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Frobenius stratification of moduli spaces of rank 3 vector bundles in positive characteristic 3, II

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Dedicated to the memory of Professor Michel Raynaud.

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$, $\mathfrak{M}^s_X(r,d)$ the moduli space of stable vector bundles of rank r and degree d on X. We study the Frobenius stratification of $\mathfrak{M}^s_X(3,d)$ in terms of Harder-Narasimhan polygons of Frobenius pull-backs of stable vector bundles and obtain the irreducibility and dimension of each non-empty Frobenius stratum in the case $(p,g) = (3,2)$ with $3 \nmid d$.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus g over k. The absolute Frobenius morphism $F_X : X \to X$ is induced by $\mathscr{O}_X \to \mathscr{O}_X$, $f \mapsto f^p$. Let $\mathfrak{M}^s_X(r,d)$ be the moduli space of stable vector bundles of rank r and degree d on X.

For any vector bundle $\mathscr{E}$ on X, by the Harder-Narasimhan filtration of $\mathscr{E}$, we can define the *Harder-Narasimhan Polygon* $\mathrm{HNP}(\mathscr{E})$, which is a convex polygon in the coordinate plane of rank-degree (cf. [18, Section 3]).

Fix integers m and n with $m > 0$. Let $\mathfrak{ConPgn}(m,n)$ be the set of all convex polygons in the coordinate plane such that their vertexes have integral coordinates, start at the origin $(0,0)$ and end at the point (m,n). Then there is a natural partial order structure, denoted by $\succcurlyeq$, on the set $\mathfrak{ConPgn}(m,n)$ (cf. [18, Section 3]).

In general, the semistability of vector bundles is possibly destabilized under the Frobenius pull-back F^*_X (cf. [3], [17]). Thus, there is a natural

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set-theoretic map

$$\begin{aligned} S^s_{\mathrm{Frob}} : \mathfrak{M}^s_X(r,d)(k) &\to \mathfrak{ConPgn}(r,pd) \\ [\mathscr{E}] &\mapsto \mathrm{HNP}(F^*_X(\mathscr{E})) \end{aligned}$$

For any $\mathscr{P} \in \mathfrak{ConPgn}(r,pd)$, we denote

$$\begin{aligned} S_X(r,d,\mathscr{P}) &:= \{[\mathscr{E}] \in \mathfrak{M}^s_X(r,d) \mid \mathrm{HNP}(F^*_X(\mathscr{E})) = \mathscr{P}\}. \\ S_X(r,d,\mathscr{P}^+) &:= \{[\mathscr{E}] \in \mathfrak{M}^s_X(r,d) \mid \mathrm{HNP}(F^*_X(\mathscr{E})) \succcurlyeq \mathscr{P}\}. \end{aligned}$$

Then we have a canonical stratification of $\mathfrak{M}^s_X(r,d)$ in terms of Harder-Narasimhan polygons of Frobenius pull-backs of stable vector bundles. We call this the *Frobenius stratification* (cf. [6]). By [18, Theorem 3], the Frobenius stratum $S_X(r,d,\mathscr{P}^+)$ is a closed subvariety of $\mathfrak{M}^s_X(r,d)$ for any $\mathscr{P} \in \mathfrak{ConPgn}(r,pd)$.

Some results about Frobenius stratification of moduli spaces of vector bundles are known in special cases for small values of p, g, r and d. Joshi-Ramanan-Xia-Yu [6] give a complete description of the Frobenius stratification of the moduli space $\mathfrak{M}^s_X(2,d)$ if $p=2$ and $g \geq 2$. In [12] the author obtains the classification of Frobenius strata in $\mathfrak{M}^s_X(3,0)$ if $p=3$ and $g=2$. It is easy to deduce the Frobenius stratification of $\mathfrak{M}^s_X(3,d)$ if $(p,g)=(3,2)$ and $3|d$. The method used in this paper, as well as [12], is a variation of the idea introduced in [6]. In the higher rank case, Joshi and Pauly [5] study the properties of the Frobenius stratum consisting of stable vector bundles whose Frobenius pull-back has maximal Harder-Narasimhan polygon if $d=0$ and $p > r(r-1)(r-2)(g-1)$. The author obtains the geometric properties of a special Frobenius stratum in $\mathfrak{M}^s_X(r,d)$ if $p|r$ and $g \geq 2$ in [11][13]. Other results about Frobenius stratification of moduli spaces of vector bundles can be found in [2][7][8][9][10][16][22] for $r=2$ and [11][14][15][22] for $r>2$.

In general, it is difficult to determine the $\mathrm{HNP}(F^*_X(\mathscr{E}))$ for some stable vector bundle $\mathscr{E}$. In the case $(p,g,r)=(3,2,3)$, we first show that there are 4 possible Harder-Narasimhan polygons $\{\mathscr{P}_1(d),\mathscr{P}_2(d),\mathscr{P}_3(d),\mathscr{P}_4(d)\}$ for any Frobenius destabilized stable vector bundles of rank 3 and degree d (see Figure 1). We show that any Frobenius destabilized stable vector bundle $[\mathscr{E}] \in \mathfrak{M}^s_X(3,d)(k)$ with $\mathrm{HNP}(F^*_X(\mathscr{E})) \in \{\mathscr{P}_2(d),\mathscr{P}_3(d),\mathscr{P}_4(d)\}$ can be embedded into $F_{X*}(\mathscr{L})$ for some line bundle $\mathscr{L}$ of degree $d-1$ on X(Proposition 3.3). Then we can determine $\mathrm{HNP}(F^*_X(\mathscr{E}))$ by analysing the intersection of $F^*_X(\mathscr{E})$ with the canonical filtration of $F^*_XF_{X*}(\mathscr{L})$. This is the key point of our method. Moreover, we show that any rank 3 and degree d subsheaf $\mathscr{E} \subset F_{X*}(\mathscr{L})$ is stable for any line bundle $\mathscr{L}$ of degree

$d-1$ with $3\nmid d$ (Proposition 3.4). Therefore we can obtain the geometric properties of Frobenius strata of $\mathfrak{M}_X^s(3,d)$ from the geometric properties of Frobenius strata of $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(-1)}(X))$ (see Section 4) by the morphism $\theta:\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X))\to\mathfrak{M}_X^s(3,d):[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\mapsto[\mathscr{E}]$.

The main goal of the paper is to study the geometric properties of Frobenius strata of $\mathfrak{M}_X^s(3,d)$ with $3\nmid d$. The main result is the following Theorem.

Theorem 1.1. (*Theorem 5.2*) *Let k be an algebraically closed field of characteristic* 3*, X a smooth projective curve of genus* 2 *over k, d an integer with $3\nmid d$. Then $S_X(3,d,\mathscr{P}_i^+(d))=\overline{S_X(3,d,\mathscr{P}_i(d))}$, and $S_X(3,d,\mathscr{P}_i^+(d))$* (*resp. $S_X(3,d,\mathscr{P}_i(d))$*) *are irreducible projective* (*resp. irreducible quasi-projective*) *varieties for $1\leq i\leq 4$,*

$$\dim S_X(3,d,\mathscr{P}_i^+(d))=\dim S_X(3,d,\mathscr{P}_i(d))=\begin{cases}5,\text{if } i=1\\5,\text{if } i=2\\4,\text{if } i=3\\2,\text{if } i=4\end{cases}$$

The method of this paper is similar to [12], and most proofs of [12] in the case $(p,g,r,d)=(3,2,3,0)$ can be applied to any degree d with little modification. However, for any line bundle $\mathscr{L}$ of degree $d-1$ on X and any rank 3 and degree d subsheaf $\mathscr{E}\subset F_{X*}(\mathscr{L})$, the main difference between the cases $3|d$ and $3\nmid d$ is that $\mathscr{E}$ is stable if $3\nmid d$, while semistable and possibly not stable if $3|d$ (see [12, Proposition 3.4] and Proposition 3.4). As a consequence of this difference, the Frobenius strata $S_X(3,d,\mathscr{P}_i^+(d))(1\leq i\leq 4)$ are irreducible projective varieties if $3\nmid d$, and $S_X(3,d,\mathscr{P}_i^+(d))(1\leq i\leq 4)$ are irreducible quasi-projective varieties if $3|d$.

