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On the four vertex theorem for curves on locally convex surfaces

Shibing Chen, Xu-Jia Wang, and Bin Zhou

The classical four vertex theorem describes a fundamental property of simple closed planar curves. It has been extended to space curves, namely a smooth, simple closed curve in $\mathbb{R}^3$ has at least four points with vanishing torsion if it lies on a convex surface. More recently, Ghomi [6] extended this property to curves lying on locally convex surfaces. In this paper we provide an alternative approach to the result via the theory of Monge-Ampere equations.

1. Introduction

The classical four-vertex theorem asserts that a simple, closed planar curve has at least four vertices, namely its curvature has at least four local extrema. This theorem was first proved by Mukhopadhyaya [14] for closed convex curves in 1909. Kneser [12] observed that the stereographic projection maps vertices of a plane curve to vertices of its image on the sphere and used this to prove the four vertex theorem for all simple closed curves in the plane. On the sphere, the torsion vanishes at the vertices of the curve. Therefore the four vertex theorem implies that every simple closed curve on the sphere has at least four points with vanishing torsion.

This property raised interest to study global behaviour of torsion of a space curve. It is easy to see that for some space curves the torsion may not vanish at all, and there are also space curves of which the torsion vanishes twice only [10]. But it has been proved that a simple closed curve on a convex surface has at least four points where the torsion vanishes [1, 2, 7, 18], and in literature this is also called four vertex theorem. An interesting question is whether every simple closed curve Γ bounding a surface of positive curvature has at least four points of vanishing torsion [4, 17].

This problem was recently studied by Ghomi [6]. He proved that the torsion of Γ either vanishes identically, or changes sign at least four times if

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Γ has no inflection points and is the boundary of a locally convex, locally nonflat (along Γ) surface $\mathcal{M}$ which is topologically a disc.

Ghomi's proof is based on the study of convex caps in the locally convex surface $\mathcal{M}$ and is very delicate. In this paper we provide an alternative approach, by adopting a classical method of stationary osculating planes and using the existence of generalised solutions to the homogeneous Monge-Ampère equation [16], or the existence of locally convex surface of vanishing Gauss curvature [8, 19]. It was proved in [8, 19] that if Γ is the boundary of a locally convex surface $\mathcal{M}$ which is strictly convex near Γ, then Γ also spans a locally convex surface $\mathcal{M}_0$ with vanishing Gauss curvature. The surface $\mathcal{M}_0$ can be decomposed as the union of disjoint line segments or planar topological discs. If $\mathcal{M}$ is topologically a disc, one infers that there are at least two boundary points where the tangent planes of $\mathcal{M}_0$ are osculating to the curve Γ. By local convexity, $\mathcal{M}_0$ stays locally on one side of its tangent plane, that means the osculating plane is stationary and so the torsion changes sign nearby (Lemma 2.5). Hence there are at least four points on the curve where the torsion vanishes.

Theorem 1.1. *Let Γ be a C^3 smooth space curve in $\mathbb{R}^3$. Assume the curvature of Γ does not vanish and Γ is the boundary of a bounded locally convex surface $\mathcal{M} \subset \mathbb{R}^3$ which is topologically a disc. Then either the torsion of Γ vanishes identically, or it changes sign at least four times.*

In the above theorem, we dropped the local non-flatness condition of $\mathcal{M}$ by Ghomi [6], allowing that Γ locally lies on a plane. We remark that the non-vanishing curvature condition of Γ is such that the principal normal vector $\mathbf{n}$ and the torsion τ are well defined and continuous, see (2.6) and (2.7) below for the definition. It was proved that for a generic space curve, non-vanishing torsion implies non-convexity [5]. Note that we do not assume any smoothness for $\mathcal{M}$, except that its boundary $\Gamma \in C^3$.

If the torsion τ changes sign four times, it has at least four local maximum or minimum. As a result, we have the four vertex theorem for the torsion.

Corollary 1.2. *Under the condition of Theorem 1.1, the torsion of Γ either vanishes identically, or has at least four local extrema.*

In Section 2, we illustrate our idea by proving Theorem 1.1 when $\mathcal{M}$ is the graph of a convex function h over a domain $\Omega \subset R^2$. In the case when h is a strictly convex function, the four vertex theorem is well-known but our approach is of its own interest. Let u be the solution to the Dirichlet problem

(2.1). Then the graph of u, denotes as $\mathcal{M}_0$, can be decomposed as a union of line segments and planar topological discs. If $\mathcal{M}$ is topologically a disc, so is $\mathcal{M}_0$, and there exist two sequences of points $p_k =: (x_k, u(x_k)), q_k =: (y_k, u(y_k)) \subset \mathcal{M}_0$ which converges to two boundary points p_0, q_0 $(p_0 \neq q_0)$, such that the support planes of $\mathcal{M}_0$ at p_k, q_k converge to tangent planes of $\mathcal{M}_0$ at p_0, q_0, respectively. Moreover, the tangent planes are osculating to the curve Γ at p_0 and q_0. Hence Theorem 1.1 is proved in the graph case.

In Section 3, we proved Theorem 1.1 under an additional condition, namely $\mathcal{M}$ satisfies a uniform cone condition. This condition ensures that for any point $p \in \mathcal{M}$, locally $\mathcal{M}$ can be represented as a graph of which the gradient is uniformly bounded. Hence we can use the Perron method to obtain a sequence of locally convex surfaces which share the same boundary Γ and converges to a locally convex surface $\mathcal{M}_0$ of vanishing Gauss curvature. Again $\mathcal{M}_0$ can be decomposed as a union of line segments and planar discs and there are at least two boundary points p_0, q_0 at which the tangent planes of $\mathcal{M}_0$ are osculating to the curve Γ. Theorem 1.1 then follows from Lemma 2.5.

In the last section we show how the uniform cone condition is satisfied for the locally convex surface $\mathcal{M}$ in Theorem 1.1. We can either extend $\mathcal{M}$ to a larger locally convex surface $\tilde{\mathcal{M}}$, such that the extended part satisfies nice geometric properties; or $\mathcal{M}$ is the graph of a convex function. Once the uniform cone condition is satisfied, the proof in Section 3 applies and Theorem 1.1 follows.

Notation and remarks:

- We use $\mathcal{M}_0$ to denote a locally convex surface with vanishing Gauss curvature, and use $\mathcal{M}$ to denote a general locally convex surface.
- $\mathcal{K}$ for the Gauss curvature and K for the contact set $L \cap \mathcal{M}$.

