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From automorphisms of Riemann surfaces to smooth 4-manifolds

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Starting from a suitable set of self-diffeomorphisms of a closed Riemann surface, we present a general branched covering method to construct surface bundles over surfaces with positive signature. Armed with this method, we study the classification problem for both surface bundles with nonzero signature and closed simply connected smooth spin 4-manifolds.

1. Introduction

Let Σ_g denote a closed Riemann surface, i.e., a compact smooth oriented real 2-dimensional manifold without boundary, having genus g. By a surface bundle (over a surface), we mean a smooth fiber bundle whose fiber and base are diffeomorphic to Σ_f and Σ_g for some nonegative integers f and g. It is well known (cf. [16] and [21]) that the signature of a surface bundle is divisible by 4 and that it vanishes when $f \leq 2$ or $g \leq 1$. The first example of a surface bundle with nonzero signature was constructed by Kodaira in [18]. His example, other early examples in [2], [15], and more recent examples in [4], [5], [6] were all constructed using branched covering methods. Inspired by these works, we will present a more general topological recipe for constructing surface bundles via branched covering method, where the initial input is a finite set of self-diffeomorphisms of a fixed surface Σ_h whose graphs are mutually disjoint. For an alternative approach using relations in mapping class groups, we refer to [8], [20], and [22].

Our first application is in the classification of surface bundles with nonzero signature. The following definition was introduced by Endo in [8].

Definition 1. The *minimal base genus function* is

$$b(f, n) = \min\{g \mid \text{there is a } \Sigma_f\text{-bundle } X \to \Sigma_g \text{ with } \sigma(X) = 4n\}.$$

Lower bounds for $b(f,n)$ were given in [8], [19], and [13]. In particular, [13] claims a lower bound

$$b(f,n) \geq \frac{3|n|}{f-1} + 1 \tag{1}$$

for all $f \geq 3$ and $n \neq 0$. In [9], it was shown that $b(f,n) \leq 8|n| + 1$ for all $f \geq 3$ and $n \neq 0$. It is natural to consider the asymptotic quantity (cf. Problem 2.18(B) in Kirby's list [17] that was posed by Mess)

$$G_f := \lim_{n\to\infty} \frac{b(f,n)}{n}.$$

From (1), we would immediately obtain $G_f \geq 3/(f-1)$. Various upper bounds for G_f were computed in [4], [5], [20], and [22]. At the time of this writing, the smallest known upper bounds seem to be $G_f \leq 6/(f-2)$ for all even $f \geq 4$ in [4], and $G_f \leq 6/(f-1)$ for all odd $f \geq 3$ in [22]. Below we will prove the following upper bound formula that covers all but finitely many cases:

$$G_f \leq \frac{5.6}{f-25} \quad \text{for all } f \geq 44. \tag{2}$$

This provides a new smaller upper bound for G_f for all f large enough (more specifically, for all odd $f \geq 363$ and all even $f \geq 348$). In fact, (2) follows from an even better set of upper bounds that depend on the congruence class of f modulo 7 (see Theorem 11 in Section 4).

Another application is in the classification of closed simply connected smooth spin 4-manifolds (the spin geography problem) with nonnegative signature. We recall the following definition from [1].

Definition 2. We say that a symmetric bilinear form has ∞^2-*property* if it is the intersection form of infinitely many pairwise nondiffeomorphic simply connected irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic simply connected irreducible nonsymplectic 4-manifolds. Given any even integer $p \geq 0$, let Λ_p denote the smallest positive odd integer such that the symmetric bilinear form $pE_8 \oplus qH$ has ∞^2-property for every odd integer $q \geq \Lambda_p$.

Note that the simply connected 4-manifolds with the same intersection form in the above definition are all homeomorphic by Freedman's work in

[10]. Here,

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Henceforth we will assume that p and q are nonnegative integers so that the rank and the signature of $pE_8 \oplus qH$ are $8p+2q$ and $8p$, respectively. Recall from [12] that a closed simply connected smooth 4-manifold with nonnegative signature is spin if and only if its intersection form is $pE_8 \oplus qH$ for some nonnegative integers p and q with p even. Also recall that if a closed simply connected smooth spin 4-manifold with intersection form $pE_8 \oplus qH$ is symplectic, then $q \equiv b_2^+ \equiv 1 \pmod 2$. (Here, $b_2^+ = 8p+q$ is the dimension of the maximal positive definite subspace of the second homology group under the intersection form.) Finally we note that Λ_p is well defined as it was shown in [26] that $pE_8 \oplus qH$ has ∞^2-property when the odd integer q is larger than some constant that depends on p.