In Section 2, we show that there are 4 possible Harder-Narasimhan polygons $\{\mathscr{P}_1(d),\mathscr{P}_2(d),\mathscr{P}_3(d),\mathscr{P}_4(d)\}$ for the Frobenius pull-backs of Frobenius destabilized stable vector bundles of rank 3 and degree d in the case $(p,g,r)=(3,2,3)$.

In Section 3, we show that any Frobenius destabilized stable bundle $[\mathscr{E}]\in\mathfrak{M}_X^s(3,d)(k)$ with $\mathrm{HNP}(F_X^*(\mathscr{E}))\in\{\mathscr{P}_2(d),\mathscr{P}_3(d),\mathscr{P}_4(d)\}$ can be embedded into $F_{X*}(\mathscr{L})$ for some line bundle $\mathscr{L}$ of degree $d-1$ on X. In addition, we show that for any line bundle $\mathscr{L}$ of degree $d-1$ on X, each rank 3 and degree d subsheaf $\mathscr{E}\subset F_{X*}(\mathscr{L})$ is stable if $3\nmid d$.

In Section 4, we study the Frobenius stratification of the Quot scheme $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X))$ and obtain the smoothness, irreducibility and dimension of each stratum.

In Section 5, we study the Frobenius stratification of moduli space $\mathfrak{M}^s_X(3,d)$ if $(p,g)=(3,2)$ with $3\nmid d$. By the morphism

$$\theta : \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X)) \to \mathfrak{M}^s_X(3,d) : [\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})] \mapsto [\mathscr{E}],$$

we obtain the geometric properties of Frobenius strata of the moduli space $\mathfrak{M}^s_X(3,d)$ from the Frobenius stratification structure of

$$\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X)).$$

2. Classification of Frobenius Harder-Narasimhan polygons

In this section, we will determine all of the possible Harder-Narasimhan polygons of $F^*_X(\mathscr{E})$ for any Frobenius destabilized stable vector bundles $[\mathscr{E}]\in\mathfrak{M}^s_X(3,d)(k)$, where X is a smooth projective curve of genus 2 over an algebraically closed field k of characteristic 3.

Theorem 2.1 (N. I. Shepherd-Barron [19] and V. Mehta, C. Pauly [15]). *Let k be an algebraically closed field of characteristic $p>0$, X a smooth projective curve of genus $g\geq 2$ over k, $\mathscr{E}$ a semistable vector bundle on X. Let $0=\mathscr{E}_0\subset\mathscr{E}_1\subset\cdots\subset\mathscr{E}_{m-1}\subset\mathscr{E}_m=F^*_X(\mathscr{E})$ be the Harder-Narasimhan filtration of $F^*_X(\mathscr{E})$. Then $\mu(\mathscr{E}_i/\mathscr{E}_{i-1})-\mu(\mathscr{E}_{i+1}/\mathscr{E}_i)\leq 2g-2$, for any $1\leq i\leq m-1$.*

By Theorem 2.1, there are 4 possible Harder-Narasimhan polygons

$$\{\mathscr{P}_1(d),\mathscr{P}_2(d),\mathscr{P}_3(d),\mathscr{P}_4(d)\}$$

for Frobenius destabilized stable vector bundles in the case $(p,g,r,d)=(3,2,3,d)$.

3. Construction of stable vector bundles

Definition 3.1. ([6][20]) Let k be an algebraically closed field of characteristic $p>0$, X a smooth projective curve over k. For any coherent sheaf $\mathscr{F}$ on X, let

$$\nabla_{\mathrm{can}} : F^*_X F_{X*}(\mathscr{F}) \to F^*_X F_{X*}(\mathscr{F})\otimes_{\mathscr{O}_X}\Omega^1_X$$

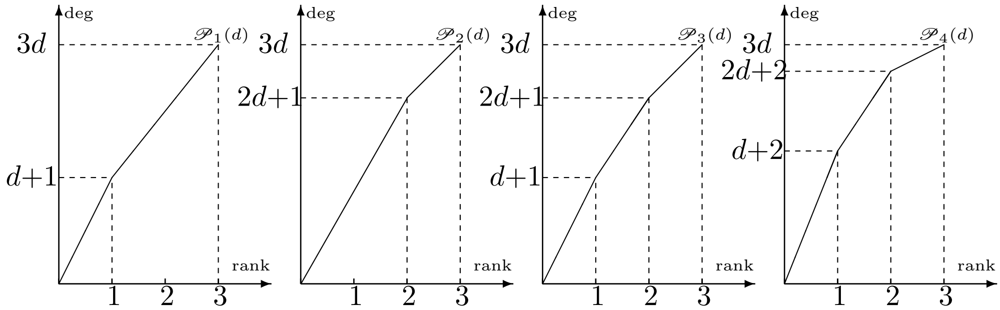

Figure 1: Classification of Harder-Narasimhan polygons if $(p,g,r,d) = (3,2,3,d)$.

be the canonical connection on $F_X^*F_{X*}(\mathscr{F})$. Set

$$V_1 := \ker(F_X^*F_{X*}(\mathscr{F}) \twoheadrightarrow \mathscr{F}),$$
$$V_{l+1} := \ker\{V_l \xrightarrow{\nabla_{\mathrm{can}}} F_X^*F_{X*}(\mathscr{F}) \otimes_{\mathscr{O}_X} \Omega_X^1 \to (F_X^*F_{X*}(\mathscr{F})/V_l) \otimes_{\mathscr{O}_X} \Omega_X^1\}$$

The filtration $\mathbb{F}^{\mathrm{can}}_{\mathscr{F}\bullet} : F_X^*F_{X*}(\mathscr{F}) = V_0 \supset V_1 \supset V_2 \supset \cdots$ is called the *canonical filtration* of $F_X^*F_{X*}(\mathscr{F})$.

Theorem 3.2 (X. Sun [20]). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus g over k, $\mathscr{E}$ a vector bundle on X. Then the canonical filtration of $F_X^*F_{X*}(\mathscr{E})$ is*

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_{l+1} \subset V_l \subset \cdots \subset V_1 \subset V_0 = F_X^*F_{X*}(\mathscr{E})$$

such that

(1) $\nabla_{\mathrm{can}}(V_{i+1}) \subset V_i \otimes_{\mathscr{O}_X} \Omega_X^1$ *for* $0 \leq i \leq p-1$.

(2) $V_l/V_{l+1} \xrightarrow{\nabla_{\mathrm{can}}} \mathscr{E} \otimes_{\mathscr{O}_X} \Omega_X^{\otimes l}$ *are isomorphic for* $0 \leq l \leq p-1$.

(3) *If $g \geq 1$, then $F_{X*}(\mathscr{E})$ is semistable whenever $\mathscr{E}$ is semistable. If $g \geq 2$, then $F_{X*}(\mathscr{E})$ is stable whenever $\mathscr{E}$ is stable.*

(4) *If $g \geq 2$ and $\mathscr{E}$ is semistable, then the canonical filtration of $F_X^*F_{X*}(\mathscr{E})$ is nothing but the Harder-Narasimhan filtration of $F_X^*F_{X*}(\mathscr{E})$.*

Proposition 3.3. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Let $\mathscr{E}$ be a rank 3 and degree*

d stable vector bundle on X with a non-trivial homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$, where $\mathscr{L}$ is a line bundle of degree $d-1$ on X. Then the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injection.