2. The graph case

In the graph case, our proof uses solutions to the Dirichlet problem of the homogeneous Monge-Ampère equation

$$\begin{cases} \det D^2 u = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$, and h is a Lipschitz continuous convex function on $\partial\Omega$. We do not assume smoothness and uniform convexity of h.

Lemma 2.1. *Assume that there is a Lipschitz continuous, convex function $\tilde{u}$ such that $\tilde{u} = h$ on $\partial\Omega$. Then there is a generalised solution u to* (2.1).

Proof. The existence of a generalised solution is well known [16]. It is given by

$$u(x) = \sup\{L(x) \mid L \text{ is linear and } L \leq h \text{ on } \partial\Omega\}. \tag{2.2}$$

The existence of a sub-solution $\tilde{u}$ means $u = h$ on $\partial\Omega$. □

If Ω is convex and $\partial\Omega, h \in C^1$, one can prove that $u \in C^1(\overline{\Omega})$. Higher regularity for the homogeneous Monge-Ampère equation (2.1) has been studied by many authors, see [13] and the references therein.

Let us first introduce some terminology related to convex functions or locally convex surfaces.

(i) *Support plane.* Let w be a convex function defined on Ω. At any point $x_0 \in \overline{\Omega}$, there is a support plane of w, namely a linear function $L(x)$ satisfying $L(x_0) = w(x_0)$ and $L(x) \leq w(x)$ in Ω. Let $\mathcal{M}$ be the graph of w, a support plane of w will also be called the support plane of $\mathcal{M}$.

(ii) Denote

$$K_{x_0,L_{x_0}}[w] = \{x \in \overline{\Omega} \mid L_{x_0}(x) = w(x)\},$$

where L_{x_0} is the support plane of w at x_0. For brevity we denote $K_{x_0} = K_{x_0,L_{x_0}}[w]$ when no confusion arises. It is a convex set in a plane.

(iii) *Tangent plane.* A support plane L_{x_0} is called a tangent plane of w if

$$|w(x) - L_{x_0}(x)| = o(|x - x_0|) \tag{2.3}$$

for x near x_0. A convex function may not have tangent plane at non-smooth points. However at a boundary point, there is always a tangent plane if the boundary is C^1 and w is C^1 on the boundary. In fact, let x_0 be a boundary and assume that $\partial\Omega$ is locally given by

$$x_2 = \rho(x_1) \quad \text{with} \quad \rho(0) = \rho'(0) = 0, \tag{2.4}$$

for some function $\rho \in C^1$. By the convexity of w, the lateral derivative

$$\partial^+_{x_2} w(0) = \lim_{t \to 0^+} \frac{1}{t}(w(te_2) - w(0))$$

exists. Since w is C^1 on $\partial\Omega$ and $\partial\Omega$ is also C^1 smooth, the derivative $\partial_{x_1} w(0)$ also exists. Hence there is a plane L_{x_0} passing through the point $p_0 = (x_0, w(x_0))$ whose gradient is equal to $(\partial_{x_1} w(0), \partial^+_{x_2} w(0))$. The plane L_{w_0} is then the tangent plane of w at x_0, or the tangent plane of $\mathcal{M}$ at p_0. The normal of L_{p_0} will be denoted as $\nu = \nu(p_0)$, and will be called the *normal* of $\mathcal{M}$ at p_0.

(iv) ς-*segment.* We call a segment ℓ in $\overline{\Omega}$ a ς-*segment* (with respect to a convex function u), if both of its endpoints lie on $\partial\Omega$ and u is linear on ℓ, and denote a ς-segment ℓ as ℓ_y if y is a point on ℓ.

A basic property from the definition (2.2) is that K_{x_0} can be spanned by $K_{x_0} \cap \partial\Omega$, namely for any point $p \in K_{x_0}$, there exists at most three points $p_1, p_2, p_3 \in K_{x_0} \cap \partial\Omega$ such that $p = \sum_{i=1}^{3} t_i p_i$, where $t_i \in [0,1]$ and $\sum t_i = 1$. For any interior point $x_0 \in \Omega$, the set K_{x_0} is either a ς-segment or it has positive measure. In the latter case, for any point $p \in \partial K_{x_0} \backslash \partial\Omega$, there exists a ς-segment ℓ_p on the boundary ∂K_{x_0}. There are at most countably many points $\{x_k;\ k = 1, 2, \dots\} \subset \Omega$ such that the sets K_{x_k} are different from each other and have positive measures.

Lemma 2.2. *Let u be the generalised solution in Lemma 2.1. Assume that $\partial\Omega \in C^1$ and $h \in C^1$. Let $\ell_{y_1} \subset K_{y_1}, \ell_{y_2} \subset K_{y_2}$ and $\ell_{y_3} \subset K_{y_3}$ be three different ς-segments. Assume that K_{y_1}, K_{y_2} and K_{y_3} are also different from each other. Then ℓ_{y_1}, ℓ_{y_2} and ℓ_{y_3} cannot share a same endpoint.*

Proof. For if ℓ_{y_1}, ℓ_{y_2} and ℓ_{y_3} share the same endpoints $x_b \in \partial\Omega$, then at least one of the segments $K_{y_1} \cap K_{y_2}$ and $K_{y_2} \cap K_{y_3}$ is transversal to $\partial\Omega$. It implies that u is not C^1 on the boundary, a contradiction to the assumption $u = h \in C^1$ on $\partial\Omega$. □

In the following, we assume that the domain Ω is a topological disc, so that a ς-segment divides Ω into two separate parts.

Lemma 2.3. *Let u be the generalised solution in Lemma 2.1. Assume that $\partial\Omega \in C^1$ and $h \in C^1$. Then there exist two sequences of ς-segments $\{\ell_{y_k}\}$ and $\{\ell_{z_k}\}$ such that $y_k \to y_0 \in \partial\Omega$, $z_k \to z_0 \in \partial\Omega$, $y_0 \neq z_0$, and the normal of ℓ_{y_k}, ℓ_{z_k} converge to that of $\partial\Omega$ at y_0, z_0, respectively.*

Proof. Let ℓ be a ς-segment. It divides Ω into two parts, $\Omega - \ell = \Omega_1 \cup \Omega_2$. We show that there is a sequence of ς-segments $\{\ell_{y_k}\}$ in Ω_1 such that $y_k \to y_0 \in \partial\Omega$ and the normal of ℓ_{y_k} converge to that of $\partial\Omega$ at y_0.

