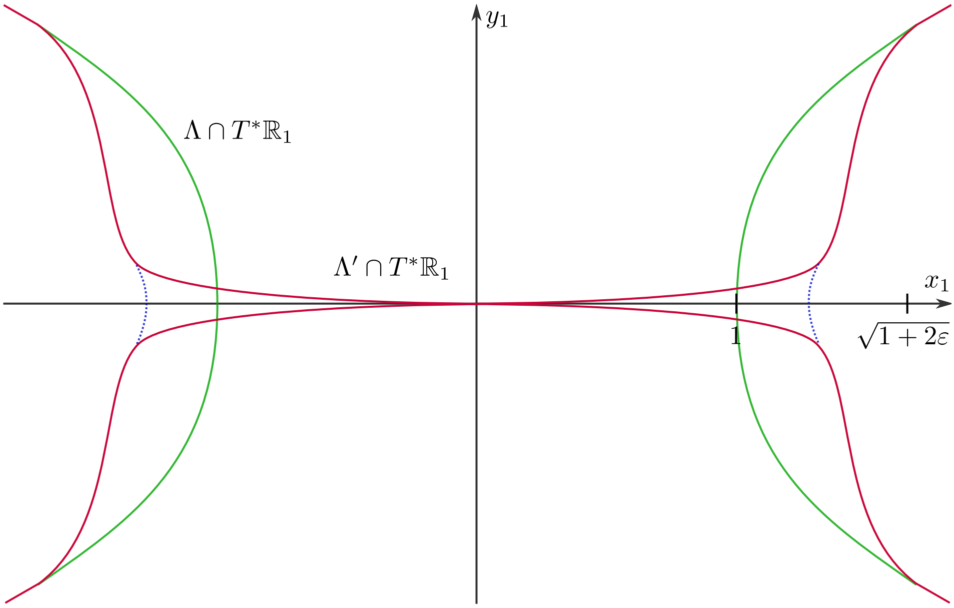

Figure 3: The figure shows the intersection of $\Lambda \subset T^*\mathbb{R}^n$, the local model for the non-singular end of antisurgery (in green), and $\Lambda' \subset T^*\mathbb{R}^n$, the local model for the singular end of antisurgery (in red), with the plane $T^*\mathbb{R}_1 \subset T^*\mathbb{R}^n$. By performing Lagrangian 0-surgery along the dotted arc, one can reverse the effect of $(n-1)$-antisurgery up to Lagrangian isotopy.

original Λ. Note that $\Lambda^\flat$ belongs to the family of resolutions modelled by

$$\lambda_+ \# \lambda_-,$$

where $\lambda_\pm = T_{(0,0)}\Lambda'_\pm$ are the tangent spaces to $\Lambda'_\pm = \{(\mathbf{x}, \pm dF_1(\mathbf{x})) \in T^*\mathbb{R}^n \mid \mathbf{x} \in \mathfrak{U}_1\}$, the sheets of the singular end of the antisurgery cobordism (cf. Section 2.1); this can be read off Figure 3, taking into account that Λ'_+ is the sheet which is contained in the first and third quadrants, while Λ_- is the sheet contained in the second and fourth quadrants.

In order to transfer this observation to the case $n > 1$, note that the local models for both $(n-1)$-antisurgery and 0-surgery are orbits of the respective one-dimensional local models (viewed as living in the (x_1, y_1)-coordinate subspace $T^*\mathbb{R}_1$ of $T^*\mathbb{R}^n$), under the $SO(n)$-action described in (18). One can use this action to extend a Lagrangian isotopy between the curves $\Lambda \cap T^*\mathbb{R}_1$ and $\Lambda^\flat \cap T^*\mathbb{R}_1$ in $T^*\mathbb{R}_1$ to a Lagrangian isotopy between Λ and $\Lambda^\flat$ in $T^*\mathbb{R}^n$ in an $SO(n)$-equivariant way.

Conversely, if we first perform Lagrangian 0-surgery at a transverse double point p of a Lagrangian L' to obtain a new Lagrangian L, we can reverse this operation by $(n-1)$-antisurgery after applying a suitable Lagrangian isotopy to L. The Lagrangian disc required for that can be constructed by applying the $SO(n)$-action described in (18) to an embedded curve $\gamma_D : [0,1] \to T^*\mathbb{R}$ connecting a point on $\mathrm{im}(\gamma)$ to $0 \in T^*\mathbb{R}$ in the local model for 0-surgery, see Figure 2.

3.3. Lagrangian vs. Hamiltonian isotopy

To obtain a more refined picture, and in particular to show that the result of successively applying Lagrangian $(n-1)$-antisurgery and 0-surgery to a Lagrangian L is generally not *Hamiltonian* isotopic to L, we will compare the symplectic areas bounded by the ends of the local model for $(n-1)$-antisurgery.

More precisely, we will compute the difference between the areas of the bounded regions enclosed by the curves $\Lambda \cap T^*\mathbb{R}_1$ resp. $\Lambda' \cap T^*\mathbb{R}_1$ and the line $\{x_1 = \sqrt{1+2\varepsilon}\}$, i.e., twice the difference between the two shaded areas in Figure 4. The description of the ends in Section 2.1 shows that

$$\begin{aligned}\Lambda \cap T^*\mathbb{R}_1 &= \{(x_1, \pm dF_0(x_1) \in T^*\mathbb{R}_1 \mid x_1 \geq 1\}\\ \Lambda' \cap T^*\mathbb{R}_1 &= \{(x_1, \pm dF_1(x_1) \in T^*\mathbb{R}_1 \mid x_1 \geq 0\},\end{aligned}$$

where $F_0(x_1) = (x_1^2 - 1)^{3/2}$ and $F_1(x_1) = (x_1^2 + (1+\varepsilon)\rho(x_1^2) - 1)^{3/2}$. Since $F_0(x_1) = F_1(x_1)$ for $x_1 \geq \sqrt{1+2\varepsilon}$, the area difference we are interested in is

$$\begin{aligned}2\left(\int_1^{\sqrt{1+2\varepsilon}} \frac{\partial F_0}{\partial x_1} dx_1 - \int_0^{\sqrt{1+2\varepsilon}} \frac{\partial F_1}{\partial x_1} dx_1\right) &= 2\Big(F_0(\sqrt{1+2\varepsilon}) - F_0(1)\Big)\\ &\quad - 2\Big(F_1(\sqrt{1+2\varepsilon}) - F_1(0)\Big)\\ &= 2\Big((2\varepsilon)^{3/2} - (2\varepsilon)^{3/2} + \varepsilon^{3/2}\Big)\\ &= 2\varepsilon^{3/2},\end{aligned}$$

using that $\rho(0) = 1$ and $\rho(1+2\varepsilon) = 0$. In particular, this computation justifies that the singular end is depicted as lying below the nonsingular end near $x_1 = \sqrt{1+2\varepsilon}$ in Figure 3.

Note that an application of Lagrangian 0-surgery to resolve the double point of the singular end Λ' leads to a Lagrangian $\Lambda^\flat$ which together with

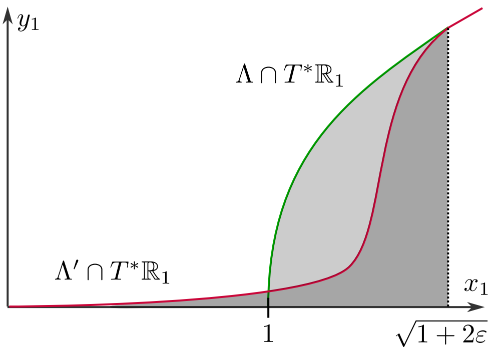

Figure 4: The areas enclosed by $\Lambda \cap T^*\mathbb{R}_1$ resp. $\Lambda' \cap T^*\mathbb{R}_1$ and the line $\{x_1 = \sqrt{1+2\varepsilon}\}$, where Λ and Λ' are the ends of model cobordism Γ corresponding to $(n-1)$-antisurgery.

the line $\{x_1 = 1 + 2\varepsilon\}$ bounds *less* area than the local model Λ' for the non-singular end. Consider now again a Lagrangian $L^\flat$ resulting from applying successively Lagrangian $(n-1)$-antisurgery and 0-surgery to a Lagrangian L. As a consequence of this local picture, any Lagrangian isotopy from L to $L^\flat$ that comes from a Lagrangian isotopy between the respective curves in the local model has non-vanishing flux and is therefore not Hamiltonian.

3.4. Lagrangian surgery of higher index

Lagrangian antisurgery produces from a Lagrangian submanifold L a new Lagrangian submanifold L' with an additional double point. Switching the rôles of input and output, we can interpret L as the result of an operation which resolves a singularity of L' by replacing an immersed copy of $D^{n-p} \times S^p \subset L'$ by an embedded copy of $S^{n-p-1} \times D^{p+1} \subset L$. In the case $p = 0$, i.e., when L' is the result of Lagrangian $(n-1)$-antisurgery on L, we have seen in Section 3.1 that the inverse operation is classical Lagrangian 0-surgery up to Lagrangian isotopy. In view of that, we propose the following definition:

Definition 3.2. We say that L can be obtained from L' by *Lagrangian p-surgery* if L' can be obtained by applying Lagrangian $(n-p-1)$-antisurgery a Lagrangian $\widetilde{L}$ which is Lagrangian isotopic to L.

