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Surfaces in the 4-disk with the same boundary and fundamental group

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We construct a family of pairs of non-isotopic symplectic surfaces in the standard symplectic 4-disk (D^4, ω_{st}) such that they are bounded by the same transverse knot in the standard contact 3-sphere and the fundamental groups of their complements are isomorphic.

1. Introduction

This paper is concerned with symplectic surfaces in the standard symplectic 4-disk (D^4, ω_{st}) bounded by the same transverse link in the standard contact 3-sphere (S^3, ξ_{st}). Such surfaces have been studied in some papers [3, 5, 6, 8, 9]. Up to the present, the provided families of distinct symplectic surfaces bounded by the same transverse knot (or link) can be distinguished by the fundamental groups of their complements. Hence it is natural to ask whether there is a pair of non-isotopic symplectic surfaces in D^4 bounded by the same transverse knot such that complements of two surfaces have isomorphic fundamental groups.

The main result of this paper is the following:

Theorem 1.1. *There is a family $\{(S_1(n), S_2(n))\}_{n \in \mathbb{Z}_{\geq 0}}$ of pairs of symplectic surfaces in the standard symplectic 4-disk (D^4, ω_{st}) with contact boundary such that:*

1) *For a fixed $n \in \mathbb{Z}_{\geq 0}$,*
   a) *their boundaries $\partial S_j(n)$ $(j = 1, 2)$ are the same transverse knot up to isotopy in the boundary (S^3, ξ_{st}),*
   b) *two fundamental groups $\pi_1(D^4 \setminus S_j(n))$ are isomorphic, and*
   c) *double branched covers $X_j(n)$ of D^4 branched along $S_j(n)$ are not homeomorphic, and therefore, two surfaces $S_j(n)$ are not isotopic;*

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2) *The boundaries* $\partial S_j(n)$ *and* $\partial S_j(n')$ *are not smoothly isotopic in* ∂D^4 *if* $n \neq n'$*.*

Rudolph exhibited two braid factorizations of a fixed 3-braid in [14]. Based on this example, we construct inequivalent braid factorizations, which provide symplectic surfaces in the above family.

We would like to point out that in his paper [1], Akbulut constructed a knot with two non-isotopic ribbon disks whose complements are diffeomorphic. Although a knot in S^3 can be deformed into a transverse knot in (S^3, ξ_{st}), it is not known whether these ribbon disks are symplectic or not.

A *Stein filling* of a contact manifold is a sublevel set of a proper, bounded below strictly plurisubharmonic function on a complex manifold whose convex boundary is contactomorphic to the given one (see [12] for more details). We obtain the following corollary from the above theorem combined with an argument about contact and Stein structures.

Corollary 1.2. *There is a family of contact* 3*-manifolds* $\{(M(n), \xi(n))\}_{n \in \mathbb{Z}_{\geq 0}}$ *such that each contact manifold admits two non-homeomorphic Stein fillings* $X_1(n)$, $X_2(n)$ *which are simply-connected and have the same homology group but non-isomorphic intersection forms.*

2. Braided surfaces

2.1. Braid groups

We here briefly review braid groups (see [15, Section 2.2] for example). Let D^2 be a closed disk in $\mathbb{R}^2$ equipped with the standard orientation and $K \subset \mathrm{Int}D^2$ a finite set. Suppose that $\#K = m$.

Definition 2.1. The *braid group* with respect to D^2 and K, denoted by $B_m[D^2, K]$, is the group of isotopy classes of orientation-preserving diffeomorphisms β of D^2 such that $\beta|_{\partial D^2} = id_{\partial D^2}$ and $\beta(K) = K$. The elements of this group are called *braids*.

Let σ be a smooth simple path in $\mathrm{Int}D^2$ with distinct end points $a, b \in K$ and $\sigma \cap K = \{a, b\}$. Choose a small tubular neighborhood $U \subset \mathrm{Int}D^2$ of σ such that $U \cap K = \{a, b\}$.

Definition 2.2. The *half-twist* $H(\sigma)$ *along* σ is an element of the braid group $B_m[D^2, K]$ which switches the end points a and b of σ by a counterclockwise 180° rotation and whose support is contained in U.

2.2. Braided surfaces and their descriptions

Let D_1^2 and D_2^2 be two oriented closed disks.

Definition 2.3. A *braided surface* in the bidisk $D_1^2 \times D_2^2$ is a properly embedded surface S in $D_1^2 \times D_2^2$ such that:

1) The restriction of the first projection $pr_1|_S : S \to D_1^2$ is a simple branched covering;
2) For each branch point $x \in S$ of $pr_1|_S$, there are complex coordinates (z, w) and ζ around x and $pr_1(x)$, respectively, compatible with orientations of $D_1^2 \times D_2^2$ and D_1^2 such that pr_1 can be written as $\zeta = pr_1(z, w) = z$ and locally the set $\{(z, w)|z = w^2\}$ coincides with S.