Proof. By adjunction, there is a non-trivial homomorphism $\mathscr{E} \to F_{X*}(\mathscr{L})$. Denote the image by $\mathscr{G}$. Suppose that $1 \leq \mathrm{rk}(\mathscr{G}) \leq 2$, then by [21, Corollary 2.4] and the stability of $F_{X*}(\mathscr{L})$, we have

$$\mu(\mathscr{G}) - \mu(F_{X*}(\mathscr{L})) \leq -\frac{3-\mathrm{rk}(\mathscr{G})}{3}.$$

By Grothendieck-Riemann-Roch theorem, we have $\deg(F_{X*}(\mathscr{L})) = d+1$ (cf. [20, Lemma 4.2]), so

$$\mu(\mathscr{G}) \leq -\frac{3-\mathrm{rk}(\mathscr{G})}{3} + \mu(F_{X*}(\mathscr{L})) = \frac{\mathrm{rk}(\mathscr{G})+d-2}{3}.$$

On the other hand, by stability of $\mathscr{E}$, we have $\mu(\mathscr{G}) > \frac{d}{3}$. This induces a contradiction. Hence, $\mathrm{rk}(\mathscr{G}) = 3$. Therefore $\mathscr{E} \cong \mathscr{G}$, i.e. the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injection. □

Proposition 3.4. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Let $\mathscr{L}$ be a line bundle of degree $d-1$ on X with $3 \nmid d$, $\mathscr{E} \subset F_{X*}(\mathscr{L})$ a subsheaf with $\mathrm{rk}(\mathscr{E}) = 3$ and $\deg(\mathscr{E}) = d$. Then $\mathscr{E}$ is a stable vector bundle.*

Proof. Let $\mathscr{G} \subset \mathscr{E}$ be a subsheaf of $\mathscr{E}$ with $\mathrm{rk}(\mathscr{G}) < \mathrm{rk}(\mathscr{E}) = 3$. By [21, Corollary 2.4] and the stability of $F_{X*}(\mathscr{L})$, we have

$$\mu(\mathscr{G}) - \mu(F_{X*}(\mathscr{L})) \leq -\frac{3-\mathrm{rk}(\mathscr{G})}{3}.$$

It follows that

$$\mu(\mathscr{G}) \leq -\frac{3-\mathrm{rk}(\mathscr{G})}{3} + \mu(F_{X*}(\mathscr{L})) = \frac{\mathrm{rk}(\mathscr{G})+d-2}{3} \leq \frac{d}{3} = \mu(\mathscr{E}).$$

If $\mathrm{rk}(\mathscr{G}) = 1$, then $\mu(\mathscr{G}) < \mu(\mathscr{E})$. If $\mathrm{rk}(\mathscr{G}) = 2$, we have $\mu(\mathscr{G}) \neq \frac{d}{3}$, since $3 \nmid d$. Thus $\mathscr{E}$ is a stable vector bundle. □

Any Frobenius destabilized stable bundle $[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k)$ with $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d), \mathscr{P}_4(d)\}$ can be embedded into $F_{X*}(\mathscr{L})$ for some line bundle $\mathscr{L}$ of degree $d-1$ on X. It can be seen from Proposition 3.3, Proposition 3.4 and the classification of Harder-Narasimhan polygons of the Frobenius pull-backs of Frobenius destabilized stable vector bundles in the case $(p,g,r)=(3,2,3)$ with $3\nmid d$ (see Figure 1).

4. Geometric properties of quot schemes

Let k be an algebraically closed field of characteristic $p>0$, X a smooth projective curve of genus g over k, $F_X: X\to X$ the absolute Frobenius morphism, r and d integers with $r>0$. Let $\mathrm{Pic}^{(t)}(X)$ be the Picard scheme parameterizes all line bundles of degree t on X, $[\mathscr{L}]\in\mathrm{Pic}^{(t)}(X)(k)$ and $\mathscr{P}\in\mathfrak{ConPgn}(r,pd)$. We first recall some notations of Quot schemes in [12, Section 4], such as $\mathrm{Quot}_X(r,d,\mathrm{Pic}^{(t)}(X))$, $\mathrm{Quot}^\sharp_X(r,d,\mathrm{Pic}^{(t)}(X))$ and so on. For simplicity, we describe these Quot schemes in the sense of closed points as the following

$$\begin{aligned}
&\mathrm{Quot}_X(r,d,\mathrm{Pic}^{(t)}(X))(k)\\
:=&\{[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\mid \mathrm{rk}(\mathscr{E})=r,\deg(\mathscr{E})=d,\mathscr{L}\in\mathrm{Pic}^{(t)}(X)\},\\
&\mathrm{Quot}_X(r,d,\mathrm{Pic}^{(t)}(X),\mathscr{P})(k)\\
:=&\{[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\in\mathrm{Quot}_X(r,d,\mathrm{Pic}^{(t)}(X))(k)\mid \mathrm{HNP}(F^*_X(\mathscr{E}))=\mathscr{P}\},\\
&\mathrm{Quot}_X(r,d,\mathrm{Pic}^{(t)}(X),\mathscr{P}^+)(k)\\
:=&\{[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\in\mathrm{Quot}_X(r,d,\mathrm{Pic}^{(t)}(X))(k)\mid \mathrm{HNP}(F^*_X(\mathscr{E}))\succcurlyeq\mathscr{P}\},\\
&\mathrm{Quot}^\sharp_X(r,d,\mathrm{Pic}^{(t)}(X))(k)\\
:=&\left\{[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\;\middle|\;\begin{array}{l}\mathrm{rk}(\mathscr{E})=r,\deg(\mathscr{E})=d,\mathscr{L}\in\mathrm{Pic}^{(t)}(X),\\ \text{adjoint homomorphism } F^*_X(\mathscr{E})\to\mathscr{L}\\ \text{is surjective.}\end{array}\right\},\\
&\mathrm{Quot}_X(r,d,\mathscr{L})(k)\\
:=&\{[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\mid \mathrm{rk}(\mathscr{E})=r,\deg(\mathscr{E})=d\},\\
&\mathrm{Quot}^\sharp_X(r,d,\mathscr{L})(k)\\
:=&\left\{[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\in\mathrm{Quot}_X(r,d,\mathscr{L})(k)\;\middle|\;\begin{array}{l}\text{adjoint homomorphism}\\ F^*_X(\mathscr{E})\to\mathscr{L}\text{ is surjective.}\end{array}\right\}
\end{aligned}$$

In this section, we will study the Frobenius stratification of the Quot scheme $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X))$ in the case $(p,g)=(3,2)$ with $3\nmid d$. In this case, the scheme $\mathrm{Quot}_X(3, d, \mathrm{Pic}^{(d-1)}(X))$ parameterizes all the rank 3 and degree d subsheaves of $F_{X*}(\mathscr{L})$ for any line bundle $\mathscr{L}$ of degree $d-1$ on X. By Proposition 3.3 and Proposition 3.4, we know that these subsheaves are stable. This induces a natural morphism

$$\theta : \mathrm{Quot}_X(3, d, \mathrm{Pic}^{(d-1)}(X)) \to \mathfrak{M}^s_X(3, d) : [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \mapsto [\mathscr{E}].$$

Now, we analysis the Frobenius stratification of the Quot scheme $\mathrm{Quot}_X(3, d, \mathrm{Pic}^{(d-1)}(X))$. Let $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})]$ be a closed point of $\mathrm{Quot}_X(3, d, \mathrm{Pic}^{(d-1)}(X))$, where $[\mathscr{L}] \in \mathrm{Pic}^{(d-1)}(X)(k)$. The non-trivial adjoint homomorphism $F^*_X(\mathscr{E}) \to \mathscr{L}$ implies that

$$\mu(F^*_X(\mathscr{E})) > \mu(\mathscr{L}) \geq \mu_{\min}(F^*_X(\mathscr{E})),$$

so $\mathscr{E}$ is a Frobenius destabilized stable vector bundle.