In [11], Furuta has shown that if $pE_8 \oplus qH$ is the intersection form of a closed smooth spin 4-manifold with $q \geq 1$, then $q \geq p+1$. Thus we obtain a necessary lower bound $\Lambda_p \geq p+1$. The famous 11/8 Conjecture (Problem 4.92 in [17]), which remains unresolved, would imply a stronger lower bound $\Lambda_p \geq \frac{3}{2}p$ when $p \geq 4$. Below we will prove a new upper bound on Λ_p which is an improvement of the upper bound in [1] for many small values of p.

Our paper is organized as follows. In Sections 2 and 3, we will give the details of the construction of surface bundles. In Section 4, we will prove the upper bound (2) in Corollary 12. In Section 5, we present a new upper bound for Λ_p in Corollary 16. Unless otherwise stated, our homology and cohomology groups will have integer coefficients.

2. Technical lemmas

For convenience, we list the key ingredients of our construction here.

1. Let Σ be a closed oriented surface of genus $g > 0$. Let $k : H_1(\Sigma; \mathbb{Z}/n) \to G$ be a surjective group homomorphism to some finite abelian group

G. Note that G must be of the form $G = (\mathbb{Z}/n_1) \times \cdots \times (\mathbb{Z}/n_r)$ for some positive integers $n_1, \ldots, n_r$ dividing n and $1 \leq r \leq 2g$.

2. Consider the surjective homomorphism

$$\pi_1(\Sigma) \xrightarrow{\phi} H_1(\Sigma; \mathbb{Z}/n) \xrightarrow{k} G,$$

where ϕ is the abelianization followed by the change of coefficients from $\mathbb{Z}$ to $\mathbb{Z}/n$. Let $p : \widetilde{\Sigma} \to \Sigma$ be the degree $|G|$ cover corresponding to k, in the sense that im $p_\# = \ker(k \circ \phi)$, where $p_\# : \pi_1(\widetilde{\Sigma}) \to \pi_1(\Sigma)$ is the induced map on the fundamental groups. Then the genus of $\widetilde{\Sigma}$ is given by

$$g(\widetilde{\Sigma}) = |G|(g - 1) + 1.$$

Note that we also have $\operatorname{im} p_* \subseteq \ker k$, where $p_* : H_1(\widetilde{\Sigma}; \mathbb{Z}/n) \to H_1(\Sigma; \mathbb{Z}/n)$ is the induced map on the homology groups.

Next suppose that $\alpha_0, \alpha_1, \ldots, \alpha_d$ are self-diffeomorphisms of Σ, where α_0 is the identity map. We define $\Gamma_0 = \Gamma_p \subseteq \widetilde{\Sigma} \times \Sigma$ to be the graph of p, and Γ_j the image of Γ_0 under $\operatorname{id} \times \alpha_j$ $(j = 0, \ldots, d)$, i.e.,

$$\Gamma_j = \{(x, \alpha_j(p(x)) \mid x \in \widetilde{\Sigma}\}.$$

Consider a divisor $D = \sum_{j=0}^{d} c_j \Gamma_j$, where each $c_j \in \mathbb{Z}$.

Our surface bundle will be an n-fold cyclic branched cover of $\widetilde{\Sigma} \times \Sigma$ with branch locus D. To apply the cyclic branched covering construction, we need to check that the homology class of D is divisible by n. Under sufficiently nice compatibility of α_j's and k, this will be true.

Lemma 3. *Let $\Sigma, k, \alpha_0, \ldots, \alpha_d$ and D be as above. Suppose that the induced maps $(\alpha_j)_* : H_1(\Sigma; \mathbb{Z}/n) \to H_1(\Sigma; \mathbb{Z}/n)$ satisfy*

$$\ker k \subseteq \ker \left(\sum_{j=0}^{d} c_j (\alpha_j)_* \right) \tag{3}$$

and that $\sum_{j=0}^{d} c_j$ is divisible by n. Then the homology class of D is divisible by n.