Suppose that S is a braided surface in $D_1^2 \times D_2^2$. Let $\Delta(S) \subset \mathrm{Int}D_1^2$ denote the set of branch points of the covering $pr_1|_S$. For a point y of $D_1^2 \setminus \Delta(S)$, the number $m = \#(S \cap pr_1^{-1}(y))$ is called the *degree* of the braided surface S.

One can read off the fundamental group of the complement of a braided surface $S \subset D_1^2 \times D_2^2$ from its braid monodromy. Fix a base point $y_0 \in \partial D_1^2$ and set $D_{y_0} = pr_1^{-1}(y_0)$ and $K(y_0) = D_{y_0} \cap S = \{x_1, \dots, x_m\}$. For a point y of $\Delta(S)$, consider a smooth simple loop $\gamma : [0, 1] \to D_1^2 \setminus \Delta(S)$ around y based at y_0 whose bounding region does not contain any other branch points. This loop lifts to $(\gamma([0, 1]) \times D_2^2) \cap S$ as a motion

$$pr_2(\{x_1(t), \dots, x_m(t) \in S \,|\, t \in [0, 1]\})$$

of m distinct points of D_2^2, where $pr_2 : D_1^2 \times D_2^2 \to D_2^2$ is the second projection. When $t = 0, 1$, it is nothing but $K(y_0)$. Hence this motion defines a braid $\beta(\gamma) \in B_m[D_{y_0}, K(y_0)]$, called a *braid monodromy* (with respect to y_0) around the branch point y. It is known that this braid is the half-twist $H(\sigma)$ along a smooth simple path σ connecting two distinct points of $K(y_0)$. One can associate an element of $B_m[D_{y_0}, K(y_0)]$ to any loop in $D_1^2 \setminus \Delta(S)$ based at y_0, and define the homomorphism

$$\varphi : \pi_1(D_1^2 \setminus \Delta(S), y_0) \to B_m[D(y_0), K(y_0)].$$

Set $\Delta(S) = \{y_1, \dots, y_k\}$. Take smooth simple loops $\gamma_i \in \pi_1(D_1^2 \setminus \Delta(S), y_0)$ around y_i, as we did before, so that the composition $\gamma_1 \cdots \gamma_k$ is homotopic to ∂D_1^2. Obviously, $\{\gamma_1, \dots, \gamma_k\}$ serves as a free basis for $\pi_1(D_1^2 \setminus \Delta(S), y_0)$,

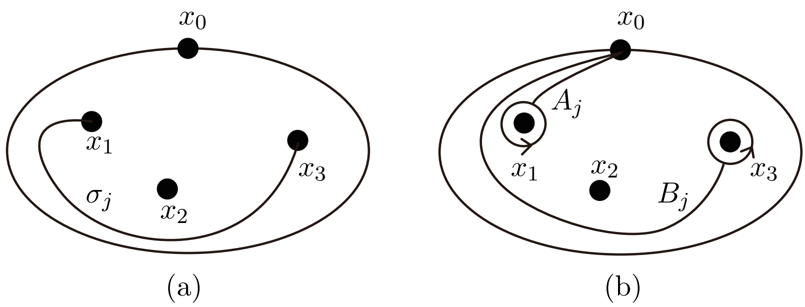

Figure 1: (a) Path σ_j. (b) Loops A_j and B_j associated to σ_j.

and it is called a *geometric basis* for the group. Then, the braid $\varphi(\partial D_1^2) = \varphi(\gamma_1 \cdots \gamma_k)$ can be factorized into k half-twists as

$$\beta(\gamma_1) \cdots \beta(\gamma_k),$$

which is a *braid monodromy factorization* of $\varphi(\partial D_1^2)$. As a similar notion, a *braid factorization* of a braid β is a factorization $\beta = \beta_1 \cdots \beta_k$ into half-twists β_j. Given a braid factorization $\beta_1 \cdots \beta_k$ of a braid, one can construct a braided surface S with k branch points whose braid monodromy around each branch point y_i is β_i for some geometric basis for $\pi_1(D_1^2 \setminus \Delta(S), y_0)$.

Now we explain how to compute the fundamental group of the complement of a braided surface as the special case of [15, Theorem 2.5]. Let S be a braided surface in $D_1^2 \times D_2^2$. Suppose that $\{\gamma_1, \ldots, \gamma_k\}$ is a geometric basis for the fundamental group $\pi_1(D_1^2 \setminus \Delta(S), y_0)$ and the ordered k-tuple $(H(\sigma_1), \ldots, H(\sigma_k))$ consists of braid monodromies $\varphi(\gamma_1), \ldots, \varphi(\gamma_k)$ of S, where each σ_j $(j = 1, \ldots, k)$ is a smooth simple path connecting two distinct points of $K(y_0)$. Fix a point x_0 of ∂D_{y_0}. Label the points of $K(y_0)$ as $x_1, \ldots, x_m$ and let $\{\gamma'_1, \ldots, \gamma'_m\}$ be a geometric basis for $\pi_1(D_{y_0} \setminus K(y_0), x_0)$ constructed in the same way we did for $\pi_1(D_1^2 \setminus \Delta(S), y_0)$. For each $j = 1, \ldots, k$, set $A_j = \gamma'_i$, where x_i is either of end points of σ_j, and $B_j = H(\sigma_j)(A_j)$ (see Figure 1). It is clear that B_j can be expressed in terms of $\gamma'_1, \ldots, \gamma'_m$ because they form a geometric basis for $\pi_1(D_{y_0} \setminus K(y_0), x_0)$. By using Zariski-Van Kampen's theorem, we have the following formula:

