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# Restrictions on submanifolds via focal radius bounds

Luis Guijarro* and Frederick Wilhelm†

We give an optimal estimate for the norm of any submanifold's second fundamental form in terms of its focal radius and the lower sectional curvature bound of the ambient manifold.

This is a special case of a similar theorem for intermediate Ricci curvature, and leads to a $C^{1,\alpha}$ compactness result for submanifolds, as well as a "soul-type" structure theorem for manifolds with nonnegative $k^{th}$–intermediate Ricci curvature that have a closed submanifold with dimension $\geq k$ and infinite focal radius.

To prove these results, we use a new comparison lemma for Jacobi fields from [18] that exploits Wilking's transverse Jacobi equation. The new comparison lemma also yields new information about group actions, Riemannian submersions, and submetries, including generalizations to intermediate Ricci curvature of results of Chen and Grove. None of these results can be obtained with just classical Riccati comparison (see Subsection 3.1 for details.)

Submanifolds restrict the Riemannian geometry of the space in which they lie, but only if they satisfy extra conditions. One constraint comes from the tubular neighborhood theorem. It asserts that given any compact submanifold $S$, there is a positive $r_0$ such that the normal disc bundle $D_{r_0}(S)$ is diffeomorphic to an open neighborhood of $S$; the diffeomorphism can be realized via the normal exponential map of $S$. This motivates the notion of *focal radius,* which is the maximum $r_0$ such that the normal exponential map is a local diffeomorphism of $D_{r_0}(S)$.

Our first result shows that we can bound the norm of the second fundamental form of any submanifold in terms of its focal radius and the ambient manifold's lower curvature bound.

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**Theorem A.** *For $\kappa = -1, 0,$ or $1$, let $M$ be a complete Riemannian $n$–manifold with sectional curvature $\geq \kappa$, and let $N$ be any submanifold of $M$ with* $\dim(N) \geq 1$. *Then the second fundamental form* $\mathrm{II}_N$ *of $N$ satisfies*

$$(1)\qquad \begin{aligned} |\,\mathrm{II}_N| &\leq \cot(\mathrm{FocalRadius}(N)) && \text{if } \kappa = 1, \\ |\,\mathrm{II}_N| &\leq \frac{1}{\mathrm{FocalRadius}(N)} && \text{if } \kappa = 0, \text{ and} \\ |\,\mathrm{II}_N| &\leq \coth(\mathrm{FocalRadius}(N)) && \text{if } \kappa = -1. \end{aligned}$$

*In particular, if $\kappa = 0$ and the focal radius of $N$ is infinite, then $N$ is totally geodesic.*

We emphasize that $N$ does not need to be closed or even complete. On the other hand, if $M$ happens to be closed, the presence of a lower curvature bound $\kappa$ is automatic, and after rescaling, we can take $\kappa$ to be $-1, 0$, or 1. So for closed manifolds, Theorem A is universal in the sense that it applies to any submanifold of any Riemannian manifold. The upper bound is, moreover, optimal. Metric balls in space forms show that for every $\kappa$ and every possible focal radius, there is a hypersurface in a space with constant curvature $\kappa$ for which Inequality (1) is an equality.

As a consequence of this result, we show that submanifolds with focal radius bounded from below and diameter bounded from above have only finitely many diffeomorphism types, a result that is of independent interest.

**Theorem B.** *Let $M$ be a compact Riemannian manifold. Given $D, r > 0$ the class $\mathcal{S}$ of closed Riemannian manifolds that can be isometrically embedded into $M$ with focal radius $\geq r$ and intrinsic diameter $\leq D$ is precompact in the $C^{1,\alpha}$–topology. In particular, $\mathcal{S}$ contains only finitely many diffeomorphism types.*

Theorem B is optimal in the sense that neither the hypothesis on the focal radius nor the hypothesis on the diameter can be removed. If either hypothesis is removed, then, after rescaling, all Riemannian $k$–manifolds can occur in a flat $n$–torus, provided $n >> k$ (see example 4.3).