It would be interesting to characterize the admissibility of a given immersed Lagrangian L' for Lagrangian p-surgery by a geometric condition analogous to the existence of an isotropic surgery disc. A necessary condition

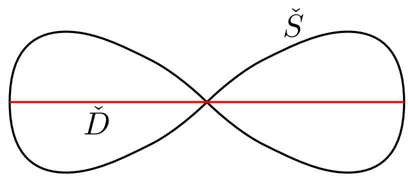

Figure 5: The central fibre of a Whitney degeneration.

is of course that L' contains an immersed copy of $D^{n-p} \times S^p$ which is obtained by implanting a suitable piece of the immersed Lagrangian $\Lambda' \subset T^*\mathbb{R}^n$ described in (8). Observe that the part of Λ' lying over $\{0\} \times \mathbb{R}^p$ is the image of a Whitney type immersion $S^p \to T^*\mathbb{R}^p \cong \{0\} \times T^*\mathbb{R}^p \subset T^*\mathbb{R}^n$, obtained from the standard Whitney immersion

$$S^p \to T^*\mathbb{R}^p, \quad (\mathbf{x}, y) \mapsto (\mathbf{x}, y\mathbf{x}) = (\mathbf{x}, \sqrt{1 - |\mathbf{x}|^2}\mathbf{x}) \tag{20}$$

for $(\mathbf{x}, y) \in S^p \subset \mathbb{R}^p \times \mathbb{R}$, by rescaling. This motivates the following definition:

Definition 3.3. Let $L' \subset M$ be a Lagrangian submanifold containing the image of a Lagrangian immersion $\iota : D^{n-p} \times S^p \to M$ which is an embedding away from $\{0\} \times S^p$ and such that $\check{S} = \iota(\{0\} \times S^p)$ has precisely one transverse double point. We call $\iota(D^{n-p} \times S^p) \subset L'$ a *Whitney degeneration* if the following holds: There exists an embedded isotropic p-disc $\check{D} \subset M$ with boundary on $\check{S}$ and containing the double point of $\check{S}$ in its interior, together with a Weinstein neighbourhood $\mathcal{N} \cong (T\check{D})^\omega / T\check{D} \oplus T^*\check{D}$ of $\check{D}$ such that upon a suitable symplectic identification of $\mathcal{N}$ with a subset of $T^*\mathbb{R}^{n-p} \times T^*\mathbb{R}^p$, $\check{S}$ is the image of a Whitney type immersion $S^p \to \{0\} \times T^*\mathbb{R}^p$ (see Figure 5).

There a two basic symplectic invariants that one can associate to a Whitney degeneration $\iota(D^{n-p} \times S^p) \subset L'$ with $p > 0$. The first one is the symplectic area of the element of $H_2(M, L')$ represented by the teardrop shown in Figure 5. The second one is the pair of Maslov indices of the discs created when resolving the double point by either of the two topologically different ways of Lagrangian 0-surgery; we will compute these Maslov indices for Whitney degenerations coming from Lagrangian antisurgery in Section 4.8. Note that, in view of these computations, the second invariant yields an obstruction to resolving a given Whitney degeneration $\iota(D^{n-p} \times S^p) \subset L'$ by Lagrangian p-surgery.

As an example for the fact that containing a Whitney degeneration is not sufficient for being able to perform Lagrangian p-surgery, consider the Whitney sphere $S^n_{Wh} \subset T^*\mathbb{R}^n$ itself. It obviously contains Whitney degenerations $\iota(D^{n-p} \times S^p)$ for every $0 \leq p \leq n-1$, but it is not possible to perform Lagrangian p-surgery on S^n_{Wh} for any $p > 0$: This would lead to a compact embedded Lagrangian $L \subset T^*\mathbb{R}^n$ with vanishing area class, which we know not to exist. Indeed, L would be diffeomorphic to $S^{n-p-1} \times S^{p+1}$, which has $H_1(L) = 0$ for $p \notin \{0, n-2\}$; in the case $p = n-2$, we would create an isotropic 2-disc whose boundary generates $H_1(L) \cong \mathbb{Z}$, and hence the area class would vanish as well.

4. Desingularization

Our aim in this section is to turn the immersed antisurgery cobordisms $V : L' \rightsquigarrow L$ constructed in Section 2.3 into embedded cobordisms $V^\natural : L^\natural \rightsquigarrow L$ by simultaneously resolving the singularities of V and L', and thus prove Theorem 1.2.

4.1. The singular loci of $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda'}$

Recall from Section 2.1 that the Lagrangian $(k+1)$-handle $\Gamma : \Lambda' \rightsquigarrow \Lambda$, which serves as the local model for the cobordisms corresponding to k-antisurgery, is defined as the union of the graphs of $\pm dF$ for a function $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. The locus Γ^s where Γ fails to be embedded is given by the points $(x_0, \mathbf{x}) \in \operatorname{int} \mathfrak{U}$ where $dF(x_0, \mathbf{x}) = 0$, which is where the graphs of $\pm dF$ intersect each other. Using this description, we obtain the following proposition.

Proposition 4.1. *The singular locus of the antisurgery handle $\Gamma : \Lambda' \rightsquigarrow \Lambda$ is given by*

$$\Gamma^s = \{((x_0, 0), (0, 0)) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \mid x_0 \geq 1 - \delta\}. \tag{21}$$

Proof. Recall from Section (2.1) that

$$dF = \frac{3}{2} f(x_0, \mathbf{x})^{1/2} \big(\sigma'(x_0)\rho(r^2)dx_0 + (1 + \sigma(x_0)\rho'(r^2))dr^2 - ds^2\big).$$

Since $f(x_0, \mathbf{x}) > 0$ for $(x_0, \mathbf{x}) \in \operatorname{int}\mathfrak{U}$, the vanishing of $dF(x_0, \mathbf{x})$ is equivalent to

$$\begin{aligned} \sigma'(x_0)\rho(r^2)dx_0 &= 0, \\ (1+\sigma(x_0)\rho'(r^2))dr^2 &= 0, \\ ds^2 &= 0. \end{aligned} \tag{22}$$

The conditions imposed on σ and ρ imply that these equations are simultaneously satisfied if and only if $x_0 \geq 1-\delta$ and $\mathbf{x} = 0$, meaning that the singular locus of Γ is as described in (21). To see this, note that the conditions $\sigma(x_0) \geq 0$ and $-1/(1+\varepsilon) < \rho'(r^2) \leq 0$ imply $1+\sigma(x_0)\rho'(r^2) > 0$, so the second equation in (22) can only hold when $dr^2 = 0$; together with the third equation $ds^2 = 0$, we conclude that $\mathbf{x} = 0$. Since $\rho(0) = 1$, the first equation simplifies to $\sigma'(x_0)dx_0 = 0$ and thus $x_0 \leq \delta$ or $x_0 \geq 1-\delta$; since $(x_0, 0) \notin \mathfrak{U}$ for $x_0 < 1-\delta$, only the second of these possibilities leads to a solution of (22). □

Recall that the part of Γ that lies over $[1-\delta, \infty) \times \mathbb{R} \subset T^*\mathbb{R}$ is cylindrical of the form

$$\Gamma|_{[1-\delta,\infty)\times\mathbb{R}} = [1-\delta, \infty) \times \{0\} \times \Lambda' \subset T^*\mathbb{R} \times T^*\mathbb{R}^n. \tag{23}$$

Proposition 4.1 therefore implies that the positive end Λ' of Γ has a double point at $\mathbf{x} = 0$ and is embedded away from that. The tangent spaces

$$\lambda_\pm = T_{(0,0)}\Lambda'_\pm \tag{24}$$

to the two sheets of Λ' at the double point are spanned by

$$\begin{aligned} &\partial_{x_i} \pm 3f_1(0)^{1/2}\partial_{y_i}, \quad i = 1, \ldots, k+1, \\ &\partial_{x_i} \mp 3f_1(0)^{1/2}\partial_{y_i}, \quad i = k+2, \ldots, n, \end{aligned} \tag{25}$$

where $f_1(0) = f(1,0)$; since $f_1(0) \neq 0$, this shows that λ^+ and λ^- intersect transversely.

The transverse double point of Λ' can be removed by Lagrangian 0-surgery such as to produce an embedded Lagrangian submanifold $\Lambda^\natural \subset T^*\mathbb{R}^n$. More interestingly, we will see below that one can resolve the singular locus Γ^s of Γ and turn the immersed cobordism $\Gamma : \Lambda' \rightsquigarrow \Lambda$ into an *embedded* Lagrangian cobordism $\Gamma^\natural : \Lambda^\natural \rightsquigarrow \Lambda$.

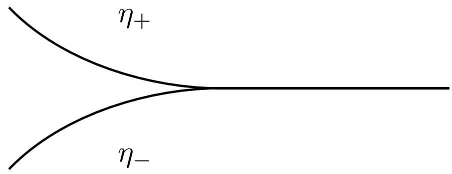

Figure 6: The singularity of Γ is modelled by $\eta_+ \times \lambda_+ \cup \eta_- \times \lambda_-$, where $\eta_\pm \subset T^*\mathbb{R}$ are the curves depicted here, and $\lambda_\pm \subset T^*\mathbb{R}^n$ are transversely intersecting Lagrangian subspaces.

4.2. Implantation of a Lagrangian 1-handle

To resolve the singular locus Γ^s of Γ, we replace a neighbourhood of Γ^s with a Lagrangian 1-handle arising as a subset of the *trace of surgery* cobordism $\mathbb{R}^n \# i\mathbb{R}^n \rightsquigarrow (\mathbb{R}^n, i\mathbb{R}^n)$ constructed in [3, Section 6.1].[3] The explanations below and Proposition 4.2 serve to justify that the cutting and isotoping in Biran–Cornea's construction can be performed in such a way as to exactly match up the resulting Lagrangian 1-handle with the given "boundary condition" that is created when one removes a neighbourhood of Γ^s from Γ.