$$\begin{aligned} (2.1)\quad \pi_1(D_1^2 \times D_2^2 \setminus S, x_0) &\cong \pi_1(D_{y_0} \setminus K(y_0), x_0) / \langle\, A_j = B_j \ (j = 1, \ldots, k) \,\rangle \\ &\cong \langle\, \gamma'_1, \ldots, \gamma'_m \mid A_j = B_j \ (j = 1, \ldots, k) \,\rangle. \end{aligned}$$

Here the point $x_0 \in D_{y_0}$ is considered as one of $D_1^2 \times D_2^2$ by the inclusion $D_{y_0} \hookrightarrow D_1^2 \times D_2^2$.

2.3. Double branched covers and Lefschetz fibrations

Let S be a braided surface of degree m in a bidisk $D_1^2 \times D_2^2$ whose braid monodromy factorization with respect to some base point y_0 and geometric basis for $\pi_1(D_1^2 \setminus \Delta(S), y_0)$ is

$$H(\sigma_1) \cdots H(\sigma_k).$$

Consider the double branched covering $p : X \to D_1^2 \times D_2^2$ whose branch set is S. The covering p restricts to the double branched covering $p|_{F_{y_0}} : F_{y_0} = p^{-1}(D_{y_0}) \to D_{y_0}$. Each path σ_j lifts, with respect to $p|_{F_{y_0}}$, to a unique simple closed curve c_j on the surface F_{y_0} up to isotopy. Then, according to [11, Proposition 1], the composition $pr_1 \circ p : X \to D_1^2$ is a *Lefschetz fibration* (see [10, Chapter 8] for the precise definition) whose fibers are diffeomorphic to the surface F_{y_0} and monodromy factorization is

$$\tau(c_k) \circ \cdots \circ \tau(c_1).$$

Here $\tau(c)$ denotes the isotopy class of a right-handed Dehn twist along c. Throughout this paper, we use the functional notation for the products in the mapping class group of F_{y_0}, i.e. $f \circ g$ means that we apply g first and then f.

3. Proof of results

3.1. Proof of Theorem 1.1

Fix an integer $n \in \mathbb{Z}_{\geq 0}$. Let $\mathbb{D}^2$ be the closed unit disk in $\mathbb{C}$ and K_{n+3} the set of $n+3$ points of $\operatorname{Int} \mathbb{D}^2$ on the real axis. Let $a, b, c_n, d_1, \ldots, d_{n+2}$ be smooth simple paths in $\mathbb{D}^2$ as shown in Figure 2. For $n \geq 1$, define two braids $\beta_1(n), \beta_2(n) \in B_{n+3}[\mathbb{D}^2, K_{n+3}]$ with factorizations given by

$$\beta_1(n) = H(a) \cdot H(b) \cdot H(d_1) \cdot H(c_n) \cdot H(d_{n+2}) \cdot \cdots \cdot H(d_3), \tag{3.1}$$

$$\beta_2(n) = H(H(d_2)(a)) \cdot H(H(d_2)(b)) \cdot H(d_1) \cdot H(c_n) \cdot H(d_{n+2}) \cdot \cdots \cdot H(d_3). \tag{3.2}$$

When $n = 0$, we set

$$\beta_1(0) = H(a) \cdot H(b) \cdot H(d_1) \cdot H(c_0),$$
$$\beta_2(0) = H(H(d_2)(a)) \cdot H(H(d_2)(b)) \cdot H(d_1) \cdot H(c_0).$$

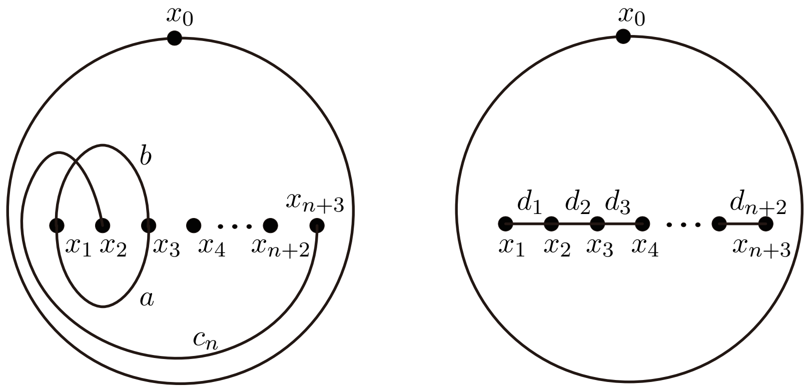

Figure 2: Arcs $a, b, c_n, d_1, \ldots, d_n$ on the disk $\mathbb{D}^2$ with K_{n+3}.

