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Infinite-time singularity type of the Kähler-Ricci flow II

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On a compact Kähler manifold with semi-ample canonical line bundle and Kodaira dimension one, we observe a relation between the infinite-time singularity type of the Kähler-Ricci flow and the characteristic indexes of singular fibers of the semi-ample fibration.

1. Introduction

We continue our study in [24] on the singularity type of long time solutions to the Kähler-Ricci flow on compact Kähler manifolds. Our main interest is the *relation between infinite-time singularity type (more precisely, long time behavior of curvature) of the Kähler-Ricci flow and the geometric/complex structure of the underlying compact Kähler manifold.*

Thanks to the maximal existence time theorem of the Kähler-Ricci flow [2, 18, 22], the existence of long time solutions is equivalent to nefness of the canonical line bundle. In this paper, as in [24], motivated by the Abundance Conjecture (which predicts that if the canonical line bundle of an algebraic manifold is nef, then it is semi-ample), we will further restrict our discussions on n-dimensional Kähler manifold, denoted by X, with semi-ample canonical line bundle K_X, and so the Kodaira dimension $kod(X) \in \{0, 1, \dots, n\}$.

Given an arbitrary Kähler metric ω_0 on X, let $\omega = \omega(t)_{t\in[0,\infty)}$ be the solution to the Kähler-Ricci flow

$$\partial_t\omega = -Ric(\omega) - \omega \tag{1.1}$$

running from ω_0. Recall from [8] that the infinite-time singularities of the Kähler-Ricci flow are divided into two types. Precisely, we say a long time solution of the Kähler-Ricci flow (1.1) is of type IIb if

$$\limsup_{t\to\infty}\left(\sup_X |Rm(\omega(t))|_{\omega(t)}\right) = \infty$$

and of type III if

$$\limsup_{t\to\infty}\left(\sup_X |Rm(\omega(t))|_{\omega(t)}\right) < \infty.$$

For X with semi-ample canonical line bundle, we let

$$f : X \to X_{can} \subset \mathbb{CP}^N \tag{1.2}$$

be the semi-ample fibration with connected fibers induced by pluricanonical system of K_X, where X_{can}, a $kod(X)$-dimensional irreducible normal projective variety, is the canonical model of X, and $V \subset X_{can}$ be the singular set of X_{can} together with the critical values of f, and $X_y = f^{-1}(y)$ be a smooth fiber for $y \in X_{can} \setminus V$. The singularity type of the Kähler-Ricci flow on such manifold X has been classified except for the following case (see e.g. [19, Section 6.3] or [24, Section 1] for an overview):

($\star$) *$0 < kod(X) < n$ ($n \geq 3$) and the generic fiber X_y is a finite quotient of a torus and $V \neq \emptyset$.*

On the one hand, as expected by Tosatti and Y.G. Zhang in [21, Section 1] (also see [19, Conjecture 6.7] for a more general conjecture by Tosatti) and confirmed by our previous work [24, Theorem 1.4], the singularity type of the Kähler-Ricci flow in case ($\star$) does not depend on the choice of the initial Kähler metric and hence should only depend on the complex structure of X. This result in particular indicates that the following problem should be very natural:

Problem 1.1. *Classify/characterize the complex structure on X in terms of infinite-time singularity types of the Kähler-Ricci flow.*

In fact, in case ($\star$), since the curvature on any compact subset of $X \setminus V$ is uniformly bounded [5, 10, 21], to determine the singularity type one only needs to analyze the long time behavior of curvature near singular set $f^{-1}(V)$ and hence the singularity type should be closely related to the properties of the singular fibers. For example, a result of Tosatti and Y.G. Zhang [21, Proposition 1.4] implies that if there exists a rational curve in some singular fiber of f in case ($\star$), then the Kähler-Ricci flow on X must develop type IIb singularity, which in particular has been applied very successfully when

X in case ($\star$) is a minimal elliptic surface, see [21, Theorem 1.6].