Proof. We recall that for any map $f : Z \to Y$, the homology class of its graph $\Gamma_f \subset Z \times Y$ is given by

$$[\Gamma_f] = \sum_i f^*(\beta_i) \times \beta^i,$$

where $\{\beta_i\}$ is a basis for $H_*(Y)$ and $\{\beta^i\}$ is the dual basis so that $\beta_i \cdot \beta^j = \delta_i^j$. As in [5], we shall write $f^*(x)$ for the homology class $PD(f^*(PD(x)))$, where PD denotes the Poincaré duality map. If $A : H_*(Y) \to H_*(Y)$ is a linear map, then we note that

$$\sum_i p^*(\beta_i) \times A\beta_i = \sum_i p^*(A^T \beta_i) \times \beta_i. \tag{4}$$

We choose the standard basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_1(\Sigma)$, oriented so that the intersection form is given by

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus g}$$

The dual basis for $H_1(\Sigma)$ is $\{b_1, -a_1, \ldots, b_g, -a_g\}$, and we have $a^i = J^T a_i$ and $b^i = J^T b_i$.

We can now compute the homology class of D. We have

$$[\Gamma_0] = p^*(\{pt\}) \times \Sigma + p^*(\Sigma) \times \{pt\} + \sum_{i=1}^{g} p^*(a_i) \times a^i + \sum_{i=1}^{g} p^*(b_i) \times b^i.$$

Write $\Delta = p^*(\{pt\}) \times \Sigma + p^*(\Sigma) \times \{pt\}$. Since $(\mathrm{id} \times \alpha_j)_* = \mathrm{id} \times (\alpha_j)_*$, we get

$$[\Gamma_j] = \Delta + \sum_{i=1}^{g} p^*(a_i) \times (\alpha_j)_* a^i + \sum_{i=1}^{g} p^*(b_i) \times (\alpha_j)_* b^i.$$

For convenience, we will denote $\sum_{j=0}^{d} c_j (\alpha_j)_*$ by F. Thus using the fact that $a^i = J^T a_i$ and $b^i = J^T b_i$ we get

$$[D] = \sum_{j=0}^{d} c_j [\Gamma_j] = \left[\sum_{j=0}^{d} c_j \right] \Delta + \sum_{i=1}^{g} \left[p^*(a_i) \times (F J^T a_i) + p^*(b_i) \times (F J^T b_i) \right].$$

Since the first part is divisible by n from our hypothesis, we only need to show the divisibility of the second part. By (4), the second part is equal to

$$\sum_{i=1}^{g} \left[p^*([JF^T]a_i) \times a_i + p^*([JF^T]b_i) \times b_i\right].$$

To check that this is divisible by n, it suffices to check that for each basis element $e \in H_1(\Sigma)$, $p^*(PD([JF^T]e))$ is divisible by n. For any $x \in H_1(\Sigma)$, we have

$$\langle p^*(PD([JF^T]e)), x\rangle = \langle PD([JF^T]e), p_*(x)\rangle,$$

and since $\operatorname{im} p_* \subseteq \ker k$ from the definition of p, it suffices to check that for all $z \in \ker k$ and basis elements e, we have

$$\langle PD([JF^T]e), z\rangle \equiv 0 \mod n.$$

Moreover, note that the intersection product on $H_1(\Sigma)$ is given exactly by

$$a \cdot b = \langle PD(a) \cup PD(b), [\Sigma]\rangle = \langle PD(a), PD(b) \cap [\Sigma]\rangle = \langle PD(a), b\rangle.$$

Since we can also express $a \cdot b$ as $a^T Jb$ in the standard basis, we can conclude that

$$\langle PD([JF^T]e), z\rangle \equiv 0 \iff e^T F(J^T J)z \equiv 0 \iff e^T Fz \equiv 0.$$

This last statement holds for all basis elements e if and only if $Fz \equiv 0$. In other words, each term of the form $p^*(PD([JF^T]e))$ is divisible by n if $\ker k \subseteq \ker F$ when we consider these as subgroups of $H_1(\Sigma; \mathbb{Z}/n)$. This was our hypothesis (3), so we conclude that $[D]$ is indeed divisible by n. □

Most times, we will be working in the following special situation.

