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On a question of Dolgachev

Marco Pacini and Damiano Testa

For each even, positive integer n, we define a rational self-map on the space of plane curves of degree n, using classical contravariants. In the case of plane quartics, we show that the degree of this map is 15. This answers a question of Dolgachev on the moduli space of curves of genus 3.

Introduction

An inspiring question of Dolgachev motivates the present paper. First, we describe a classical construction that associates to a general plane quartic curve C another plane quartic curve $H_4(C)$.

Let k be a field of characteristic relatively prime to 6 and fix a smooth plane quartic $C \subset \mathbb{P}^2_k$ over k. A line $\ell \subset \mathbb{P}^2_k$, transverse to C, intersects C in a configuration of 4 distinct points. The double cover C_ℓ of ℓ branched above these 4 points is a smooth curve of genus one. We let $H_4(C) \subset \mathbb{P}^{2\vee}_k$ denote the closure of the locus of lines $\ell \subset \mathbb{P}^2_k$ such that the j-invariant of the curve C_ℓ vanishes. The closed subset $H_4(C) \subset \mathbb{P}^{2\vee}_k$ is a plane curve of degree 4. We thus obtain a rational self-map $H_4 \colon \mathbb{P}^{14}_k \dashrightarrow \mathbb{P}^{14}_k$ of the space of plane quartics, assigning to the quartic $C \subset \mathbb{P}^2_k$ the quartic $H_4(C) \subset \mathbb{P}^{2\vee}_k$.

Question 1 (Dolgachev). Is the rational map H_4 *generically finite*? If so, what is its *degree*?

We answer this question in Theorem 2.4: the map H_4 is generically finite of degree 15. The second author found the answer to Question 1, involving the use of a computer in an essential way. We propose here a proof that we check entirely by hand. Nevertheless, determining the degree is still the outcome of a lengthy computation: we do not have an interpretation for the fibers of the map H_4.

In an attempt to obtain a more conceptual understanding of the fibers of the map H_4, we computed the monodromy group of H_4. Theorem 2.7 shows that this group is the full symmetric group $\mathfrak{S}_{15}$. Unfortunately we have not been able to use this information: we still lack an understanding of how the 15 quartics in a general fiber of H_4 arise.

To put Question 1 into perspective, observe that the map H_4 is equivariant with respect to the group $\mathbb{P}GL_3$ of projective changes of coordinates. Moreover, the quotient of the space of plane quartic curves by the group $\mathbb{P}GL_3$ is birational to the moduli space $\mathcal{M}_3$ of curves of genus 3. After checking that the map H_4 is generically finite, we deduce that it descends to a generically finite rational map $\overline{H}_4 \colon \mathcal{M}_3 \dashrightarrow \mathcal{M}_3$. We show that the degree of $\overline{H}_4$ is also 15.

More generally, Dolgachev considers rational self-maps of moduli spaces of curves of low genus (such as $\overline{H}_4$) and of hypersurfaces (such as H_4) in [Dol16]. He provides several examples and constructions of dominant rational self-maps of such spaces to themselves of degree strictly larger than 1.

In line with Dolgachev's general strategy, for each even degree n, we introduce a rational self-map

$$H_n \colon \mathbb{P}^{\binom{n+2}{2}-1} \dashrightarrow \mathbb{P}^{\binom{n+2}{2}-1}$$

on the projective space of plane curves of degree n (for odd n, the map is not defined). As in the case $n = 4$, the map H_n is equivariant with respect to the group $\mathbb{P}GL_3$. When H_n is generically finite, it descends to a generically finite rational self-map $\overline{H}_n$ on the moduli space of plane curves of degree n. We can extend Question 1 to any even positive integer n.

Question 2. Is the rational map H_n *generically finite*? If so, what are the *degrees* of H_n and $\overline{H}_n$?

For $n = 2$, the map H_2 assigns to a general conic its dual conic and is therefore birational. For $n = 4$, the map H_4 is the one defined above, of degree 15. For $n \in \{2, 4\}$, the degrees of H_n and $\overline{H}_n$ coincide: we do not know if they are always equal.

In Section 1, we briefly set up the notation and basic facts about invariants, contravariants and Lie algebras for ternary forms. Here, we define the rational maps H_n on the space of plane curves of degree n to itself. We recommend [Dol12, Chapter 1] as a general introduction to the topic; the whole book contains a wealth of information and details about this beautiful subject and beyond. In Equation (3), we introduce the trilinear form $t_n(-,-,-)$: it provides a fundamental link between an invariant of ternary forms and the maps H_n via the Lie algebra $\mathfrak{sl}_3$. The heart of the Section is devoted to the proof of Theorem 1.1, providing a crucial symmetry of the trilinear form t_n. In Section 2, we determine explicitly the scheme-theoretic fiber of H_4 over the Fermat quartic (see Theorem 2.1). Theorem 2.4 exploits the structure of this fiber: the degree of H_4 is 15 and the monodromy group of H_4 contains the alternating group $\mathfrak{A}_{15}$ (see also Theorem 2.7). Knowing that the monodromy group is 2-transitive, allows us to prove that the degree of the induced quotient map $\overline{H}_4$ is also 15. In Section 3, we define an invariant ρ_n vanishing on the locus where the differential of the map H_n is not an isomorphism. In the case $n = 4$, the polynomial ρ_4 is an invariant of degree 15 of ternary quartic forms: Theorem 3.1 gives the expression of ρ_4 in terms of the Dixmier-Ohno invariants. The proof uses in an essential way the Magma [BCP97] package `g3twists` described in [LRS]. We include in the source of the file [PT], the code that computes and checks our assertions. In the Appendix, we determine the degree of a scheme that plays an important role in our argument. The calculation could alternatively be carried out using a computer.

1. Preliminary identities

In this section, we prove the main identities used in the paper working over the ring $\mathbb{Z}$ of integers. In the later sections, we specialize to fields of characteristic coprime to 6.

