

Complex Finsler vector bundles with positive Kobayashi curvature

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In this short note, we prove that a complex Finsler vector bundle with positive Kobayashi curvature must be ample, which partially solves a problem of S. Kobayashi posed in 1975. As applications, a strongly pseudoconvex complex Finsler manifold with positive Kobayashi curvature must be biholomorphic to the complex projective space; we also show that all Schur polynomials are numerically positive for complex Finsler vector bundles with positive Kobayashi curvature.

Introduction

Let $\pi : E \rightarrow M$ be a holomorphic vector bundle over compact complex manifold M . In this paper, we always assume that $\text{rank} E = r$, $\dim M = n$. It is well-known that E is ample in the sense of Hartshorne if and only if the hyperplane line bundle $\mathcal{O}_{P(E^*)}(1)$ is a positive line bundle over

$$P(E^*) = (E^* \setminus \{0\})/\mathbb{C}^*$$

(see [13, Proposition 3.2]), i.e. $\mathcal{O}_{P(E^*)}(1)$ admits a positive curvature metric. Understanding the relations between the algebraic positivity and the geometric positivity is an important problem. When E itself admits a Hermitian metric of Griffiths positive curvature (see e.g. [15, Definition 2.1]), then E is an ample vector bundle. In [12], Griffiths conjectured that its converse also holds, namely there exists a Hermitian metric of Griffiths positive curvature on E if E is ample. If M is a curve, H. Umemura [21] and Campana-Flenner [7] gave an affirmative answer to this conjecture. For the general case, B. Berndtsson [4] proved that $E \otimes \det E$ is Nakano positive, C. Mourougane

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and S. Takayama [16] proved that $S^k E \otimes \det E$ is Griffiths positive for any $k > 0$.

In 1975, S. Kobayashi [14] obtained the following equivalent description on the ampleness in Finsler setting. The related notations will be introduced in Section 1 in this paper. More precisely,

Theorem 0.1 (Kobayashi [14, Theorem 5.1]). *E is ample if and only if there exists a strongly pseudoconvex complex Finsler metric on E^* with negative Kobayashi curvature.*

Furthermore, in [14, Section 5, Page 162], S. Kobayashi posed the following problem:

Problem 0.2. It is reasonable to expect that E is ample if and only if it admits a complex Finsler structure of positive curvature. The question is whether E admits a complex Finsler structure of positive curvature if and only if E^* admits a complex Finsler structure of negative curvature.

In this paper, we partially solve this problem affirmatively and obtain

Theorem 0.3. *Let $\pi : E \rightarrow M$ be a holomorphic vector bundle over the compact complex manifold M . If E admits a strongly pseudoconvex complex Finsler metric with positive Kobayashi curvature, then E is ample.*

It is easy to see that E admitting a Hermitian metric of Griffiths positive curvature is equivalent to the existence of a Griffiths negative Hermitian metric on the dual bundle E^* . However, in the Finslerian case, it is very difficult to find such a simple duality as in the Hermitian situation. A natural and direct way suggested by S. Kobayashi [14, Section 5, page 162] is: for a given complex Finsler structure G in E , considering the complex Finsler structure G^* on E^* defined by

$$G^*(z, \zeta^*) = \sup_{G(z, \zeta)=1} |\langle \zeta^*, \zeta \rangle|^2,$$

and trying to check that G has positive curvature if and only if G^* has negative curvature. Apparently, this is very hard to be achieved due to the difficulty in finding more computable relationships between G and G^* . On the other hand, we know that if the Finsler metric G is (fiberwise) strictly convex and has positive (resp. negative) curvature, then the dual metric G^* has negative (resp. positive) curvature (see [9, Theorem 2.5] or [18]).

However, it is unknown whether the ampleness of E guarantees the existence of a convex strictly plurisubharmonic Finsler metric on E^* .

In the following we first briefly introduce our approach to Theorem 0.3.