We obtain two braided surfaces $S_1(n)$ and $S_2(n)$ whose braid monodromy factorizations are the given braid factorizations (3.1) and (3.2), respectively. By a result of Rudolph [13], these braided surfaces can be considered as symplectic surfaces in the round 4-disk (D^4, ω_{st}) with transverse link boundaries. In particular, the boundaries $\partial S_1(n)$ and $\partial S_2(n)$ are knots. This can be seen as follows. The transverse links $\partial S_1(n)$ and $\partial S_2(n)$ in (S^3, ξ_{st}) are represented by the closure of braids $\beta_1(n)$ and $\beta_2(n)$, respectively. Here the braids are thought as geometric ones although we defined them as in Definition 2.1 (see [7, Chapter 2]). Consider the canonical projection $B_{n+3} \to \mathfrak{S}_{n+3}$, where $\mathfrak{S}_{n+3}$ denotes the symmetric group of degree n. The image of $\beta_1(n)$ and $\beta_2(n)$ under this projection is equal to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n+2 & n+3 \\ 3 & 1 & 4 & 5 & \ldots & n+3 & 2 \end{pmatrix} \in \mathfrak{S}_{n+3}.$$

The order of this element is $n+3$, and hence the closure of the two braids are knots.

Let us see that $\partial S_1(n)$ and $S_2(n)$ are transverse isotopic. It can be easily checked that $H(d_2)$ commutes with the product $H(a) \cdot H(b)$. Indeed, since $H(a) = H^{-1}(d_2) \cdot H(d_1) \cdot H(d_2)$ and $H(b) = H(d_2) \cdot H(d_1) \cdot H^{-1}(d_2)$, underling each triple of half-twists where we apply a braid relation, we have

$$
\begin{aligned}
& H(d_2) \cdot H(a) \cdot H(b) \\
&= H(d_2) \cdot (H^{-1}(d_2) \cdot H(d_1) \cdot H(d_2)) \cdot (H(d_2) \cdot H(d_1) \cdot H^{-1}(d_2)) \\
&= H^{-1}(d_2) \cdot \underline{H(d_2) \cdot H(d_1) \cdot H(d_2)} \cdot (H(d_2) \cdot H(d_1) \cdot H^{-1}(d_2)) \\
&= H^{-1}(d_2) \cdot H(d_1) \cdot H(d_2) \cdot \underline{H(d_1) \cdot H(d_2) \cdot H(d_1)} \cdot H^{-1}(d_2) \\
&= H^{-1}(d_2) \cdot H(d_1) \cdot H(d_2) \cdot H(d_2) \cdot H(d_1) \cdot H(d_2) \cdot H^{-1}(d_2) \\
&= (H^{-1}(d_2) \cdot H(d_1) \cdot H(d_2)) \cdot (H(d_2) \cdot H(d_1) \cdot H^{-1}(d_2)) \cdot H(d_2) \\
&= H(a) \cdot H(b) \cdot H(d_2).
\end{aligned}
$$

The commutativity proves $\beta_1(n) = \beta_2(n)$. Hence two boundaries are transversely isotopic. Hereafter, for the sake of simplicity, set $\beta(n) = \beta_1(n) = \beta_2(n)$.

Next, we show that the fundamental groups of complements $D^4 \setminus S_1(n)$ and $D^4 \setminus S_2(n)$ are isomorphic. Fixing a base point x_0 in a fiber of the projection pr_1 of the bidisk, by the formula (2.1), $\pi_1(D^4 \setminus S_1(n), x_0)$ is isomorphic to the group generated by $\gamma_1, \ldots, \gamma_{n+3}$, as shown in Figure 3, with relations

$$
\begin{gathered}
\gamma_1 = \gamma_2\gamma_3\gamma_2^{-1}, \quad \gamma_1 = \gamma_3, \quad \gamma_1 = \gamma_2, \\
(\gamma_1\gamma_2 \cdots \gamma_{n+2})\gamma_{n+3}(\gamma_1\gamma_2 \cdots \gamma_{n+2})^{-1} = \gamma_2, \quad \gamma_j = \gamma_{j+1} \ (j = 3, \ldots, n+2).
\end{gathered}
$$

Hence

$$
\pi_1(D^4 \setminus S_1(n), x_0) \cong \langle \gamma_1 | - \rangle \cong \mathbb{Z}.
$$