Let $\eta_\pm : \mathbb{R} \to T^*\mathbb{R}$ be curves given by $\eta_\pm(x) = (x, \pm y(x))$, where $y : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a smooth function such that $y(x) > 0$ for $x < 0$ and $y(x) = 0$ for $x \geq 0$ (see Figure 6). Let $\lambda_\pm \subset T^*\mathbb{R}^n$ be two transversely intersecting Lagrangian subspaces. Then

$$W = \eta_+ \times \lambda_+ \cup \eta_- \times \lambda_-$$

is an immersed Lagrangian submanifold of $T^*\mathbb{R} \times T^*\mathbb{R}^n$ whose singular locus is $\mathbb{R}_{\geq 0} \times \{0\}$. We view W as an immersed Lagrangian cobordism $W : \lambda_- \cup \lambda_+ \rightsquigarrow (\lambda_-, \lambda_+)$ and will use it as our local model for a neighbourhood of Γ^s.

Proposition 4.2. *There exists an embedded Lagrangian cobordism $W^\natural : \lambda_- \# \lambda_+ \rightsquigarrow (\lambda_-, \lambda_+)$, which topologically is a 1-handle $\cong D^1 \times D^n$, such that $W^\natural$ and W coincide outside of an arbitrarily small neighbourhood of the singular locus $\mathbb{R}_{\geq 0} \times \{0\}$ of W.*

Proof. We will show how to perform the construction of Biran–Cornea's trace cobordism [3] corresponding to the Lagrangian surgery of $\lambda_\pm$ in such

[3] A Lagrangian 1-handle as needed here could also be obtained by modifying our construction of the 1-handle corresponding to $(n-1)$-antisurgery given in Section 2.1 in such a way that the components of the positive end lie over disjoint curves in $T^*\mathbb{R}$, thus making the 1-handle embedded.

way that the result agrees with W outside of an arbitrarily small neighbourhood of the singular locus $\mathbb{R}_{\geq 0} \times \{0\}$ of W. We assume that $\lambda_- = \mathbb{R}^n \times \{0\}$ and $\lambda_+ = \{0\} \times \mathbb{R}^n$ for notational simplicity.

Choose a curve $\gamma : \mathbb{R} \to T^*\mathbb{R}$, $\gamma(t) = (a(t), b(t))$, as in Section 3.1 and let $L = \lambda_- \# \lambda_+ \subset T^*\mathbb{R}^n$ be the result of the corresponding 0-surgery of $\lambda_\pm$. Then define

$$\phi_\gamma : \mathbb{R} \times S^n \to T^*\mathbb{R}^{n+1} \tag{26}$$

to be the composition of the map $\mathbb{R} \times S^n \to T^*\mathbb{R}^{n+1}$, $(t, \mathbf{x}) \mapsto (a(t)\mathbf{x}, b(t)\mathbf{x})$, with a rotation of the first factor of $T^*\mathbb{R}^{n+1} = T^*\mathbb{R} \times T^*\mathbb{R}^n$ by $\frac{\pi}{4}$. Let

$$W' = (\operatorname{im} \phi_\gamma)|_{\{(x,y) \in T^*\mathbb{R} | x \leq 0\}}$$

be the part of the image of ϕ_γ that lies over the half-plane $\{(x, y) \in T^*\mathbb{R} \mid x \leq 0\}$ (using the notation introduced in Section 1.3). Note that W' is a manifold with boundary, and the boundary $\partial W'$ is given by

$$L_0 = \{(0, 0)\} \times L.$$

In the rest of the proof, we will describe how to adjust W' such that a cylindrical end $\mathbb{R}_{\geq 0} \times L$ can be glued on, and such that the resulting Lagrangian looks like $W = \eta_+ \times \lambda_+ \cup \eta_- \times \lambda_-$ outside of a small neighbourhood of $\mathbb{R}_{\geq 0} \times \{0\} \subset T^*\mathbb{R} \times T^*\mathbb{R}^n$.

To start, let $\mathcal{N}$ be a Weinstein neighbourhood of the Lagrangian $\mathbb{R} \times L \subset T^*\mathbb{R}^{n+1}$ which is of the form $\mathcal{N} = T^*\mathbb{R} \times \mathcal{N}_L$, where $\mathcal{N}_L \subset T^*\mathbb{R}^n$ is a Weinstein neighbourhood of L, and such that the map $\pi_\mathcal{N} : \mathcal{N} \to \mathbb{R} \times L$ induced by the canonical projection in the cotangent bundle is of the form

$$\pi_\mathcal{N} = \pi_{T^*\mathbb{R}} \times \pi_{\mathcal{N}_L}, \tag{27}$$

where $\pi_{T^*\mathbb{R}}$ and $\pi_{\mathcal{N}_L}$ are the corresponding maps for $\mathbb{R} \subset T^*\mathbb{R}$ and $L \subset \mathcal{N}$ (in particular, $\pi_{T^*\mathbb{R}} : T^*\mathbb{R} \to \mathbb{R}$ is simply the projection onto the first factor of $T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$).

Let $U_0' \subset W'$ be a neighbourhood of $L_0 = \partial W'$ in W'. By shrinking it if necessary, we may assume that U_0' lies entirely in the Weinstein neighbourhood $\mathcal{N}$ and that it is the graph of a closed 1-form α_0 over the subset $U_0 = (-3\varepsilon_0, 0] \times L \subset \mathbb{R} \times L$ for some small $\varepsilon_0 > 0$. Note that α_0 is exact because its restriction to L_0 vanishes (as L_0 is contained in $\mathbb{R} \times L$) and because U_0' retracts onto L_0. Moreover, α_0 vanishes on L_0 and hence any primitive of α_0 is constant on L_0. We denote by $g_0 : U_0 \to \mathbb{R}$ the primitive of α_0 which vanishes on L_0.

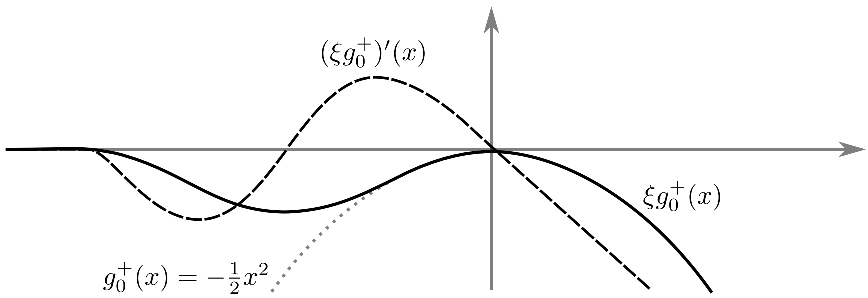

Figure 7: Construction of the function ξg_0 whose Hamiltonian flow is used in the proof of Proposition 4.2 to adjust Biran–Cornea's Lagrangian 1-handle W' to match the local model W of the singular locus of the antisurgery cobordism away from a small neighborhood of its singular locus.

Consider now the subset

$$U_0' \cap \left(T^*\mathbb{R} \times (B^n_{2\kappa}(\lambda_+)^c \cup B^n_{2\kappa}(\lambda_-)^c)\right) \tag{28}$$

of W', where $B^n_{2\kappa}(\lambda_\pm)^c$ denote the complements in $\lambda_\pm$ of the balls $B^n_{2\kappa}(\lambda_\pm) \subset \lambda_\pm$ of radius 2κ (it could also be written as $U_0' \cap \left(T^*\mathbb{R}^n \times B_{2\kappa}(T^*\mathbb{R}^n)^c\right)$, where $B_{2\kappa}(T^*\mathbb{R}^n)^c$ is the complement of a ball in $T^*\mathbb{R}^n$). This subset has two components which are contained in $\ell_\pm \times B^n_{2\kappa}(\lambda_\pm)^c \subset T^*\mathbb{R} \times L$, where $\ell_\pm$ are the lines in $T^*\mathbb{R}$ given by $y = \mp x$, i.e. the graphs of $d(\mp\frac{1}{2}x^2)$; in fact, these components are the graphs of $\alpha_0 = dg_0$ over $(-3\varepsilon_0, 0] \times B^n_{2\kappa}(\lambda_\pm)^c \subset U_0$. Using this and the split nature (27) of $\pi_{\mathcal{N}}$, it follows that the restrictions of g_0 to $(-3\varepsilon_0, 0] \times B^n_{2\kappa}(\lambda_\pm)^c$, which we denote by $g_0^\pm$, depend only on $x \in \mathbb{R}$ and are in fact given by $g_0^\pm(x) = \mp\frac{1}{2}x^2$, see Figure 7.

Let now $\xi : U_0 \to \mathbb{R}$ be a cut-off function which depends only on $x \in \mathbb{R}$ and which satisfies $\xi(x) \equiv 0$ for $x \in (-3\varepsilon_0, -2\varepsilon_0]$, $\xi(x) \equiv 1$ for $x \in [-\varepsilon_0, 0]$, and

$$\pm\left(\xi(x) g_0^\pm(x)\right)' \leq \pm(g_0^\pm)'(x) \tag{29}$$

for all x; it is not hard to verify that this last condition can be satisfied. Then denote by $\Phi : \mathcal{N} \to \mathcal{N}$ the time-one map of the Hamiltonian flow of $-\xi g_0$. We will use Φ to adjust W' as required.