Since Inequality (1) applies to any submanifold of any Riemannian manifold, the Gauss Equation implies that the class $\mathcal{S}$ in Theorem B, has uniformly bounded sectional curvature. Theorem B follows from this and Cheeger's Finiteness Theorem ([5]), provided the class $\mathcal{S}$ also has a uniform lower bound for its volume. We achieve the lower volume bound as a consequence of Heintze and Karcher's tube formula ([20], see Lemma 4.1, below).

Since Theorem A gives a new way to estimate curvature, it has many corollaries. For example, using again the Gauss Equation, Theorem A provides us with a simple proof of the following two statements, that are valid for submanifolds of arbitrary codimension.

**Corollary C.**

- *A submanifold of* $\mathbb{S}^n$ *is positively curved if its focal radius is* $> \frac{\pi}{4}$.
- *A submanifold of any hyperbolic manifold is nonpositively curved if it has infinite focal radius.*

The Clifford torus in $\mathbb{S}^3$ has focal radius $\frac{\pi}{4}$, so the first statement of the corollary is optimal.

Theorem A is obtained as a consequence of a more general bound on the second fundamental form of a submanifold, that is true in the more general context of bounds on the intermediate Ricci curvature (see Theorem 3.1 below). As a consequence, we recapture all of the rigidity of the Soul Theorem ([2], [16],[25], [31], [33]), provided a manifold with $\mathrm{Ric}_k \geq 0$ contains a closed submanifold with infinite focal radius. (See [18] or [35] for the definition of intermediate Ricci curvature.)

**Theorem D.** *Let* $M$ *be a complete Riemannian* $n$*–manifold with* $\mathrm{Ric}_k \geq 0$, *and let* $N$ *be any closed submanifold of* $M$ *with* $\dim(N) \geq k$ *and infinite focal radius. Then:*

1. $N$ *is totally geodesic.*
2. *The normal bundle* $\nu(N)$ *with the pull back metric* $\left(\exp_N^{\perp}\right)^*(g)$ *is a complete manifold with* $\mathrm{Ric}_k \geq 0$.
3. $\exp_N^{\perp} : \left(\nu(N), \left(\exp_N^{\perp}\right)^* g\right) \longrightarrow (M, g)$ *is a Riemannian cover.*
4. *The zero section* $N_0$ *is totally geodesic in* $\left(\nu(N), \left(\exp_N^{\perp}\right)^*(g)\right)$.
5. *The projection* $\pi : \left(\nu(N), \left(\exp_N^{\perp}\right)^*(g)\right) \longrightarrow N$ *is a Riemannian submersion.*
6. *If* $c : I \longrightarrow N$ *is a unit speed geodesic in* $N$, *and* $V$ *is a parallel normal field along* $c$, *then*

$$\Phi : I \times \mathbb{R} \longrightarrow M, \qquad \Phi(s,t) = \exp_{c(s)}^{\perp}(tV(s))$$

*is a totally geodesic immersion whose image has constant curvature* $0$.

7. *All radial sectional curvatures from $N$ are nonnegative. That is, for $\gamma(t) = \exp_N^\perp(tv)$ with $v \in \nu(N)$, the curvature of any plane containing $\gamma'(t)$ is nonnegative.*
8. *If $n \geq 3$ and $k \leq n-2$, then for all $r > 0$, the intrinsic metric on $\exp_N^\perp(S(N_0, r))$ has $\mathrm{Ric}_k \geq 0$, where $S(N_0, r)$ is the metric $r$–sphere around the zero section $N_0$ in $\nu(N)$.*

The version of Part 8 of Theorem D for nonnegative sectional curvature and small $r$ is similar to Theorem 2.5 of [17]. In the latter result, $N$ needs to be a soul of $M$ but can have any focal radius.

In the case of Ricci curvature, Theorem D is Theorem 3 of [9], but in the sectional curvature case, it yields new information about open nonnegatively curved manifolds.

**Corollary E.** *Let $N$ be a closed submanifold in a complete, noncompact, simply connected nonnegatively curved manifold $(M, g)$. If $N$ has infinite focal radius, then $N$ is a soul of $M$.*

While examples show that souls need not have infinite focal radius, using the main theorem of [15], we can always modify the metric of $M$ so that its soul has infinite focal radius.