On the other hand, $\pi_1(D^4 \setminus S_2(n), x_0)$ is isomorphic to the group generated by $\gamma_1, \ldots, \gamma_{n+3}$ (see Figure 3) with relations

$$
\begin{gathered}
\gamma_2 = \gamma_3^{-1}\gamma_2^{-1}\gamma_1\gamma_2\gamma_3, \quad \gamma_1 = \gamma_2, \quad \gamma_1 = \gamma_2, \\
(\gamma_1\gamma_2 \cdots \gamma_{n+2})\gamma_{n+3}(\gamma_1\gamma_2 \cdots \gamma_{n+2})^{-1} = \gamma_2, \quad \gamma_j = \gamma_{j+1} \ (j = 3, \ldots, n+2).
\end{gathered}
$$

Thus,

$$
\pi_1(D^4 \setminus S_2(n), x_0) \cong \langle \gamma_1 | - \rangle \cong \mathbb{Z}
$$

that is isomorphic to $\pi_1(D^4 \setminus S_1(n), x_0)$.

For each $j = 1, 2$ let $p_j(n) : X_j(n) \to D_1^2 \times D_2^2$ be the double branched covering whose branch set is $S_j(n)$. As we discussed in Section 2.3, $X_j(n)$ is considered as the total space of the Lefschetz fibration $f_j(n) = pr_1 \circ p_j(n)$. Let A, B, C_n, D_i be lifts of arcs a, b, c_n, d_i, respectively, with respect to the

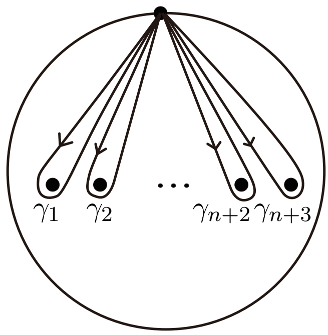

Figure 3: Generators $\gamma_1, \ldots, \gamma_{n+3}$.

covering $p_j(n)|_{F_{y_0}}$, where F_{y_0} is the preimage of $y_0 \in D_1^2$ under $f_j(n)$ (see Figure 4). Fibers of the Lefschetz fibration $f_j(n)$ are diffeomorphic to F_{y_0} and its monodromy factorization is

$$\tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) \circ \tau(D_1) \circ \tau(B) \circ \tau(A) \qquad \text{if } j = 1, \tag{3.3}$$

$$\tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) \circ \tau(D_1) \circ \tau(\tau(D_2)(B)) \circ \tau(\tau(D_2)(A)) \qquad \text{if } j = 2. \tag{3.4}$$

Note that the curve $\tau(D_2)(B)$ is isotopic to D_1 because the arc $H(d_2)(b)$ is isotopic to d_1. From these data, one can draw handle diagrams (or Kirby diagrams) of $X_1(n)$ and $X_2(n)$ as in Figure 5. Here we use the standard Seifert surface for the $(2, n+3)$-torus link as the fiber surface to see the monodromy curves more easily. We should note that the surface framing of each curve does not always coincide with its blackboard framing (see [10, Section 6.3], which explains the way to draw handle diagrams of Milnor fibers in the same manner as ours). The surface framing of each curve is given as the linking number with its push-off in the positive normal direction of F_{y_0} in $\mathbb{R}^3$ (see Figure 6). After sliding 2-handles and cancelling 1-/2-handle pairs as indicated in Figures 7 and 8, we obtain handle diagrams of $X_1(n)$ and $X_2(n)$ each of which consists of only one 0-handle and two 2-handles.

From the bottom left diagram of Figure 8 one can see that $X_2(n)$ contains a smooth surface with self-intersection number -2. In contrast, we will show below that the double cover $X_1(n)$ contains no such surfaces: Let $\{e_1, e_2\}$ be the basis for the homology group $H_2(X_1(n); \mathbb{Z})$, where each e_j is the homology class represented by the 2-handle depicted in the bottom left digram of Figure 7. The arrows in the figure indicate the orientation of