Corollary 4. *Suppose that $\alpha : \Sigma \to \Sigma$ is a self-diffeomorphism with induced map $\alpha_* : H_1(\Sigma; \mathbb{Z}/n) \to H_1(\Sigma; \mathbb{Z}/n)$. If we perform the above construction with $\alpha_j = \alpha^j$ $(j = 0, \ldots, d)$, $k = \sum_{j=0}^{d} c_j \alpha_*^j$, and $G = \operatorname{im} k$, where $c_j \in \mathbb{Z}$ are chosen so that $\sum_{j=0}^{d} c_j \equiv 0 \mod n$, then $[D]$ is divisible by n.*

Remark 5. A particular instance of this corollary is the example in Section 3 of [5], in which we have $\alpha = \tau$, $k = \mathrm{id} - \tau_*$, $c_0 = 1$, $c_1 = -1$ and $d = 1$. One can also check that in this case, the expression

$$\sum_{i=1}^{g} \big[p^*([JF^T]a_i) \times a_i + p^*([JF^T]b_i) \times b_i\big]$$

does in fact give the correct formula for the homology class $[D]$.

We now compute the self-intersection number of $[D]$.

Lemma 6. *Let $\alpha : \Sigma \to \Sigma$ be an orientation preserving self-diffeomorphism, and set $\alpha_j = \alpha^j$ $(j = 0, \ldots, d)$. Further assume that $\alpha, \alpha^2, \ldots, \alpha^d$ are all fixed point free. If $c = \sum_{j=0}^{d} c_j$ and $\tau = \sum_{j=0}^{d} c_j^2$, then the self-intersection number of D in $\widetilde{\Sigma} \times \Sigma$ is given by:*

$$[D]^2 = -(c^2 + 2\tau(g-1))|G|.$$

Proof. As before, we write $F = \sum_{j=0}^{d} c_j \alpha_*^j$. Since p is a degree $|G|$ cover, we readily compute that

$$[D]^2 = c^2|G| + \left[\sum_{i=1}^{g} \big[p^*([JF^T]a_i) \times a_i + p^*([JF^T]b_i) \times b_i\big]\right]^2. \tag{5}$$

Let $\Theta = [D]^2 - c^2|G|$ denote the last square term of (5). Since $(\xi_1 \times \eta_1) \cdot (\xi_2 \times \eta_2) = (-1)^{\deg(\xi_2)\deg(\eta_1)}(\xi_1 \cdot \xi_2) \times (\eta_1 \cdot \eta_2)$, and self-intersections are all zero, Θ is equal to

$$\begin{aligned}
&\sum_{i=1}^{g} \big[- p^*(JF^T a_i) \cdot p^*(JF^T b_i) \times (a_i \cdot b_i) \\
&\qquad - p^*(JF^T b_i) \cdot p^*(JF^T a_i) \times (b_i \cdot a_i)\big] \\
&= \sum_{i=1}^{g} \left[-p^*\left(a_i^T F J^T J J F^T b_i\right) \times (a_i \cdot b_i) - p^*\left(b_i^T F J^T J J F^T a_i\right) \times (b_i \cdot a_i)\right].
\end{aligned}$$

Since $a_i \cdot b_i = -b_i \cdot a_i = 1$, we can write

$$\Theta = p^* \left[\sum_{i=1}^{g} b_i^T F J F^T a_i - a_i^T F J F^T b_i\right] \times \{pt\}.$$