Let x, y, z be homogeneous coordinates on the projective plane $\mathbb{P}^2_{\mathbb{Z}}$ and let u, v, w be the dual coordinates on the dual projective plane. Let n be a non-negative integer and denote by $\mathbb{Z}[x, y, z]_n$ and $\mathbb{Z}[u, v, w]_n$ the $\mathbb{Z}$-modules of ternary forms of degree n in respective variables x, y, z and u, v, w. Set $N = \binom{n+2}{2} - 1$; thus the rank of the free $\mathbb{Z}$-module $\mathbb{Z}[x, y, z]_n$ is $N + 1$. We identify $\mathbb{P}(\mathbb{Z}[x, y, z]_n)$ and $\mathbb{P}(\mathbb{Z}[u, v, w]_n)$ with $\mathbb{P}^N_{\mathbb{Z}}$ equivariantly with respect to the action of the group-scheme $GL_{3,\mathbb{Z}}$. The projective space $\mathbb{P}^N_{\mathbb{Z}}$ is the space of plane curves of degree n.

We define the *polar pairing* $\langle -, - \rangle : \mathbb{Z}[u, v, w]_n \times \mathbb{Z}[x, y, z]_n \to \mathbb{Z}$ by

$$\langle q_1, q_2 \rangle = q_1(\partial_x, \partial_y, \partial_z)\, q_2(x, y, z).$$

Equivalently, the polar pairing $\langle -, - \rangle$ is the unique bilinear form taking the values

$$(1) \qquad (u^{a_1} v^{b_1} w^{c_1}\,,\, x^{a_2} y^{b_2} z^{c_2}) \longmapsto \begin{cases} a_1!\, b_1!\, c_1! & \text{if } (a_1, b_1, c_1) = (a_2, b_2, c_2), \\ 0 & \text{otherwise}, \end{cases}$$

on monomials. The polar pairing allows us to associate to each ternary form $q \in \mathbb{Z}[u, v, w]_n$ a linear form $\langle q, - \rangle : \mathbb{Z}[x, y, z]_n \longrightarrow \mathbb{Z}$.

We define a bilinear map

$$J_n \colon \mathbb{Z}[x,y,z]_n \times \mathbb{Z}[x,y,z]_n \longrightarrow \mathbb{Z}[u,v,w]_n$$

as follows:

$$\begin{aligned} &J_n(q_1(x,y,z), q_2(x,y,z)) \\ &= q_1\left(\begin{vmatrix} \partial_y & \partial_z \\ v & w \end{vmatrix}, -\begin{vmatrix} \partial_x & \partial_z \\ u & w \end{vmatrix}, \begin{vmatrix} \partial_x & \partial_y \\ u & v \end{vmatrix}\right) q_2(x,y,z). \end{aligned} \tag{2}$$

For a ternary form $q(x,y,z) \in \mathbb{Z}[x,y,z]_n$ of degree n, we let $H_n(q)(u,v,w)$ be the ternary form

$$H_n(q)(u,v,w) = J_n(q(x,y,z), q(x,y,z))$$

of degree n; we call $H_n(q)(u,v,w)$ the *harmonic form* associated to q. If the integer n is odd, then $H_n(q)$ is always zero. The function $H_n \colon \mathbb{Z}[x,y,z]_n \to \mathbb{Z}[u,v,w]_n$ is a contravariant of ternary forms with respect to the group-scheme $SL_{3,\mathbb{Z}}$ (see [Dol12, Section 3.4.2 and Example 3.4.2]).

Combining the polar pairing with the bilinear map J_n, we define a trilinear form $t_n \colon (\mathbb{Z}[x,y,z]_n)^{\times 3} \to \mathbb{Z}$:

$$t_n(q_1, q_2, q_3) = \langle J_n(q_1, q_2), q_3 \rangle\,. \tag{3}$$

Theorem 1.1. *For every permutation $\sigma \in \mathfrak{S}_3$, the trilinear form t_n satisfies*

$$t_n(q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)}) = (\operatorname{sign}\sigma)^n t_n(q_1, q_2, q_3),$$

where $\operatorname{sign}\sigma$ *denotes the sign of the permutation σ.*

We now give a proof of Theorem 1.1. The argument is entirely combinatorial and relies on a few identities that we prove first. We use an alternative definition of the bilinear map J_n.

Let $m_1, m_2 \in \mathbb{Z}[x,y,z]_n$ be monomials and set

$$m_3 = \frac{(xyz)^n}{m_1 m_2} \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

For $i \in \{1,2,3\}$, write $m_i = x^{a_i} y^{b_i} z^{c_i}$; by construction, the integers a_1, b_1, c_1, a_2, b_2, c_2 are non-negative, while a_3, b_3 and c_3 need not be. Define an integer $[m_1 \,|\, m_2]$ by

$$[m_1 \,|\, m_2] = (-1)^{a_2+b_3+c_1}\, a_1!\, b_1!\, c_1! \sum (-1)^{\alpha} \binom{a_2}{\alpha} \binom{b_2}{c_1-\alpha} \binom{c_2}{b_3-\alpha}, \tag{4}$$

and observe that the summands vanish if α is outside of the range $0 \le \alpha \le \min\{a_2, c_1\}$.

Proposition 1.2 (Properties of $[-\,|\,-]$). *Let $m_1 = x^{a_1}y^{b_1}z^{c_1}, m_2 = x^{a_2}y^{b_2}z^{c_2} \in \mathbb{Z}[x,y,z]_n$ be monomials. Set $m_3 = \frac{(xyz)^n}{m_1m_2} = x^{a_3}y^{b_3}z^{c_3}$, with a_3, b_3, c_3 integers.*

i) $[1 \,|\, 1] = 1$.

ii) *If x divides m_1, then* $[m_1 \,|\, m_2] = b_2 \left[\frac{m_1}{x} \,|\, \frac{m_2}{y}\right] - c_2 \left[\frac{m_1}{x} \,|\, \frac{m_2}{z}\right]$.
If y divides m_1, then $[m_1 \,|\, m_2] = c_2 \left[\frac{m_1}{y} \,|\, \frac{m_2}{z}\right] - a_2 \left[\frac{m_1}{y} \,|\, \frac{m_2}{x}\right]$.
If z divides m_1, then $[m_1 \,|\, m_2] = a_2 \left[\frac{m_1}{z} \,|\, \frac{m_2}{x}\right] - b_2 \left[\frac{m_1}{z} \,|\, \frac{m_2}{y}\right]$.

iii) $[m_2 \,|\, m_1] = (-1)^n [m_1 \,|\, m_2]$.

iv) *If at least one of a_3, b_3, c_3 is strictly negative, then, for every α, the product $\binom{a_2}{\alpha}\binom{b_2}{c_1-\alpha}\binom{c_2}{b_3-\alpha}$ vanishes. In particular, the identity $[m_1 \,|\, m_2] = 0$ holds.*

v) *If the integers a_3, b_3, c_3 are non-negative, then the identity*

$$a_3!b_3!c_3!\,[m_1 \,|\, m_2] = (-1)^n a_1!b_1!c_1!\,[m_3 \,|\, m_2]$$

holds.