Let y_1 be a point on ℓ. If $\Omega_1 = K_{y_1}$, then u is linear in Ω_1 and we may choose any sequence y_k which converges to a boundary point $y_0 \notin \ell$. Otherwise there exists a ς-segment ℓ_y for some point $y \in \Omega_1 \backslash K_{y_1}$. The segments ℓ_{y_1} and ℓ_y bound a sub-domain of Ω_1, denoted as $\Omega_{\ell_{y_1},\ell_y}$. That is

$$\Omega_{\ell_{y_1},\ell_y} = \{x \in \Omega_1 \mid \exists \text{ an open smooth curve } c \subset \Omega_1 \text{ connecting } y \text{ to } y_1, \\ \text{passing through } x, \text{ such that } c \cap \ell_{y_1} = \emptyset,\ c \cap \ell_y = \emptyset\ \}.$$

Here we say a smooth curve is open if it does not include its two endpoints. Then the measure $|\Omega_{\ell_{y_1},\ell_y}| > 0$. Denote $A = \sup\{|\Omega_{\ell_{y_1},\ell_y}| \mid \ell_y$ is a ς-segment in $\Omega_1\}$. Choose a sequence of ς-segments $\ell_{y_k} \subset \Omega_1$ such that

$$\lim_{k\to\infty} |\Omega_{\ell_{y_1},\ell_{y_k}}| = A. \tag{2.5}$$

We claim that y_k fulfils the requirements in Lemma 2.3.

By choosing a subsequence, we may assume that $|\Omega_{\ell_{y_1},\ell_{y_k}}|$ is strictly increasing. By choosing a sub-sequence, we may assume that ℓ_{y_k} converges to a limit. Then the limit is either a ς-segment in $\overline{\Omega}$ or a point on $\partial\Omega$. Hence the measure of $\Omega_1 - \Omega_{\ell_{y_1},\ell_{y_k}}$ converges to 0.

If the limit is a single point x_b on the boundary $\partial\Omega$, the C^1 regularity of $\partial\Omega$ implies that the normal of ℓ_{y_k} converges to that of $\partial\Omega$ at x_b. If the limit is a segment, then apparently the segment lies on the boundary $\partial\Omega$, and the normal of ℓ_{y_k} converges to that of $\partial\Omega$ too.

Similarly there is a sequence of ς-segments $\{\ell_{z_k}\}$ in Ω_2 such that $z_k \to z_0 \in \partial\Omega$ and the normal of ℓ_{z_k} converge to that of $\partial\Omega$ at z_0. By Lemma 2.2, we see that $y_0 \neq z_0$. □

For clarity let us assume that y_k and z_k are the middle points of ℓ_{y_k} and ℓ_{z_k}. Let L_{y_k} and L_{z_k} be the support planes of u at y_k and z_k. Then L_{y_k} and L_{z_k} sub-converge to support planes L_{y_0} and L_{z_0} of u at y_0 and z_0, respectively.

Lemma 2.4. *Assume that $\partial\Omega$ and h are C^3 smooth, and the curvature of the curve $\Gamma = \{(x, h(x)) \in \mathbb{R}^3 \mid x \in \partial\Omega\}$ does not vanish. Then L_{y_0} and L_{z_0} are respectively osculating planes of Γ at y_0 and z_0.*

Proof. Let $\{\mathbf{r}(s) \mid s \in [0, T]\}$ be a parametrization of Γ by its arclength. Denote by $\mathbf{t} = \mathbf{r}'(s) =: \frac{\partial \mathbf{r}(s)}{ds}$ the unit tangent vector of Γ at $\mathbf{r}(s)$, and by

$$\mathbf{n} = \kappa^{-1}\mathbf{t}'(s) =: \kappa^{-1}\frac{d\mathbf{t}(s)}{ds} \tag{2.6}$$

the principal normal vector, where κ is the curvature of Γ. By the condition that the curvature κ does not vanish, $\mathbf{n}$ is well-defined.

By choosing a proper coordinates we may assume that $y_0 = 0$ and $\partial\Omega$ is locally given by (2.4), so that $e_2 = (0, 1, 0)$ is the inner normal of $\partial\Omega$ at y_0. We may also assume that $u(0) = 0$ and that $L_{y_0} = \{x_3 = 0\}$. Hence to prove that L_{y_0} is an osculating plane of Γ at y_0, it suffices to prove that $\mathbf{n} = e_2$.

Let us express the support plane L_{y_k} by

$$x_3 = L_k(x) = a_k x_1 + b_k x_2 + c_k.$$

Let $p^k = (p_1^k, p_2^k, p_3^k)$ and $q^k = (q_1^k, q_2^k, q_3^k)$ be the endpoints of the ς-segments ℓ_k, with $p^k \subset \{x_1 < 0\}$ and $q^k \subset \{x_1 > 0\}$. Noting that Γ is tangent to the x_1-axis at 0, we have $p_3^k = o\big(|p_1^k|\big)$ and $q_3^k = o\big(|q_1^k|\big)$, which implies that $a_k \to 0$. Noting also that $L_{y_k} \to L_{y_0} = \{x_3 = 0\}$, we also have $b_k \to 0$.

By definition, we have $\mathbf{n} = \lim_{k\to\infty}(\mathbf{t}(p^k) - \mathbf{t}(0))/|p^k|$, where $\mathbf{t}(p)$ is the unit tangent vector of Γ at p. Denote $\mathbf{n} = (n_1, n_2, n_3)$ and $\mathbf{t} = (t_1, t_2, t_3)$. By our choice of the coordinates, we have $\mathbf{t}(0) = e_1 = (1, 0, 0)$. By definition, $\mathbf{n} \perp \mathbf{t}$. Hence $n_1 = 0$. It is also easy to see that $n_3 = \lim_{k\to\infty} t_3(p^k)/|p^k| = 0$. Indeed, since $\mathbf{t}(p_k)$ is a vector contained in the support plane L_{y_k} and $a_k \to 0, b_k \to 0$, we have $n_3 \to 0$ as $k \to 0$. Therefore we have proved that $\mathbf{n} = e_2$ at 0. □

The following lemma is a classical result [1](see also Lemma 6.12 in [6]). It says that the torsion changes sign at a point where the osculating plane is a support plane, namely locally the curve lies on one side of the plane.