For a strongly pseudoconvex complex Finsler metric G on E (cf. Definition 1.1), we still denote by G the induced metric on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$. Then the $(1, 1)$ -form $\sqrt{-1}\partial\bar{\partial}\log G$ admits a decomposition (cf. [10, 14], also Proposition 1.5):

$$\sqrt{-1}\partial\bar{\partial}\log G = -\Psi + \omega_{FS},$$

where ω_{FS} is a vertical $(1, 1)$ -form and positive definite along each fiber of $p : P(E) \rightarrow M$, and Ψ , the *Kobayashi curvature* of the Finsler metric G named in [10], is a horizontal $(1, 1)$ -form. The Finsler metric G is of *positive (negative) Kobayashi curvature* if $\Psi > 0 (< 0)$ along horizontal directions (cf. Definition 1.3).

If E admits a strongly pseudoconvex complex Finsler metric with positive Kobayashi curvature, we prove that $P(E^*)$ is projective (cf. Lemma 2.1). In order to prove E is ample or $\mathcal{O}_{P(E^*)}(1)$ is positive, from our Lemma 2.2, it suffices to show for any holomorphic line bundle F on $P(E^*)$, there exists a positive integer m_0 such that

$$(0.1) \quad H^i(P(E^*), \mathcal{O}_{P(E^*)}(m) \otimes F) = 0, \quad i > 0, \quad m \geq m_0.$$

Let $p : P(E) \rightarrow M$ and $p_1 : P(E^*) \rightarrow M$ denote the natural projections. Since the Picard group of $P(E^*)$ has the following simple structure

$$\text{Pic}(P(E^*)) \simeq \text{Pic}(M) \oplus \mathbb{Z}\mathcal{O}_{P(E^*)}(1),$$

there exist a line bundle F_1 on M and an integer $a \in \mathbb{Z}$ such that $F = p_1^*F_1 \otimes \mathcal{O}_{P(E^*)}(a)$. Now by the Serre duality and [3, Theorem 5.1], for any integer $m \geq -a$, we get (cf. Proposition 2.4) for the proof of the following isomorphisms)

$$\begin{aligned} H^i(P(E^*), \mathcal{O}_{P(E^*)}(m) \otimes F) &\cong H^i(M, S^{m+a}E \otimes F_1) \\ &\cong H^{n-i}(M, S^{m+a}E^* \otimes F_1^* \otimes K_M) \\ &\cong H^{n-i}(P(E), \mathcal{O}_{P(E)}(m+a) \otimes p^*(F_1^* \otimes K_M)). \end{aligned}$$

As pointed by Demailly [9, Section 5.9, Page 247], for any locally free sheaf \mathcal{F} , it holds that

$$H^q(P(E), \mathcal{O}_{P(E)}(m) \otimes \mathcal{F}) = 0, \quad q \neq n, \quad m \geq m_0$$

for some integer $m_0 > 0$. By taking $\mathcal{F} = p^*(F_1^* \otimes K_M)$, we finally get (0.1) and therefore the ampleness of E .

Now we give some applications on Theorem 0.3. The following two direct corollaries follow from the famous theorems of S. Mori [17, Theorem 8], W. Fulton and R. Lazarsfeld [11, Theorem I]:

Corollary 0.4. *If (M, G) is a strongly pseudoconvex complex Finsler manifold with positive Kobayashi curvature, then M is biholomorphic to \mathbb{P}^n .*

Note that when the above Finsler metric G is induced from a Kähler metric on M , Y.-T. Siu and S.-T. Yau in [20] proved this result in a direct geometric way.

Corollary 0.5. *All Schur polynomials are numerically positive for complex Finsler bundles with positive Kobayashi curvature.*

This article is organized as follows. In Section 1, we shall fix the notations and recall some basic definitions and facts on complex Finsler vector bundles, positive (negative) Kobayashi curvature. In Section 2, we will prove our main Theorem 0.3. In Section 3, we will give two applications on Theorem 0.3 and prove Corollary 0.4, 0.5.

1. Complex Finsler vector bundle

In this section, we shall fix the notations and recall some basic definitions and facts on complex Finsler vector bundles. For more details we refer to [6, 9, 10, 14, 22].