Since $\alpha_* \in \mathrm{Sp}(2g, \mathbb{Z})$, we have the relation $\alpha_* J \alpha_*^T = J$. Thus we have

$$\begin{aligned} FJF^T &= \sum_{0 \le j,l \le d} c_j c_l \alpha_*^j J (\alpha_*^T)^l \\ &= \sum_{j=0}^{d} c_j^2 \alpha_*^j J (\alpha_*^T)^j + \sum_{0 \le j < l \le d} c_j c_l \alpha_*^j J (\alpha_*^T)^l + \sum_{0 \le l < j \le d} c_j c_l \alpha_*^j J (\alpha_*^T)^l \\ &= \tau J + \sum_{0 \le j < l \le d} c_j c_l (\alpha_*^{j-l} + \alpha_*^{l-j}) J \\ &= \tau J + \sum_{0 \le j < l \le d} c_j c_l \Phi_{jl} J, \end{aligned}$$

where we introduce the notation $\Phi_{jl} = \alpha_*^{j-l} + \alpha_*^{l-j}$. Hence Θ is equal to

$$\begin{aligned} & p^* \left[\sum_{i=1}^{g} \left\{ b_i^T \left(\tau J + \sum_{j<l} c_j c_l \Phi_{jl} J \right) a_i - a_i^T \left(\tau J + \sum_{j<l} c_j c_l \Phi_{jl} J \right) b_i \right\} \right] \times \{pt\} \\ &= p^* \left[\sum_{i=1}^{g} \left\{ 2\tau b_i^T J a_i - \sum_{j<l} c_j c_l a_i^T \Phi_{jl} J b_i + \sum_{j<l} c_j c_l b_i^T \Phi_{jl} J a_i \right\} \right] \times \{pt\}. \end{aligned}$$

Note that by convention, $a_i = e_{2i-1}$ and $b_i = e_{2i}$, where e_i denotes the i-th element in the basis $\{a_1, b_1, \ldots, a_g, b_g\}$. Since $e_i^T A e_j$ is the ij-th entry $A_{i,j}$ of the matrix A, we get

$$\begin{aligned} \Theta &= p^* \left[\left[-2g\tau - \sum_{j<l} c_j c_l \sum_{i=1}^{g} \{ [\Phi_{jl} J]_{2i-1,2i} - [\Phi_{jl} J]_{2i,2i-1} \} \right] \{pt\} \right] \times \{pt\} \\ &= p^* \left[\left[-2g\tau - \sum_{j<l} c_j c_l \sum_{i=1}^{g} \{ [\Phi_{jl}]_{2i-1,2i-1} + [\Phi_{jl}]_{2i,2i} \} \right] \{pt\} \right] \times \{pt\} \\ &= p^* \left[\left[-2g\tau - \sum_{j<l} c_j c_l \operatorname{tr}(\Phi_{jl}) \right] \{pt\} \right] \times \{pt\}. \end{aligned}$$

If $A \in \mathrm{Sp}(2g, \mathbb{Z})$, then $\mathrm{tr}(A) = \mathrm{tr}(A^{-1})$, and thus $\mathrm{tr}(\Phi_{jl}) = 2\mathrm{tr}(\alpha_*^{l-j})$. Since each α^{l-j} is fixed point free by our hypothesis, the Lefschetz fixed-point theorem implies that $\mathrm{tr}(\alpha_*^{l-j}) = 2$ for all $0 \leq j < l \leq d$. Thus we have

$$\begin{aligned}\Theta &= \left[-2g\tau - \left(\frac{c^2-\tau}{2}\right)\cdot 2\cdot 2\right] p^*(\{pt\}) \times \{pt\} \\ &= -2(c^2 + \tau(g-1))|G|.\end{aligned}$$

Here, we have used the fact that

$$c^2 = \sum_{j,l} c_j c_l = \sum_j c_j^2 + 2\sum_{j<l} c_j c_l = \tau + 2\sum_{j<l} c_j c_l,$$

which implies that $\sum_{j<l} c_j c_l = (c^2 - \tau)/2$. □

One can check that the formula in the last lemma agrees with the special case given in [5].

Remark 7. It was shown in [7] and [23] that if α is an orientation preserving self-homeomorphism of a surface with genus $g \geq 2$, then at least one of the powers $\alpha, \alpha^2, \ldots, \alpha^{2g-2}$ has a fixed point. Thus we must have $d \leq 2g - 3$ in the hypothesis of Lemma 6.