Theorem 4.1 (S. S. Shatz [18] Theorem 2 and Theorem 3). *Let k be an algebraically closed field, X a smooth projective variety over k, H an ample divisor on X. Consider the Harder-Narasimhan filtrations of torsion free sheaves on X in the sense of Mumford's semistability with respect to H. Then*

(1) *For any torsion free sheaf $\mathscr{E}$ and any subsheaf $\mathscr{F} \subseteq \mathscr{E}$, we have the point $(\mathrm{rk}(\mathscr{F}), \deg(\mathscr{F}))$ lies below $\mathrm{HNP}(\mathscr{E})$.*

(2) *Let $\mathcal{E}$ be a flat family of torsion free sheaves of rank r and degree d on $S \times_k X$ parameterized by a scheme S of finite type over k. Then for any convex polygon $\mathscr{P} \in \mathfrak{ConPgn}(r, d)$, the subset*

$$S_{\mathscr{P}} = \{s \in S \mid \mathrm{HNP}(\mathcal{E}|_{\{s\}\times_k X}) \succcurlyeq \mathscr{P}\}$$

is a closed scheme of S.

Proposition 4.2. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Let $\mathscr{L}$ be a line bundle of degree $d-1$ on X with $3 \nmid d$, $0 = E_3 \subset E_2 \subset E_1 \subset E_0 = F^*_X F_{X*}(\mathscr{L})$ the canonical filtration of $F^*_X F_{X*}(\mathscr{L})$. Let $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3, d, \mathscr{L})(k)$. Then*

$$\mathrm{HNP}(F^*_X(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d), \mathscr{P}_4(d)\}.$$

Moreover, we have

(1) $\mathrm{HNP}(F^*_X(\mathscr{E})) = \mathscr{P}_4(d)$ *if and only if* $\deg(F^*_X(\mathscr{E}) \cap E_2) = d + 2$ *if and only if the adjoint homomorphism* $F^*_X(\mathscr{E}) \to \mathscr{L}$ *is not surjective. In this case, the Harder-Narasimhan filtration of* $F^*_X(\mathscr{E})$ *is*

$$0 \subset F^*_X(\mathscr{E}) \cap E_2 \subset F^*_X(\mathscr{E}) \cap E_1 \subset F^*_X(\mathscr{E}).$$

(2) $\mathrm{HNP}(F^*_X(\mathscr{E})) = \mathscr{P}_3(d)$ *if and only if* $\deg(F^*_X(\mathscr{E}) \cap E_2) = d + 1$. *In this case,* $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}^{\sharp}_X(3, d, \mathscr{L})(k)$, *and the Harder-Narasimhan filtration of* $F^*_X(\mathscr{E})$ *is*

$$0 \subset F^*_X(\mathscr{E}) \cap E_2 \subset F^*_X(\mathscr{E}) \cap E_1 \subset F^*_X(\mathscr{E}).$$

(3) $\mathrm{HNP}(F^*_X(\mathscr{E})) = \mathscr{P}_2(d)$ *if and only if* $\deg(F^*_X(\mathscr{E}) \cap E_2) = d$. *In this case,* $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}^{\sharp}_X(3, d, \mathscr{L})(k)$, *and the Harder-Narasimhan filtration of* $F^*_X(\mathscr{E})$ *is*

$$0 \subset F^*_X(\mathscr{E}) \cap E_1 \subset F^*_X(\mathscr{E}).$$

Proof. For any $1 \leq i \leq 2$, consider the commutative diagram of abelian sheaves

$$\begin{array}{ccccccc}
E_{i+1} & & \xrightarrow{\nabla_{\mathrm{can}}} & & E_i \otimes_{\mathscr{O}_X} \Omega^1_X & & \\
\uparrow & \searrow & & & \uparrow & \searrow & \\
& & E_i & & \xrightarrow{\nabla_{\mathrm{can}}} & & E_{i-1} \otimes_{\mathscr{O}_X} \Omega^1_X \\
& & \uparrow & & & & \uparrow \\
F^*_X(\mathscr{E}) \cap E_{i+1} & & \xrightarrow{\nabla_{\mathrm{can}}} & & (F^*_X(\mathscr{E}) \cap E_i) \otimes_{\mathscr{O}_X} \Omega^1_X & & \\
& \searrow & & & & \searrow & \\
& & F^*_X(\mathscr{E}) \cap E_i & & \xrightarrow{\nabla_{\mathrm{can}}} & & (F^*_X(\mathscr{E}) \cap E_{i-1}) \otimes_{\mathscr{O}_X} \Omega^1_X
\end{array}.$$

we can get the commutative diagram of vector bundles

$$\begin{array}{ccc}
E_i/E_{i+1} & \xrightarrow{\cong} & (E_{i-1}/E_i) \otimes_{\mathscr{O}_X} \Omega^1_X \\
\uparrow & & \uparrow \\
(F^*_X(\mathscr{E}) \cap E_i)/(F^*_X(\mathscr{E}) \cap E_{i+1}) & \longrightarrow & (F^*_X(\mathscr{E}) \cap E_{i-1})/(F^*_X(\mathscr{E}) \cap E_i) \otimes_{\mathscr{O}_X} \Omega^1_X.
\end{array}$$

Thus $(F^*_X(\mathscr{E}) \cap E_i)/(F^*_X(\mathscr{E}) \cap E_{i+1}) \hookrightarrow (F^*_X(\mathscr{E}) \cap E_{i-1})/(F^*_X(\mathscr{E}) \cap E_i) \otimes_{\mathscr{O}_X} \Omega^1_X$ is injective for any $1 \leq i \leq 2$. Therefore we have the following inequalities

$$\deg(F^*_X(\mathscr{E}) \cap E_2) \leq d + 3,$$

$$\deg\left(\frac{F_X^*(\mathscr{E})\cap E_1}{F_X^*(\mathscr{E})\cap E_2}\right)\le d+1,$$

$$\deg\left(\frac{F_X^*(\mathscr{E})}{F_X^*(\mathscr{E})\cap E_1}\right)\le d-1,$$

$$(*)\quad \deg(F_X^*(\mathscr{E})\cap E_2)\le \deg(\frac{F_X^*(\mathscr{E})\cap E_1}{F_X^*(\mathscr{E})\cap E_2})+2\le \deg(\frac{F_X^*(\mathscr{E})}{F_X^*(\mathscr{E})\cap E_1})+4,$$

$$(**)\qquad \deg(F_X^*(\mathscr{E})\cap E_2)+\deg(\frac{F_X^*(\mathscr{E})\cap E_1}{F_X^*(\mathscr{E})\cap E_2})+\deg(\frac{F_X^*(\mathscr{E})}{F_X^*(\mathscr{E})\cap E_1})=3d.$$

By computation, we can get

$$d\le \deg(F_X^*(\mathscr{E})\cap E_2)\le d+2.$$

Suppose that $[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\in \mathrm{Quot}_X(3,d,\mathscr{L})(k)$ such that $F_X^*(\mathscr{E})$ is semistable or $\mathrm{HNP}(F_X^*(\mathscr{E}))=\mathscr{P}_1(d)$, then we have $\mu_{\min}(F_X^*(\mathscr{E}))\ge\frac{2d-1}{2}$ by Figure 1. This contradicts to the fact $\deg(F_X^*(\mathscr{E})/(F_X^*(\mathscr{E})\cap E_1))\le d-1$. Hence

$$\mathrm{HNP}(F_X^*(\mathscr{E}))\in\{\mathscr{P}_2(d),\mathscr{P}_3(d),\mathscr{P}_4(d)\}.$$