Proof. (i) Follows directly from the definition.

(ii) We only argue the case in which x divides m_1; the remaining cases are analogous. The identities

$$b_2\binom{b_2-1}{c_1-\alpha} = (b_2-c_1+\alpha)\binom{b_2}{c_1-\alpha}$$
$$c_2\binom{c_2-1}{b_3-1-\alpha} = (b_3-\alpha)\binom{c_2}{b_3-\alpha}$$

and

$$b_2 - c_1 + \alpha + b_3 - \alpha = a_1$$

hold. Combining these identities with Equation (4) we find

$$\begin{aligned}
& b_2 \left[\frac{m_1}{x} \,|\, \frac{m_2}{y}\right] - c_2 \left[\frac{m_1}{x} \,|\, \frac{m_2}{z}\right] \\
&= (-1)^{a_2+b_3+c_1} (a_1-1)!\, b_1!\, c_1! \sum (-1)^\alpha \binom{a_2}{\alpha} b_2 \binom{b_2-1}{c_1-\alpha} \binom{c_2}{b_3-\alpha} \\
&\quad - (-1)^{a_2+b_3-1+c_1} (a_1-1)!\, b_1!\, c_1! \sum (-1)^\alpha \binom{a_2}{\alpha} \binom{b_2}{c_1-\alpha} c_2 \binom{c_2-1}{b_3-1-\alpha} \\
&= (-1)^{a_2+b_3+c_1} a_1!\, b_1!\, c_1! \sum (-1)^\alpha \binom{a_2}{\alpha} \binom{b_2}{c_1-\alpha} \binom{c_2}{b_3-\alpha} \\
&= [m_1 \,|\, m_2]
\end{aligned}$$

as needed.

(iii) Let α be any integer and set $\alpha' = b_3 - \alpha$. Expanding the binomials in the product $a_1!\, b_1!\, c_1! \binom{a_2}{\alpha}\binom{b_2}{c_1-\alpha}\binom{c_2}{b_3-\alpha}$ we find

$$\frac{a_1!\, b_1!\, c_1!\, a_2!\, b_2!\, c_2!}{\alpha!(a_2-\alpha)!(c_1-\alpha)!(b_2-c_1+\alpha)!(b_3-\alpha)!(c_2-b_3+\alpha)!}.$$

We use the equalities

$$a_2 - b_3 + \alpha' = b_1 - c_2 + \alpha' \qquad b_2 - c_1 + b_3 - \alpha' = a_1 - \alpha'$$

to find

$$\begin{aligned}
a_1!\, b_1!\, c_1! \binom{a_2}{\alpha}\binom{b_2}{c_1-\alpha}\binom{c_2}{b_3-\alpha} \\
= a_2!\, b_2!\, c_2! \binom{a_1}{\alpha'}\binom{b_1}{c_2-\alpha'}\binom{c_1}{b_3-\alpha'}.
\end{aligned} \tag{5}$$

Finally, from the identity

$$a_2 + b_3 + c_1 = n - a_1 + 2b_3 - c_2$$

we deduce

$$(-1)^{a_2+b_3+c_1+\alpha} = (-1)^n (-1)^{a_1+b_3+c_2-\alpha'}. \tag{6}$$

Combining Equations (5) and (6) and summing over all α, we obtain the required identity

$$[m_1 \,|\, m_2] = (-1)^n \,[m_2 \,|\, m_1]\,.$$

(iv) If α is not in the interval $[0, \min\{a_2, c_1\}]$, then the product $\binom{a_2}{\alpha}\binom{b_2}{c_1-\alpha}$ vanishes. Suppose therefore that α satisfies the inequalities $0 \leq \alpha \leq \min\{a_2, c_1\}$.

- If $a_3 < 0$, then, using $c_2 - b_3 = a_3 - c_1$, we obtain that $\binom{c_2}{b_3-\alpha} = \binom{c_2}{c_2-b_3+\alpha} = \binom{c_2}{a_3-c_1+\alpha}$ vanishes.
- If $b_3 < 0$, then $\binom{c_2}{b_3-\alpha}$ vanishes.
- If $c_3 < 0$, then, using $b_2 - c_1 = c_3 - a_2$, we obtain that $\binom{b_2}{c_1-\alpha} = \binom{b_2}{b_2-c_1+\alpha} = \binom{b_2}{c_3+\alpha-a_2}$ vanishes.

The vanishing of $[m_1 \,|\, m_2]$ follows, since we just proved that every summand in Equation (4) is zero.

(v) Using the identities $c_2 - b_3 = b_1 - a_2$ and $b_2 - c_1 = c_3 - a_2$, we find, for any integer α, the equalities

$$\binom{c_2}{b_3-\alpha} = \binom{c_2}{c_2-b_3+\alpha} = \binom{c_2}{b_1-(a_2-\alpha)}$$

and

$$\binom{b_2}{c_1-\alpha} = \binom{b_2}{b_2-c_1+\alpha} = \binom{b_2}{c_3-(a_2-\alpha)}.$$

Moreover, also the equality $b_3 + c_1 - (b_1 + c_3) = n + a_2 - 2(b_1 + c_3)$ holds. Set $\alpha' = a_2 - \alpha$; we deduce the equality

$$\begin{aligned}&(-1)^{a_2+b_3+c_1+\alpha}\binom{a_2}{\alpha}\binom{b_2}{c_1-\alpha}\binom{c_2}{b_3-\alpha}\\ &= (-1)^n(-1)^{a_2+b_1+c_3+\alpha'}\binom{a_2}{\alpha'}\binom{b_2}{c_3-\alpha'}\binom{c_2}{b_1-\alpha'}\end{aligned}$$

and we conclude summing over all α. □

Let m_1, m_2 be monomials in $\mathbb{Z}[x, y, z]_n$. The ratio $\frac{(xyz)^n}{m_1 m_2}$ is a Laurent monomial in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ and we let $a_3, b_3, c_3 \in \mathbb{Z}$ be its exponents: $\frac{(xyz)^n}{m_1 m_2} = x^{a_3} y^{b_3} z^{c_3}$.