3. Construction of surface bundles

We will now finish the general construction. Let $\alpha : \Sigma \to \Sigma$ satisfy the hypotheses of Lemma 6. Suppose that we have an epimorphism

$$k : H_1(\Sigma; \mathbb{Z}/n) \to G$$

satisfying condition (3) in Lemma 3 with respect to $\alpha_j = \alpha^j$ $(j = 0, \ldots, d)$, and that $c = \sum_{j=0}^d c_j$ is divisible by n so that the homology class $[D]$ is also divisible by n. Further suppose that

$$c_j = \pm 1 \quad \text{for every } j = 0, 1, \ldots, d. \tag{6}$$

Let $\varphi_n : X_n \to \widetilde{\Sigma} \times \Sigma$ be the cyclic $\mathbb{Z}/n$ branched cover that is ramified on D. Since Γ_j's are mutually disjoint in $\widetilde{\Sigma} \times \Sigma$ and each Γ_j occurs with multiplicity ± 1 in D by (6), the branched cover X_n will be a smooth surface bundle and hence a symplectic 4-manifold (cf. p. 1022 of [14]). If α is holomorphic, then the resulting branched cover X_n will be a complex surface (a double Kodaira fibration in the terminology of [6]) as explained on p. 261 of

[15]. By Hirzebruch's signature formula [15], since $\sigma(\widetilde{\Sigma} \times \Sigma) = 0$, we have

$$\sigma(X_n) = -\frac{n^2-1}{3n}[D]^2 = \frac{n^2-1}{3n}(c^2 + 2(d+1)(g-1))|G|, \tag{7}$$

where $\tau = \sum_{j=0}^{d} c_j^2 = d+1$. We can compute the genera of the fibers of the surface bundles $X_n \to \Sigma$ and $X_n \to \widetilde{\Sigma}$ as follows.

Lemma 8. *The genera of the surface bundles are given by:*

bundle	base genus	fiber genus
$\mathrm{pr}_2 \circ \varphi_n : X_n \to \Sigma$	g	$\|G\|(n(g-1) + \frac{(d+1)(n-1)}{2}) + 1$
$\mathrm{pr}_1 \circ \varphi_n : X_n \to \widetilde{\Sigma}$	$\|G\|(g-1)+1$	$n(g + \frac{d-1}{2}) - (\frac{d-1}{2})$

Proof. Since the divisor $D|_{pt\times\Sigma}$ is $c_0 p(pt) + c_1\alpha(p(pt)) + \cdots + c_d\alpha^d(p(pt))$, the fibers of $X_n \to \widetilde{\Sigma}$ are $\mathbb{Z}/n$ cyclic covers of Σ ramified over $d+1$ points. The Riemann-Hurwitz formula then gives

$$2g(\text{fiber}) - 2 = n(2g-2) + (d+1)(n-1)$$

so the the fiber genus is $n(g + \frac{d-1}{2}) - (\frac{d-1}{2})$. Similarly, $D|_{\widetilde{\Sigma}\times pt} = c_0 p^{-1}(pt) + c_1 p^{-1}(\alpha^{-1}(pt)) + \cdots + c_d p^{-1}(\alpha^{-d}(pt))$ so the fibers of $X_n \to \Sigma$ are $\mathbb{Z}/n$ covers of $\widetilde{\Sigma}$ ramified over $(d+1)|G|$ points. The Riemann-Hurwitz formula gives

$$2g(\text{fiber}) - 2 = n(2g(\widetilde{\Sigma}) - 2) + (d+1)(n-1)|G|,$$

and so in this case the fiber genus is $|G|(n(g-1) + \frac{(d+1)(n-1)}{2}) + 1$. □

Note that in the special case considered in [5], we had $g = 3$, $d = 1$, $c_0 = 1$, $c_1 = -1$, $|G| = n^2$, and the formulas for the fiber genera agree ($3n$ and $3n^3 - n^2 + 1$, respectively).

Lemma 9. *If n is a positive odd integer, then X_n is spin.*

Proof. By a formula of Brand [3], the second Stiefel-Whitney class of X_n is

$$w_2(X_n) = \frac{n-1}{n}(\varphi_n^*(PD[D])),$$

and consequently, X_n is spin when n is odd. □

4. Application to the minimal base genus problem

We start with the following lemma due to Nielsen.