We will use $z = (z^1, \dots, z^n)$ to denote a local holomorphic coordinate system on M and use $\{e_i\}_{1 \leq i \leq r}$ to denote a local holomorphic frame of E . Then any element v in E can be written as

$$v = v^i e_i \in E,$$

where we adopt the summation convention of Einstein. In this way, one gets a local holomorphic coordinate system of the complex manifold E :

$$(1.1) \quad (z; v) = (z^1, \dots, z^n; v^1, \dots, v^r).$$

Definition 1.1 ([14]). A Finsler metric G on the holomorphic vector bundle E is a continuous function $G : E \rightarrow \mathbb{R}$ satisfying the following conditions:

F1) G is smooth on $E^o = E \setminus \{0\}$;

F2) $G(z, v) \geq 0$ for all $(z, v) \in E$ with $z \in M$ and $v \in \pi^{-1}(z)$, and $G(z, v) = 0$ if and only if $v = 0$;

F3) $G(z, \lambda v) = |\lambda|^2 G(z, v)$ for all $\lambda \in \mathbb{C}$.

Moreover, G is called strongly pseudoconvex if it satisfies

F4) the Levi form $\sqrt{-1}\partial\bar{\partial}G$ on E^o is positive definite along each fiber $E_z = \pi^{-1}(z)$ for $z \in M$.

Clearly, any Hermitian metric on E is naturally a strongly pseudoconvex complex Finsler metric on it.

We write

$$G_i = \partial G / \partial v^i, \quad G_{\bar{j}} = \partial G / \partial \bar{v}^j, \quad G_{i\bar{j}} = \partial^2 G / \partial v^i \partial \bar{v}^j, \\ G_{i\alpha} = \partial^2 G / \partial v^i \partial z^\alpha, \quad G_{i\bar{j}\bar{\beta}} = \partial^3 G / \partial v^i \partial \bar{v}^j \partial \bar{z}^\beta, \quad \text{etc.},$$

to denote the differentiation with respect to v^i, \bar{v}^j ($1 \leq i, j \leq r$), z^α, \bar{z}^β ($1 \leq \alpha, \beta \leq n$).

If G is a strongly pseudoconvex complex Finsler metric on M , then there is a canonical h-v decomposition of the holomorphic tangent bundle TE^o of E^o (see [6, §5] or [10, §1]).

$$TE^o = \mathcal{H} \oplus \mathcal{V}.$$

In terms of local coordinates,

$$(1.2) \quad \mathcal{H} = \text{span}_{\mathbb{C}} \left\{ \frac{\delta}{\delta z^\alpha} = \frac{\partial}{\partial z^\alpha} - G_{\alpha\bar{j}} G^{\bar{j}k} \frac{\partial}{\partial v^k}, 1 \leq \alpha \leq n \right\}, \\ \mathcal{V} = \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial v^i}, 1 \leq i \leq r \right\}.$$

The dual bundle T^*E^o also has a smooth h-v decomposition $T^*E^o = \mathcal{H}^* \oplus \mathcal{V}^*$:

$$(1.3) \quad \mathcal{H}^* = \text{span}_{\mathbb{C}} \{ dz^\alpha, 1 \leq \alpha \leq n \}, \\ \mathcal{V}^* = \text{span}_{\mathbb{C}} \{ \delta v^i = dv^i + G^{\bar{j}i} G_{\alpha\bar{j}} dz^\alpha, 1 \leq i \leq r \}.$$

With respect to the h-v decomposition (1.2), the $(1, 1)$ -form $\sqrt{-1}\partial\bar{\partial} \log G$ has the following decomposition.