We define a bilinear map $J'_n \colon \mathbb{Z}[x, y, z]_n \times \mathbb{Z}[x, y, z]_n \to \mathbb{Z}[u, v, w]_n$ by setting

$$J'_n(m_1, m_2) = [m_1 \,|\, m_2]\, u^{a_3} v^{b_3} w^{c_3}$$

on monomials and extending by bilinearity.

Corollary 1.3. *For all non-negative integers n, the bilinear maps J_n and J'_n coincide.*

Proof. We proceed by induction on n. The case $n = 0$ follows from the definitions. Suppose that $n > 0$ is an integer and that the maps J_{n-1} and J'_{n-1} coincide. Let m_1, m_2 be monomials in $\mathbb{Z}[x, y, z]_n$. To prove the result, it suffices to show that the identity $J_n(m_1, m_2) = J'_n(m_1, m_2)$ holds. Let m_3 denote the Laurent monomial $\frac{(xyz)^n}{m_1 m_2}$. For $i \in \{1, 2, 3\}$, write $m_i = x^{a_i} y^{b_i} z^{c_i}$. If x divides m_1, that is, if a_1 is strictly positive, then Equation (2) implies the identity

$$J_n(m_1, m_2) = b_2 J_{n-1}\left(\frac{m_1}{x}, \frac{m_2}{y}\right) w - c_2 J_{n-1}\left(\frac{m_1}{x}, \frac{m_2}{z}\right) v,$$

with the convention that if b_2 or c_2 vanish, then the corresponding term vanishes as well. Using the inductive hypothesis, we obtain

$$J_n(m_1, m_2) = b_2 \left[\frac{m_1}{x} \mid \frac{m_2}{y}\right] u^{a_3} v^{b_3} w^{c_3 - 1} w - c_2 \left[\frac{m_1}{x} \mid \frac{m_2}{z}\right] u^{a_3} v^{b_3 - 1} w^{c_3} v$$

and this last expression equals $J'_n(\frac{m_1}{x}, \frac{m_2}{z})$ by Proposition 1.2 (ii). Arguing similarly if y or z divides m_1, we conclude the proof of the induction step. The result follows by induction. $\square$

Proof of Theorem 1.1. Let $q_1, q_2, q_3 \in \mathbb{Z}[x, y, z]_n$ be three forms. By Corollary 1.3 and Proposition 1.2 (iii), the identity

$$J_n(q_2, q_1) = (-1)^n J_n(q_1, q_2)$$

holds. In particular, to prove the result, it suffices to show the identity

$$t_n(q_1, q_2, q_3) = t_n(q_2, q_3, q_1).$$

Using the linearity of t_n in its three arguments, it suffices to prove the result in the case in which q_1, q_2, q_3 are monomials. For $i \in \{1, 2, 3\}$, write $q_i = x^{a_i} y^{b_i} z^{c_i}$. Using the definition of the polar pairing, we deduce that $t_n(q_1, q_2, q_3)$ vanishes if the monomial $q_1 q_2 q_3$ is not equal to $(xyz)^n$. Thus, suppose that $q_1 q_2 q_3$ equals $(xyz)^n$. Corollary 1.3 allows us to deduce the equality $J_n(q_1, q_2) = [q_1 \mid q_2]\, u^{a_3} v^{b_3} w^{c_3}$. We compute

$$\begin{array}{ll}
\text{Equation (3)} & t_n(q_1, q_2, q_3) = [q_1 \mid q_2] \left\langle u^{a_3} v^{b_3} w^{c_3}, x^{a_3} y^{b_3} z^{c_3} \right\rangle \\
\text{Equation (1)} & \qquad = a_3! b_3! c_3!\, [q_1 \mid q_2] \\
\text{Proposition 1.2 (v)} & \qquad = (-1)^n a_1! b_1! c_1!\, [q_3 \mid q_2] \\
\text{Corollary 1.3} & \qquad = \langle (-1)^n J_n(q_3, q_2), q_1 \rangle \\
\text{Proposition 1.2 (iii)} & \qquad = \langle J_n(q_2, q_3), q_1 \rangle \\
\text{Equation (3)} & \qquad = t_n(q_2, q_3, q_1)
\end{array}$$

and we are done. □

We make use of the relationship between the contravariant H_n and an invariant A_n under the action of $SL_{3,\mathbb{Z}}$. The expression

$$A_n(q) = t_n(q, q, q) = \langle H_n(q), q \rangle \in \mathbb{Z}$$

is an invariant of ternary forms q under $SL_{3,\mathbb{Z}}$; the degree of A_n in the coefficients of the form q is 3. If n is odd, then the harmonic form H_n vanishes identically; therefore, the same is true of the invariant A_n. Denote by $\mathfrak{gl}_3$ and by $\mathfrak{sl}_3$ the Lie algebras of $GL_{3,\mathbb{Z}}$ and $SL_{3,\mathbb{Z}}$ respectively.

Theorem 1.4. *For every derivation* $\mathfrak{g}$ *in* $\mathfrak{sl}_3$*, the identity*

$$\langle H_n(q), \mathfrak{g}q \rangle = 0, \tag{7}$$

holds.

Proof. If $\mathfrak{g}$ is a derivation in $\mathfrak{gl}_3$ and q_1, q_2, q_3 are forms in $\mathbb{Z}[x, y, z]_n$, then the equality

$$\mathfrak{g}t_n(q_1, q_2, q_3) = \langle J_n(\mathfrak{g}q_1, q_2), q_3 \rangle + \langle J_n(q_1, \mathfrak{g}q_2), q_3 \rangle + \langle J_n(q_1, q_2), \mathfrak{g}q_3 \rangle$$

holds. Using Theorem 1.1, we obtain the identity $\mathfrak{g}A_n(q) = 3\,\langle J_n(q, q), \mathfrak{g}q \rangle$. Suppose now that $\mathfrak{g}$ is in $\mathfrak{sl}_3$. Since A_n is invariant under $SL_{3,\mathbb{Z}}$, we deduce that $\mathfrak{g}A_n(q) = 0$. Combining these formulas, we obtain the required identity. □

2. The computation of the degree

We now restrict our attention to the case $n = 4$. To perform the main computations, we work over a general field k of characteristic zero. An easy argument appearing in Remark 3.2 shows that this restriction on the characteristic can be weakened.