Lemma 10. *There exists an orientation preserving self-diffeomorphism α of a surface Σ for each genus $g \geq 2$ such that $\alpha, \alpha^2, \ldots, \alpha^{2g-3}$ are all fixed point free.*

Proof. An example of such α was given in [23]. One views Σ as the union of a cylinder and a genus $g-1$ surface Σ' with two small disks removed. Next one thinks of Σ' as a regular $4(g-1)$-gon with small holes at the center and the outer vertex. One can then define α on the Σ' part to be the rotation by angle $\frac{2\pi}{4(g-1)}$ about the center, and extend it smoothly onto the remaining cylinder part. We refer to p. 222–223 of the English translation [24] for more details. □

Theorem 11. *Let $f \geq 44$ and $2 \leq i \leq 8$ be integers. If $f \equiv 4i \pmod 7$, then*

$$G_f \leq \frac{5.6}{f - 4i + 7}. \tag{8}$$

Proof. To prove this, we will construct an infinite family of surface bundles and then employ the usual trick of pulling back to unramified covers. We start with the triple of positive integers (g, n, d), where $g \geq 2$, $n \geq 2$, and d is an odd integer such that $1 \leq d \leq 2g-3$. Following the construction in the previous section, construct surface bundles X_n associated to the sum

$$k = \mathrm{id} - \alpha_* + \alpha_*^2 - \cdots - \alpha_*^d,$$

where α is as in Lemma 10 and $G = \mathrm{im}\, k$. For convenience, we introduce the variable $u = (d-1)/2$, so that $0 \leq u \leq g-2$. From (7) and Lemma 8, these bundles have the following data:

base genus	fiber genus	signature
$\lvert G\rvert(g-1)+1$	$n(g+u)-u$	$\frac{4}{3} \cdot \frac{n^2-1}{n}(u+1)(g-1)\lvert G\rvert$

Given a genus h surface, there is an m-fold unramified cover $\pi : \Sigma_{m(h-1)+1} \to \Sigma_h$ by a genus $m(h-1)+1$ surface. By pulling back the bundle X_n via π, we get a new family of surface bundles $X_{n,m}$ with the following data (for any $g \geq 2$, $n \geq 2$, $0 \leq u \leq g-2$, and $m \geq 1$):

base genus	fiber genus	signature
$m\|G\|(g-1)+1$	$n(g+u)-u$	$\frac{4}{3}\cdot\frac{n^2-1}{n}m(u+1)(g-1)\|G\|$

Let $f = n(g+u) - u$. Then we obtain

$$G_f \leq \lim_{m\to\infty} \frac{m|G|(g-1)+1}{\frac{1}{4}\cdot\frac{4}{3}\cdot\frac{n^2-1}{n}m(u+1)(g-1)|G|} = \frac{3n}{n^2-1}\cdot\frac{1}{u+1}. \tag{9}$$

This gives a wide family of possible upper bounds for G_f, depending on our choice of the parameters g, n and u.

We consider the special case when $n = 4$, $g \geq 8$, and $u \in \{g-8, g-7, \ldots, g-2\}$. When $u = g - i$ for $2 \leq i \leq 8$, we have $f = 7g - 3i = 7u + 4i$. Thus $u = (f - 4i)/7$ and so (9) becomes

$$G_f \leq \frac{3\cdot 4}{4^2-1}\cdot\frac{1}{\frac{f-4i}{7}+1} = \frac{5.6}{f-4i+7}.$$

Note that as $g \geq 8$ and i ranges between $\{2, 3, \ldots, 8\}$, $7g - 3i$ covers all residue classes modulo 7 bigger than 43, and so this bound for G_f does indeed hold for any $f \geq 44$. □

Corollary 12. *Let $f \geq 44$ be an integer. Then $G_f \leq 5.6/(f - 25)$.*

Proof. The worst possible upper bound given by (8) occurs when $i = 8$. □

Remark 13. Different choices of parameters (g, n, u) yield similar upper bounds when $f < 44$ as well, but these do not seem to improve upon the known bounds.

5. Application to spin geography problem

We will need the following theorem that was proved in [1].