In this note we will focus on the $kod(X) = 1$ case, in which the canonical model X_{can} of X is a closed smooth Riemann surface. Let $f : X \to X_{can}$ be the semi-ample fibration in (1.2), $V = \{s_0, \ldots, s_L\}$ the finite set of isolated critical values of f and X_{s_l} the singular fiber over s_l. Therefore, in $kod(X) = 1$ case one may more precisely formulate Problem 1.1 as: *classify/characterize the singular fibers X_{s_l} in terms of infinite-time singularity types of the Kähler-Ricci flow.*

Our main goal here is to find some restrictions on singular fibers when X admits type III solution to the Kähler-Ricci flow. To begin with, we first observe one more application of [21, Proposition 1.4] as follows.

Proposition 1.2. *Assume X as in case ($\star$), Kodaira dimension $kod(X)$ is one and X is projective. Let $V = \{s_0, \ldots, s_L\} \subset X_{can}$ is the set of critical values of f in* (1.2)*. If the Kähler-Ricci flow on X is of type III, then every singular fiber X_{s_l} has only one irreducible component, say F_l, where F_l is a normal projective variety of only canonical singularities, satisfies $K_{F_l} \sim_{\mathbb{Q}} 0$ and contains no rational curve.*

To state the next restriction on singular fibers, recall that for every singular fiber X_{s_l} we can naturally define (see Section 2 for precise definitions) the characteristic index of X_{s_l} along X, which is a pair of two numbers, denoted as (β_l, N_l), measuring the singularities of the singular fiber X_{s_l}. In general, $\beta_l \in \mathbb{Q}_{>0}$ and $N_l \in \{1, 2, \ldots, n\}$. With these notations, the following is our main result, which provides an interesting relation between the singularity type of the Kähler-Ricci flow and the characteristic indexes of singular fibers.

Theorem 1.3. *Assume X as in case ($\star$) and Kodaira dimension $kod(X)$ is one. Let $V = \{s_0, \ldots, s_L\} \subset X_{can}$ be the set of critical values of f in* (1.2)*. If the Kähler-Ricci flow on X is of type III , then for every singular fiber X_{s_l} the characteristic index $(\beta_l, N_l) = (\frac{1}{m_l}, 1)$ for some positive integer m_l.*

Note that in Theorem 1.3 we do *not* need to assume X is projective. Our proof for Theorem 1.3 mainly makes use of a metric geometric approach, which in fact proves a slightly stronger result, see Theorem 2.1 below.

Proofs will be provided in Section 2.

2. Proofs

To prove Theorem 1.3 let's first recall some definitions, following [4, Definition 2.3.1] (also see [3, Section 2.2] and [7, Definition 6.2.1]). Assume X is an n-dimensional ($n \geq 2$) compact Kähler manifold with semi-ample canonical line bundle and Kodaira dimension one. Let $f : X \to X_{can}$ and $V = \{s_1, ..., s_L\}$ as before. For any given $s_l \in V$, by applying a Hironaka's resolution of singularities, one obtains a birational morphism

$$\pi : \hat{X} \to X$$

such that $\hat{X}$ is smooth, $\pi : \hat{X} \setminus \pi^{-1}(X_{s_l}) \to X \setminus X_{s_l}$ is biholomorphic and

$$\hat{f} := f \circ \pi : \hat{X} \to X_{can}$$

is a holomorphic surjective map with connected fibers and the singular fiber

$$\hat{X}_{s_l} = \sum a_j E_j,$$

where $a_i \in \mathbb{N}_{\geq 1}$, E_i's are smooth irreducible divisors in $\hat{X}$ and have simple normal crossings. We also have the following ramification formula