Lemma 2.5. *Let $\{\mathbf{r}(s) \mid s \in [-a, a]\}$ be a local parametrization of Γ by its arclength. Assume the curvature of Γ does not vanish, and $L_0 = \{x_3 = 0\}$ is the osculating plane of Γ at $s = 0$. Assume that Γ lies above L_0, and $\mathbf{r}(\pm a)$ are disjoint from L_0. Then the torsion of Γ changes sign along $\mathbf{r}(s)$ from $s = -a$ to $s = a$. In particular, the torsion vanishes at $s = 0$.*

At $s = 0$ the torsion of Γ vanishes and L_0 is a stationary osculating plane. A point with vanishing torsion is called inflection point in literature [2, 11] (the terminology of inflection has different meaning in [6]). Lemma 2.5 can be easily seen by the definition of the torsion, which is given by

$$\tau = -\mathbf{n} \cdot \mathbf{b}',$$

where $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is the binormal vector and $\mathbf{b}' = \frac{d}{ds}\mathbf{b}(s)$. It can be explicitly written as

$$\tau = \frac{\det(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{\|\mathbf{r}' \times \mathbf{r}''\|^2} = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2}. \tag{2.7}$$

Theorem 1.1 in the graphic case now follows immediately.

Theorem 2.6. *Let Γ be a C^3-smooth curve in $\mathbb{R}^3$. Assume that the projection of Γ on $\{x_3 = 0\}$ is a simple curve which bounds a topological disc Ω. Assume that there is a Lipschitz continuous, convex function $\tilde{u}$ on Ω such that Γ is the boundary of the graph of $\tilde{u}$. Then the torsion of Γ either vanishes identically, or changes sign four times.*

Proof. Let y_0, z_0 be the points in Lemma 2.4. Consider the Dirichlet problem (2.1). If the boundary value function h is a linear function, then the torsion of Γ vanishes identically. Otherwise K_{y_0} is a proper subset of $\partial\Omega$. Let J_{y_0} and J_{z_0} be disjoint neighbourhoods of $K_{y_0} \cap \partial\Omega$ and $K_{z_0} \cap \partial\Omega$, respectively. By Lemma 2.5, there exist two points $p_1, p_2 \in J_{y_0}$ such that the torsion $\tau(p_1) > 0$ and $\tau(p_2) < 0$. Similarly there exist two points $q_1, q_2 \in J_{z_0}$ such that the torsion $\tau(q_1) > 0$ and $\tau(q_2) < 0$. Moreover the points $y_0, p_1, q_2, z_0, q_1, p_2$ are in a monotone, anti-clockwise order. Hence there is a point between p_1 and q_2, and a point between p_2 and q_1, where the torsion vanishes. $\square$

Remark 2.1. The property that $y_0, p_1, q_2, z_0, q_1, p_2$ in a monotone, anti-clockwise order is due to the convexity of the surface. If one can find osculating planes of Γ at k different points $y_1, \ldots, y_k \in \Gamma$, then the torsion changes sign at these points and we infer that the torsion vanishes at least at $2k$ points on Γ. This property is also true for locally convex surfaces treated in the subsequent sections.

3. Proof of Theorem 1.1 under condition (H)

First we recall the definition for locally convex hypersurfaces [19].

Definition 3.1. A locally convex hypersurface $\mathcal{M}$ in $\mathbb{R}^{n+1}$ is an immersion of an n-dimensional oriented and connected manifold $\mathcal{N}$ (possibly with boundary) in $\mathbb{R}^{n+1}$, i.e., a mapping $f : \mathcal{N} \to \mathcal{M} \subset \mathbb{R}^{n+1}$, such that $\forall\, x \in \mathcal{N}$, $\exists$ a neighbourhood $\omega_x \subset \mathcal{N}$ which satisfies the proprties: (i) f is a homeomorphism from ω_x to $f(\omega_x)$; (ii) $f(\omega_x)$ is a convex graph; (iii) the convexity of $f(\omega_x)$ agrees with the orientation.

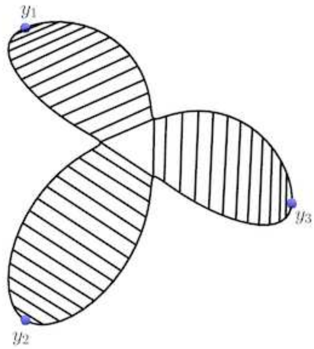

Figure 0.

Since the surface is immersed, self-intersection may occur. Therefore it is convenient to identify a point on $\mathcal{M}$ with the corresponding point on the manifold $\mathcal{N}$. Namely when we refer to a point $p \in \mathcal{M}$, we actually mean a point $x \in \mathcal{N}$ such that $p = f(x)$. Similarly we say $\omega_p \subset \mathcal{M}$ is a neighbourhood of p if it is the image of a neighbourhood in $\mathcal{N}$ of x, and a set $E \subset \mathcal{M}$ is connected if it is the image of a connected set in $\mathcal{N}$, and so on. Let U be a subset of $\mathbb{R}^{n+1}$ containing $p \in \mathcal{M}$, we denote by $U \cap_p \mathcal{M}$ the (intrinsic) connected component of $U \cap \mathcal{M}$ containing p. For any interior point $p \in \mathcal{M}$, by Definition 3.1 there is a neighbourhood $\omega_p \subset \mathcal{M}$, which can be represented as a graph of a convex function.

Definition 3.2. The cone

$$\mathcal{C}_{p,\xi,r,\alpha} := \{q \in \mathbb{R}^{n+1} \mid |q-p| < r, \langle q-p, \xi\rangle \geq |q-p|\cos\alpha\} \tag{3.1}$$

is called an inner contact cone of $\mathcal{M}$ at p if it lies on the concave side of ω_p (i.e. the cone and ω_p lie on the same side of a support hyperplane of ω_p at p) and

$$\mathcal{C}_{p,\xi,r,\alpha} \cap \omega_p = \{p\},$$

where ξ is the axis of the cone. We say $\mathcal{M}$ satisfies the uniform cone condition with radius r and aperture α if $\mathcal{M}$ has an inner contact cone at all points with the same r and α.

For the convenience of our statement, we introduce a notation. Let $\mathcal{M}$ be a locally convex surface. Assume $\omega \subset \mathcal{M}$ can be represented as a graph of a convex function v defined in a domain Ω. Let u be another convex function in Ω satisfying $u \geq v$ in Ω and $u = v$ on $\partial\Omega$. Removing ω from $\mathcal{M}$ and replacing it by the graph of u, we obtain a new locally convex hypersurface $\mathcal{M}_1$. We will call this replacement an operation. If a locally convex hypersurfaces $\mathcal{M}_1$ can be obtained from $\mathcal{M}$ by performing finitely many times operations, we will denote $\mathcal{M}_1 \preceq \mathcal{M}$. The set of all locally convex hypersurfaces $\mathcal{M}' \preceq \mathcal{M}$ will be denoted as $Q_{\mathcal{M}}$. Note that for any $\mathcal{M}' \subset Q_{\mathcal{M}}$, we have $\partial\mathcal{M}' = \partial\mathcal{M}$.