Theorem D also imposes rigidity on compact nonnegatively curved manifolds that contain closed submanifolds with no focal points (see Corollary 3.2).

To prove Theorems A and D we use the new Jacobi field comparison lemma from [18]. It also has consequences for Riemannian submersions, isometric group actions, and Riemannian foliations of manifolds with positive intermediate Ricci curvature. To state them succinctly, we recall the definition of "manifold submetry" from [6].

**Definition.** A submetry

$$\pi : M \longrightarrow X$$

of a Riemannian manifold is called a "manifold submetry" if and only if $\pi^{-1}(x)$ is a closed smooth submanifold for all $x \in X$ and every geodesic of $M$ that is initially perpendicular to a fiber of $\pi$ is everywhere perpendicular to the fibers of $\pi$.

If the leaves of a singular Riemannian foliation are closed, then as pointed out in [6], its quotient map is a manifold submetry. Thus the following result applies to singular Riemannian foliations with closed leaves. In particular, it applies to quotient maps of isometric group actions and to Riemannian submersions. In it, we use the term "geodesic" to mean a curve that locally minimizes distances but need not be a globally shortest path.

**Theorem F.** *Let* $\pi : M \longrightarrow X$ *be a manifold submetry of a complete Riemannian* $n$*–manifold with* $\mathrm{Ric}_k \geq k$. *Suppose that for some* $x \in X$, $\dim \pi^{-1}(x) \geq k$.

1) *For every geodesic* $\gamma$ *emanating from* $x$, *either* $\gamma$ *does not extend to a geodesic on any interval that properly contains* $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, *or* $\gamma$ *has a conjugate point to* $x$ *in* $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. *In particular, if* $X$ *is smooth and* $\pi$ *is a Riemannian submersion, then the conjugate radius of* $X$ *at* $x$ *is* $\leq \frac{\pi}{2}$.

2) *If all geodesics emanating from* $x$ *extend to geodesics on* $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ *and are free of conjugate points on* $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, *then* $\pi^{-1}(x)$ *is totally geodesic in* $M$, *and the universal cover of* $M$ *is isometric to the sphere or a projective space with the standard metrics.*

3) *If* $\dim \pi^{-1}(x) \geq k$ *for some* $x \in X$ *for which* $\max\{\mathrm{dist}_x\} = \mathrm{diam}(X)$, *then the diameter of* $X$ *is* $\leq \frac{\pi}{2}$.

The relevant definition of conjugate points in length spaces is given in 5.5.

Projective spaces viewed as the bases of Hopf fibrations show that the conjugate radius estimate in Part 1 is optimal. The conclusion about the extendability of $\gamma$ is also optimal.

**Example.** Let $SO(n)$ act reducibly on the unit sphere, $\mathbb{S}^n$, in the usual way, by cohomogeneity one. Let $x \in \mathbb{S}^n / SO(n)$ be the orbit of the equator. The geodesic passing through $x$ at time 0 extends to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, where it is free of conjugate points, but it does not extend to any larger interval.

This example also shows that for Part 3 of Theorem $F$, it is not enough to know that $\dim \pi^{-1}(x) \geq k$ for *some* $x \in X$; we must also assume that $x$ realizes the diameter of $X$.

The remainder of the paper is organized as follows. In Section 1 we establish notations and conventions. In Section 2, we review the comparison lemma and focal radius theorems of [18]. Theorems A and D are proven in Section 3. In Section 4, we prove Theorem B, provide examples that show it is optimal, and state another finiteness theorem whose proof is essentially the same as the proof of Theorem B. The paper concludes with Section 5 where we prove Theorem F and state some of its corollaries for isometric groups actions.

**Remark.** To keep the exposition simple, we have stated all of our results with the global hypothesis $\mathrm{Ric}_k\ M \geq k \cdot \kappa$; however, most of them also hold with only the corresponding hypothesis about radial intermediate Ricci curvatures. That is, for any geodesic $\gamma$ that leaves our submanifold orthogonally at time 0, we only need

$$\sum_{i=1}^{k} \sec(\dot{\gamma}, E_i) \geq k \cdot \kappa$$

for any orthonormal set $\{\dot{\gamma}, E_1, \ldots, E_k\}$. This remark applies to Theorems A, D, and $F$, except for Part 2 of Theorem $F$ for which our proof still requires the global hypothesis.