$$K_{\hat{X}} = \pi^* K_X + \sum k_j E_j,$$

where $k_j \in \mathbb{N}_{\geq 0}$ since X is smooth. Then the *log-canonical threshold* of X_{s_l} along X is defined to be $\beta_l = \min\{\frac{k_j+1}{a_j}\}$, the *log-canonical multiplicity* of X_{s_l} along X is defined to be the maximal integer N_l such that there are N_l distinct E_j's with $\frac{k_j+1}{a_j} = \beta_l$ and non-empty intersection, and the *characteristic index* of X_{s_l} along X is defined to be the pair (β_l, N_l). The characteristic index (β_l, N_l) measures the singularities of the singular fiber X_l along X. Note that the above definitions do not depend on the choice of resolution π, see [3, Lemma 2.1, Definition 2.1] and [4, Section 2.3].

Then we will prove the following, which implies our main result Theorem 1.3 immediately.

Theorem 2.1. *Assume X as in case ($\star$) and Kodaira dimension $kod(X)$ is one. Let $V = \{s_0, \ldots, s_L\} \subset X_{can}$ be the set of critical values of f in* (1.2). *If there exists a singular fiber $X_{s_{l_0}}$ such that the characteristic index (β_{l_0}, N_{l_0}) is not of the form $(\frac{1}{m}, 1)$ for some positive integer m, then for every neighborhood U of $X_{s_{l_0}}$ and every solution $\omega(t)$ to the Kähler-Ricci flow on X we* *have*

$$\liminf_{t\to\infty}\left(\sup_U sec_{\omega(t)}\right)=\infty.$$

In particular, every solution is of type IIb.

To prove Theorem 2.1 we will make use of a metric geometric approach, in which a key point is that, thanks to [23, Theorem 1], the informations about the characteristic indexes of singular fibers will be saved in Gromov-Hausdorff limits of the Kähler-Ricci flow solution, via Song-Tian's generalized Kähler-Einstein current.

Proof of Theorem 2.1. Firstly, we just assume X is a compact Kähler manifold with semi-ample canonical line bundle and Kodaira dimension one. Let $\omega(t)_{t\in[0,\infty)}$ be a solution to the Kähler-Ricci flow (1.1) on X. Thanks to Song and Tian's fundamental works [14, 15] there exists a generalized Kähler-Einstein current ω_{GKE} on X_{can} which is a closed positive $(1,1)$-current on X_{can} and a smooth Kähler metric on $X_{can}\setminus V$ such that, as $t\to\infty$, $\omega(t)$ converges to $f^*\omega_{GKE}$ as currents on X; moreover, for any $K\subset\subset X\setminus f^{-1}(V)$ there holds that $\omega(t)\to f^*\omega_{GKE}$ in $C^0(K,\omega_0)$-topology [5, 10, 20].

On the other hand, the numbers β_l and N_l arise naturally in the metric behaviors of the limiting singular metric ω_{GKE} near its singular locus. Precisely, by [23, Theorem 1] (also see [14, Lemma 3.4] and [9, Section 3.3] for the special case that X is a minimal elliptic surface), there exists a constant $C\geq 1$ such that, for any $s_l\in V$, we can choose a local holomorphic chart (Δ,s) in X_{can} near s_l (in which we identify Δ with $\{s\in\mathbb{C}||s|\leq 1\}$ and s_l with $0\in\mathbb{C}$) with the property that

$$\begin{aligned}(2.1)\qquad C^{-1}|s|^{-2(1-\beta_l)}(-\log|s|)^{N_l-1} &\leq \frac{\omega_{GKE}}{\sqrt{-1}ds\wedge d\bar{s}}(s)\\ &\leq C|s|^{-2(1-\beta_l)}(-\log|s|)^{N_l-1}\end{aligned}$$

holds on $\Delta\setminus\{0\}$. Here, β_l and N_l are the log-canonical threshold and the log-canonical multiplicity of the singular fiber X_{s_l}, respectively. Consequently, (X_∞,d_∞), the metric completion of $(X_{can}\setminus V,\omega_{GKE})$, is a compact length metric space homeomorphic to X_{can}.