Condition (H). The uniform cone condition holds for all locally convex hypersurfaces in $Q_{\mathcal{M}}$, with uniform α and r for all members in $Q_{\mathcal{M}}$.

The uniform cone condition, introduced in [19], is a key ingredient in our treatment of locally convex hypersurfaces. It enables us to prove the existence of locally convex hypersurafces with vanishing Gauss curvature (or constant Gauss curvature [8, 19]). For any point $p \in \mathcal{M}$, if the uniform cone condition holds, then there exists a constant $r > 0$ such that $\omega_p := B_r(p) \cap_p \mathcal{M}$ can be represented as a graph of a Lipschitz continuous, convex function v over a domain Ω. Let u be the solution to

$$\begin{cases} \det D^2 u = 0 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Now we remove ω_p from $\mathcal{M}$ and replace it by the graph of u over Ω, and denote the resulting hypersurface by $\mathcal{M}_1$. Then $\mathcal{M}_1 \preceq \mathcal{M}$ and $A(\mathcal{M}_1) \le A(\mathcal{M})$, where $A(\cdot)$ denotes the area functional of hypersurfaces. We choose the point p such that $A(\mathcal{M}) - A(\mathcal{M}_1)$ is maximized.

Assume that condition (H) holds. Then we can repeat the above operation, and obtain a sequence of locally convex hypersurfaces $\mathcal{M}_k$, with the monotonicity $\mathcal{M}_{k+1} \preceq \mathcal{M}_k$ for all $k \ge 0$. For any sequence of points $p_k \in \mathcal{M}_k$, the uniform cone condition ensures that $B_r(p_k) \cap_{p_k} \mathcal{M}_k$ is the graph of a uniformly Lipschitz continues convex function. Hence $\mathcal{M}_k$ subconverges to a locally convex hypersurface $\mathcal{M}_0$. One easily verifies that $\mathcal{M}_0$ has vanishing Gauss curvature and the boundary $\partial\mathcal{M}_0 = \partial\mathcal{M}$.

If hypersurfaces in $Q_{\mathcal{M}}$ does not satisfy the uniform cone condition, then singularities, such as sharp edges and multiplicity two planes, may occurs in the above process. Under condition (H), $\mathcal{M}_k$ is locally Lipschitz continuous and no further regularity is required. Therefore we have the following existence of locally convex hypersurfaces of vanishing Gauss curvature.

Theorem 3.3. *Given a locally convex hypersurface $\mathcal{M}$, assume the condition (H) holds. Then there exists a locally convex immersion $\mathcal{M}_0$ such that the Gauss curvature $\mathcal{K}_{\mathcal{M}_0} = 0$ and $\partial\mathcal{M}_0 = \partial\mathcal{M}$.*

We point out that the condition (H) is satisfied in the following three situations.

(I) $\mathcal{M}$ is the graph of a locally convex, Lipschitz continuous function v on a bounded domain Ω. In this case the axis in the cone (3.1) is e_3, and α depends only on the Lipschitz constant of v. Note that if a convex function u satisfies $u \geq v$ in a sub-domain $\omega \subset \Omega$ and $u = v$ on $\partial\omega$, then $\|u\|_{Lip(\omega)} \leq \|v\|_{Lip(\omega)}$. We may also allow that v is multi-valued (the graph of v is a locally convex hypersurface).

(II) $\mathcal{M}$ can be extended across its boundary to a locally convex hypersurface $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}}$ is locally strictly convex near $\partial\tilde{\mathcal{M}}$. This is the case discussed in [19]. We will explain the idea briefly after Lemma 3.4.

(III) $\mathcal{M}$ can be extended across its boundary to a locally convex hypersurface $\tilde{\mathcal{M}}$ with the following property. There is a constant $\theta_0 > 0$, such that for any point $p_0 \in \partial\mathcal{M}$, the angle between L_{p_0} and L'_{p_0} is greater than θ_0, where L_{p_0} is the tangent plane of $\mathcal{M}$ at p_0 (defined in (2.3)), L'_{p_0} is the tangent plane of $\tilde{\mathcal{M}} - \mathcal{M}^o$ at p_0, and $\mathcal{M}^o = \mathcal{M} - \partial\mathcal{M}$.

Note that in case (III), if $\mathcal{M}_1 \in Q_{\mathcal{M}}$ and $L_{p_0,1}$ is the tangent plane of $\mathcal{M}_1$ at p, the angle between $L_{p_0,1}$ and L'_{p_0} is also greater than θ_0.

The uniform cone condition in cases (II) and (III) is based on the following property of locally convex hypersurfaces [3, 9, 19].

Lemma 3.4. *Let $\mathcal{M}$ be a bounded, locally convex surface with boundary $\partial\mathcal{M}$. Assume that $\partial\mathcal{M}$ is on a plane L. Then $\mathcal{M}$ is convex, namely it is on the boundary of a convex body.*

Lemma 3.4 was proved by the technique of moving parallel planes [9, 19]. Let $L = \{x_3 = 0\}$. Move the paraboloid $x_3 = \epsilon(x_1^2 + x_2^2)$ from a low position upwards until it touches $\mathcal{M}$ at some point p_0. Then $\mathcal{M}$ is strictly convex at p_0. Let L_{p_0} be a local support plane of $\mathcal{M}$ at p_0. One can move L_{p_0} slightly to cut off a convex cap. One can keep moving the parallel plane and show that the cap is always convex, until it touches the boundary. As ϵ can be arbitrarily small, the lemma is proved.

We can apply the above technique to prove the uniform cone condition. For any interior point $p_0 \in \mathcal{M}$, if $K_{p_0} = \mathcal{M} \cap_{p_0} L_{p_0}$ contains no boundary point of $\mathcal{M}$, we can move L_{p_0} slightly to cut off a convex cap, and keep moving the plane and show that the cap is always convex, until it touches the boundary. From the conditions in (II) or in (III), there is an inner contact cone at p_0, see Figure 1. Moreover, the aperture α and radius r of the cone can be determined by the geometry of $\tilde{\mathcal{M}}$ near the boundary $\partial\mathcal{M}$ and the diameter of $\mathcal{M}$. Therefore the aperture α and radius r of the inner contact cone is uniformly bounded from below for all locally convex surface $\mathcal{M}' \subset Q_{\mathcal{M}}$. We refer the reader to [19] for details.