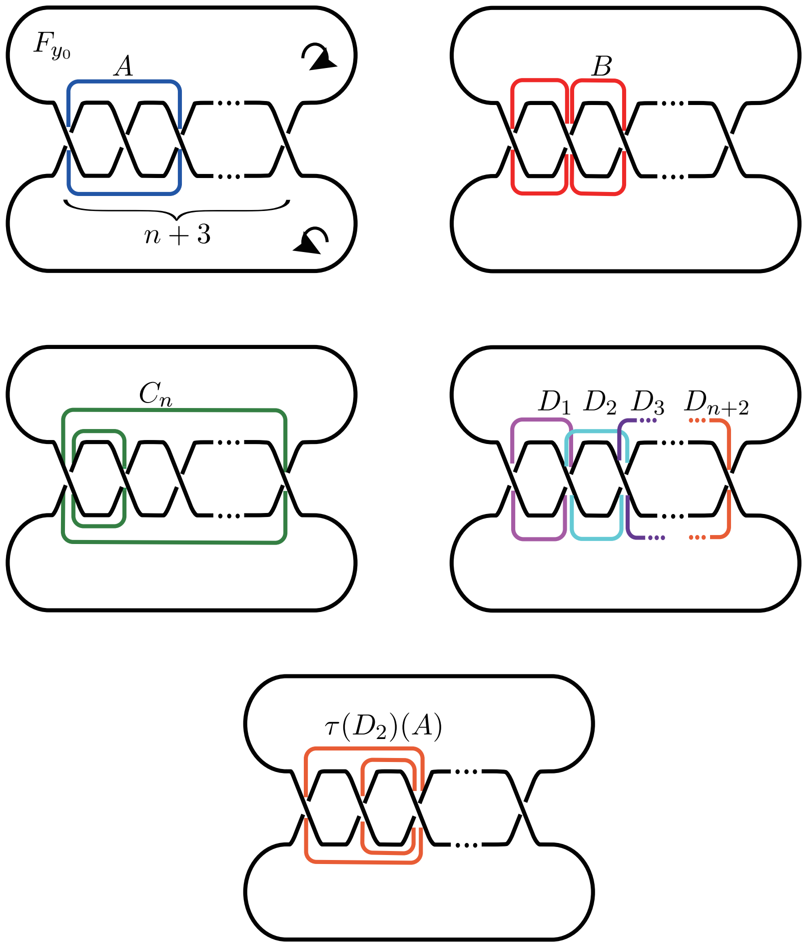

Figure 4: Surface F_{y_0} as the double branched cover and lifts A, B, C_n, D_i ($i = 1, \ldots, n+2$) and $\tau(D_2)(A)$: Each rounded arrow indicates the orientation of F_{y_0}.

these generators. Note that the following consequence is independent of the choice of orientation.

Suppose for the sake of contradiction that there are integers $\alpha_1, \alpha_2 \in \mathbb{Z}$ such that $(\alpha_1 e_1 + \alpha_2 e_2)^2 = -2$. The matrix $Q(n)$ of the intersection form $Q_{X_1(n)}$ with respect to the basis $\{e_1, e_2\}$ can be read off from the handle diagram of $X_1(n)$, and

$$Q(n) = \begin{bmatrix} -2n-4 & 1 \\ 1 & -8 \end{bmatrix}.$$

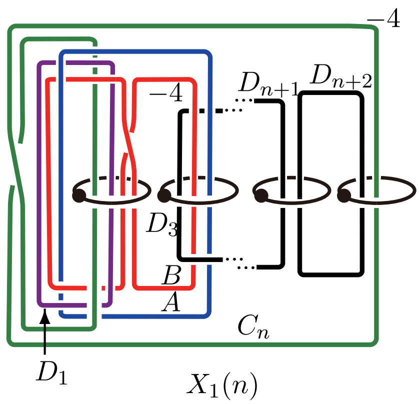

Figure 5: Handle diagrams of $X_1(n)$ and $X_2(n)$: All 2-handle framings which are not written here are -2.

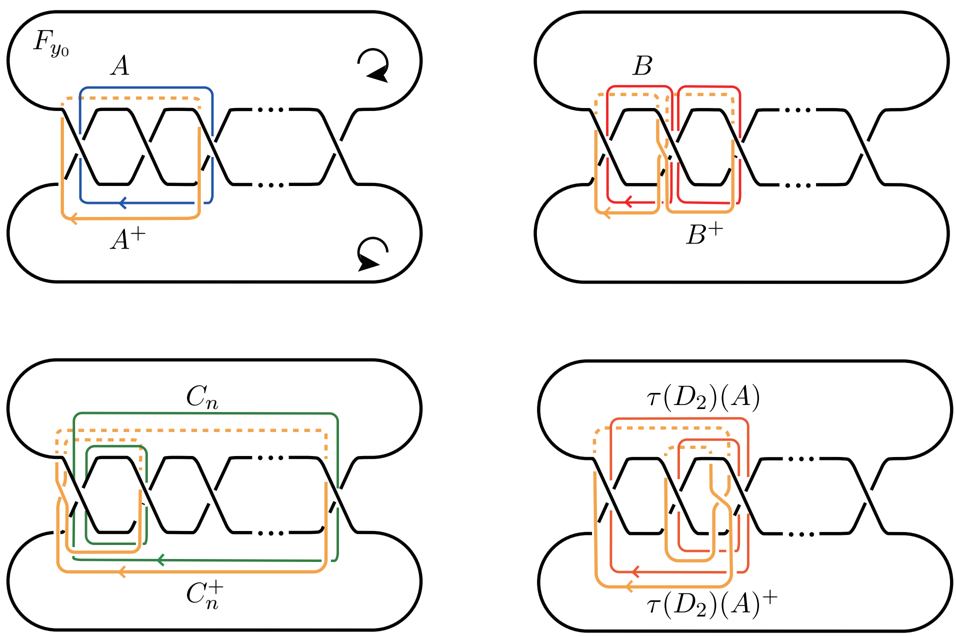

Figure 6: Simple closed curves A, B, C_n, $\tau(D_2)(A)$ and their positive push-off A^+, B^+, C_n^+, $\tau(D_2)(A)^+$: The linking numbers are $lk(A, A^+) = -1$, $lk(B, B^+) = lk(C_n, C_n^+) = lk(\tau(D_2)(A), \tau(D_2)(A)^+) = -3$. Also, similarly to the case of A, $lk(D_i, D_i^+) = -1$.