Denote $(X_{can},d_{can}):=(X_\infty,d_\infty)$.

From now on, we further assume the generic fiber of f is a finite quotient of a torus. Thanks to a recent work of Tian and Z.L. Zhang [17, Theorem 1.5, Corollary 1.6] (also see our previous work [23, Theorem 2] for a weaker version), for X in case $(\star)$ of Kodaira dimension one, $(X,\omega(t))\to(X_{can},d_{can})$ in Gromov-Hausdorff topology, as $t\to\infty$.

Now assume Theorem 2.1 fails. We may assume without loss of generality that $s_{l_0} = s_0$. Then we have a solution $\omega(t)$ to the Kähler-Ricci flow (1.1) on X, a small open neighborhood U of X_{s_0}, a time sequence $t_i \to \infty$ and a constant $A' \geq 1$ such that

$$\sup_U sec_{\omega(t_i)} \leq A'. \tag{2.2}$$

We may assume $U \cap X_{s_l} = \emptyset$ for every $s_l \neq s_0$.

Since by a well-known result of Hamilton the scalar curvature of $\omega(t)$ is uniformly bounded from below, it follows from (2.2) that there exists a constant $A'' \geq 1$ with

$$\inf_U sec_{\omega(t_i)} \geq -A'' \tag{2.3}$$

for every i.

On the other hand, let's try to understand the limiting metric space (X_{can}, d_{can}) from a metric geometric viewpoint. In fact, by combining (2.2) and (2.3) we know $(X, \omega(t_i)$ has uniformly bounded sectional curvature on U; also, the diameter of $(X, \omega(t_i))$ is uniformly bounded from above. We fix an arbitrary point $x_0 \in X_{s_0}$, then $(X, \omega(t_i), x_0) \to (X_{can}, d_{can}, s_0)$ in pointed Gromov-Hausdorff topology. We will need the following

Lemma 2.2. *There exist a sufficiently small constant $\rho > 0$ and a sufficiently large constant $T > 0$, such that for any $t_i \geq T$ there holds*

$$B_{\omega(t_i)}(x_0, \rho) \subset U.$$

Proof of Lemma 2.2. This is essentially contained in [17, Section 5, proof of Theorem 1.5] (also see [23, Theorem 2, Section 4]). In fact, we may firstly fix a small positive number δ such that $f^{-1}(B_{d_{can}}(s_0, \delta)) \subset U$, where $B_{d_{can}}(s_0, \delta)$ is the ball in (X_{can}, d_{can}). We then set $\rho := \frac{\delta}{4}$, By Gromov-Hausdorff convergence of the Kähler-Ricci flow one can fix a $T > 0$ such that for any $t_i \geq T$, $f : (X, \omega(t_i)) \to (X_{can}, d_{can})$ is a $\frac{\delta}{4}$-Gromov-Hausdorff approximation. Now, for any $x \in B_{\omega(t_i)}(x_0, \rho)$, i.e. $d_{\omega(t_i)}(x, x_0) < \frac{\delta}{4}$, we have

$$d_{can}(f(x), s_0) \leq d_{\omega(t_i)}(x, x_0) + \frac{\delta}{4} \leq \frac{\delta}{2},$$

and so $f(x) \in B_{d_{can}}(s_0, \delta)$, i.e. $x \in f^{-1}(B_{d_{can}}(s_0, \delta)) \subset U$. This lemma is proved. □

(We may also mention an observation which is not necessary in this note: for sufficiently t_i, $X_{s_0} \subset B_{\omega(t_i)}(x_0, \rho)$ since the diameter of fiber X_{s_0} tends to zero as $t_i \to \infty$, see [17, Section 5, proof of Theorem 1.5].)