## 1. Notations and conventions

Let $\gamma : (-\infty, \infty) \longrightarrow M$ be a unit speed geodesic in a complete Riemannian $n$–manifold $M$. Call an $(n-1)$–dimensional subspace $\Lambda$ of normal Jacobi fields along $\gamma$, *Lagrangian,* if the restriction of the Riccati operator to $\Lambda$ is self adjoint, that is, if

$$\left\langle J_1(t), J_2'(t)\right\rangle = \left\langle J_1'(t), J_2(t)\right\rangle$$

for all $t$ and for all $J_1, J_2 \in \Lambda$.

For a subspace $V \subset \Lambda$ we write

$$V(t) \equiv \{ J(t) \mid J \in V\} \oplus \left\{ J'(t) \;\middle|\; J \in V \text{ and } J(t) = 0 \right\}. \tag{2}$$

Given a submanifold $N$ of the Riemannian manifold $M$, we let $\nu(N)$ be its normal bundle. We use $\pi$ for the projection of $\nu(N)$ onto $N$, and $N_0$ for the 0–section of $\nu(N)$. If $\gamma$ is a geodesic with $\gamma'(0) \perp N$, we consider variations of $\gamma$ by geodesics that leave $N$ orthogonally at time 0. We let $\Lambda_N$ be the the corresponding variations fields; note that $\Lambda_N$ is Lagrangian. Lemma 4.1 on page 227 of [7] says that $\Lambda_N$ is the set of normal Jacobi fields $J$ given by:

$$\begin{aligned} \Lambda_N \equiv &\left\{ J | J(0) = 0,\ J'(0) \in \nu_{\gamma(0)}(N)\right\} \\ &\oplus \left\{ J | J(0) \in T_{\gamma(0)}N \text{ and } J'(0) = \mathrm{S}_{\gamma'(0)} J(0)\right\}, \end{aligned} \tag{3}$$

where $\mathrm{S}_{\gamma'(0)}$ is the shape operator of $N$ in the direction of $\gamma'(0)$, that is,

$$\begin{aligned}\mathrm{S}_{\gamma'(0)} &: T_{\gamma(0)}N \longrightarrow T_{\gamma(0)}N \text{ is}\\ \mathrm{S}_{\gamma'(0)} &: w \longmapsto \left(\nabla_w \gamma'(0)\right)^{TN}.\end{aligned}$$

For every $t \in \mathbb{R}$, we let $\mathcal{E}_t : \Lambda \longrightarrow T_{\gamma(t)}M$, be the evaluation map $\mathcal{E}_t(J) = J(t)$. Unless otherwise indicated, we suppose that $\mathcal{E}_t$ is injective on $(t_0, t_{\max})$. When this occurs, we say that $\Lambda$ is nonsingular on $(t_0, t_{\max})$.

Geodesics are parameterized by arc length, except if we say otherwise. $\gamma_v$ will be the unique geodesic tangent to $v$ at time 0.

Finally, we use sec to denote sectional curvature.

## 2. The comparison lemmas and their consequences

To prove Theorems A and D we exploit the new Jacobi field comparison lemmas from [18]. We review these here, and refer the reader to [18] for a full exposition.

Lagrangian subspaces in 2–dimensional constant curvature spaces are spanned by single Jacobi fields of the form $\tilde{f}E$, where $E$ is a parallel field. After rescaling the metric, $\tilde{f}$ is one of the following

$$\tilde{f}(t) = \begin{cases} (c_1 \sin t + c_2 \cos t) & \text{if } \kappa = 1, \\ (c_1 t + c_2) & \text{if } \kappa = 0, \\ (c_1 \sinh t + c_2 \cosh t) & \text{if } \kappa = -1, \end{cases} \tag{4}$$

for a choice of $c_1, c_2 \in \mathbb{R}$.