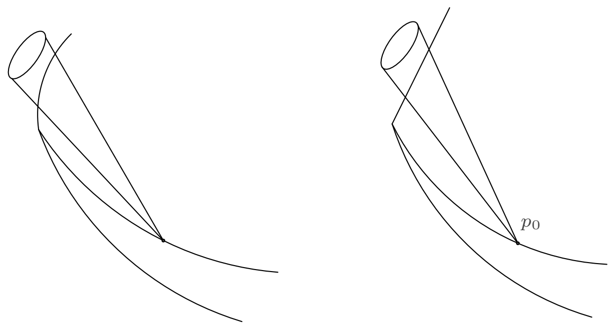

Figure 1.

By Theorem 3.3, we can adapt the method in Section 2 to prove Theorem 1.1, under condition (H). In the following we assume $n = 2$, and $\mathcal{N}$ is a topological disk.

As in Section 2 we denote, for any interior point p_0 of $\mathcal{M}_0$,

$$K_{p_0,L_{p_0}} = L_{p_0} \cap_{p_0} \mathcal{M}_0,$$

where L_{p_0} is a local support plane of $\mathcal{M}_0$ at p_0. As before we denote, for brevity, that $K_{p_0} = K_{p_0,L_{p_0}}$ when no confusion arises. A basic property is that K_{p_0} can be spanned by $K_{p_0} \cap \partial\mathcal{M}$, otherwise the Gauss curvature of $\mathcal{M}_0$ wouldn't vanish everywhere. Therefore K_{p_0} is either a segment or a planar disc in $\mathcal{M}_0$. If it is a segment, both endpoints are boundary points of $\mathcal{M}_0$. If it is a planar disc, then for any point $p \in \partial K_{p_0,L_{p_0}} \backslash \partial\mathcal{M}$, there is a line segment $\ell_p \subset K_{p_0}$ of which both endpoints are boundary points. For convenience we call a segment ℓ in $\mathcal{M}_0$ a ς-segment if both of its endpoints lie on $\partial\mathcal{M}_0$.

Similarly to Lemma 2.2 we have

Lemma 3.5. *Let $f : \mathcal{N} \to \mathcal{M}_0 \subset \mathbb{R}^3$ be a locally convex immersion such that the Gauss curvature $\mathcal{K}_{\mathcal{M}_0} = 0$. Assume that $\partial\mathcal{M}_0$ is C^1 smooth. Let $\ell_{p_1} \subset K_{p_1}, \ell_{p_2} \subset K_{p_2}$ and $\ell_{p_3} \subset K_{p_3}$ be three different ς-segments. Assume* *that K_{p_1}, K_{p_2} and K_{p_3} are also different from each other. Then ℓ_{p_1}, ℓ_{p_2} and ℓ_{p_3} cannot share a same endpoint.*

We can also establish the following lemma in correspondence to Lemma 2.3.

Lemma 3.6. *Let $f : \mathcal{N} \to \mathcal{M}_0 \subset \mathbb{R}^3$ be a locally convex immersion such that $\mathcal{K}_{\mathcal{M}_0} = 0$. Suppose that $\partial\mathcal{M}_0$ is C^1 smooth curve. Then, there exists two sequences of ς-segments $\{\ell_{p_k}\}$ and $\{\ell_{q_k}\}$ such that $p_k \to p_0$, $q_k \to q_0$, $p_0 \neq q_0$, and L_{p_k}, L_{q_k} converge to L_{p_0}, L_{q_0}, the support planes of $\mathcal{M}_0$ at p_0, q_0, respectively.*

Proof. The argument is similar to the proof of Lemma 2.3. Let ℓ be a ς-segment such that $f^{-1}(\ell)$ divides $\mathcal{N}$ into two parts, $\mathcal{N} - f^{-1}(\ell) = \mathcal{N}_1 \cup \mathcal{N}_2$. Let p_1 be the middle point of ℓ. If $\mathcal{N}_1 = f^{-1}(K_{p_1} \cap_{p_1} \mathcal{M}_0)$, then $\mathcal{M}_0$ is linear in $f(\mathcal{N}_1)$ and we are through. Otherwise for any point $p \in f(\mathcal{N}_1) - K_{p_1}$, there is a ς-segment ℓ_p in K_p. The curves $f^{-1}(\ell_{p_1})$ and $f^{-1}(\ell_p)$ bound a subdomain of $\mathcal{N}_1$, which we denote as $\mathcal{N}_{\ell_{p_1},\ell_p}$. Denote $A = \sup\{|\mathcal{N}_{\ell_{p_1},\ell_p}| \mid \ell_p$ is a ς-segment in $f(\mathcal{N}_1)\}$. Choose a sequence of ς-segments $\ell_{p_k} \subset \mathcal{M}_0$ such that $\lim_{k\to\infty} |\mathcal{N}_{\ell_{p_1},\ell_{p_k}}| = A$. By Lemma 3.5 we can follow the proof of Lemma 2.3 to show that $\{p_k\}$ fulfils the requirements in Lemma 3.6. Similarly, we can find the desired sequence $\{q_k\}$ in $f(\mathcal{N}_2)$. $\square$

Now, we can prove Theorem 1.1 under the assumption (H).

Proof of Theorem 1.1 under condition (H). By Theorem 3.3, there exists a locally convex surface $\mathcal{M}_0$ such that $\mathcal{K}_{\mathcal{M}_0} = 0$ and $\partial\mathcal{M}_0 = \Gamma$. Using Lemma 3.5 and 3.6, and by the same proof of Lemma 2.4 (because locally $\mathcal{M}_0$ is a graph), we see that L_{p_0} and L_{q_0} are osculating planes of Γ at p_0 and q_0, respectively. Hence the torsion of Γ vanishes at p_0 and q_0. Moreover, by Lemma 2.5, there exist points p_1, p_2, q_1, q_2 such that $\tau(p_1) < 0, \tau(p_2) > 0, \tau(q_1) < 0, \tau(q_2) > 0$, and $p_1, p_0, p_2, q_1, q_0, q_2$ are cyclicly oriented on Γ. Therefore τ changes sign four times. $\square$

4. Verification of Condition (H)

Let $\mathcal{M}$ be a locally convex surface with C^3 smooth boundary Γ. Assume that Γ has non-vanishing curvature. In this section we show that either condition (H) is fulfilled, or there exist two points $p_0, q_0 \in \Gamma$ and two (local) tangent planes L_{p_0} and L_{q_0}, which are osculating planes of Γ at p_0 and q_0, respectively. In the former case, Theorem 1.1 was proved in Section 3. In the latter case, Theorem 1.1 holds automatically (by Lemma 2.5).