By Lemma 2.2, we can fix a sufficiently large constant A such that $(X, \omega(t_i), x_0)$ are $\{A\}$-regular at x_0 in the sense of Naber and Tian [11, Definition 1.1]. Then, we can apply theory of Naber and Tian [11, Theorem 1.1, Remark 1.2] to see that (X_{can}, d_{can}) is a C^1 Riemannian orbifold in a neighborhood of s_0 (note that X_{can} is of real 2-dimension and hence there is no non-orbifold point in a neighborhood of s_0). It is known that real 2-dimensional orbifold singularities are completely classified, see e.g. [13, Sections 1 and 2]. In our case, for the given s_0 in the singular locus V, it is an *isolated* singular point; then by the forementioned classification results we know the associated orbifold group must be the finite cyclic group generated by a rotation, namely, $\mathbb{Z}_{m_0}$ for some integer $m_0 \geq 2$ depending on s_0 (see e.g. [13, Sections 1 and 2]). Consequently, locally near s_0, (X_{can}, d_{can}) is isometric to the quotient of (Δ, g) by $\mathbb{Z}_{m_0}$, where $\Delta = \{u \in \mathbb{C} | |u| < 1\}$ and g is some Riemannian metric on Δ which is $\mathbb{Z}_{m_0}$-invariant.

Let's first proceed by assuming an additional condition that the Riemannian metric g is compatible with the complex structure of (Δ, u). Then we may write $g = h(u) du \otimes d\bar{u}$ for $u \in \Delta$. We want to rewrite g with respect to the holomorphic coordinate s in Δ, which is the holomorphic coordinate used in (2.1). To this end, we first set a holomorphic coordinate $v = u^{m_0}$ in Δ and rewrite g with respect to v. Then we easily conclude (see e.g. [12, Section 6D]) that the metric g in local holomorphic chart (Δ, v) reads $h(v^{\frac{1}{m_0}}) m_0^{-2} \frac{dv \otimes d\bar{v}}{|v|^{2(1-\frac{1}{m_0})}}$ (note that $h(v^{\frac{1}{m_0}})$ is well-defined). After possibly shrinking Δ, we may choose a biholomorphism $v = k(s) : (\Delta, s) \to (\Delta, v)$; moreover, by theory on functions of one complex variable, there exists a holomorphic function $\tilde{k} = \tilde{k}(s)$ such that $|\tilde{k}(s)|$ is a positive function on Δ and $v = k(s) = \tilde{k}(s) s$. Therefore, the metric g in local holomorphic chart (Δ, s) reads

$$\frac{h(k(s)^{\frac{1}{m_0}}) |\frac{\partial k}{\partial s}|^2}{m_0^2 |\tilde{k}(s)|^{2(1-\frac{1}{m_0})}} \frac{ds \otimes d\bar{s}}{|s|^{2(1-\frac{1}{m_0})}},$$

and there exists a constant $\tilde{C} \geq 1$ such that

$$\tilde{C}^{-1} \frac{ds \otimes d\bar{s}}{|s|^{2(1-\frac{1}{m_0})}} \leq g \leq \tilde{C} \frac{ds \otimes d\bar{s}}{|s|^{2(1-\frac{1}{m_0})}} \tag{2.4}$$

on $\Delta \setminus \{0\}$. Here we have used that all the involved functions $h, |\frac{\partial k}{\partial s}|$ and $|\tilde{k}|$ are continuous, positive and bounded on Δ.

Next we shall show that the inequality (2.4) holds generally, even when g is not necessarily compatible with the complex structure. In fact, we can always fix on Δ a $\mathbb{Z}_{m_0}$-invariant Riemannian metric $\tilde{g}$ which is quasi-isometric to g (i.e. there is a constant $C \geq 1$ such that $C^{-1}\tilde{g} \leq g \leq C\tilde{g}$ holds on Δ) and compatible with the complex structure (for example, choose $\tilde{g}$ to be the standard Euclidean metric). Then the above arguments prove the inequality (2.4) for $\tilde{g}$, and so for g since g is quasi-isometric to $\tilde{g}$. In conclusion, g satisfies (2.4) generally.