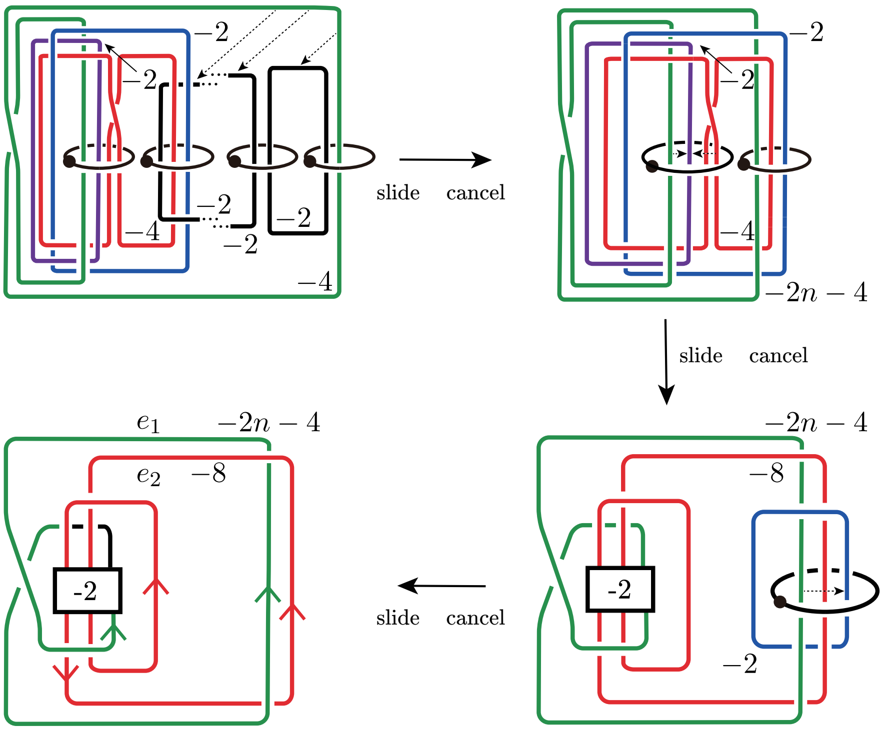

Figure 7: Handle calculus for $X_1(n)$: Each dashed arrow in the diagram indicates how we slide a 2-handle over another one.

Then, by using this matrix, we have $(-2n-4)\alpha_1^2 + 2\alpha_1\alpha_2 - 8\alpha_2^2 = -2$. The left-hand side also can be written as the form

$$-(2n+4)\left(\alpha_1 - \frac{1}{2n+4}\alpha_2\right)^2 - \left(8 - \frac{1}{2n+4}\right)\alpha_2^2.$$

Since $-(2n+4)(\alpha_1 - \alpha_2/(2n+4))^2$ and $-(8-1/(2n+4))\alpha_2^2$ are non-positive, we conclude that the latter should be greater than or equal to -2, that is,

$$(8 - 1/(2n+4))\alpha_2^2 \leq 2.$$

The coefficient $8 - 1/(2n+4)$ is greater than 2, and hence $\alpha_2^2 < 1$, namely $\alpha_2 = 0$. Thus

$$-(2n+4)\alpha_1^2 = -2 \quad \text{and} \quad \alpha_1 \in \mathbb{Z} \setminus \{0\}. \tag{3.5}$$