Theorem 14. *Let X be a spin 4-manifold that is the total space of a genus f surface bundle over a genus b surface. Assume that $\sigma(X) = 16s$, and X has a section whose image is a genus b surface of self-intersection $-2t$ for* *some integer t. Let r be a positive integer satisfying*

$$1 - t \leq r \leq \min\{s,\ f + b + 1 - t\}.$$

If p and q are nonnegative integers satisfying

$$\begin{aligned} p &\equiv 0 \pmod 2, \quad 0 \leq p \leq 2(s - r), \\ q &\equiv 1 \pmod 2, \quad q \geq 2fb + 12s - 1 - 10p, \end{aligned}$$

then the symmetric bilinear form $pE_8 \oplus qH$ has ∞^2-property (cf. Definition 2) and

$$\Lambda_p \leq 2fb + 12s - 1 - 10p.$$

We will now apply Theorem 14 to the following example.

Example 15. Let $g=2$, and let α be the fixed point free self-diffeomorphism of a genus 2 surface in Lemma 10 that was constructed by Nielsen. One can verify (cf. p. 231 of [24]) that the induced map on the first homology is representable by the matrix

$$\alpha_* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By setting $k = \mathrm{id} - \alpha_*$ and $G = \mathrm{im}\ k$, we get $c = 0$, $d = 1$, and $|G| = n^3$. According to (7) and Lemma 8, the corresponding surface bundle $\mathrm{pr}_1 \circ \varphi_n : X_n \to \widetilde{\Sigma}$ has base genus $b = n^3 + 1$, fiber genus $f = 2n$, and signature $\sigma = 16s = \frac{4}{3}(n^4 - n^2) > 0$. There are two sections, each image with self-intersection equal to $-2t = \frac{[D]^2}{(d+1)n} = -2n^2$. By Lemma 9, X_n is spin if n is odd.

Setting $r = 1$ in Theorem 14 (note that $f + b + 1 - t = n^3 - n^2 + 2n + 2 > 0$), the surface bundles in Example 15 lead to the following upper bound on Λ_p.

Corollary 16. *Let $p \geq 0$ be an even integer. If $n \geq 3$ is any odd integer satisfying $p \leq \frac{1}{6}(n^4 - n^2) - 2$, then we have*

$$\Lambda_p \leq 5n^4 - n^2 + 4n - 1 - 10p. \tag{10}$$

We can now compare the upper bound in (10) with that in Theorem 5 of [1] for some low values of p, where $n = 3$, 5 or 7:

bound in [1]	bound in (10)
$\Lambda_p \leq 275 - 10p$ when $0 \leq p \leq 6$	$\Lambda_p \leq 407 - 10p$ when $0 \leq p \leq 10$
$\Lambda_p \leq 1259 - 10p$ when $8 \leq p \leq 38$	$\Lambda_p \leq 3119 - 10p$ when $12 \leq p \leq 98$
$\Lambda_p \leq 3443 - 10p$ when $40 \leq p \leq 110$	$\Lambda_p \leq 11983 - 10p$ when $100 \leq p \leq 390$

We note that (10) is an improvement over the bound in [1] when $p = 8$, 10, and when $40 \leq p \leq 98$. For example, when $p = 8$, (10) implies that for every odd integer $q \geq 327$, the topological 4-manifold

$$4(\overline{K3})\#(q-12)(S^2 \times S^2), \tag{11}$$

the connected sum of four copies of $\overline{K3}$ and $q - 12$ copies of $S^2 \times S^2$, has infinitely many pairwise nondiffeomorphic irreducible symplectic smooth structures. Here, $S^2 \times S^2$ is the cartesian product of two 2-spheres, and $\overline{K3}$ denotes the complex $K3$ surface equipped with the noncomplex orientation and thus with the intersection form $2E_8 \oplus 3H$. Note that the existence of infinitely many pairwise nondiffeomorphic irreducible *nonsymplectic* smooth structures on (11) is immediately obtained by just reversing the orientation of the simply connected irreducible nonsymplectic spin 4-manifolds with signature -64 that were constructed in [25].

At the moment it is not clear to the authors how to optimize these bounds across many possible choices of surface bundles X_n. We hope to clarify this issue in a future work.

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