The invariant $A_4(q)$ has degree 3 in the coefficients of q: it is, up to scaling, the unique non-constant invariant of smallest degree of plane quartics. Salmon denotes the contravariant H_4 by σ [Sal73, p. 264, §292] and the invariant $A_4(q)$ by A [Sal73, p. 264, §293]; Dolgachev denotes the contravariant H_4 by $\Omega_{2,4}$ and the invariant $A_4(q)$ by I_3.

Let $\mathrm{Fer} \subset \mathbb{P}^2_k$ denote the Fermat quartic with equation $\mathrm{Fer}\colon x^4 + y^4 + z^4 = 0$; similarly, let $\mathrm{Fer}' \subset \mathbb{P}^{2\vee}_k$ denote the Fermat quartic with equation $\mathrm{Fer}'\colon u^4 + v^4 + w^4 = 0$. We also define the four quartics $C_0, C_1, C_2, C_3 \subset \mathbb{P}^2_k$ with equations

$$\begin{aligned}
C_0&\colon (x^4 + y^4 + z^4) - 6(x^2y^2 + x^2z^2 + y^2z^2) = 0\\
C_1&\colon (x^4 + y^4 + z^4) - 6(x^2y^2 - x^2z^2 - y^2z^2) = 0\\
C_2&\colon (x^4 + y^4 + z^4) - 6(-x^2y^2 + x^2z^2 - y^2z^2) = 0\\
C_3&\colon (x^4 + y^4 + z^4) - 6(-x^2y^2 - x^2z^2 + y^2z^2) = 0.
\end{aligned}$$

The curves C_0, C_1, C_2, C_3 are all isomorphic: permutations of the coordinates induce projective equivalences among C_1, C_2, C_3; rescaling z by a square root of -1 transforms C_0 into C_1. An easy check shows that they are smooth.

Let $\widetilde{\mathbb{P}}^{14}_k \subset \mathbb{P}^{14}_k \times \mathbb{P}^{14\vee}_k$ be the closure of the graph of the rational map H_4. The second projection $\mathbb{P}^{14}_k \times \mathbb{P}^{14\vee}_k \to \mathbb{P}^{14\vee}_k$ restricts to a morphism

$$H\colon \widetilde{\mathbb{P}}^{14}_k \longrightarrow \mathbb{P}^{14\vee}_k.$$

A plane quartic C in the indeterminacy locus of H_4 must be singular: see [PT19, Proposition 2.5] for a more precise statement. Let (C, D) be a pair in $\widetilde{\mathbb{P}}^{14}_k$. If the rational map H_4 is defined at C, then the projection $\mathbb{P}^{14}_k \times \mathbb{P}^{14\vee}_k \to \mathbb{P}^{14}_k$ restricts to an isomorphism on an open subset of $\widetilde{\mathbb{P}}^{14}_k$ containing (C, D). When this happens, to simplify the notation, we identify the pair $(C, D) \in \widetilde{\mathbb{P}}^{14}_k$ with C, since D can be obtained as $H_4(C)$.

Theorem 2.1. *The fiber of the morphism H above the Fermat quartic* $\mathrm{Fer}' \subset \mathbb{P}^{2\vee}_k$ *consists of the five quartics* Fer, C_0, C_1, C_2, $C_3 \subset \mathbb{P}^2_k$, *where the Fermat quartic* Fer *appears with multiplicity* 11 *and each one of the remaining four quartics appears with multiplicity* 1.

Proof. Let C be a plane quartic and let $q(x,y,z) = \sum a_{ijk} x^i y^j z^k \in k[x,y,z]$ be an equation for C. If the pair (C, Fer') is contained in $\widetilde{\mathbb{P}}^{14}_k$, then Theorem 1.4 implies that, for every element $\mathfrak{g}$ of $\mathfrak{sl}_3$, the identity

$$\langle u^4 + v^4 + z^4, \mathfrak{g}q \rangle = 0$$

holds. Specializing this identity with $\mathfrak{g}$ in the list

1. $x\partial_x - y\partial_y$	3. $y\partial_x$	5. $x\partial_y$	7. $x\partial_z$
2. $y\partial_y - z\partial_z$	4. $z\partial_x$	6. $z\partial_y$	8. $y\partial_z$,

we obtain the identities

1. $48(a_{400} - a_{040}) = 0$	3. $12a_{130} = 0$	5. $12a_{013} = 0$	7. $12a_{301} = 0$
2. $48(a_{040} - a_{004}) = 0$	4. $12a_{103} = 0$	6. $12a_{310} = 0$	8. $12a_{031} = 0$.

We deduce that q is of the form

$$\begin{aligned} q(x,y,z) = {} & \rho(x^4+y^4+z^4) + \sigma_3 x^2y^2 + \sigma_2 x^2z^2 + \sigma_1 y^2z^2 \\ & + xyz(\tau_1 x + \tau_2 y + \tau_3 z), \end{aligned}$$

where $\rho = a_{400} = a_{040} = a_{004}$, $\sigma_1 = a_{022}$, $\sigma_2 = a_{202}$, $\sigma_3 = a_{220}$ and $\tau_1 = a_{211}$, $\tau_2 = a_{121}$, $\tau_3 = a_{112}$. The pair (C, Fer') is contained in $\widetilde{\mathbb{P}}^{14}_k$ if $H_4(q)$ is an equation for Fer'. Using the expression that we obtained for q we find

$$\begin{aligned} H_4(q) = {} & (12\rho^2+\sigma_1^2)u^4 + (12\rho^2+\sigma_2^2)v^4 + (12\rho^2+\sigma_3^2)w^4 \\ & - 2(\sigma_3\tau_1 vw^3 + \sigma_3\tau_2 uw^3 + \sigma_2\tau_3 uv^3 + \sigma_2\tau_1 v^3w + \sigma_1\tau_2 u^3w + \sigma_1\tau_3 u^3v) \\ & + (12\rho\sigma_1 + 2\sigma_2\sigma_3 + \tau_1^2)v^2w^2 + (12\rho\sigma_2 + 2\sigma_1\sigma_3 + \tau_2^2)u^2w^2 \\ & + (12\rho\sigma_3 + 2\sigma_1\sigma_2 + \tau_3^2)u^2v^2 \\ & + (4\sigma_1\tau_1 - \tau_2\tau_3)u^2vw + (4\sigma_2\tau_2 - \tau_1\tau_3)uv^2w + (4\sigma_3\tau_3 - \tau_1\tau_2)uvw^2. \end{aligned}$$

The condition that $u^4+v^4+w^4$ and $H_4(q)$ be proportional determines a subscheme F_0 of $\mathbb{P}^{14}$. The scheme F_0 is isomorphic to the scheme F defined in (A.2) and the result follows from Lemma A.1. □

Remark 2.2. The Fermat curve Fer is not isomorphic to any one of the curves C_0, C_1, C_2, C_3. This is an immediate consequence of Theorem 2.1: the map H_4 is contravariant and hence projectively equivalent curves appear with the same multiplicity in fibers of H_4.