At any given boundary point $p_0 \in \Gamma$, we can uniquely define five unit vectors, the unit tangent vector $\mathbf{t}$, the principal normal vector $\mathbf{n}$ (see proof of Lemma 2.4), the binormal vector $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, the normal ν of the surface $\mathcal{M}$ (see definition after (2.4)), and $\gamma = \nu \times \mathbf{t}$. Here γ is perpendicular to Γ, tangential to the surface $\mathcal{M}$ and pointing to the side of $\mathcal{M}$. There are two separate cases:

Case (a): $\gamma \cdot \mathbf{n} > -1$, namely $\gamma \neq -\mathbf{n}$;

Case (b): $\gamma \cdot \mathbf{n} = -1$, namely $\gamma = -\mathbf{n}$.

The proof of Theorem 1.1 will be carried out in two subsections.

4.1. Case (a) is true everywhere on Γ. In this case we show that $\mathcal{M}$ can be extended along Γ such that the extended part is strictly convex. This is the case (II) discussed in Section 3. Hence the uniform cone condition (H) is satisfied and Theorem 1.1 holds by Section 3.

4.2. Case (b) occurs somewhere on Γ. In this case we will divide $\mathcal{M}$ into two parts, corresponding respectively to the cases (I) and (III) in Section 3. Hence the uniform cone condition (H) is satisfied and Theorem 1.1 also holds.

4.1. Case (a) is true everywhere on Γ

Observe that the quantity $\gamma \cdot \mathbf{n}$ is lower semicontinuous, as a function on the curve Γ. Indeed, let L_p be the tangent plane of ω_{p_0} at $p \in \Gamma$, as defined above. If $p \to p_0$ and L_p converges to L_0, then L_0 is a support plane of ω_{p_0} at p_0, namely ω_{p_0} stays on one side of L_0. Therefore if Case (a) holds everywhere on Γ, there is a positive constant $\delta_0 > 0$ such that

$$\gamma \cdot \mathbf{n} \geq -1 + \delta_0 \tag{4.1}$$

everywhere on Γ.

As Γ is C^3 smooth, the normal vector $\mathbf{n}$ is C^1 smooth only ($\mathbf{n}$ is C^2 if Γ is a planar curve). We choose two C^2 smooth unit vector fields $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ on Γ, satisfying $\hat{\mathbf{n}} \cdot \mathbf{n} > 1 - \frac{1}{4}\delta_0$ and $\hat{\mathbf{b}} \cdot \mathbf{b} > 1 - \frac{1}{4}\delta_0$, respectively. For any point $p_0 \in \Gamma$, choose the coordinates such that p_0 is the origin, and at p_0, $\hat{\mathbf{n}} = e_2$, $\hat{b} = e_3$, such that $\mathbf{t} \cdot e_1 \geq 1 - \frac{1}{2}\delta_0$. Then by (4.1), locally ω_{p_0} can be represented as the graph of a convex function. Replacing the parameter s by $-s$ if needed, we may assume that ω_{p_0} lies above the plane $\{x_3 \geq 0\}$.

Corresponding to the point p_0, we define a curve ℓ_{p_0}, which is given by

$$\ell_{p_0} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_3 = -\frac{\delta_0}{4} x_2 + M x_2^2, x_2 \in [0, M^{-2}] \right\}, \tag{4.2}$$

where $M > 1$ is a large constant. Then ℓ_{p_0} is a curve in the plane spanned by the vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$. Note that the point p_0 is arbitrarily chosen, so for any point $p \in \Gamma$ we have defined a curve ℓ_p.

The extended part of $\mathcal{M}$ is then the union of all the curves ℓ_p,

$$\mathcal{M}' = \bigcup_{p \in \Gamma} \ell_p. \tag{4.3}$$

As $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ are C^2 smooth, the extended part $\mathcal{M}'$ is also C^2-smooth.

Near the origin p_0, in the above coordinates, $\mathcal{M}'$ is the graph of a function v. We calculate the Hessian matrix of v at 0,

$$\begin{aligned} & v_{x_1 x_1}(0) \geq \frac{1}{8} \delta_0 \kappa, \\ & |v_{x_1 x_2}(0)| \leq C, \\ & v_{x_2 x_2}(0) = 2M. \end{aligned} \tag{4.4}$$

Hence $D^2 v(0)$ is positive definite if M is chosen large. By the C^2 smoothness of $\mathcal{M}'$, we see that $D^2 v$ is positive definite and v is uniformly convex in a neighbourhood of Γ.

4.2. Case (b) occurs somewhere on Γ

Note that if Case (b) holds at some point $p_0 \in \Gamma$, then the tangent plane L_{p_0} is an osculating plane of Γ at p_0. Hence by Lemma 2.5, Theorem 1.1 holds if Case (b) occurs at two different points. Therefore we may assume that Case (a) holds everywhere on Γ except one point $p_0 \in \Gamma$, or an arc $\mathcal{A} \subset \Gamma$ containing p_0, and contained in the tangent plane L_{p_0} of $\mathcal{M}$ at p_0.

Choose the coordinates such that $L_{p_0} = \{x_3 = 0\}$ and locally $\mathcal{M}$ lies above the plane $\{x_3 = 0\}$. Denote $\mathcal{M}_h^- = \mathcal{M} \cap_{p_0} \{x_3 < h\}$. Denote $K_0 = \mathcal{M} \cap_{p_0} L_{p_0}$. We have two cases:

(i) $K_0 \cap (\partial \mathcal{M} \backslash \mathcal{A}) = \emptyset$, and

(ii) $K_0 \cap (\partial \mathcal{M} \backslash \mathcal{A}) \neq \emptyset$.