For a subspace $W \subset \Lambda$, write

$$W(t) = \{ J(t) \mid J \in W\} \oplus \left\{ J'(t) \;\middle|\; J \in W \text{ and } J(t) = 0 \right\},$$

and

$$P_{W,t} : \Lambda(t) \longrightarrow W(t)$$

for orthogonal projection. If $S$ is the Riccati operator associated to $\Lambda$, then to abbreviate we write

$$\text{Trace}\, S_t|_W \text{ for } \text{Trace}\left(P_{W,t} \circ S_t|_W\right).$$

Finally, recall that a subspace $\mathcal{V}$ of $\Lambda$ has full index on an interval $I$ if it contains any Jacobi field in $\Lambda$ that vanishes at some point of $I$.

We can now state the comparison lemmas from [18] that we use here.

**Lemma 2.1 (Intermediate Ricci Comparison).** *For $\kappa = -1, 0,$ or $1$, let $\gamma : (-\infty, \infty) \longrightarrow M$ be a unit speed geodesic in a complete Riemannian $n$–manifold $M$ with $\mathrm{Ric}_k \geq k \cdot \kappa$. Let $\tilde{\lambda}_\kappa : [t_0, t_{\max}) \longrightarrow \mathbb{R}$ be any solution of the scalar Riccati equation*

$$\tilde{\lambda}_\kappa' + \tilde{\lambda}_\kappa^2 + \kappa = 0. \tag{5}$$

*Let $\Lambda$ be a Lagrangian subspace of normal Jacobi fields along $\gamma$ with Riccati operator $S$, and let $W_{t_0} \perp \gamma'(t_0)$ be some $k$–dimensional subspace such that*

$$\mathrm{Trace}\, S_{t_0}|_{W_{t_0}} \leq k \cdot \tilde{\lambda}_\kappa\left(t_0\right). \tag{6}$$

*Denote by $\mathcal{V}$ the subspace of $\Lambda$ formed by those Jacobi fields that are orthogonal to $W_{t_0}$ at $t_0$ and by $H(t)$ the subspace of $\gamma'(t)^{\perp}$ that is orthogonal to $\mathcal{V}(t)$ at each $t \in (t_0, t_{max})$. Assume that $\mathcal{V}$ is of full index in the interval $[t_0, t_{max})$.*

*Then for all $t \in [t_0, t_{\max})$,*

$$\mathrm{Trace}\, S_t|_{H(t)} \leq k \cdot \tilde{\lambda}_\kappa\left(t\right). \tag{7}$$

*Moreover, if $\lim_{t \to t_{\max}^-} \tilde{\lambda}_\kappa\left(t\right) = -\infty$ then the Jacobi equation splits orthogonally along $\gamma$ in the interval $[t_0, t_{max})$ as*

$$\Lambda = \mathcal{V} \oplus \mathcal{H}$$

*where every nonzero Jacobi field $J \in \mathcal{H}$ is equal to $J = \tilde{f} \cdot E$, where $E$ is a unit parallel field with $E(t_0) \in W_{t_0}$, and $\tilde{f}$ is the function from (4) that satisfies $\tilde{f}\left(t_0\right) = \left|J\left(t_0\right)\right|$.*

**Lemma 2.2.** *Let $\gamma : [t_0, \infty) \longrightarrow M$ be a unit speed geodesic in a complete Riemannian $n$–manifold $M$ with $\mathrm{Ric}_k \geq 0$. Let $\Lambda$ be a Lagrangian subspace of normal Jacobi fields along $\gamma$ with Riccati operator $S$. Suppose that for some $k$–dimensional subspace $W_{t_0} \perp \gamma'(t_0)$ ,*

$$\mathrm{Trace}\, S_{t_0}|_{W_{t_0}} \leq 0. \tag{8}$$

*With $\mathcal{V}$ and $H(t)$ as in Lemma 2.1, the Jacobi equation splits orthogonally along $\gamma$ in the interval $[t_0, \infty)$ as*

$$\Lambda = \mathcal{V} \oplus \mathcal{H}$$

*where every nonzero Jacobi field $J \in \mathcal{H}$ is equal to $J = \tilde{f} \cdot E$, where $E$ is a unit parallel field with $E(t_0) \in W_{t_0}$, and $\tilde{f}$ is the function from (4) that satisfies $\tilde{f}(t_0) = |J(t_0)|$.*

We also need the focal radius theorem from [18].