By construction, Φ takes $U_0' \cap ((-\varepsilon_0, 0] \times T^*\mathbb{R}^n)$ to $(-\varepsilon_0, 0] \times L$ and leaves $U_0' \cap ((-3\varepsilon_0, -2\varepsilon_0] \times T^*\mathbb{R}^n)$ fixed. We can therefore extend Φ to a map

$$\hat{\Phi} : W' \to T^*\mathbb{R} \times T^*\mathbb{R}^n$$

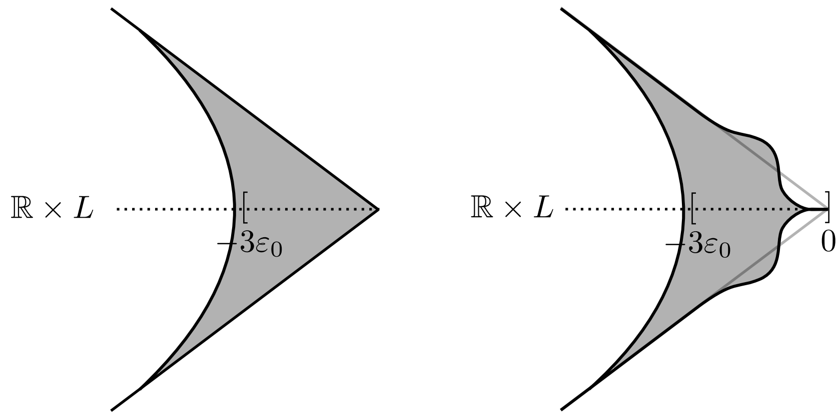

Figure 8: Projection of W' resp. $\Phi(W')$ to $T^*\mathbb{R}$, where W' is Biran–Cornea's Lagrangian 1-handle and Φ is the time-one map of the Hamiltonian flow of the function ξg_0 constructed in the proof of Proposition 4.2 in order to adjust W' as required.

defined on all of W' that leaves $W' \cap ((-\infty, -2\varepsilon_0] \times T^*\mathbb{R}^n)$ fixed and that takes $W' \cap ((-\varepsilon_0, 0] \times T^*\mathbb{R}^n)$ to $\mathbb{R} \times L$. The last statement implies that we can extend $\hat{\Phi}(W')$ by a cylindrical end such that it becomes a valid Lagrangian cobordism.

Note that the restrictions of Φ to the components of (28) are of the form $\phi_\pm \times \mathrm{id}$, where $\phi_\pm$ are the time-one maps of the Hamiltonian flows of the restrictions of ξg_0 to the corresponding components of $U_0 \cap (T^*\mathbb{R} \times (B^n_{2\kappa}(\lambda_+)^c \cup B^n_{2\kappa}(\lambda_-)^c))$. In other words, Φ moves these sets in the direction of the fibres of $T^*\mathbb{R}$. The extended map $\hat{\Phi}$ therefore takes $W' \cap (T^*\mathbb{R} \times (B^n_{2\kappa}(\lambda_+)^c \cup B^n_{2\kappa}(\lambda_-)^c))$ to $\tilde{\eta}_+ \times B^n_{2\kappa}(\lambda_+)^c \cup \tilde{\eta}_- \times B^n_{2\kappa}(\lambda_-)^c$, where $\tilde{\eta}_\pm = \phi_\pm(\eta_\pm)$. It follows from the inequalities (29) that the curves $\tilde{\eta}_\pm$ are entirely contained in the upper resp. lower half-planes and only intersect along the x-axis; that is, up to a horizontal shift they are of the type describe right before this proposition. Using an appropriate symplectomorphism $\psi : T^*\mathbb{R} \to T^*\mathbb{R}$ of the form $(x, y) \mapsto (f(x), g(x, y))$, we can match up the curves $\tilde{\eta}_\pm$ with $\eta_\pm$. (Such a symplectomorphism exists provided that the functions $y, \tilde{y}$ defining $\eta_\pm, \tilde{\eta}_\pm$ via $\eta_\pm(x) = (x, \pm y(x))$ etc. satisfy $\lim_{x\to 0} y(x)/\tilde{y}(x) \to 1$, see Lemma 4.3; this condition can be guaranteed by choosing the cut-off function ξ appropriately.)

Applying $\psi \times \mathrm{id}$ to $\hat{\Phi}(W')$ and gluing on a cylindrical end hence yields a Lagrangian cobordism $W^\natural$ that agrees with W outside of an arbitrarily small neighbourhood of the singular locus of W. $\square$

Lemma 4.3. *Let $\eta^i : \mathbb{R} \to T^*\mathbb{R}$, $x \mapsto (x, y^i(x))$, $i = 0, 1$, be curves as described above for which* $\lim_{x\to 0-}(y^0(x)/y^1(x)) = 1$. *Then there exists a symplectomorphism $\Psi : T^*\mathbb{R} \to T^*\mathbb{R}$ taking $\eta^0_\pm$ to $\eta^1_\pm$.*

Proof. We will construct a diffeomorphism of the form $\Psi(x, y) = (f(x), g(x, y))$. For such Ψ to be a symplectomorphism, we need

$$\left|\begin{pmatrix} \partial_x f & 0 \\ \partial_x g & \partial_y g \end{pmatrix}\right| = \partial_x f \cdot \partial_y g = 1$$

everywhere, which implies

$$g(x, y) = \frac{y}{\partial_x f(x)}$$

up to an additive constant. In order for Ψ to take $\pm\eta^0$ to $\pm\eta^1$, we need

$$g(x, \pm y^0(x)) = \pm y^1(f(x)),$$

for $x < 0$ and $f(x) = x$ for $x \geq 0$. In view of the previous equation, this is equivalent to the ordinary differential equation

$$\partial_x f(x) = \frac{y^0(x)}{y^1(f(x))}$$

for $x < 0$. Provided that $\lim_{x\to 0-}(y^0(x)/y^1(x)) = 1$, this ODE has a solution which extends to a function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$ for $x \geq 0$. $\square$

4.3. Surgery of the singular locus of Γ

In suitable Darboux coordinates, a neighbourhood $\mathcal{U}(\Gamma^s)$ of the singular locus Γ^s of Γ looks like $(\eta_- \times B^n_r(\lambda_-)) \cup (\eta_+ \times B^n_r(\lambda_+))$, where $B^n_r(\lambda_\pm)$ denotes a ball of radius r in $\lambda_\pm = T_{(0,0)}\Lambda'_\pm$ and with curves $\eta_\pm$ as described in the previous subsection, up to a shift to the right by $1 - \delta$ and up to restricting them from $\mathbb{R}$ to $[-\nu, \infty)$ for some small $\nu > 0$. Explicitly, we can take $\eta_\pm(x) = (x, \pm\frac{3}{2}(\sigma(x) - 1)^{1/2}\sigma'(x))$.

Having set up such an identification, we can use Proposition 4.2 to replace this neighbourhood with a corresponding piece of the 1-handle $W^\natural$: $\lambda_-\#\lambda_+ \rightsquigarrow (\lambda_-, \lambda_+)$. More precisely, we can take a 1-handle $W^\natural$ as described there with $2\kappa < r$ and replace

$$\mathcal{U}(\Gamma^s) \cap \Big(T^*\mathbb{R} \times \big(B^n_{2\kappa}(\lambda_+)^c \cup B^n_{2\kappa}(\lambda_-)^c\big)\Big)^c \tag{30}$$

with

$$W^\natural \cap \Big(T^*\mathbb{R} \times \big(B^n_{2\kappa}(\lambda_+)^c \cup B^n_{2\kappa}(\lambda_-)^c\big)\Big)^c. \tag{31}$$

We assume here tacitly that the 1-handle $W^\natural$ has been pruned analogously to restricting the curves $\eta_\pm$ from $\mathbb{R}$ to $[-\nu, \infty)$, i.e., by removing the parts lying over $\eta_\pm((-\infty, \nu))$. With this in mind, the boundaries of (30) and (31) are both given by $\partial^v \cup \partial^h$ with

$$\begin{aligned} \partial^v &= \{\eta_+(-\nu)\} \times B^n_{2\kappa}(\lambda_+) \cup \{\eta_-(-\nu)\} \times B^n_{2\kappa}(\lambda_-) \\ \partial^h &= \eta_+([-\nu, \infty)) \times \partial B^n_{2\kappa}(\lambda_+) \cup \eta_-([-\nu, \infty)) \times \partial B^n_{2\kappa}(\lambda_-), \end{aligned} \tag{32}$$

which makes the cutting and pasting along this “attaching region” possible.

The outcome of this operation is an embedded Lagrangian cobordism

$$\Gamma^\natural : \Lambda^\natural \rightsquigarrow \Lambda, \tag{33}$$

where $\Lambda^\natural$ is the result of resolving the double point of Λ' by an application of Lagrangian 0-surgery which is locally modelled on $\lambda_- \# \lambda_+$. By choosing the curve γ used in the proof of Proposition 4.2 sufficiently small, we can guarantee that $\Gamma^\natural$ and Γ coincide outside of an arbitrarily small neighbourhood of Γ^s in $T^*\mathbb{R}^{n+1}$.