Finally, recall that d_{can} on X_{can} is induced by ω_{GKE} which satisfies (2.1) in local holomorphic coordinate (Δ, s) near s_0. By comparing (2.1) with (2.4) there must hold $\beta_0 = \frac{1}{m_0}$ and $N_0 = 1$.

Theorem 2.1 is proved. □

Remark 2.3. (i) Obviously, Theorem 1.3 is an immediate consequence of Theorem 2.1. Alternatively, we can prove Theorem 1.3 directly. Indeed, if we assume a type III solution $\omega(t)$ on X, then the curvature of $\omega(t)$ is uniformly bounded on X and hence we can apply the same arguments as above, in which the role of [17, Theorem 1.5] can be played by a weaker result [23, Theorem 2] since a type III solution automatically has a uniform lower bound for Ricci curvature on X.

(ii) We mention that while the proof of Gromov-Hausdorff convergence of the Kähler-Ricci flow uses only the upper bound of ω_{GKE} given in (2.1) (see [17, Theorem 1.5] or [23, Theorem 2]), the above proof of Theorem 2.1 really needs the *asymptotics* of ω_{GKE} given in (2.1).

(iii) From its proof, Theorems 2.1 and 1.3 also hold when X is of dimension two, i.e. a minimal elliptic surface. Let X be a minimal elliptic surface with elliptic fibration $f : X \to X_{can}$. Let $V = \{s_0, \ldots, s_L\}$ be the critical values set of f. According to Kodaira's table of singular fibers [1], if $\beta_l = \frac{1}{m_l}$ for some positive integer m_l and $N_l = 1$, then the singular fiber X_{s_l} must be one of Kodaira's types $mI_0, I_0^*, \mathrm{II}^*, \mathrm{III}^*$ and IV^*. Therefore, if there exists a singular fiber which is not of Kodaira's type $mI_0, I_0^*, \mathrm{II}^*, \mathrm{III}^*$ nor IV^*, then the Kähler-Ricci flow on X must be of type IIb. This result partially recovers [21, Theorem 1.6].

(iv) The proof of Theorem 2.1 is essentially a contradiction argument. A natural question is: can we have an effective argument for it? Precisely, given the condition in Theorem 2.1, can we estimate the blowup speed of curvature effectively?

(v) In the study of long time solutions to the Kähler-Ricci flow, two of the most important aspects are the convergence and curvature behavior at time-infinity. We know from previous works (e.g. [5, 6, 17, 20, 21, 23]) that the curvature behavior at time-infinity is useful in obtaining/improving the convergence at time-infinity. Our above arguments conversely show that the convergence at time-infinity is also useful in understanding the curvature behavior at time-infinity. It should be very interesting to find more interplays between these two aspects of the Kähler-Ricci flow.

Finally, we give a

Proof of Proposition 1.2. As we recalled in Section 1, if there exists a type III solution to the Kähler-Ricci flow on X, then by [21, Proposition 1.4] every singular fiber X_{s_l} must contain no rational curve, which in particular implies X_{s_l} must contain no uniruled irreducible component. In this case, since X is projective, according to Takayama's classification result, X_{s_l} falls into the case (1.2) in [16, Theorem 1.1], namely, $X_{s_l} = d_l F_l$, where F_l is the only one irreducible component of X_{s_l} (d_l is the multiplicity of F_l), and F_l is a normal projective variety of only canonical singularities, satisfies $K_{F_l} \sim_{\mathbb{Q}} 0$ (here "$\sim_{\mathbb{Q}}$" means $\mathbb{Q}$-linear equivalence) and contains no rational curve. Lastly, we remark that, when X is projective, the smooth fiber of f is a flat projective Kähler manifold [21] and hence an Abelian variety up to a finite unramified covering. □

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