In case (i), for any given point $q_0 \in \mathcal{A}$, we choose the (x_1, x_2)-axes such that q_0 is the origin, $\mathbf{t}(q_0) = e_1$, $\mathbf{n}(q_0) = -e_2$ and $\gamma = e_2$. We restrict to the piece $\omega =: \mathcal{M}_h^- \cap_{q_0} \{x_2 \geq 0\}$. Then, the positive curvature of Γ and $\mathcal{A} \subset L_{p_0}$ implies that $\mathcal{A}$ is locally strictly convex (if $\mathcal{A}$ is not a single point). Hence $\mathcal{A} \cap_{q_0} \{x_2 \geq 0\} = \{q_0\}$ is a single point. For sufficiently small constant $\epsilon > 0$, the plane $\{x_3 = \epsilon x_2\}$ (denote this plane by K_{q_0}) cuts off a cap $\mathcal{C}_\epsilon$ from ω, and $\partial\mathcal{C}_\epsilon \cap \partial\omega = \{q_0\}$. Then, let

$$\mathcal{M}_0 := \mathcal{M}_h^- \bigcap \left(\bigcup_{q_0 \in \mathcal{A}} \{x \in \mathbb{R}^3 : x \text{ is above } K_{q_0}\} \right).$$

Hence the slope of the tangent plane of $\mathcal{M}_0$ at q_0 is greater than $\epsilon > 0$. It implies that Case (a) holds at q_0 for the locally convex surface $\mathcal{M}_0$. Hence for $\mathcal{M}_0$, Case (a) is true everywhere on Γ. Therefore by the argument in §4.1, Theorem 1.1 holds.

In case (ii), the set $\mathcal{A}' = K_0 \cap (\Gamma - \mathcal{A})$ is not empty, and $\forall\ q_0 \in \mathcal{A}'$, the vector $\mathbf{n}(q_0)$ is transversal to the plane L_{p_0}. Hence by the positive curvature of Γ, q_0 must be an isolated point in $\mathcal{A}'$, namely $\mathcal{A}'$ consists of finitely many isolated points. Hence K_0 is either a ζ-segment, or a planar (topological) disc in the plane L_{p_0}.

Lemma 4.1. *For $\delta > 0$ sufficiently small, the locally convex surface $\mathcal{M} \cap_{p_0} \{x_3 < \delta\}$ satisfies the uniform cone condition (H).*

Proof. Denote $\Gamma_1 = \Gamma \cap_{p_0} B_\delta(p_0)$ and $\Gamma_2 = \Gamma \cap_{q_0} B_\delta(q_0)$, where $q_0 \in \mathcal{A}'$ is any given point, and $\delta > 0$ is a small constant. Since $\mathbf{n}(q_0)$ is transversal to the plane L_{p_0} but $\mathbf{n}(p_0) \subset L_{p_0}$. The convex hull of Γ_1, Γ_2 and the segment $\overline{p_0 q_0}$ contains a ball, whose centre is close to the middle of the segment. Hence $\mathcal{M} \cap_{p_0} \{x_3 < \delta\}$ can be represented is a radial graph with respect to center of the ball. □

Denote

$$\begin{aligned} \mathcal{M}_h^- &= \mathcal{M} \cap_{p_0} \{x_3 < h\}, \\ \mathcal{M}_h^+ &= \mathcal{M} - \mathcal{M}_{h/2}^-. \end{aligned}$$

Note that $\mathcal{M}_h^-$ and $\mathcal{M}_h^+$ overlaps in the part $\{h/2 \leq x_3 \leq h\}$. When $h > 0$ is small, by Lemma 4.1 the uniform cone condition is satisfied for $\mathcal{M}_h^-$. Note that the boundary $\partial\mathcal{M}_h^+$ consists of two parts, one in $\partial\mathcal{M}_h^+ \cap \{x_3 \neq \frac{h}{2}\}$ and the other one on $\partial\mathcal{M}_h^+ \cap \{x_3 = \frac{h}{2}\}$. In §4.1 we have made an extension of $\mathcal{M}_h^+$ on the part in $\partial\mathcal{M}_h^+ \cap \{x_3 \neq \frac{h}{2}\}$. For the part $\partial\mathcal{M}_h^+ \cap \{x_3 = \frac{h}{2}\}$, the plane $L_h = \{x_3 = \frac{h}{2}\}$ makes a local expansion for $\mathcal{M}_h^+$. They corresponds respectively to the cases (II) and (III) in Section 3. Therefore the uniform cone condition holds for $\mathcal{M}_h^+$ too.

Therefore by Theorem 3.3, there is a locally convex surface $\mathcal{N}^+$ of vanishing Gauss curvature with the boundary $\partial\mathcal{M}_h^+$. Removing $\mathcal{M}_h^+$ from $\mathcal{M}$ and gluing back by $\mathcal{N}^+$, we get a locally convex surface $\mathcal{M}_1$. Next we apply Theorem 3.3 to $\mathcal{M}_{1,h}^- =: \{x_3 < h\} \cap_{p_0} \mathcal{M}_1$ and obtain a locally convex surface $\mathcal{N}^-$ of vanishing Gauss curvature with the boundary $\partial\mathcal{M}_{1,h}^-$. Removing $\mathcal{M}_{1,h}^-$ from $\mathcal{M}_1$ and gluing back by $\mathcal{N}^-$, we get a locally convex surface $\mathcal{M}_2$.

Assume that we have the locally convex surfaces $\mathcal{M}_1, \ldots, \mathcal{M}_k$. Denote $\mathcal{M}_{k,h}^- = \mathcal{M}_k \cap_{p_0} \{x_3 < h\}$ and $\mathcal{M}_{k,h}^+ = \mathcal{M}_k - \mathcal{M}_{k,h/2}^-$. Carrying out the above operations, we therefore obtain a sequence of locally convex surfaces $\{\mathcal{M}_k\}_{k=1}^{\infty}$. They satisfies the monotonicity $\mathcal{M}_{k+1} \preceq \mathcal{M}_k$ for all $k \geq 1$. By Lemma 4.1, the condition (H) is satisfied uniformly for this sequence. Hence the sequence converges to a limit surface $\mathcal{M}_0$ of vanishing Gauss curvature with the boundary $\partial\mathcal{M}_0 = \Gamma$. Therefore Theorem 1.1 follows from the argument in Section 3.

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School of Mathematical Sciences
University of Science and Technology of China
Jinzhai Road 96, Hefei 230026, China
E-mail address: chenshib@ustc.edu.cn

Centre for Mathematics and Its Applications
The Australian National University
Canberra, ACT 2601, Australia
E-mail address: Xu-Jia.Wang@anu.edu.au

School of Mathematical Science, Peking University
Yiheyuan Road 5, Beijing 100871, China
E-mail address: bzhou@pku.edu.cn

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