(1). I. If $\mathrm{HNP}(F_X^*(\mathscr{E}))=\mathscr{P}_4(d)$, there exists a unique maximal destabilizing sub-line bundle $E'\subset F_X^*(\mathscr{E})$ with $\deg(E')=d+2$. Suppose that $E'\nsubseteq F_X^*(\mathscr{E})\cap E_1$, then the composition

$$E'\hookrightarrow F_X^*(\mathscr{E})\hookrightarrow F_X^*F_{X*}(\mathscr{L})\twoheadrightarrow F_X^*F_{X*}(\mathscr{L})/E_1\cong\mathscr{L}$$

is non-trivial. This induces a contradiction since $\deg(E')>\deg(\mathscr{L})$. Suppose that $E'\subset F_X^*(\mathscr{E})\cap E_1$ and $E'\nsubseteq F_X^*(\mathscr{E})\cap E_2$, then the composition

$$E'\hookrightarrow F_X^*(\mathscr{E})\cap E_1\hookrightarrow E_1\twoheadrightarrow E_1/E_2$$

is non-trivial. This induces a contradiction since $\deg(E')>\deg(E_1/E_2)$. Hence $E'\subset F_X^*(\mathscr{E})\cap E_2$. Thus $\deg(F_X^*(\mathscr{E})\cap E_2)=d+2$. In fact $E'=F_X^*(\mathscr{E})\cap E_2$.

II. If $\deg(F_X^*(\mathscr{E})\cap E_2)=d+2$, then by $(*)$ and $(**)$, we have

$$\deg(\frac{F_X^*(\mathscr{E})\cap E_1}{F_X^*(\mathscr{E})\cap E_2})\ge d,\deg(\frac{F_X^*(\mathscr{E})}{F_X^*(\mathscr{E})\cap E_1})\le d-2<\deg(\mathscr{L}).$$

Hence the composition $F_X^*(\mathscr{E})\twoheadrightarrow F_X^*(\mathscr{E})/(F_X^*(\mathscr{E})\cap E_1)\hookrightarrow F_X^*F_{X*}(\mathscr{L})/E_1\cong\mathscr{L}$ is not surjective.

III. If $F_X^*(\mathscr{E}) \to \mathscr{L}$ is not surjective, then $\mu_{\min}(F_X^*(\mathscr{E})) \leq d-2$. Then we must have $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_4(d)$ by Figure 1.

Combining the proofs of I, II and III, we have $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_4(d)$ if and only if $\deg(F_X^*(\mathscr{E}) \cap E_2) = d+2$ if and only if the adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ is not surjective. In this case, we have $\deg((F_X^*(\mathscr{E}) \cap E_1)/(F_X^*(\mathscr{E}) \cap E_2)) = d$ and $\deg(F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1)) = d-2$ by $(*)$ and $(**)$. Hence, the Harder-Narasimhan filtration of $F_X^*(\mathscr{E})$ is

$$0 \subset F_X^*(\mathscr{E}) \cap E_2 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

(2). I. If $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_3(d)$, then there exists a unique maximal destabilizing sub-line bundle $E' \subset F_X^*(\mathscr{E}) \cap E_1$ with $\deg(E') = d+1$. Suppose that $E' \nsubseteq F_X^*(\mathscr{E}) \cap E_2$, then the composition

$$E' \hookrightarrow F_X^*(\mathscr{E}) \cap E_1 \hookrightarrow E_1 \twoheadrightarrow E_1/E_2$$

is non-trivial. This implies $E' \cong E_1/E_2$ since E' and E_1/E_2 are line bundles with same degree. Then $E_1 = E' \oplus E''$ for some line bundle E'' of degree $d+3$. This induces a contradiction by Lemma 4.3. Hence $E' \subseteq F_X^*(\mathscr{E}) \cap E_2$. Thus $\deg(F_X^*(\mathscr{E}) \cap E_2) = d+1$. In fact $E' = F_X^*(\mathscr{E}) \cap E_2$.

II. If $\deg(F_X^*(\mathscr{E}) \cap E_2) = d+1$, then the adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ is surjective by (1), i.e. $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(3,d,\mathscr{L})(k)$. Moreover, by $(*)$ and $(**)$, we have

$$\mu(F_X^*(\mathscr{E}) \cap E_2) = d+1,\ \mu\Big(\frac{F_X^*(\mathscr{E}) \cap E_1}{F_X^*(\mathscr{E}) \cap E_2}\Big) = d,\ \deg\Big(\frac{F_X^*(\mathscr{E})}{F_X^*(\mathscr{E}) \cap E_1}\Big) = d-1.$$

Hence, $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_3(d)$ by Figure 1.

Combining the proofs of I and II, we have $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_3(d)$ if and only if $\deg(F_X^*(\mathscr{E}) \cap E_2) = d+1$. In this case, the Harder-Narasimhan filtration of $F_X^*(\mathscr{E})$ is

$$0 \subset F_X^*(\mathscr{E}) \cap E_2 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

(3). By the proofs of (1) and (2), we can conclude that $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_2(d)$ if and only if $\deg(F_X^*(\mathscr{E}) \cap E_2) = d$. In this case, the adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ is surjective by (1), i.e.

$$[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(3,d,\mathscr{L})(k),$$

and the Harder-Narasimhan filtration of $F_X^*(\mathscr{E})$ is

$$0 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}). \qquad \square$$

Lemma 4.3 (A. Grothendieck, M. Raynaud). *Let k be an algebraically closed field of characteristic $p>2$, X a smooth projective curve of genus $g\geq 2$ over k and $\mathscr{L}$ a line bundle on X. If $p\nmid(g-1)$, we have*

$$F_X^*F_{X*}(\mathscr{L})\ncong(\Omega_X^{\otimes p-1}\otimes\mathscr{L})\oplus(\Omega_X^{\otimes p-2}\otimes\mathscr{L})\oplus\cdots\oplus(\Omega_X^1\otimes\mathscr{L})\oplus\mathscr{L}.$$

In particular, in the case $p=3$ and $g=2$, we have

$$F_X^*F_{X*}(\mathscr{L})\ncong(\Omega_X^{\otimes 2}\otimes\mathscr{L})\oplus(\Omega_X^1\otimes\mathscr{L})\oplus\mathscr{L}.$$

Proof. Suppose that

$$F_X^*F_{X*}(\mathscr{L})\cong(\Omega_X^{\otimes p-1}\otimes\mathscr{L})\oplus(\Omega_X^{\otimes p-2}\otimes\mathscr{L})\oplus\cdots\oplus(\Omega_X^1\otimes\mathscr{L})\oplus\mathscr{L}.$$

For any $0\leq i\leq p-1$, the composition

$$\begin{aligned}\nabla|_{\Omega_X^{\otimes i}\otimes\mathscr{L}}:\Omega_X^{\otimes i}\otimes\mathscr{L}&\hookrightarrow F_X^*F_{X*}(\mathscr{L})\\ &\stackrel{\nabla_{\text{can}}}{\to}F_X^*F_{X*}(\mathscr{L})\otimes\Omega_X^1\twoheadrightarrow(\Omega_X^{\otimes i}\otimes\mathscr{L})\otimes\Omega_X^1\end{aligned}$$

induces a connection on $\Omega_X^{\otimes i}\otimes\mathscr{L}$, whose p-curveture is zero. Then by Katz's theorem, we know that $\Omega_X^{\otimes i}\otimes\mathscr{L}\cong F_X^*(L_i)$ for some line bundle L_i on X, for any $0\leq i\leq p-1$. Hence $p|\deg(\Omega_X^{\otimes i}\otimes\mathscr{L})$, i.e. $p|i(2g-2)+\deg(\mathscr{L})$, $0\leq i\leq p-1$.