We want to compute the monodromy of the morphism H. For this, we use the following result, due to Jordan (see [Isa08, Theorem 8.23]). For a positive integer n, denote by $\mathfrak{S}_n$ the symmetric group on n elements and by $\mathfrak{A}_n$ the alternating group.

Theorem 2.3 (Jordan). *Let n be a positive integer and let G be a primitive subgroup of $\mathfrak{S}_n$. If p is a prime satisfying $p < n-2$ and G contains a p-cycle, then G contains $\mathfrak{A}_n$.*

Theorem 2.4. *The morphism $H\colon \widetilde{\mathbb{P}}^{14}_k \to \mathbb{P}^{14\vee}_k$ is generically finite of degree 15. The monodromy group of H contains the alternating group $\mathfrak{A}_{15}$.*

Proof. Let F be the fiber of H over the Fermat quartic curve $\mathrm{Fer}' \subset \mathbb{P}^{2\vee}_k$. By Theorem 2.1, the scheme F is finite of degree 15. Thus, the morphism H is quasi-finite in a neighbourhood of Fer' and therefore finite, since it is projective. To conclude that the degree of H is 15, it suffices to argue that H is flat at Fer'. By the Miracle Flatness Theorem [Mat89, Theorem 23.1], it is enough to check that $\widetilde{\mathbb{P}}^{14}_k$ is smooth at F. This is true, since the rational map H_4 is defined at the points of F and its domain, $\mathbb{P}^{14}_k$, is smooth (recall that the graph morphism is an immersion, see [GW10, Proposition 9.5]).

By what we just proved, the monodromy group of H is isomorphic to a subgroup G of the symmetric group $\mathfrak{S}_{15}$. Since $\widetilde{\mathbb{P}}^{14}_k$ is irreducible, the group G is transitive. Since the fiber F contains four reduced points and one point of multiplicity 11, we deduce that G contains a subgroup with an orbit consisting of 11 elements. Hence, G also contains a cycle of length 11 and is therefore primitive (see [Isa08, Lemma 8.20]). Theorem 2.3 shows that G contains the alternating group $\mathfrak{A}_{15}$ and we are done. □

Remark 2.5. Denote by $\mathbb{P}^N_k /\!\!/ \mathbb{P}GL_3$ the GIT-quotient of $\mathbb{P}^N_k$ by $\mathbb{P}GL_3$. We check that, for even n, the contravariant H_n induces a rational map

$$H_n\colon \mathbb{P}^N_k \dashrightarrow \mathbb{P}^N_k,$$

descending to a rational map on the quotient

$$\overline{H}_n\colon \mathbb{P}^N_k /\!\!/ \mathbb{P}GL_3 \dashrightarrow \mathbb{P}^N_k /\!\!/ \mathbb{P}GL_3.$$

Indeed, it suffices to find a ternary form q of even degree n defining a plane curve, such that $H_n(q)$ a GIT-stable curve of degree n. For this, we compute

$$H_n(x^n + y^n + z^n) = (1 + (-1)^n) \cdot n!(u^n + v^n + w^n) = 2 \cdot n!(u^n + v^n + w^n),$$

and we are done, since smooth curves are GIT-stable.

In the case $n = 4$, the quotient $\mathbb{P}_k^{14} /\!\!/ \mathbb{P}GL_3$ is birational to the moduli space $\mathcal{M}_3$ of smooth curves of genus 3 and we obtain

$$\overline{H}_4 \colon \mathcal{M}_3 \dashrightarrow \mathcal{M}_3.$$

Corollary 2.6. *The rational map $\overline{H}_4$ is generically finite of degree* 15.

Proof. Denote by $\mathbb{P}^{(2)}$ the locally closed subset of $\widetilde{\mathbb{P}}_k^{14} \times \widetilde{\mathbb{P}}_k^{14}$ consisting of pairs (C, D), with C, D distinct smooth plane quartics with $H(C) = H(D)$. By Theorem 2.4, the monodromy group of the morphism H is 2-transitive on fibers and the scheme $\mathbb{P}^{(2)}$ is irreducible. The pair (Fer, C_0) is in $\mathbb{P}^{(2)}$ and consists of two smooth non-projectively equivalent plane quartics. By the irreducibility of $\mathbb{P}^{(2)}$, we deduce that the fiber of H over a general point of $\mathbb{P}_k^{14\vee}$ consists of 15 pairwise non-projectively equivalent smooth plane quartics. In particular, the rational map $\overline{H}_4$ is generically finite of the same degree 15 as H, as stated. □

So far, we proved all the results without using a computer. The next results, though, involve more lengthier calculations that we find too tedious to check by hand.

Theorem 2.7. *The monodromy group of H is the symmetric group $\mathfrak{S}_{15}$.*

Proof. By Theorem 2.4, it suffices to show that the monodromy group of H contains a transposition. For this, we exhibit a plane quartic $D \subset \mathbb{P}_k^{2\vee}$ such that $H^{-1}(D)$ is contained in the smooth locus of $\widetilde{\mathbb{P}}_k^{14}$ and consists of 13 reduced points and a single non-reduced of multiplicity 2 (see [Har79, Lemma on p. 698]). Thus, it is sufficient to find a plane quartic D for which the fiber $H^{-1}(D)$ consists of 14 distinct pairs (C, D) with $C \subset \mathbb{P}_k^2$ a smooth plane quartic. Using the computer algebra program Magma [BCP97], we check that the curve D with equation

$$D\colon \qquad u^3(v + w) + v^3(u + w) + w^3(u + v) = 0 \tag{8}$$

has the required properties and we are done. □

Remark 2.8. In the proof of Theorem 2.7, we saw that above the curve D of (8) the morphism H has a unique simple ramification point. This point corresponds to the smooth plane quartic $Q \subset \mathbb{P}^2_k$ with equation

$$Q\colon \begin{cases} (x^4+y^4+z^4) - 4(x^3(y+z)+y^3(x+z)+z^3(x+y)) \\ \qquad +6(x^2y^2+x^2z^2+y^2z^2) - 12xyz(x+y+z) = 0. \end{cases}$$

3. A geometric invariant for plane curves

Let n be an even, positive integer; recall that we set $N+1 = \binom{n+2}{2}$. We define an invariant ρ_n of degree $N+1$ associated to plane curves of degree n. In the case of plane quartics, we obtain an expression for ρ_4 in terms of the Dixmier-Ohno invariants. For background on invariants of plane quartics, we refer to [Dix87, Ohn]. We used the package developed in [LRS] for computations with Dixmier-Ohno invariants.