Proposition 1.2 ([2, 14]). *Let G be a strongly pseudoconvex complex Finsler metric on E , one has*

$$\sqrt{-1}\partial\bar{\partial}\log G = -\Psi + \omega_V$$

on E^o , where Ψ and ω_V are given by

$$(1.4) \quad \Psi = \sqrt{-1}R_{i\bar{j}\alpha\bar{\beta}}\frac{v^i\bar{v}^j}{G}dz^\alpha \wedge d\bar{z}^\beta, \quad \omega_V = \sqrt{-1}\frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} \delta v^i \wedge \delta \bar{v}^j,$$

with

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha \partial \bar{z}^\beta} + G^{\bar{l}k} \frac{\partial G_{i\bar{l}}}{\partial z^\alpha} \frac{\partial G_{k\bar{j}}}{\partial \bar{z}^\beta}.$$

Definition 1.3 ([10, Definition 1.2]). The form Ψ defined by (1.4) is called the *Kobayashi curvature* of the complex Finsler vector bundle (E, G) . A strongly pseudoconvex complex Finsler metric G is said to be of *positive* (respectively, *negative*) *Kobayashi curvature* if

$$\left(R_{i\bar{j}\alpha\bar{\beta}} \frac{v^i\bar{v}^j}{G} \right)$$

is a positive (respectively, negative) definite matrix on E^o .

Remark 1.4. Note that the positive (resp. negative) Kobayashi curvature is a natural generalization of Griffiths positive (resp. negative) of a Hermitian vector bundle (cf. [15, Definition 2.1]). In fact, if a Finsler metric comes from a Hermitian metric, then the Finsler metric has positive (resp. negative) Kobayashi curvature is equivalent to Griffiths positive (resp. negative).

Let q denote the natural projection

$$(1.5) \quad \begin{aligned} q : E^o &\rightarrow P(E) := E^o/\mathbb{C}^* \\ (z; v) &\mapsto (z; [v]) := (z^1, \dots, z^n; [v^1, \dots, v^r]), \end{aligned}$$

which gives a local coordinate system of $P(E)$ by

$$(1.6) \quad \begin{aligned} (z; w) &:= (z^1, \dots, z^n; w^1, \dots, w^{r-1}) \\ &= \left(z^1, \dots, z^n; \frac{v^1}{v^k}, \dots, \frac{v^{k-1}}{v^k}, \frac{v^{k+1}}{v^k}, \dots, \frac{v^r}{v^k} \right) \end{aligned}$$

on

$$U_k := \{(z, [v]) \in P(E), v^k \neq 0\}.$$

Denote by $((\log G)^{\bar{b}a})_{1 \leq a, b \leq r-1}$ the inverse of the matrix

$$\left((\log G)_{a\bar{b}} := \frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b} \right)_{1 \leq a, b \leq r-1}$$

and set

$$(1.7) \quad \delta w^a = dw^a + (\log G)_{\alpha\bar{b}} (\log G)^{\bar{b}a} dz^\alpha.$$

By using the coframe $\{\delta w^a\}$ of \mathcal{V}^* , there is a well-defined vertical $(1, 1)$ -form on $P(E)$ by

$$(1.8) \quad \omega_{FS} := \sqrt{-1} \frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b} \delta w^a \wedge \delta \bar{w}^b.$$

From [22, Lemma 1.4, Remark 1.5, Proposition 1.6] one has

Proposition 1.5. (i) $q^* \omega_{FS} = \omega_V$;

(ii) $\sqrt{-1} \partial \bar{\partial} \log G = -\Psi + \omega_{FS}$ on $P(E)$;

(iii) *A Finsler metric G is strongly pseudoconvex if and only if ω_{FS} is positive definite along each fiber of $p : P(E) \rightarrow M$.*

Proof. (i) is exactly [22, Lemma 1.4] and (ii) follows directly from (i). The proof of (iii) can be found in [22, Proposition 1.6]. For readers' convenience, we give the proof here.