4.4. Proof of Theorem 1.2

Suppose that we have a Lagrangian $L \subset M$ with an embedded sphere $S \subset L$ and an isotropic surgery disc D for S. Repeating the construction in Section 2.3, but replacing Γ by $\Gamma^\natural$, the desingularized Lagrangian cobordism constructed in Section 4.3, and consequently Λ by $\Lambda^\natural$, we produce a Lagrangian cobordism

$$V^\natural : L^\natural \rightsquigarrow L,$$

where $L^\natural$ is a Lagrangian obtained by resolving the double point created when performing antisurgery on L along the isotropic disc D by an application of Lagrangian 0-surgery. The cobordism $V^\natural$ is embedded if L is embedded, and it can be arranged to agree with with the corresponding $V : L' \rightsquigarrow L$

outside of an arbitrarily small neighbourhood of its singular locus by choosing the parameter κ sufficiently small. Topologically, this cobordism is the concatenation of the traces corresponding to first performing k-surgery on L and then 0-surgery on the result L' of the first step; in other words, $V^\natural$ is obtained from $[0,1] \times L$ by first attaching a $(k+1)$-handle and then a 1-handle.

This completes the proof of Theorem 1.2. $\square$

4.5. The Lagrangian isotopy class of the resolution

Recall from Section 3.1 that there are *two* families of resolutions of the double point of Λ' by Lagrangian 0-surgery which depend on a choice of order of sheets near the double point (and correspond to Lagrangian isotopy classes of 0-surgery models). It is important to note that in the construction leading to the desingularized antisurgery cobordism $V^\natural : L^\natural \rightsquigarrow L$ whose existence is asserted by Theorem 1.2 we do *not* have a choice regarding which of these families the desingularized end $\Lambda^\natural$ belongs to, as Proposition 4.2 only gives the existence of a Lagrangian cobordism $(\lambda_-, \lambda_+) \rightsquigarrow \lambda_- \# \lambda_+$ (as opposed to $(\lambda_-, \lambda_+) \rightsquigarrow \lambda_+ \# \lambda_-$).

Recall from Section 3.2 that in the case $k = n-1$, both L and $L^\natural$ are resolutions of $p \in L'$, the double point created when applying $(n-1)$-antisurgery to L, by Lagrangian 0-surgery. In this situation, we have:

Proposition 4.4. *The ends of the desingularized cobordism $V^\natural : L^\natural \rightsquigarrow L$ resulting from $(n-1)$-antisurgery on L belong to* distinct *families of resolutions of $p \in L'$ by Lagrangian 0-surgery.*

Proof. As observed in Section 3.2, Λ belongs to the family of resolutions of Λ' locally given by $\lambda_+ \# \lambda_-$, while $L^\natural$ belongs to the family of resolutions locally given by $\lambda_- \# \lambda_+$. $\square$

4.6. The size of the resolution.

Let L' be the singular end of a Lagrangian cobordism $V : L' \rightsquigarrow L$ arising from antisurgery with parameter ε. In the following we present evidence that the size of a Lagrangian 0-surgery (in the sense of Definition 3.1) which can be applied to L in such a way that the cobordism can be desingularized simultaneously, is upper bounded by $2\varepsilon^{3/2}$.

Consider the intersection of the model cobordism Γ with the plane $T^*\mathbb{R} \times \{(0,0)\} \subset T^*\mathbb{R} \times T^*\mathbb{R}^n$. By setting $\mathbf{x} = (0,0) \in T^*\mathbb{R}^n$ in (4), one can see that

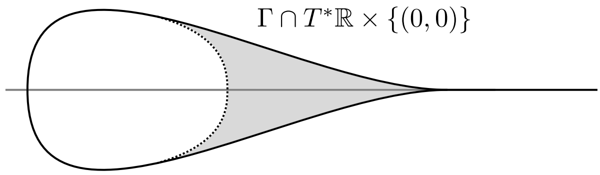

Figure 9: The intersection of the antisurgery cobordism Γ with the plane $T^*\mathbb{R} \times \{(0,0)\} \subset T^*\mathbb{R} \times T^*\mathbb{R}^n$. The dotted line shows the intersection of the Lagrangian 1-handle used for desingularization with the plane $T^*\mathbb{R} \times \{(0,0)\}$.

this intersection consists of a curve that bounds a teardrop-shaped region as shown in Figure 9. The Lagrangian 1-handle that we use in order to resolve the singularity of Γ also intersects $T^*\mathbb{R}^n \times \{(0,0)\}$ in a curve, which is obtained by restricting the map $\phi_\gamma : \mathbb{R} \times S^n \to T^*\mathbb{R}^{n+1}$ (26) modelling this handle to $I \times \{(1, 0, \dots, 0)\} \subset \mathbb{R} \times S^n$ for a suitable interval $I \supset [-\kappa, \kappa]$ (with κ as in the description of the local model for Lagrangian 0-surgery (16)); after the implanting the 1-handle, this curve lies inside the teardrop (the part of the 1-handle that projects to the coordinate axes in the local model projects to the boundary curve of the teardrop after implantation).

This picture suggest that the area of the teardrop is an upper bound for the size of the Lagrangian 0-surgery (the area of the shaded region in Figure 9) that we can perform on the positive end of the antisurgery cobordism in such a way that we can simultaneously desingularize the latter. A simple computation using again (4) shows that the area of the teardrop is $2\varepsilon^{3/2}$.

4.7. Orientability

The result of abstract k-surgery on an orientable manifold L is always orientable if $k \geq 1$, since the $D^{k+1} \times S^{n-k-1}$ we glue in has a connected boundary in that case (or rather, every component has a connected boundary—this includes the case $k = n-1$). In the case $k = 0$, orientability depends on whether $D^1 \times S^{n-1}$ is glued consistently along its two boundary components. Let

$$P^n = S^1 \times S^{n-1} \quad \text{and} \quad Q^n = D^1 \times S^{n-1}/\sim,$$

where $\sim$ identifies $\{1\} \times S^{n-1}$ with $\{-1\} \times S^{n-1}$ using an orientation-reversing involution of S^{n-1} such as $(x_1, x_2, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$ (so Q^n is the mapping torus of such an involution). The result of 0-surgery on

L is diffeomorphic to $L\#P^n$ in the orientable case and to $L\#Q^n$ in the non-orientable case.

Returning to the Lagrangian setting, assume that the L we start with is orientable. When passing from L to L' by k-antisurgery, we replace an embedded copy of $S^k \times D^{n-k}$ (a subset of Λ) by an immersed copy of $D^{k+1} \times S^{n-k-1}$ with a transverse double point (a subset of Λ'), and we resolve this double point by 0-surgery when passing from L' to $L^\natural$. Thus $L^\natural$ is obtained from L by replacing an embedded copy of $S^k \times D^{n-k-1}$ by an embedded copy (a subset of $\Lambda^\natural$) of either

$$(D^{k+1} \times S^{n-k-1})\#P^n \quad \text{or} \quad (D^{k+1} \times S^{n-k-1})\#Q^n$$

in the case $k < n-1$ (for $k = n-1$, $D^{k+1} \times S^{n-k-1} = D^n \times S^0$ is not connected). The first possibility leads to $L^\natural$ being orientable, the second to $L^\natural$ being non-orientable; the next proposition states when which of these alternatives holds.

Proposition 4.5. *Let L be an orientable Lagrangian and let $L^\natural$ be the result of k-antisurgery on L and subsequent desingularization, for some k with $0 < k < n-1$. Then $L^\natural$ is orientable if k is odd and non-orientable if k is even.*

Proof. Note first that the result of abstract k-surgery on L is always orientable when $0 < k < n-1$. Whether $L^\natural$ is orientable or not is a local question that depends on whether the copy of $\mathbb{R} \times S^{n-1}$ we glue in when performing Lagrangian 0-surgery on Λ' matches up orientations of the sheets $\Lambda'_\pm$ which induce the same orientation of Λ'. Assuming that such orientations for $\Lambda'_\pm$ have been chosen, it follows from [14, Theorem 4][4] that the result is orientable iff $(-1)^{n(n-1)/2+1}\Lambda'_+ \cdot \Lambda'_- = 1$, where $\Lambda'_+ \cdot \Lambda'_-$ denotes the intersection index with respect to the symplectic orientation $\mathfrak{o}_\omega$ of the ambient manifold.

To find such orientations for $\Lambda'_\pm$, consider first the orientations $\mathfrak{o}_\pm$ which the projections $\Lambda'_\pm \to \mathbb{R}^n \times \{0\}$ to the zero-section match up with the standard orientation of $\mathbb{R}^n \times \{0\}$. These are represented by the ordered bases of $\lambda_\pm = T_{(0,0)}\Lambda'_\pm$ listed in (25). By continuously deforming these to bases of $T_{(\mathbf{x},\mathbf{y})}\Lambda'$ with $(\mathbf{x},\mathbf{y})$ in the boundary of $\Lambda'_\pm$ (such as $(\mathbf{x},\mathbf{y}) = ((0,\dots,0,\sqrt{\varepsilon}),$

[4] When applying Polterovich's theorem, one needs to take into account that our convention for the direction of surgery handles is opposite to the one in [14]; according to which our handle is directed "from λ_+ to λ_-"; this becomes clear by inspection of the description of the model handle in [14, Section 1.8].

$(0,\dots,0)))$, one can see that $\mathfrak{o}_\pm$ induce *different* orientations of Λ'; a choice of orientations inducing the *same* orientation of Λ' is hence given by $\mathfrak{o}_+$ and $-\mathfrak{o}_-$.