Let $R = \mathbb{Z}[a_{ijk}]$ denote the polynomial ring over the integers with $N+1$ indeterminates, corresponding to the coefficients of the monomials of degree n in x, y, z. Let $q \in R[x,y,z]_n$ be the universal ternary form $q = \sum a_{ijk}x^iy^jz^k$ of degree n. We define an $(N+1)\times(N+1)$ symmetric matrix R_n with rows and columns indexed by the $N+1$ monomials of degree n in x, y, z. The entry of R_n corresponding to the pair of monomials (m_1, m_2) is

$$(R_n)_{m_1,m_2} = t_n(q, m_1, m_2).$$

We give two different interpretations for the matrix R_n. First, the matrix R_n determines a $\mathbb{Z}$-module homomorphism $\mathbb{Z}[x,y,z]_n \to \mathbb{Z}[x,y,z]_n^\vee$ given by $q_1 \mapsto t_n(q, q_1, -)$. Alternatively, the differential of the map H_n at the form q is the linear transformation $\mathbb{Z}[x,y,z]_n \to \mathbb{Z}[u,v,w]_n$ given by $q_1 \mapsto J_n(q, q_1)$. Identifying $\mathbb{Z}[u,v,w]_n$ with $\mathbb{Z}[x,y,z]_n^\vee$ via the polar pairing, we obtain that the linear transformation R_n is the differential of the map H_n at q:

$$\mathrm{d}_q H_n = R_n. \tag{9}$$

The determinant of the matrix R_n is a polynomial of degree $N+1$ in the $N+1$ variables of R. From either of the two descriptions above, it is clear that $\det R_n$ is an invariant for the action of SL_3. We let

$$\kappa_n = \prod_{\substack{i,j,k\geq 0\\ i+j+k=n}} i!j!k!$$

be the product of the factorials of all the exponents of all the monomials of degree n in x, y, z; the first few values of κ_n for even n are

$$\kappa_2 = 2^3, \quad \kappa_4 = 2^{24} \cdot 3^9, \quad \kappa_6 = 2^{84} \cdot 3^{33} \cdot 5^9, \quad \kappa_8 = 2^{201} \cdot 3^{81} \cdot 5^{30} \cdot 7^9.$$

We define

$$\rho_n = \frac{1}{\kappa_n} \det R_n. \tag{10}$$

We deduce that the differential of map H_n is not an isomorphism at the vanishing set of ρ_n and therefore ρ_n is an SL_3-invariant.

In the case $n = 4$ of ternary quartic forms, the ring of invariants under SL_3 is completely explicit. It is generated by 13 invariants, called the *Dixmier-Ohno invariants*:

- (Dixmier) I_{3d}, for $d \in \{1, \dots, 7\}$,
- (Ohno) J_{3d}, for $d \in \{3, \dots, 7\}$, and
- the discriminant I_{27}.

The indices represent the degree of each invariant as a polynomial in the coefficients of the quartic form. We follow the notation of [LRS].

Theorem 3.1. *The invariant ρ_4 of degree* 15 *satisfies the identity*

$$\frac{2 \cdot 5^4 \cdot 7}{24^{15}} \rho_4 = \left\{ \begin{array}{rrcrr} 2 \cdot 3^3 \cdot 5 \cdot 7^2 & J_{15} & - & 2 \cdot 3^3 \cdot 5 \cdot 7 & I_{15} \\ -3^2 \cdot 5 \cdot 109 & I_3 J_{12} & + & 2^3 \cdot 3^5 \cdot 5 & I_3 I_{12} \\ +2 \cdot 3^2 \cdot 137 & I_3^2 J_9 & + & 3 \cdot 271 & I_3^2 I_9 \\ +2^3 \cdot 3^3 \cdot 5 \cdot 7^2 & I_6 J_9 & - & 2^4 \cdot 3^3 \cdot 5 \cdot 7^2 & I_6 I_9 \\ -2^3 \cdot 5 \cdot 7 \cdot 149 & I_3^3 I_6 & + & 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 & I_3 I_6^2. \end{array} \right.$$

Proof. The argument is a direct computer calculation. There are 11 monomials of degree 15 in the Dixmier-Ohno invariants and the invariant ρ_4 is a linear combination of these 11 monomials. By choosing 11 sufficiently general ternary quartic forms, we check that the identity in the statement of the theorem is the unique solution. Note that the monomial I_3^5 is the unique monomial of degree 15 in the Dixmier-Ohno invariants not appearing the expression of ρ_4. □

Remark 3.2. So far, the characteristic of the ground field k was zero. Nevertheless, the map H_4 is defined over $\operatorname{Spec} \mathbb{Z}$ and hence over a field of arbitrary characteristic. We now assume that the characteristic of the ground field is coprime with 6 and we check that the map H_4 is generically finite of degree 15. First, we evaluate the invariant ρ_4 on the quartic form $q = x^3y + y^3z + z^3x$, vanishing on the Klein plane quartic. We obtain $\rho_4(q) = 2^{34} \cdot 3^{24}$, which does not vanish in k. Therefore, the map H_4 is generically smooth over k and hence generically finite. We conclude, by generic flatness, that the degree of H_4 is also 15.

Appendix A. The scheme F and its degree

The proof of Lemma A.1 appearing in this Appendix is entirely independent of the results of the rest of the paper. We compute without using the computer the degree of a zero-dimensional scheme F, isomorphic to a scheme that appears in the proof of Theorem 2.1. The proof could just as well be carried out over the field of rational numbers by a computer algebra system, such as Magma.