By Definition 1.1, G is strongly pseudoconvex if $(G_{i\bar{j}})$ is a positive definite matrix, which gives a Hermitian metric $\langle \cdot, \cdot \rangle$ on the vertical subbundle \mathcal{V} . Denote $v = v^i \frac{\partial}{\partial v^i}$. If G is strongly pseudoconvex, then for any $u = u^i \frac{\partial}{\partial v^i} \in \mathcal{V}$,

$$\begin{aligned} (-\sqrt{-1})\omega_V(u, \bar{u}) &= \frac{1}{G^2} (GG_{i\bar{j}} - G_i G_{\bar{j}}) u^i \bar{u}^j \\ &= \frac{1}{G^2} (\|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2) \geq 0, \end{aligned}$$

the equality holds if and only if $u = \lambda v$ for some constant $\lambda \in \mathbb{C}$. So ω_V has $r - 1$ positive eigenvalues and one zero eigenvalue. Since $\omega_V(v, \bar{v}) = 0$ and $q_*(v) = 0$ and by (i), ω_{FS} is positive definite along each fiber of $p : P(E) \rightarrow$

M . Conversely, if ω_{FS} is positive definite along each fiber, then $\omega_V = q^*\omega_{FS}$ has $r - 1$ positive eigenvalues and one zero eigenvalue. So $\omega_V(v, \bar{v}) = 0$ and

$$G_{i\bar{j}}u^i\bar{u}^j = \frac{1}{G}|G_i u^i|^2 + G(-\sqrt{-1})\omega_V(u, \bar{u}) \geq 0.$$

Moreover, $G_{i\bar{j}}u^i\bar{u}^j = 0$ if and only if $u = \lambda v$ and $G_i v^i = 0$, if and only if $\lambda = 0$. So $(G_{i\bar{j}})$ is a positive definite matrix. □

2. Positive Kobayashi curvature

In this section, we will prove a complex Finsler vector bundle with positive Kobayashi curvature must be ample.

Let G be a strongly pseudoconvex complex Finsler metric on E with positive Kobayashi curvature, that is,

$$(2.1) \quad \sqrt{-1}\partial\bar{\partial}\log G = -\Psi + \omega_{FS}$$

with $\Psi > 0$ on horizontal subbundle \mathcal{H} , and $\omega_{FS} > 0$ along each fiber of $p : P(E) \rightarrow M$. Then

Lemma 2.1. *If E admits a strongly pseudoconvex complex Finsler metric with positive Kobayashi curvature, then $P(E^*)$ is projective.*

Proof. From [10, Lemma 2.3], the first Chern class $c_1(\det E)$ satisfies

$$(2.2) \quad \begin{aligned} c_1(\det E) &= c_1(E) = - \int_{P(E)/M} c_1(\mathcal{O}_{P(E)}(1))^r \\ &= \left[- \int_{P(E)/M} \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\log G \right)^r \right] \\ &= \left[\frac{r}{(2\pi)^r} \int_{P(E)/M} \Psi \wedge \omega_{FS}^{r-1} \right]. \end{aligned}$$

By assumption, $\int_{P(E)/M} \Psi \wedge \omega_{FS}^{r-1}$ is a positive $(1, 1)$ -form on M , which yields that $\det E$ is a positive line bundle. By take k large enough, the line bundle

$$p_1^*(\det E)^k \otimes \mathcal{O}_{P(E^*)}(1) \rightarrow P(E^*)$$

is also a positive line bundle, where $p_1 : P(E^*) \rightarrow M$. By Kodaira embedding theorem, $P(E^*)$ is projective. □

The following lemma can be found in the proof of [19, Lemma 5.2].

Lemma 2.2. *Let $L \rightarrow M$ be a line bundle over projective manifold M , and satisfies for any line bundle F over M , there exists an integer $m_0 > 0$, $H^i(M, F \otimes L^m) = 0$ for $i > 0$, $m \geq m_0$. Then L is a positive line bundle over M .*