On the other hand, if $p>2$ and $p\nmid(g-1)$, there exists some $i\in\{1,2,\ldots,p-1\}$ such that $p\nmid i(2g-2)+\deg(\mathscr{L})$. This contradicts to the assumption.

Therefore, $p>2$ and $p\nmid(g-1)$ imply

$$F_X^*F_{X*}(\mathscr{L})\ncong(\Omega_X^{\otimes p-1}\otimes\mathscr{L})\oplus(\Omega_X^{\otimes p-2}\otimes\mathscr{L})\oplus\cdots\oplus(\Omega_X^1\otimes\mathscr{L})\oplus\mathscr{L}.$$

In particular, in the case $p=3$ and $g=2$, we have

$$F_X^*F_{X*}(\mathscr{L})\ncong(\Omega_X^{\otimes 2}\otimes\mathscr{L})\oplus(\Omega_X^1\otimes\mathscr{L})\oplus\mathscr{L}.$$

□

Proposition 4.4. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Then*

$$\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d))$$

are smooth irreducible projective varieties for $2 \leq i \leq 4$*, and*

$$\begin{aligned}&\dim \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d))\\ &= \dim \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i(d)) = \begin{cases}5, \text{if } i=2\\ 4, \text{if } i=3\\ 3, \text{if } i=4\end{cases}\end{aligned}$$

Proof. By [4], there is a morphism

$$\begin{aligned}\varPi : \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X)) &\to X \times \mathrm{Pic}^{(d-1)}(X)\\ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] &\mapsto (\mathrm{Supp}(F_{X*}(\mathscr{L})/\mathscr{E}),\mathscr{L}).\end{aligned}$$

For any point $x \in X(k)$ and any $[\mathscr{L}] \in \mathrm{Pic}^{(d-1)}(X)(k)$, we denote the fiber of $\varPi$ over $(x,[\mathscr{L}])$ by $\mathrm{Quot}_X(3,d,x,\mathscr{L})$. Then there is a one to one correspondence between the set of closed points $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})]$ of $\mathrm{Quot}_X(3,d,x,\mathscr{L})(k)$ and the set of $\mathscr{O}_x$-submodules V of the stalk $F_{X*}(\mathscr{L})_x$ such that

$$F_{X*}(\mathscr{L})_x/V \cong k,$$

the latter has a natural structure of algebraic variety which is isomorphic to projective space $\mathbb{P}^2_k$. Hence $\varPi$ is surjective and $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X))$ is a smooth irreducible projective variety of dimension 5. Without loss of generality, we can assume that $\mathscr{O}_x \cong k[[t^3]]$, then $F_{X*}(\mathscr{L})_x \cong k[[t]]$ endows with $k[[t^3]]$-module structure induced by injection $k[[t^3]] \hookrightarrow k[[t]]$ and

$$F_X^* F_{X*}(\mathscr{L})_x \cong k[[t]] \otimes_{k[[t^3]]} k[[t]].$$

Suppose that the $\mathscr{O}_x$-submodule $\mathscr{E}_x$ of $F_{X*}(\mathscr{L})_x$ corresponds to the $k[[t^3]]$-submodule $V_{\mathscr{E}}$ of $k[[t]]$, then the $\mathscr{O}_x$-submodule $F_X^*(\mathscr{E})_x$ of $F_X^*F_{X*}(\mathscr{L})_x$ corresponds to the $k[[t]]$-submodule $V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$ of $k[[t]] \otimes_{k[[t^3]]} k[[t]]$.

Consider the decomposition of $k[[t]] = k[[t^3]] \oplus k[[t^3]] \cdot t \oplus k[[t^3]] \cdot t^2$ as a $k[[t^3]]$-module. Then the $k[[t^3]]$-submodule $V_{\mathscr{E}} \subset k[[t]]$ with $k[[t]]/V_{\mathscr{E}} \cong k$ implies that

$$k[[t^3]] \cdot t^3 \oplus k[[t^3]] \cdot t^4 \oplus k[[t^3]] \cdot t^5 \subset V_{\mathscr{E}}.$$

Now, we investigate the intersection of $F_X^*(\mathscr{E})$ with the canonical filtration

$$0 \subset E_2 \subset E_1 \subset F_X^* F_{X*}(\mathscr{L}).$$

Locally, the stalk E_{1x} has a basis $\{t\otimes 1-1\otimes t,(t\otimes 1-1\otimes t)^2\}$ and E_{2x} has a basis $\{(t\otimes 1-1\otimes t)^2\}$ as $k[[t]]$-submodules of

$$F_X^*F_{X*}(\mathscr{L})_x\cong k[[t]]\otimes_{k[[t^3]]}k[[t]]$$

by [20, Lemma 3.2]. Let $[\mathscr{E}\hookrightarrow F_{X*}(\mathscr{L})]\in \mathrm{Quot}_X(3,d,x,\mathscr{L})(k)$, we claim that

(a) $(t\otimes 1-1\otimes t)^2\notin V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]]$

(b) $(t\otimes 1-1\otimes t)^2t\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]]$ if and only if $\{t,t^2\}\subset V_{\mathscr{E}}$.

(c) $(t\otimes 1-1\otimes t)^2t^2\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]]$ if and only if $t^2\in V_{\mathscr{E}}$.

(d) $(t\otimes 1-1\otimes t)^2t^3\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]]$.

Suppose that $(t\otimes 1-1\otimes t)^2\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]]$, then we have $F_X^*(\mathscr{E})\cap E_2=E_2$ as $F_X^*(\mathscr{E})_x\cap E_{2x}=E_{2x}$. So $\deg(F_X^*(\mathscr{E})\cap E_2)=d+3$, and it contradicts to Proposition 4.2. Therefore the claim of (a) is proved.

Since

$$\begin{aligned}(t\otimes 1-1\otimes t)^2t&=t^2\otimes t-2t\otimes t^2+1\otimes t^3\\&=t^2\otimes t-2t\otimes t^2+t^3\otimes 1\end{aligned}$$

and $\{t^3\}\subset V_{\mathscr{E}}$ by $k[[t^3]]\cdot t^3\oplus k[[t^3]]\cdot t^4\oplus k[[t^3]]\cdot t^5\subset V_{\mathscr{E}}$, then we have

$$(t\otimes 1-1\otimes t)^2t\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]] \text{ iff } t^2\otimes t-2t\otimes t^2\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]],$$

which is equivalent to $\{t,t^2\}\subset V_{\mathscr{E}}$. Therefore the claim of (b) is proved.

Since

$$\begin{aligned}(t\otimes 1-1\otimes t)^2t^2&=t^2\otimes t^2-2t\otimes t^3+1\otimes t^4\\&=t^2\otimes t^2-2t^4\otimes 1+t^3\otimes t\end{aligned}$$

and $\{t^3,t^4\}\subset V_{\mathscr{E}}$ by $k[[t^3]]\cdot t^3\oplus k[[t^3]]\cdot t^4\oplus k[[t^3]]\cdot t^5\subset V_{\mathscr{E}}$, then we have

$$(t\otimes 1-1\otimes t)^2t^2\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]] \text{ iff } t^2\otimes t^2\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]],$$

which is equivalent to $t^2\in V_{\mathscr{E}}$. Therefore the claim of (c) is proved.

Since

$$\begin{aligned}(t\otimes 1-1\otimes t)^2t^3 &= t^2\otimes t^3-2t\otimes t^4+1\otimes t^5\\ &= t^5\otimes 1-2t^4\otimes t+t^3\otimes t^2\end{aligned}$$

and $\{t^3,t^4,t^5\}\subset V_{\mathscr{E}}$ by $k[[t^3]]\cdot t^3\oplus k[[t^3]]\cdot t^4\oplus k[[t^3]]\cdot t^5\subset V_{\mathscr{E}}$, then we have

$$(t\otimes 1-1\otimes t)^2t^3\in V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]].$$

Therefore the claim of (d) is proved.