**Theorem 2.3.** *For $k \geq 1$, suppose that $M$ is a complete Riemannian $n-$manifold with $\mathrm{Ric}_k \geq k$ and $N$ is any submanifold of $M$ with $\dim(N) \geq k$.*

1. *Counting multiplicities, every unit speed geodesic $\gamma$ that leaves $N$ orthogonally at time 0 has at least $\dim(N) - k + 1$ focal points for $N$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In particular, the focal radius of $N$ is $\leq \frac{\pi}{2}$.*
2. *If $N$ has focal radius $\frac{\pi}{2}$, then it is totally geodesic.*
3. *If $N$ is closed and has focal radius $\frac{\pi}{2}$, then the universal cover of $M$ is isometric to the sphere or a projective space with the standard metrics, and $N$ is totally geodesic in $M$.*

## 3. Second fundamental form, focal radius, and lower curvature bounds

In this section, we prove Theorems A and D. The first is a special case of the following result.

**Theorem 3.1.** *For $\kappa = -1, 0,$ or 1, let $M$ be a complete Riemannian $n-$manifold with $\mathrm{Ric}_k \geq k\kappa$, and let $N$ be any submanifold of $M$ with $\dim(N) \geq k$. Then for any unit normal vector $v$ to $N$, the shape operator of $N$ for $v$ satisfies*

$$\begin{aligned} \left|\sum_{i=1}^{k} \langle \mathrm{S}_v(e_i), e_i \rangle\right| &\leq k \cot(\mathrm{FocalRadius}(N, \gamma_v)) && \text{if } \kappa = 1, \\ \left|\sum_{i=1}^{k} \langle \mathrm{S}_v(e_i), e_i \rangle\right| &\leq \frac{k}{\mathrm{FocalRadius}(N, \gamma_v)} && \text{if } \kappa = 0, \qquad (9) \\ \left|\sum_{i=1}^{k} \langle \mathrm{S}_v(e_i), e_i \rangle\right| &\leq k \coth(\mathrm{FocalRadius}(N, \gamma_v)) && \text{if } \kappa = -1, \end{aligned}$$

*where $\{e_i\}_{i=1}^{k} \subset T_{\gamma_v(0)}N$ is any orthonormal set and $\mathrm{FocalRadius}(N, \gamma_v)$ is the focal radius of $N$ along $\gamma_v$.*

*Proof.* We set

$$\mathrm{ct}_\kappa(t) = \begin{cases} \cot(t) & \text{if } \kappa = 1, \\ 1/t & \text{if } \kappa = 0, \\ \coth(t) & \text{if } \kappa = -1. \end{cases}$$

Then $\mathrm{ct}_\kappa$ is an odd function that satisfies $\lim_{t\to 0^-} \mathrm{ct}_\kappa = -\infty$.

Let $\Lambda_N$ be the Lagrangian family along $\gamma_v$ from Equation (3). Let $S$ be the corresponding Riccati operator. Observe that at $t = 0$, the restriction of $S$ to the second summand in (3) coincides with the shape operator $S_v$ of $N$.

If the conclusion is false, there are $\{J_1, \ldots, J_k\} \subset \Lambda_N$ with $\{J_i(0)\}_{i=1}^k$ orthonormal and tangent to $N$ so that

$$J_i'(0)^T = S(J_i(0)) \text{ and}$$
$$\left|\Sigma_{i=1}^k \langle S(J_i(0)), J_i(0)\rangle\right| \geq k \cdot \mathrm{ct}_\kappa(\mathrm{FocalRadius}(N) - \alpha),$$

for some $\alpha \in (0, \mathrm{FocalRadius}(N))$. After possibly replacing $v$ with $-v$, we may assume that

$$\begin{aligned} &\Sigma_{i=1}^k \langle S(J_i), J_i\rangle|_0 \\ &\leq -k \cdot \mathrm{ct}_\kappa(\mathrm{FocalRadius}(N) - \alpha) \\ &= k \cdot \mathrm{ct}_\kappa(\alpha - \mathrm{FocalRadius}(N)), \text{ since } \mathrm{ct}_\kappa \text{ is an odd function} \\ &< 0. \end{aligned} \tag{10}$$