To compute $\Lambda'_+ \cdot \Lambda'_-$ when $\Lambda'_\pm$ carry the orientations $\pm\mathfrak{o}_\pm$, note first that $\mathfrak{o}_+ \oplus \mathfrak{o}_-$ is represented by the ordered basis

$$\begin{aligned}\big(&\partial_{x_1}+\partial_{y_1},\dots,\partial_{x_{k+1}}+\partial_{y_{k+1}},\partial_{x_{k+2}}-\partial_{y_{k+2}},\dots,\partial_{x_n}-\partial_{y_n},\\ &\partial_{x_1}-\partial_{y_1},\dots,\partial_{x_{k+1}}-\partial_{y_{k+1}},\partial_{x_{k+2}}+\partial_{y_{k+2}},\dots,\partial_{x_n}+\partial_{y_n}\big)\end{aligned}$$

of $\mathbb{R}^{2n} \cong T_{(0,0)}T^*\mathbb{R}^n$. The linear map taking this basis to the symplectic basis

$$\big(\partial_{x_1},\partial_{y_1},\dots,\partial_{x_n},\partial_{y_n}\big)$$

has determinant $(-1)^{n(n-1)/2+k+1}2^n$, and thus $\mathfrak{o}_+ \oplus \mathfrak{o}_- = (-1)^{n(n-1)/2+k+1}\mathfrak{o}_\omega$. It follows that

$$\mathfrak{o}_+ \oplus (-\mathfrak{o}_-) = (-1)^{n(n-1)/2+k}\mathfrak{o}_\omega,$$

and therefore

$$(-1)^{n(n-1)/2+1}\Lambda'_+ \cdot \Lambda'_- = (-1)^{k+1},$$

from which the claimed statement follows. □

Remark 4.6. For $k=0$, it is possible that the manifold L' resulting from abstract 0-surgery on an orientable L is already non-orientable, in which case $L^\natural$ is also non-orientable; if, however, L' is orientable, the statement of Proposition 4.5 applies to the resulting $L^\natural$. For $k=n-1$, it is possible that $L \smallsetminus (S^{n-1}\times D^1)$ is disconnected, in which case $L^\natural$ is orientable.

4.8. Computation of Maslov indices

Consider again $\Lambda^\natural$, i.e., the resolution of the singular end Λ' of the $(k+1)$-handle $\Gamma : \Gamma' \rightsquigarrow \Gamma$ corresponding to k-antisurgery for which a desingularized cobordism $\Gamma^\natural : \Lambda^\natural \rightsquigarrow \Lambda$ (33) exists. Note that $\Lambda^\natural$ is diffeomorphic to either $(D^{k+1}\times S^{n-k-1})\#P^n$ or $(D^{k+1}\times S^{n-k-1})\#Q^n$, with P^n and Q^n as defined in Section 4.7. For k satisfying $0 \le k < n-2$, we have

$$H_2(T^*\mathbb{R}^n,\Lambda^\natural) \cong \mathbb{Z}$$

and there exists a preferred generator $\sigma \in H_2(T^*\mathbb{R}^n,\Lambda^\natural)$ characterized by the positivity of its symplectic area. In the proof Proposition 4.7 below we will describe σ by describing a loop ℓ on $\Lambda^\natural$ representing $\partial\sigma \in H_1(\Lambda^\natural) \cong H_2(T^*\mathbb{R}^n,\Lambda^\natural)$ (this description also extends to the case $k=n-2$).

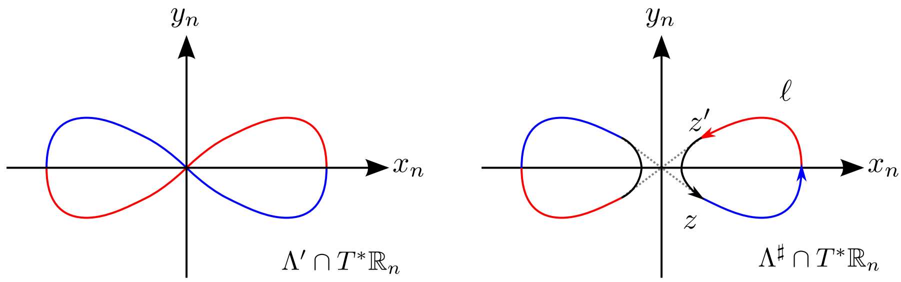

Figure 10: The intersections of the singular end Λ' of the antisurgery cobordism and its resolution $\Lambda^\natural$ with the plane $T^*\mathbb{R}_n \subset T^*\mathbb{R}^n$. The loop ℓ represents the boundary of the generator $\sigma \in H_2(T^*\mathbb{R}^n, \Lambda^\natural)$ whose Maslov index is computed in Proposition 4.7.

Proposition 4.7 computes the Maslov index $\mu(\sigma)$ of this generator, and in particular shows that $\mu(\sigma)$ is non-positive for every k satisfying $1 \leq k \leq n-2$. As a consequence, Lagrangians resulting from k-antisurgery and subsequent desingularization for k in that range are never monotone.

Proposition 4.7. *Assume that $0 \leq k < n-1$. The Maslov index of $\sigma \in H_2(T^*\mathbb{R}^n, \Lambda^\natural)$ is given by $\mu(\sigma) = 1 - k$.*

Proof. Denote by $\mathbb{R}_i$ the x_i-coordinate subspace of $\mathbb{R}^n$ and by $T^*\mathbb{R}_i$ the (x_i, y_i)-coordinate subspace of $T^*\mathbb{R}^n$, for $i = 1, \ldots, n$. To compute $\mu(\sigma)$, we will represent $\partial\sigma$ by a loop ℓ in $\Lambda^\natural \cap T^*\mathbb{R}_n$ and compute how the tangent spaces to $\Lambda^\natural$ twist as we traverse ℓ.

Recall from Section 2.1 that we described the handle $\Gamma : \Lambda' \rightsquigarrow \Lambda$ as the union of the graphs of $\pm dF$ for a certain function F. Specializing the formula (4) for dF to the case $x_0 = 1$, we see that the differential of $F_1 = F(1, \cdot)$ is

$$dF_1 = \frac{3}{2} f_1(\mathbf{x})^{1/2} \big((1 + (1+\varepsilon)\rho'(r^2)) dr^2 - ds^2 \big) \tag{34}$$

with $f_1(\mathbf{x}) = f(1, \mathbf{x}) = r^2 + (1+\varepsilon)\rho(r^2) - s^2 - 1$ (see Section 2). In particular, at points of the form $\mathbf{x} = (0, \ldots, 0, x_n) \in \mathbb{R}_n$ is given by $dF_1(\mathbf{x}) = -3(\varepsilon - x_n^2)^{1/2} x_n dx_n$, and hence

$$\Lambda' \cap T^*\mathbb{R}_n = \{(x_n, \mp 3(\varepsilon - x_n^2)^{1/2} x_n) \in T^*\mathbb{R}_n \mid x_n^2 \leq \varepsilon\},$$

as depicted in the left part of Figure 10, where the blue segment corresponds to $+dF_1$ and the red segment to $-dF_1$. Differentiation of (34) shows that the tangent space to $\Lambda'_\pm = \mathrm{graph}(\pm dF_1)$ over $\mathbf{x} = (0, \ldots, 0, x_n) \in \mathbb{R}_n$ is spanned by

$$\begin{aligned} \partial_{x_i} \pm 3 f_1(\mathbf{x})^{1/2} \partial_{y_i}, &\quad i = 1, \ldots, k+1 \\ \partial_{x_i} \mp 3 f_1(\mathbf{x})^{1/2} \partial_{y_i}, &\quad i = k+2, \ldots, n-1 \\ \partial_{x_n} \mp 3 \left(f_1(\mathbf{x})^{1/2} - f_1(\mathbf{x})^{-1/2} x_n^2 \right) \partial_{y_n}, &\quad i = n. \end{aligned} \tag{35}$$

Note that the last vector is proportional to $f_1(\mathbf{x})^{1/2}\partial_{x_n} \mp 3 \left(f_1(\mathbf{x}) - x_n^2 \right) \partial_{y_n}$, so it approaches a multiple of ∂_{y_n} as $x_n^2 \to \varepsilon$ (which is the conormal direction to $\{f_1(\mathbf{x}) = 0\} \subset \mathbb{R}^n$ at $\mathbf{x} = (0, \ldots, 0, \pm\varepsilon^{1/2})$). Moreover, (35) shows that all these tangent spaces split as direct sums of 1-dimensional subspaces of the 2-dimensional planes $T^*\mathbb{R}_i \subset T^*\mathbb{R}^n$; in particular, the tangent spaces to $\Lambda'_\pm$ at the origin are of the form $\lambda_\pm = \lambda^1_\pm \times \cdots \times \lambda^n_\pm$, where $\lambda^i_\pm$ is a 1-dimensional subspaces of $T^*\mathbb{R}_i$ for $i = 1, \ldots, n$.