Proof. For any coherent sheaf \mathcal{F} over the projective algebraic manifold M , there is a resolution

$$0 \rightarrow \mathcal{K} \rightarrow \bigoplus_{s_n} E_n \rightarrow \cdots \rightarrow \bigoplus_{s_2} E_2 \rightarrow \bigoplus_{s_1} E_1 \rightarrow \mathcal{F} \rightarrow 0,$$

where the E_i are holomorphic line bundles over M . By the Hilbert syzygy theorem, $\mathcal{K} = V$ for some holomorphic vector bundle V on M . Hence for m sufficiently large, we obtain

$$(2.3) \quad H^i(M, L^m \otimes \mathcal{F}) = H^{n+i}(M, L^m \otimes V) = 0, \quad i > 0, \quad m \geq m_0$$

for some positive integer m_0 . By Cartan-Serre-Grothendieck theorem, see e.g. [19, Theorem 5.1], L is a positive line bundle. \square

The following vanishing theorem appeared in [9, Section 5.9], which can be proved by Andreotti-Grauert theorem [1, Theorem 14].

Lemma 2.3. *If E admits a strongly pseudoconvex complex Finsler metric with positive Kobayashi curvature, then for any holomorphic line bundle F on $P(E)$ there exists an integer $m_0 > 0$ such that*

$$(2.4) \quad H^q(P(E), \mathcal{O}_{P(E)}(m) \otimes F) = 0, \quad q < n, \quad m \geq m_0.$$

Proof. By Serre duality, one has

$$(2.5) \quad \begin{aligned} & H^q(P(E), \mathcal{O}_{P(E)}(m) \otimes F) \\ & \cong H^{n+r-1-q}(P(E), \mathcal{O}_{P(E)}(-m) \otimes F^* \otimes K_{P(E)}). \end{aligned}$$

Since the curvature form of $\mathcal{O}_{P(E)}(-1)$ is

$$(2.6) \quad \sqrt{-1} \bar{\partial} \partial \log G = \Psi - \omega_{FS},$$

which has n positive eigenvalues at each point of $P(E)$, so $\mathcal{O}_{P(E)}(-1)$ is $(r-1)$ -positive. By Andreotti-Grauert theorem [1, Theorem 14] (see also

[8, Proposition 2.1]), $\mathcal{O}_{P(E)}(-1)$ is $(r - 1)$ -ample, that is, for any coherent sheaf \mathcal{F} on $P(E)$ there exists a positive integer m_0 such that

$$H^i(P(E), \mathcal{F} \otimes \mathcal{O}_{P(E)}(-m)) = 0, \quad i > r - 1, \quad m \geq m_0.$$

By taking $\mathcal{F} = F^* \otimes K_{P(E)}$ one gets

$$(2.7) \quad \begin{aligned} H^{n+r-1-q}(P(E), \mathcal{O}_{P(E)}(-m) \otimes F^* \otimes K_{P(E)}) &= 0, \\ q < n, \quad m \geq m_0. \end{aligned}$$

Combining with (2.5) completes the proof. □

Our main result in this section is the following.

Theorem 2.4. *If E admits a strongly pseudoconvex complex Finsler metric with positive Kobayashi curvature, then E is ample.*

Proof. From Lemma 2.1, 2.2, it suffices to show for any holomorphic line bundle F on $P(E^*)$, there exists a positive integer m_0 such that

$$(2.8) \quad H^i(P(E^*), \mathcal{O}_{P(E^*)}(m) \otimes F) = 0, \quad i > 0, \quad m \geq m_0.$$

Denote $p : P(E) \rightarrow M$ and $p_1 : P(E^*) \rightarrow M$. Since the Picard group of $P(E^*)$ has the following simple structure,

$$(2.9) \quad \text{Pic}(P(E^*)) \simeq \text{Pic}(M) \oplus \mathbb{Z}\mathcal{O}_{P(E^*)}(1),$$

so there exist a line bundle F_1 on M and an integer $a \in \mathbb{Z}$ such that

$$(2.10) \quad F = p_1^*F_1 \otimes \mathcal{O}_{P(E^*)}(a).$$

It follows that

$$(2.11) \quad \mathcal{O}_{P(E^*)}(m) \otimes F = \mathcal{O}_{P(E^*)}(m + a) \otimes p_1^*F_1.$$