In summary, by above claims, we have

$$1\leq \dim E_{2x}/((V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]])\cap E_{2x})\leq 3,$$

$$\dim E_{2x}/((V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]])\cap E_{2x})=\begin{cases}1 & \text{if ond only if } \{t,t^2\}\subset V_{\mathscr{E}}\\ 2 & \text{if ond only if } t\notin V_{\mathscr{E}} \text{ and } t^2\in V_{\mathscr{E}}\\ 3 & \text{if ond only if } t^2\notin V_{\mathscr{E}}.\end{cases}$$

Consider the exact sequence of $\mathscr{O}_X$-modules

$$0\to F_X^*(\mathscr{E})\cap E_2\to E_2\to \frac{E_2}{F_X^*(\mathscr{E})\cap E_2}\to 0.$$

Notice that $E_2/(F_X^*(\mathscr{E})\cap E_2)\cong E_{2x}/((V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]])\cap E_{2x})$. Therefore, by Proposition 4.2, we have

$$\begin{aligned}\mathrm{HNP}(F_X^*(\mathscr{E}))=\mathscr{P}_2(d) &\Leftrightarrow \deg(F_X^*(\mathscr{E})\cap E_2)=d\\ &\Leftrightarrow \deg(E_{2x}/((V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]])\cap E_{2x}))=3\\ &\Leftrightarrow t^2\notin V_{\mathscr{E}}.\end{aligned}$$

$$\begin{aligned}\mathrm{HNP}(F_X^*(\mathscr{E}))=\mathscr{P}_3(d) &\Leftrightarrow \deg(F_X^*(\mathscr{E})\cap E_2)=d+1\\ &\Leftrightarrow \deg(E_{2x}/((V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]])\cap E_{2x}))=2\\ &\Leftrightarrow t\notin V_{\mathscr{E}} \text{ and } t^2\in V_{\mathscr{E}}.\end{aligned}$$

$$\begin{aligned}\mathrm{HNP}(F_X^*(\mathscr{E}))=\mathscr{P}_4(d) &\Leftrightarrow \deg(F_X^*(\mathscr{E})\cap E_2)=d+2\\ &\Leftrightarrow \deg(E_{2x}/((V_{\mathscr{E}}\otimes_{k[[t^3]]}k[[t]])\cap E_{2x}))=1\\ &\Leftrightarrow \{t,t^2\}\subset V_{\mathscr{E}}.\end{aligned}$$

For $i = 2,3,4$, we denote the closed subschemes $\mathrm{Quot}_X(3,d,x,\mathscr{L},\mathscr{P}_i^+(d))$ of $\mathrm{Quot}_X(3,d,x,\mathscr{L})$ consisting of closed points

$$\begin{aligned}&\mathrm{Quot}_X(3,d,x,\mathscr{L},\mathscr{P}_i^+(d))(k)\\ &= \{[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3,d,x,\mathscr{L})(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \succcurlyeq \mathscr{P}_i(d)\}\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{Quot}_X(3,d,x,\mathscr{L},\mathscr{P}_2^+(d))\\ &\cong \{V \mid k[[t^3]]\text{-submodule } V \subset k[[t]], k[[t]]/V \cong k\} \cong \mathbb{P}_k^2,\\ &\mathrm{Quot}_X(3,d,x,\mathscr{L},\mathscr{P}_3^+(d))\\ &\cong \{V \mid k[[t^3]]\text{-submodule } V \subset k[[t]], k[[t]]/V \cong k, t^2 \in V_{\mathscr{E}}\} \cong \mathbb{P}_k^1,\\ &\mathrm{Quot}_X(3,d,x,\mathscr{L},\mathscr{P}_4^+(d))\\ &\cong \{V \mid k[[t^3]]\text{-submodule } V \subset k[[t]], k[[t]]/V \cong k, \{t,t^2\} \subset V_{\mathscr{E}}\} \cong \{pt\}.\end{aligned}$$

So $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d))$ are smooth irreducible projective varieties for $2 \leq i \leq 4$, and

$$\dim \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d)) = \begin{cases} 5, \text{if } i = 2\\ 4, \text{if } i = 3\\ 3, \text{if } i = 4.\end{cases}$$

By Theorem 4.1(2), it is easy to see that $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i(d))$ is an open subvariety of $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d))$ for any $2 \leq i \leq 4$. Therefore, for any $2 \leq i \leq 4$, we have

$$\begin{aligned}&\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d))\\ &= \overline{\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i(d))},\\ &\dim \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d))\\ &= \dim \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i(d)).\end{aligned}$$

□

5. Geometric properties of Frobenius strata

We now study the geometric properties of Frobenius strata in the moduli space $\mathfrak{M}_X^s(3,d)$ with $3 \nmid d$, where X is a smooth projective curve of genus 2 over an algebraically closed field k of characteristic 3.

Proposition 5.1. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Let d be an integer with $3 \nmid d$. Then the image of the morphism*

$$\theta : \mathrm{Quot}_X(3, d, \mathrm{Pic}^{(d-1)}(X)) \to \mathfrak{M}_X^s(3,d)$$
$$[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \mapsto [\mathscr{E}]$$

is the subset

$$\{[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d), \mathscr{P}_4(d)\}\}.$$

Moreover, the restriction $\theta|_{\mathrm{Quot}_X^\sharp(3,d,\mathrm{Pic}^{(d-1)}(X))}$ is an injective morphism and the image of $\theta|_{\mathrm{Quot}_X^\sharp(3,d,\mathrm{Pic}^{(d-1)}(X))}$ is the subset

$$\{[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d)\}\}.$$

Proof. Let $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3, d, \mathrm{Pic}^{(d-1)}(X))(k)$, then we have

$$\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d), \mathscr{P}_4(d)\}$$

by Proposition 4.2. It follows that the image of θ lies in the following subset

$$\{[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d), \mathscr{P}_4(d)\}\}.$$

On the other hand, let $[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k)$ such that $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_i(d)$ for some $2 \leq i \leq 4$. Then $F_X^*(\mathscr{E})$ has a quotient line bundle $\mathscr{L}'$ of $\deg(\mathscr{L}') \leq d-1$. Embedding $\mathscr{L}'$ into some line bundle $\mathscr{L}$ of $\deg(\mathscr{L}) = d-1$, we can get the non-trivial homomorphism

$$F_X^*(\mathscr{E}) \twoheadrightarrow \mathscr{L}' \hookrightarrow \mathscr{L}.$$

Then the adjunction $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injection by Proposition 3.3. Hence, the image of θ is just the subset

$$\{[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d), \mathscr{P}_4(d)\}\}.$$