To see how the 0-surgery by which we pass from Λ' to $\Lambda^\natural$ affects the picture, recall from Section 3.1 that in order to obtain $\Lambda^\natural$, we use a symplectomorphism $\Phi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ which identifies neighbourhoods of the origin in $\mathbb{R}^n \times \{0\}$ resp. $\{0\} \times \mathbb{R}^n$ with neighbourhoods of the origin in $\Lambda'_- = \{(\mathbf{x}, -dF_1(\mathbf{x})) \in T^*\mathbb{R}^n \mid \mathbf{x} \in \mathfrak{U}_1\}$ resp. $\Lambda'_+ = \{(\mathbf{x}, +dF_1(\mathbf{x})) \in T^*\mathbb{R}^n \mid \mathbf{x} \in \mathfrak{U}_1\}$ (cf. the explanation in Section 4.5). After perturbing Λ' a bit such that it agrees with $\lambda_- \cup \lambda_+$ near the origin, we may assume that Φ is linear and of the form $\phi^1 \times \cdots \times \phi^n$ with linear maps $\phi^i : T^*\mathbb{R} \to T^*\mathbb{R}_i$ that take $\mathbb{R} \times \{0\}$ to λ^i_- and $\{0\} \times \mathbb{R}$ to λ^i_+. Using such an identification, we glue in a Lagrangian copy of $D^1 \times S^{n-1}$ which is given by a part of the image of the map $h_\gamma : \mathbb{R} \times S^{n-1} \to T^*\mathbb{R}^n$ defined in (17). Figure 10 shows what $\Lambda^\natural \cap T^*\mathbb{R}_n$ looks like.

We now compute $\mu(\sigma)$. Let ℓ be the loop on $\Lambda^\natural \cap T^*\mathbb{R}_n$ shown in Figure 10 traversed in counterclockwise direction, which is a representative of $\partial\sigma \in H_1(\Lambda^\natural)$. The black segment of ℓ is the image of the restriction of $\Phi \circ h_\gamma : I \times S^{n-1} \to T^*\mathbb{R}^n$ to $I \times \{(0, \ldots, 0, -1)\}$, where $I \subset \mathbb{R}$ is a small interval containing 0. Differentiation of (17) shows that the tangent space of $\Lambda^\natural$ at $\Phi \circ h_\gamma(t, (0, \ldots, 0, -1)) = \Phi(-a(t)e_n, -b(t)e_n)$ is spanned by

$$\begin{aligned} \Phi \circ Dh_\gamma(\partial_{x_i}) &= \phi^i(a(t)e_i, b(t)e_i), \quad i = 1, \ldots, n-1, \\ \Phi \circ Dh_\gamma(\partial_t) &= \phi^n(-\dot{a}(t)e_n, -\dot{b}(t)e_n); \end{aligned} \tag{36}$$

here $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.

The formulas (35) and (36) show that the loop in the Lagrangian Grassmannian $Gr_L(T^*\mathbb{R}^n)$ induced by ℓ is contained in $Gr_L(T^*\mathbb{R}_1) \times \cdots \times$

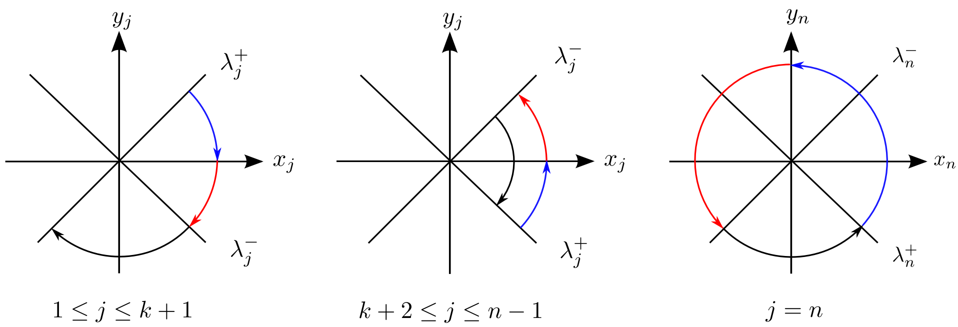

Figure 11: Computation of the Maslov index $\mu(\sigma)$ of the generator $\sigma \in H_2(T^*\mathbb{R}^n, \Lambda^\natural)$ whose boundary is represented by the loop ℓ in Figure 10. The corresponding loop in $Gr_L(T^*\mathbb{R}^n)$ is a direct sum of loops in $Gr_L(T^*\mathbb{R}_j)$, $j = 1, \ldots, n$, which are shown here.

$Gr_L(T^*\mathbb{R}_n) \subset Gr_L(T^*\mathbb{R}^n)$, i.e., it is a direct sum of loops in $Gr_L(T^*\mathbb{R}_j)$. Figure 11 indicates what the pieces of these loops corresponding to the pieces of ℓ look like; the pictures can be deduced from formulas (35), (36). One can read off from these pictures the Maslov indices of the loops in $Gr_L(T^*\mathbb{R}_j)$, which for $j = 1, \ldots, k+1$ are -1, for $j = k+2, \ldots, n-1$ are 0, and for $j = n$ is 2; the Maslov index of σ is their sum, $\mu(\sigma) = 1 - k$. $\square$

5. Cobordisms between Clifford and Chekanov tori

Assume that a Lagrangian L possesses a *Lagrangian* surgery disc D and let L' be the result of $(n-1)$-antisurgery on L along D. As discussed in Section 3.1, this implies that L can be obtained by resolving a double point of L' and then applying a Lagrangian isotopy. The positive end of the desingularized antisurgery cobordism $V : L^\natural \rightsquigarrow L$ is also a resolution of the same double point of L' by Lagrangian 0-surgery; as stated in Proposition 4.4, L and $L^\natural$ belong to distinct families of such resolutions.

5.1. Clifford and Chekanov tori

We now specialize to the case in which L is the Whitney sphere S^2_{Wh} in $\mathbb{C}^2$. Resolving its double point by Lagrangian 0-surgery produces a Lagrangian torus which is monotone, since the boundary of the Lagrangian disc created when performing this surgery generates one summand of H_1 of the torus and has Maslov index 0.

In fact, the two topologically different types of resolving the double point yield a Clifford type torus T^2_{Cl} in one case and a Chekanov type torus T^2_{Ch} in the other case. One can see that by viewing each of the three Lagrangians as obtained by rotating certain curves $\gamma : S^1 \to \mathbb{C}$, i.e., as

$$L_\gamma = \{(\gamma(e^{is})e^{it}, \gamma(e^{is})e^{-it}) \in \mathbb{C}^2 \mid s,t \in [0,2\pi]\}.$$

To obtain S^2_{Wh} in that way, one uses a figure-8 curve γ_{Wh} with a double point at the origin and symmetric with respect to the involution $z \mapsto -z$ (to be precise, this yields the image of the standard Whitney immersion $S^2 \to T^*\mathbb{R}^2 \cong \mathbb{C}^2$ given in (20) under a linear symplectomorphism). Resolving the double point of S^2_{Wh} by 0-surgery has the same result as resolving the double point of the figure-8 curve in such a way that it stays symmetric with respect to $z \mapsto -z$, and then rotating the resulting curve. The two different ways of performing this surgery yield a connected curve γ_{Cl} enclosing the origin in one case, and a disconnected curve γ_{Ch} whose components do not enclose the origin in the other case, see Figure 12. The corresponding Lagrangian tori are T^2_{Cl} resp. T^2_{Ch} up to Hamiltonian isotopy, see e.g. [8, 9].

For better compatibility of this description with our setting, it is useful to note that one can also view the Lagrangians S^2_{Wh}, T^2_{Cl} and T^2_{Ch} as orbits of curves $\gamma \subset T^*\mathbb{R}_1$ under the standard $SO(2)$-action on $T^*\mathbb{R}^2$ given by $A \cdot (\mathbf{x},\mathbf{y}) = (A\mathbf{x}, A\mathbf{y})$ for $A \in SO(2)$. Namely, there exists a linear symplectomorphism $\mathbb{C}^2 \to T^*\mathbb{R}^2$ which is equivariant with respect to the S^1-action on $\mathbb{C}^2$ given by $e^{it} \cdot (z_1,z_2) = (e^{it}z_1, e^{-it}z_2)$ and the standard $SO(2)$-action on $T^*\mathbb{R}^2$ (identifying $SO(2) \cong S^1$ in the usual fashion), and that takes the plane $\{(z,z) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}$ to the (x_1,y_1)-plane $T^*\mathbb{R}_1 \subset T^*\mathbb{R}^2$. Explicitly,

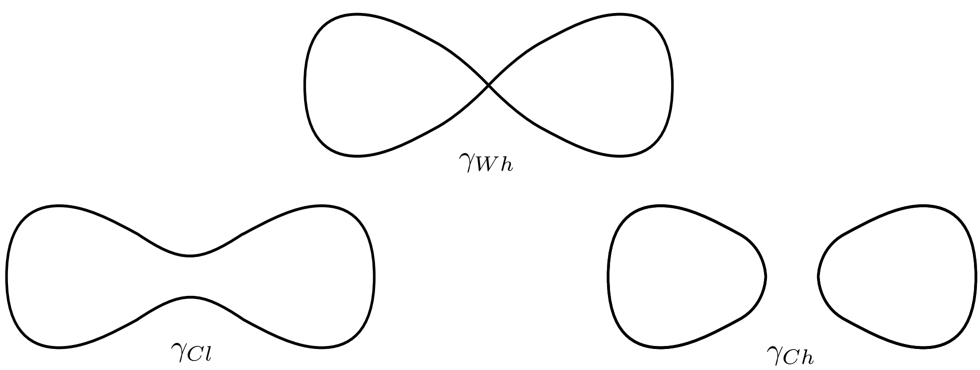

Figure 12: The curves used for constructing S^2_{Wh}, T^2_{Cl} and T^2_{Ch}.