For any integer $m \geq -a$, one has

$$\begin{aligned}
 (2.12) \quad & H^i(P(E^*), \mathcal{O}_{P(E^*)}(m) \otimes F) \\
 & \cong H^i(P(E^*), \mathcal{O}_{P(E^*)}(m+a) \otimes p_1^*F_1) \quad \text{by (2.11)} \\
 & \cong H^i(M, S^{m+a}E \otimes F_1) \quad \text{by [3, Theorem 5.1]} \\
 & \cong H^{n-i}(M, S^{m+a}E^* \otimes F_1^* \otimes K_M) \\
 & \quad \text{by Serre duality, see e.g. [19, Corollary 2.11]} \\
 & \cong H^{n-i}(P(E), \mathcal{O}_{P(E)}(m+a) \otimes p^*(F_1^* \otimes K_M)) \\
 & \quad \text{by [3, Theorem 5.1]} \\
 & = 0, \quad i > 0, \quad m \geq m_0 \quad \text{by Lemma 2.3.}
 \end{aligned}$$

Thus, E is ample. □

3. Applications

In this section, we will give some applications of Theorem 0.3. From Theorem 0.3, it is possible that many related results under the assumption of ampleness could be valid by changing the assumption of ampleness into the existence of a strongly pseudoconvex complex Finsler metric with positive Kobayashi curvature. Among them we only mention here the following two famous theorems due to S. Mori [17, Theorem 8], W. Fulton and R. Lazarsfeld [11, Theorem I].

In [17], Mori proved the following theorem, which solves Hartshorne's conjecture.

Theorem 3.1 ([17, Theorem 8]). *Every irreducible n -dimensional non-singular projective variety with ample tangent bundle defined over an algebraically closed field k of characteristic ≥ 0 is isomorphic to the projective space \mathbb{P}_k^n .*

In the case $k = \mathbb{C}$, by Theorem 0.3, we obtain

Corollary 3.2. *If (M, G) is a strongly pseudoconvex complex Finsler manifold with positive Kobayashi curvature, then M is biholomorphic to \mathbb{P}^n .*

Note that when the above Finsler metric G is induced from a Kähler metric on M , Y.-T. Siu and S.-T. Yau in [20] proved this result in a direct geometric way.

Let $P \in \mathbb{Q}[c_1, \dots, c_r]$ be a weighted homogeneous polynomail of degree n , the variable c_i being assigned weight i . We say that P is numerically positive for ample vector bundles (resp. complex Finsler vector bundles of positive Kobayashi curvature) if for every projective variety M of dimension n , and every ample vector bundle E (resp. complex Finsler vector bundles E of positive Kobayashi curvature) over M , the Chern number

$$(3.1) \quad \int_M P(c_1(E), \dots, c_r(E)) > 0,$$

where $r = \text{rank}E$. Denote by $\Lambda(n, r)$ the set of all partitions of n by non-negative integers $< r$. Thus an element $\lambda \in \Lambda(n, r)$ is specified by a sequence

$$(3.2) \quad r \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0 \quad \text{with} \quad \sum \lambda_i = n.$$

Each $\lambda \in \Lambda(n, r)$ gives rise to a Schur polynomial $P_\lambda \in \mathbb{Q}[c_1, \dots, c_r]$ of degree n , defined as the $n \times n$ determinant

$$(3.3) \quad P_\lambda = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+n-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_n-n+1} & c_{\lambda_n-n+2} & \cdots & c_{\lambda_n} \end{vmatrix},$$

where by convention $c_0 = 1$ and $c_i = 0$ if $i \notin [0, r]$. The Schur polynomials $P_\lambda (\lambda \in \Lambda(n, r))$ form a basis for the \mathbb{Q} -vector space of weighted homogeneous polynomials of degree n in r variables. Given such a polynomail P , write

$$(3.4) \quad P = \sum_{\lambda \in \Lambda(n, r)} a_\lambda(P) P_\lambda \quad (a_\lambda(P) \in \mathbb{Q})$$

In [11], W. Fulton and R. Lazarsfeld obtained the following theorem, which generalized the result of S. Bloch and D. Gieseker [5].