Let $\mathbb{P}^6_k$ be the projective space over the field k with homogeneous coordinates $\rho, \sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3$. Let $\mathscr{G}_0$ be the set of 14 forms

$$\mathscr{G}_0 = \left\{ \begin{array}{ccc} A_1 = \sigma_3^2 - \sigma_2^2 & & A_2 = \sigma_3^2 - \sigma_1^2 \\ \sigma_1\tau_2 & \sigma_2\tau_1 & \sigma_3\tau_1 \\ \sigma_1\tau_3 & \sigma_2\tau_3 & \sigma_3\tau_2 \\ S_1 = 12\rho\sigma_1 + 2\sigma_2\sigma_3 + \tau_1^2 & & T_1 = 4\sigma_1\tau_1 - \tau_2\tau_3 \\ S_2 = 12\rho\sigma_2 + 2\sigma_1\sigma_3 + \tau_2^2 & & T_2 = 4\sigma_2\tau_2 - \tau_1\tau_3 \\ S_3 = 12\rho\sigma_3 + 2\sigma_1\sigma_2 + \tau_3^2 & & T_3 = 4\sigma_3\tau_3 - \tau_1\tau_2 \end{array} \right\}. \tag{A.1}$$

We introduce the subscheme F of $\mathbb{P}^6_k$

$$F \colon V(\mathscr{G}_0) \subset \mathbb{P}^6_k \tag{A.2}$$

defined by the vanishing set of $\mathscr{G}_0$ in $\mathbb{P}^6_k$.

Lemma A.1. *The scheme F has dimension* 0 *and degree* 15*. The support of F consists of* 5 *points: the point* $[1,0,0,0,0,0,0]$ *of multiplicity* 11 *and the* 4 *points* $[1,-6,-6,-6,0,0,0]$, $[1,-6,6,6,0,0,0]$, $[1,6,-6,6,0,0,0]$, $[1,6,6,-6,0,0,0]$ *of multiplicity* 1*.*

Proof. Let $I = \langle \mathscr{G}_0 \rangle$ be the ideal generated by $\mathscr{G}_0$. As a first step, we determine a Gröbner basis for the ideal I. Let $\mathscr{G}_1$ be the set of 4 forms

$$\text{(A.3)} \qquad \mathscr{G}_1 = \left\{ \begin{array}{c} \frac{1}{2}(\sigma_2 S_1 + 2\sigma_3 A_2 - \tau_1 \cdot \sigma_2\tau_1) \\ \frac{1}{2}(\sigma_3 S_1 - \tau_1 \cdot \sigma_3\tau_1) \\ \frac{1}{2}(\sigma_3 S_2 - \tau_2 \cdot \sigma_3\tau_2) \\ \frac{1}{4}(\sigma_3 T_3 + \tau_2 \cdot \sigma_3\tau_1) \end{array} \right\} = \left\{ \begin{array}{c} 6\rho\sigma_2\sigma_1 + \sigma_3^3 \\ 6\rho\sigma_3\sigma_1 + \sigma_3^2\sigma_2 \\ 6\rho\sigma_3\sigma_2 + \sigma_3^2\sigma_1 \\ \sigma_3^2\tau_3 \end{array} \right\}.$$

By construction, the forms in $\mathscr{G}_1$ are contained in the ideal I. Let $\mathscr{G}$ be the set of 18 forms $\mathscr{G} = \mathscr{G}_0 \cup \mathscr{G}_1 \subset I$.

Assign the following weights to the variables:

Variable:	ρ	σ_3	σ_2	σ_1	τ_1	τ_2	τ_3
Weight:	1	3	4	4	5	5	5

and resolve ties among monomials using the lexicographic ordering with

$$\rho < \sigma_3 < \sigma_2 < \sigma_1 < \tau_1 < \tau_2 < \tau_3.$$

Using Buchberger's Criterion, it is straightforward to check that $\mathscr{G}$ is a Gröbner basis of I with respect to the monomial order just defined. We omit this routine computation.

Let I_0 be the initial ideal of I. Since $\mathscr{G}$ is a Gröbner basis of I, the monomial ideal I_0 is the ideal generated by the 18 initial monomials of the elements of $\mathscr{G}$:

$$I_0 = \left\langle \begin{array}{cccccc} \sigma_3^3 & \sigma_3^2\tau_2 & \sigma_3^2\tau_1 & \sigma_3^2\tau_3 & \sigma_2^2 & \sigma_1^2 \\ \sigma_3\tau_1 & \sigma_2\tau_3 & \sigma_1\tau_2 & \sigma_3\tau_2 & \sigma_2\tau_1 & \sigma_1\tau_3 \\ \tau_1^2 & \tau_2^2 & \tau_3^2 & \tau_1\tau_2 & \tau_1\tau_3 & \tau_2\tau_3 \end{array} \right\rangle.$$

The 15 monomials

$$\begin{array}{cccccccc} 1 & \sigma_1 & \sigma_2 & \sigma_3 & \tau_1 & \tau_2 & \tau_3 & \\ \sigma_3^2 & \sigma_2\sigma_3 & \sigma_1\sigma_3 & \sigma_1\sigma_2 & \sigma_3\tau_3 & \sigma_2\tau_2 & \sigma_1\tau_1 & \sigma_1\sigma_2\sigma_3 \end{array}$$

are all the monomials not divisible by ρ and not contained in the ideal I_0. Thus, the Hilbert polynomial of I_0 is the constant polynomial 15, hence the same is true for the ideal I. We conclude that the scheme F has dimension 0 and degree 15, as stated.

A direct calculation of the Jacobian of the given equations shows that the points satisfying the inequality $\sigma_1\sigma_2\sigma_3 \neq 0$ are reduced points of the scheme F.

Let F_{red} be the reduced subscheme associated to F. Note that τ_1, τ_2, τ_3 vanish on F_{red}. We obtain that F_{red} consists of the 5 points $[1,-6,6,6,0,0,0]$, $[1,6,-6,6,0,0,0]$, $[1,6,6,-6,0,0,0]$, $[1,-6,-6,-6,0,0,0]$, $[1,0,0,0,0,0,0]$. Since the points different from $[1,0,0,0,0,0,0]$ are reduced and the total degree is 15, the result follows. $\square$

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I.M., Universidade Federal Fluminense
Niterói, Rio de Janeiro, Brazil
E-mail address: `pacini.uff@gmail.com, pacini@impa.br`

Mathematics Institute, University of Warwick
Coventry, CV4 7AL, UK
E-mail address: `adomani@gmail.com`

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