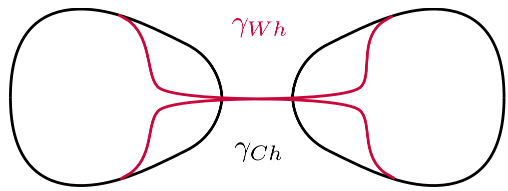

Figure 13: Antisurgering the Chekanov torus.

this symplectomorphism is given by

$$\big(a_1 + ib_1, a_2 + ib_2\big) \mapsto \frac{1}{2}\big(a_1 + a_2, b_1 - b_2, b_1 + b_2, -a_1 + a_2\big)$$

for $(z_1 = a_1 + ib_1, z_2 = a_2 + ib_2) \in \mathbb{C}^2$; on the right-hand side, the first two coordinates correspond to the 0-section, and the last two to the fibre.

5.2. Cobordisms between the tori

In what follows, we denote by $T^2_{Cl}(A)$ and $T^2_{Ch}(A)$ Clifford and Chekanov tori for which a disc of Maslov index 2 has area $A > 0$ (i.e., for which the monotonicity constant is $\frac{1}{2}A$); moreover, we denote by $S^2_{Wh}(A)$ the Whitney sphere for which a generator of $H_2(T^*\mathbb{R}^2, S^2_{Wh}) \cong \mathbb{Z}$ has area A. In all cases, A is half of the area bounded by the respective curves in Figure 12.

The following theorem constructs cobordisms between a Chekanov and a Clifford torus by first applying 1-antisurgery to the Chekanov torus such as to obtain a Whitney sphere, and then desingularizing the corresponding antisurgery cobordism to get a cobordism whose positive end is a Clifford torus.

Theorem 5.1. *For every choice of $a < A$ with a/A sufficiently close to 1, there exists a Lagrangian cobordism $T^2_{Cl}(a) \rightsquigarrow T^2_{Ch}(A)$ which as smooth a manifold is obtained from $[0,1] \times T^2$ by successively attaching a 2-handle and a 1-handle.*

Proof. Consider a curve γ_{Ch} of the type required for constructing a Chekanov torus $T^2_{Ch}(A)$, i.e., as in the lower right part of Figure 12 and such that the area bounded by each component is A. By performing 0-antisurgery on γ_{Ch}, we obtain an immersed curve γ_{Wh} which after rotation yields a Whitney sphere $S^2_{Wh}(\alpha)$, see Figure 13. It follows from the computation in Section 3.3 that the areas A and α are related by $\alpha = A - 2\varepsilon^{3/2}$, where

ε is the size parameter of the antisurgery model we implant. If we subsequently resolve the double point of γ_{Wh} by implanting a local model for Lagrangian 0-surgery of size δ (in the sense of Definition 3.1), we obtain a curve γ_{Cl} bounding area $2(\alpha + \delta)$, which after rotating yields a Clifford torus $T^2_{Cl}(\alpha + \delta)$.

By inspecting the definitions of the antisurgery models in Section 2.1, one sees easily that the model for 1-antisurgery in $T^*\mathbb{R}^2$ is the $SO(2)$-orbit of the model for 0-antisurgery in $T^*\mathbb{R}$, viewing the latter as living in $T^*\mathbb{R}_1 \subset T^*\mathbb{R}^2$. Similarly, the Lagrangians $T^2_{Ch}(A)$ and $S^2_{Wh}(\alpha)$ are $SO(2)$-orbits of the curves γ_{Ch}, γ_{Wh}, as noted at the end of Section 5.1. Combining these statements and the fact that γ_{Wh} is the result of 0-antisurgery on γ_{Ch}, we see that $S^2_{Wh}(\alpha)$ is the result of 1-antisurgery on $T^2_{Ch}(A)$ (and thus $T^2_{Ch}(A)$ is the result of 0-surgery on $S^2_{Wh}(\alpha)$).

Consider now the desingularized antisurgery cobordism correponding to the 1-antisurgery that takes $T^2_{Ch}(A)$ to $S^2_{Wh}(\alpha)$. This cobordism has as its negative end $T^2_{Ch}(A)$ and as its positive end a resolution by 0-surgery of $S^2_{Wh}(\alpha)$ of the form $T^2_{Cl}(\alpha + \delta)$ for small $\delta > 0$ (using Proposition 4.4), i.e., it is a cobordism $T^2_{Cl}(\alpha + \delta) \rightsquigarrow T^2_{Ch}(A)$. Moreover, it has the claimed topology by Theorem 1.2. Recalling the relation $\alpha = A - 2\varepsilon^{3/2}$, one sees that with this construction one can obtain a cobordism $T^2_{Cl}(a) \rightsquigarrow T^2_{Ch}(A)$ for any $0 < a < A$ for which a/A is sufficiently close to 1 by making ε and δ sufficiently small. □

It seems likely that one can deform the curve γ_{Ch} and the antisurgery model in such a way that the area parameter α describing the size of the Whitney sphere $S^2_{Wh}(\alpha)$ that comes up in the proof of Theorem 5.1 is arbitrarily close to 0. This would imply the existence of a cobordism $T^2_{Cl}(a) \rightsquigarrow T^2_{Cl}(A)$ for *any* choice of $0 < a < A$ (cf. the discussion in Section 1.6).

On the other hand, it is *not* possible to construct a cobordism $T^2_{Cl}(A) \rightsquigarrow T^2_{Ch}(A)$, i.e., between Clifford and Chekanov type tori of the same size, by our method. Indeed, such a cobordism would be *monotone* by Proposition 5.2 below, which would imply equality of counts of pseudoholomorphic discs of Maslov index 2 through a given point on both ends, as first observed in [5] (see also [3, 4]). However, it is well known that these counts are different for T^2_{Cl} and T^2_{Ch}. The same argument shows that one cannot build a Lagrangian cobordism between the *monotone* Clifford and Chekanov tori in $\mathbb{C}P^2$ or $S^2 \times S^2$ by this method, since the monotonicity constant of any monotone Lagrangian there is determined by that of the ambient manifold (in particular, it is the same for Clifford and Chekanov type tori).

Proposition 5.2. *Let $V^\natural : L^\natural \rightsquigarrow L$ be a Lagrangian cobordism obtained by desingularizing a cobordism arising from $(n-1)$-antisurgery on an embedded Lagrangian submanifold L of a symplectic manifold (M, ω). If $H_1(M) = 0$, then the map $H_2(M, L) \oplus H_2(M, L^\natural) \to H_2(T^*\mathbb{R} \times M, V^\natural)$ induced by the inclusions of the ends is surjective. In particular, if L and $L^\natural$ are both monotone with the same monotonicity constant, then $V^\natural$ is monotone as well.*

Proof. As a smooth manifold, $V^\natural$ is obtained from the cylinder $[0, 1] \times L$ by successively attaching an n-handle and a 1-handle. One can see from this description that the map $H_1(L) \oplus H_1(L^\natural) \to H_1(V^\natural)$ induced by the inclusion of the ends is surjective; in fact, there exist a generator $\gamma \in H_1(L^\natural)$ such that the restriction of this map to $H_1(L) \oplus \mathbb{Z}\gamma \to H_1(V^\natural)$ is surjective. Consider now the following commutative diagram, where the horizontal arrows come from the exact sequences of the various pairs and the vertical ones are induced by inclusions:

$$\begin{tikzcd}
H_2(M)^{\oplus 2} \arrow[r] \arrow[d] & H_2(M, L) \oplus H_2(M, L^\natural) \arrow[r] \arrow[d] & H_1(L) \oplus H_1(L^\natural) \arrow[r] \arrow[d] & H_1(M)^{\oplus 2} \arrow[d] \\
H_2(T^*\mathbb{R} \times M) \arrow[r] & H_2(T^*\mathbb{R} \times M, V^\natural) \arrow[r] & H_1(V^\natural) \arrow[r] & H_1(T^*\mathbb{R} \times M)
\end{tikzcd}$$

Since the first and third vertical maps are surjective and the fourth is an isomorphism as $H_1(M) = 0$ by assumption, it follows from the 4-lemma that the map $H_2(M, L) \oplus H_2(M, L^\natural) \to H_2(T^*\mathbb{R} \times M, V^\natural)$ is also surjective.

The last statement follows easily by observing that the pullbacks of the area and Maslov homomorphisms $H_2(T^*\mathbb{R} \times M, V^\natural) \to \mathbb{R}$ by the natural inclusions $L, L^\natural \hookrightarrow V^\natural$ of the ends are the corresponding homomorphisms $H_2(M, L) \to \mathbb{R}$ resp. $H_2(M, L^\natural) \to \mathbb{R}$. □

5.3. Successive antisurgery/surgery for Clifford and Chekanov tori

As discussed in Sections 3.2 and 3.3, the result of successively applying $(n-1)$-antisurgery to a Lagrangian L and then 0-surgery to the resulting L' leads to a Lagrangian $L^\natural$ which is Lagrangian isotopic to the original L (for one of the two topologically different ways of implanting the local model for 0-surgery), but not Hamiltonian isotopic to L. In particular, if L is a torus of Clifford or Chekanov type, then the resulting $L^\natural$ obtained that way is again a torus of the same type. Assume that we perform the two operations as described in Section 5.2, i.e., by first operating on the level of curves and then rotating. Note that if we start with a Chekanov torus $T^2_{Cl}(A)$, then the Whitney sphere obtained by antisurgery has area less than A, as can be seen from Figure 13, and hence the resulting $T^2_{Ch}(a)$ has a *smaller* area parameter than the original one. On the other hand, if we start with a Clifford torus $T^2_{Cl}(a)$, then the resulting $T^2_{Cl}(A)$ has *larger* area parameter. It would be interesting to investigate if this observation can be used to give another proof of the fact that $T^2_{Cl}(A)$ and $T^2_{Ch}(A)$ are not Hamiltonian isotopic.

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