Theorem 3.3 ([11, Theorem I]). *The polynomial P is numerically positive for ample vector bundles if and only if P is non-zero and*

$$a_\lambda(P) \geq 0 \quad \text{for all} \quad \lambda \in \Lambda(n, r).$$

Combining with Theorem 0.3, we obtain

Corollary 3.4. *All Schur polynomials are numerically positive for complex Finsler bundles with positive Kobayashi curvature.*

We should stress that how to prove these two famous theorems in a purely differential geometrical method is still widely open.

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References

- [1] A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.
- [2] T. Aikou, *Finsler Geometry on complex vector bundles*, Riemann-Finsler Geometry, MSRI Publications **50** (2004), 83–105.
- [3] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 4, Springer-Verlag, Berlin, (1984).
- [4] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. Math. **169** (2009), 531–560.
- [5] S. Bloch and D. Gieseker, *The positivity of the Chern classes of an ample vector bundle*, Invent. Math. **12** (1971), 112–117.
- [6] J. Cao and P.-M. Wong, *Finsler geometry of projectivized vector bundles*, J. Math. Kyoto Univ. **43** (2003), no. 2, 369–410.
- [7] F. Campana and H. Flenner, *A characterization of ample vector-bundles on a curve*, Mathematische Annalen **287** (1990), no. 4, 571–575.
- [8] J.-P. Demailly, T. Peternell, and M. Schneider, *Holomorphic line bundles with partially vanishing cohomology*, in: Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (RamatGan, 1993), pp. 165–198, Israel Math. Conf. Proc., 9, BarIlan Univ (1996).
- [9] J.-P. Demailly, *Pseudoconvex-concave duality and regularization of currents*, in: Several Complex Variables (Berkeley, CA, 1995-1996), pp. 233–271, Math. Sci. Res. Inst. Publ. **37**, Cambridge Univ. Press, Cambridge, (1999).
- [10] H. Feng, K. Liu, and X. Wan, *Chern forms of holomorphic Finsler vector bundles and some applications*, Inter. J. Math. **27** (2016), no. 4, 1650030.

- [11] W. Fulton and R. Lazarsfeld, *Positive polynomials for ample vector bundles*, *Annals of Mathematics* **118** (1983), no. 1, 35–60.
- [12] P. Griffiths, *Hermitian differential geometry, Chern classes and positive vector bundles*, *Global Analysis, Papers in Honor of K. Kodaira*, Princeton Univ. Press, Princeton (1969), pp. 181–251.
- [13] R. Hartshorne, *Ample vector bundles*, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 29, 63–94.
- [14] S. Kobayashi, *Negative vector bundles and complex Finsler structures*, *Nagoya Math. J.* Vol. **57** (1975), 153–166.
- [15] K. Liu, X. Sun, and X. Yang, *Positivity and vanishing theorems for ample vector bundles*, *J. Algebraic Geom.* **22** (2013), no. 2, 303–331.
- [16] C. Mourougane and S. Takayama, *Hodge metrics and positivity of direct images*, *J. Reine Angew. Math.* **606** (2007), 167–178.
- [17] S. Mori, *Projective manifolds with ample tangent bundles*, *Ann. of Math.* (2) **110** (1979), no. 3, 593–606.
- [18] A.-J. Sommese, *Concavity theorems*, *Math. Ann.* **235** (1978), 37–53.
- [19] B. Shiffman and A. J. Sommese, *Vanishing Theorems on Complex Manifolds*, *Progress in Mathematics* **56**, Birkhäuser Boston, Inc., Boston, MA, (1985).
- [20] Y.-T. Siu and S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, *Invent. Math.* **59** (1980), no. 2, 189–204.
- [21] H. Umemura, *Some results in the theory of vector bundles*, *Nagoya Math. J.* **52** (1973), 97–128.
- [22] X. Wan, *Positivity preserving along a flow over projective bundle*, [arXiv:1801.09886](https://arxiv.org/abs/1801.09886), (2018).

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