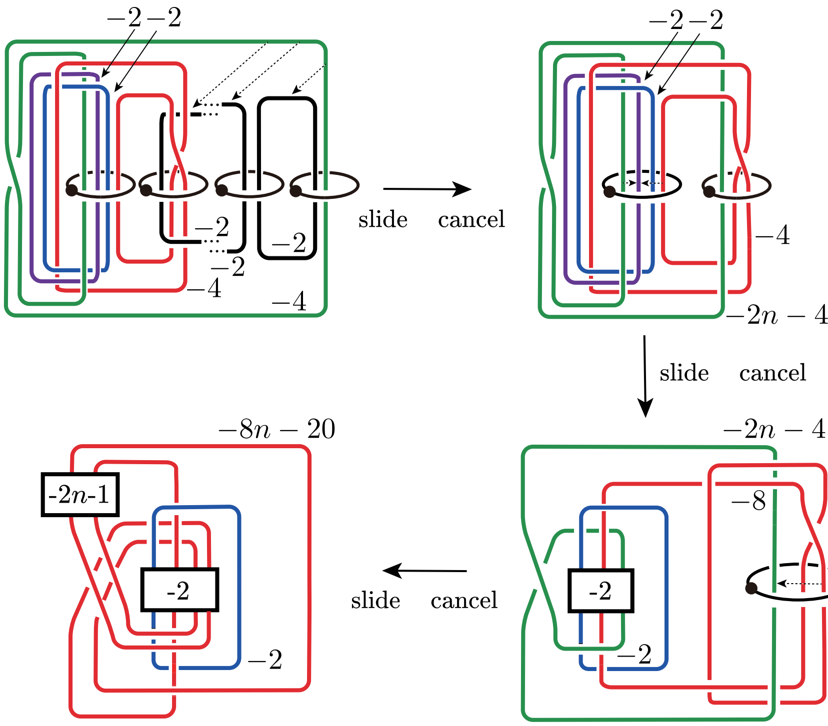

Figure 8: Handle calculus for $X_2(n)$: Each dashed arrow in the diagram indicates how we slide a 2-handle over another one.

However, $-(2n+4)\alpha_1^2 \leq -(2n+4) \leq -4$ for $\alpha_1 \in \mathbb{Z} \setminus \{0\}$, which contradicts the equation (3.5). Thus, we conclude that $X_1(n)$ and $X_2(n)$ are not homeomorphic.

To distinguish the closure $\widehat{\beta(n)}$ from $\widehat{\beta(n')}$ for $n \neq n'$, we use the *determinant* of a knot, defined by $|\det(V+V^T)|$, where V is a Seifert matrix for the knot. It is known that it equals the order of the first homology group of the double branched cover of S^3 branched along the knot. Moreover, let X be a compact 4-manifold admitting a handle decomposition with only one 0-handle and 2-handles. Then, the determinant of a matrix for the intersection form Q_X coincides with $|H_1(\partial X;\mathbb{Z})|$ up to sign (see [10, Corollary 5.3.12]). Thus the determinant of the closure $\widehat{\beta(n)}$ is

$$\det Q(n) = \det \begin{bmatrix} -2n-4 & 1 \\ 1 & -8 \end{bmatrix} = 16n+31,$$

which proves that all $\widehat{\beta(n)}$ ($n \in \mathbb{Z}_{\geq 0}$) are mutually non-isotopic. This finishes the proof.

Remark 3.1. Braid factorizations

$$H(a) \cdot H(b) \quad \text{and} \quad H(H(d_2)(a)) \cdot H(H(d_2)(b))$$

we used above are essentially found by Rudolph in [14, Example 1.13], where he showed that the two factorizations represent the same braid. This pair was also used in [3, Proposition 3.1] (see also Example 4.2 in the same paper).

Remark 3.2. Two mapping class factorizations (3.3) and (3.4) are related by a *partial conjugation*, twisting the last two factors by $\tau(D_2)$. This implies that two corresponding double covers are related by a Luttinger surgery along a torus built by parallel transport of the curve D_2 along a loop in D_1^2 (see [4]).

3.2. Proof of Corollary 1.2

Let each $S_j(n)$ ($j = 1, 2$) be the braided surface constructed above and $p_j(n) : X_j(n) \to D^4$ ($j = 1, 2$) the double branched covering whose branch set is $S_j(n)$. As we mentioned before, the covering $p_j(n)$ induces the Lefschetz fibration $f_j(n)$ on $X_j(n)$. In particular, this Lefschetz fibration is *allowable*, that is, its all vanishing cycles are homologically non-trivial in a reference fiber. According to [2, 11], $X_j(n)$ admits a Stein structure, and the contact structure $\xi_j(n)$ on the boundary $M_j(n) = \partial X_j(n)$ induced from the Stein structure is compatible with the open book determined by the Lefschetz fibration. We see that the monodromy $\phi_j(n)$ of this open book is isotopic to the composition

$$\phi_j(n) = \begin{cases} \tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) \circ \tau(D_1) \circ \tau(B) \circ \tau(A) & (j = 1), \\ \tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) & \\ \qquad \circ \tau(D_1) \circ \tau(\tau(D_2)(B)) \circ \tau(\tau(D_2)(A)) & (j = 2). \end{cases}$$

Since $\tau(D_2)$ commutes with $\tau(B) \circ \tau(A)$, we have

$$\tau(B) \circ \tau(A) = \tau(\tau(D_2)(B)) \circ \tau(\tau(D_2)(A)).$$

Hence $\phi_1(n) = \phi_2(n)$, which proves that the contact manifolds $(M_1(n), \xi_1(n))$ and $(M_2(n), \xi_2(n))$ are mutually contactomorphic. Therefore, $X_1(n)$ and

$X_2(n)$ serve as Stein fillings of the contact manifold $(M(n), \xi(n)) \coloneqq (M_1(n), \xi_1(n))$ whose intersection forms are non-isomorphic by Theorem 1.1. Moreover, it follows from handle diagrams depicted in Figures 7 and 8 that $X_1(n)$ and $X_2(n)$ are simply-connected and have the same homology group. This completes the proof.

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