Now, we will prove $\theta|_{\mathrm{Quot}_X^\sharp(3,d,\mathrm{Pic}^{(d-1)}(X))}$ is an injective morphism. Let

$$e_i := [\mathscr{E}_i \hookrightarrow F_{X*}(\mathscr{L}_i)] \in \mathrm{Quot}_X^\sharp(3, d, \mathrm{Pic}^{(d-1)}(X))(k),$$

where $[\mathscr{L}_i] \in \mathrm{Pic}^{(d-1)}(X)(k)$, $i = 1, 2$. Suppose that

$$\theta(e_1) = \theta(e_2) \in \mathfrak{M}_X^s(3,d)(k)$$

, i.e. $\mathscr{E}_1 \cong \mathscr{E}_2$. Since $\mathrm{HNP}(F_X^*(\mathscr{E}_i)) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d)\}$, we have

$$\mu_{\min}(F_X^*(\mathscr{E}_i)) = d-1, \quad i = 1,2.$$

So the surjection $F_X^*(\mathscr{E}_i) \to \mathscr{L}_i$ implies that $\mathscr{L}_i$ is the quotient line bundle of $F_X^*(\mathscr{E}_i)$ with minimal slope in the Harder-Narasimhan filtration of $F_X^*(\mathscr{E}_i)$. By the uniqueness of Harder-Narasimhan filtration, there exists an isomorphism $\psi : \mathscr{L}_1 \to \mathscr{L}_2$ making the following diagram

$$\begin{array}{ccccc} F_X^*(\mathscr{E}_1) & \longrightarrow & \mathscr{L}_1 & \longrightarrow & 0 \\ {\scriptstyle \phi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \psi}\downarrow{\scriptstyle \cong} & & \\ F_X^*(\mathscr{E}_2) & \longrightarrow & \mathscr{L}_2 & \longrightarrow & 0 \end{array}$$

commutative, where the isomorphism ϕ is induced from an isomorphism $\mathscr{E}_1 \xrightarrow{\cong} \mathscr{E}_2$. By adjunction, we have commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{E}_1 & \longrightarrow & F_{X*}(\mathscr{L}_1) \\ & & \downarrow{\scriptstyle \cong} & & {\scriptstyle F_{X*}(\psi)}\downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & \mathscr{E}_2 & \longrightarrow & F_{X*}(\mathscr{L}_2) \end{array}$$

where the horizontal homomorphisms is the isomorphism

$$F_{X*}(\psi) : F_{X*}(\mathscr{L}_1) \xrightarrow{\cong} F_{X*}(\mathscr{L}_2).$$

This implies $\mathscr{E}_1 = \mathscr{E}_2$ as subsheaves of $F_{X*}(\mathscr{L})$, where $[\mathscr{L}] = [\mathscr{L}_1] = [\mathscr{L}_2] \in \mathrm{Pic}^{(d-1)}(X)(k)$. Thus, e_1 and e_2 are the same point in the

$$\mathrm{Quot}_X^{\sharp}(3, d, \mathrm{Pic}^{(d-1)}(X))(k).$$

Hence the morphism $\theta|_{\mathrm{Quot}_X^{\sharp}(3,d,\mathrm{Pic}^{(d-1)}(X))}$ is injective.

Let $[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k)$ and $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d)\}$. Then $F_X^*(\mathscr{E})$ has a quotient line bundle $\mathscr{L}$ of degree $d-1$. Then by Proposition 3.3, the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injective homomorphism. Therefore

$$e := [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^{\sharp}(3, d, \mathrm{Pic}^{(d-1)}(X))(k)$$

and $\theta(e) = [\mathscr{E}]$. Hence the image of $\theta|_{\mathrm{Quot}_X^{\sharp}(3,d,\mathrm{Pic}^{(d-1)}(X))}$ is just the subset

$$\{[\mathscr{E}] \in \mathfrak{M}_X^s(3,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2(d), \mathscr{P}_3(d)\}\}. \qquad \square$$

Theorem 5.2. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k, d an integer with $3 \nmid d$. Then*

$$S_X(3,d,\mathscr{P}_i^+(d)) = \overline{S_X(3,d,\mathscr{P}_i(d))},$$

and $S_X(3,d,\mathscr{P}_i^+(d))$ (resp. $S_X(3,d,\mathscr{P}_i(d))$) are irreducible projective (resp. irreducible quasi-projective) varieties for $1 \leq i \leq 4$,

$$\dim S_X(3,d,\mathscr{P}_i^+(d)) = \dim S_X(3,d,\mathscr{P}_i(d)) = \begin{cases} 5, \text{if } \ i=1 \\ 5, \text{if } \ i=2 \\ 4, \text{if } \ i=3 \\ 2, \text{if } \ i=4 \end{cases}$$

Proof. The morphism

$$\theta : \mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X)) \to \mathfrak{M}^s_X(3,d)$$
$$[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \mapsto [\mathscr{E}]$$

maps $\mathrm{Quot}_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i^+(d))$ onto $S_X(3,d,\mathscr{P}_i^+(d))$ for $2 \leq i \leq 4$. Then by Proposition 4.4, $S_X(3,d,\mathscr{P}_i^+(d))$ are irreducible projective varieties and

$$\overline{S_X(3,d,\mathscr{P}_i(d))} = S_X(3,d,\mathscr{P}_i^+(d))$$

for $2 \leq i \leq 4$, since $S_X(3,d,\mathscr{P}_i(d))$ is an open subvariety of $S_X(3,d,\mathscr{P}_i^+(d))$ by Thm. 4.1(2). Moreover, by Prop. 5.1, the injection $\theta|_{\mathrm{Quot}^\sharp_X(3,d,\mathrm{Pic}^{(d-1)}(X))}$ maps $\mathrm{Quot}^\sharp_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i(d))$ onto $S_X(3,d,\mathscr{P}_i(d))$ for $i=2,3$. Then by Proposition 4.4, we have

$$\begin{aligned} \dim S_X(3,d,\mathscr{P}_i^+(d)) &= \dim S_X(3,d,\mathscr{P}_i(d)) \\ &= \dim \mathrm{Quot}^\sharp_X(3,d,\mathrm{Pic}^{(d-1)}(X),\mathscr{P}_i(d)) \\ &= \begin{cases} 5, \text{if } \ i=2. \\ 4, \text{if } \ i=3. \end{cases} \end{aligned}$$

The isomorphism

$$\iota : \mathfrak{M}^s_X(3,d) \to \mathfrak{M}^s_X(3,-d) : [\mathscr{E}] \mapsto [\mathscr{E}^\vee]$$

maps $S_X(3,d,\mathscr{P}_1(d))$ (resp. $S_X(3,d,\mathscr{P}_1^+(d))$) onto $S_X(3,-d,\mathscr{P}_2(-d))$ (resp. $S_X(3,-d,\mathscr{P}_2^+(-d))$). So we have $\overline{S_X(3,d,\mathscr{P}_1(d))} = S_X(3,d,\mathscr{P}_1^+(d))$ is an irreducible projective variety and

$$\dim S_X(3,d,\mathscr{P}_1^+(d)) = \dim S_X(3,d,\mathscr{P}_1(d)) = 5.$$

Now we study the properties of subvariety $S_X(3,d,\mathscr{P}_4(d))$. By [11, Lemma 3.1], we know that any vector bundle $[\mathscr{E}] \in S_X(3,d,\mathscr{P}_4(d))(k)$ has the form $F_{X*}(\mathscr{L}')$ for some line bundle $\mathscr{L}'$ of degree $d-2$ on X. Moreover, by [11, Theorem 2.5], the morphism

$$\begin{aligned} P^s_{\mathrm{Frob}} : \mathfrak{M}^s_X(1,d-2) &\to \mathfrak{M}^s_X(3,d) \\ [\mathscr{L}'] &\mapsto [F_{X*}(\mathscr{L}')] \end{aligned}$$

is a closed immersion and the image of P^s_{Frob} is just the $S_X(3,d,\mathscr{P}_4(d))$. Thus $S_X(3,d,\mathscr{P}_4(d)) = S_X(3,d,\mathscr{P}_4^+(d))$ is isomorphic to Jacobian variety Jac_X of X which is a smooth irreducible projective variety of dimension 2. □

By Theorem 5.2, we know that the subvariety Z of Frobenius destabilized stable vector bundles is a reducible closed subvariety consisting two irreducible closed subvarieties of dimension 5 in $\mathfrak{M}^s_X(3,d)$. Notice that $\dim \mathfrak{M}^s_X(3,d) = 10$, we have $\mathrm{codim}_Z \mathfrak{M}^s_X(3,d) = 5$.

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