We apply Lemma 2.1 with $\Lambda = \Lambda_N$ and $W_{t_0} = \mathrm{span}\{J_i(0)\}$. To see that the hypotheses of Lemma 2.1 are satisfied, we note that:

- Inequality (10) gives us that Inequality (6) holds with
$$\tilde{\lambda}_\kappa(t) = \mathrm{ct}_\kappa(\alpha - \mathrm{FocalRadius}(N) + t) \text{ and } t_0 = 0.$$
- Since $\Lambda_N$ is nonsingular on $(0, \mathrm{FocalRadius}(N))$, its subspace $\mathcal{V}$ has full index on the interval $(0, \mathrm{FocalRadius}(N))$.

Thus it follows from Lemma 2.1 that for all $t_1 \in (0, \mathrm{FocalRadius}(N))$ there is a $k$–dimensional subspace $H(t_1) \subset \gamma'(t_1)$ so that

$$\mathrm{Tr}\, S|_{H(t_1)} \leq k \cdot \mathrm{ct}_\kappa(\alpha - \mathrm{FocalRadius}(N) + t_1).$$

Since $\alpha - \mathrm{FocalRadius}(N) < 0$ and $\lim_{t\to 0^-} \mathrm{ct}_\kappa = -\infty$, $\Lambda_N$ has a singularity by time $\mathrm{FocalRadius}(N) - \alpha$. This is a contradiction because $\Lambda_N$ is nonsingular on the interval $(0, \mathrm{FocalRadius}(N))$. □

*Proof of Theorem D.* Let $M$ be a complete Riemannian $n$–manifold with $\mathrm{Ric}_k \geq 0$, and let $N$ be any closed submanifold of $M$ with $\dim(N) \geq k$ and infinite focal radius. Let $v$ be any unit normal vector to $N$. As in Equation (3) we let

$$\Lambda_N \equiv \left\{ J | J(0) = 0,\ J'(0) \in \nu_{\gamma_v(0)} N \right\} \\ \oplus \left\{ J | J(0) \in T_{\gamma_v(0)} N \text{ and } J'(0) = \mathrm{S}_v J(0) \right\}.$$

We set

$$\mathcal{V} \equiv \left\{ J | J(0) = 0,\ J'(0) \in \nu_{\gamma_v(0)} N \right\}, \text{ and} \\ W \equiv \left\{ J | J(0) \in T_{\gamma_v(0)} N \text{ and } J'(0) = \mathrm{S}_v J(0) \right\}.$$

Since $N$ has no focal points, $\Lambda_N$ has no singularities on $\mathbb{R} \setminus \{0\}$. Thus for all $J \in \Lambda_N \setminus \{0\}$ and all $t \in \mathbb{R} \setminus \{0\}$, $J(t) \neq 0$. By replacing $v$ with $-v$, if necessary, we may assume that

$$\mathrm{Tr}\left(\mathrm{S}|_W(0)\right) \leq 0. \tag{11}$$

By Lemma 2.2, it follows that $t \longmapsto \Lambda(t)$ splits orthogonally into the parallel distributions

$$\Lambda(t) \equiv W(t) \oplus \mathcal{V}(t),$$

and every field in $W$ is parallel. Since we started with an arbitrary normal vector, $N$ is totally geodesic, and Parts 1 and 4 are proven. Part 2 is a consequence of the Hopf-Rinow Theorem (see Part (e) of Theorem 2.8 on page 147 of [7]).

Part 3 follows by observing that $\exp_N^\perp : \left(\nu(N), \left(\exp_N^\perp\right)^* (g)\right) \longrightarrow (M, g)$ is a local isometry, so as in the proof of Cartan-Hadamard, $\exp_N^\perp$ is a cover (see Lemma 3.3 on page 150 of [7] or Lemma 5.6.4 of [26]). Part 5 follows from the fact that every field in $W$ is parallel.

To prove Part 6, let II be the second fundamental form of image$(\Phi)$. Since

$$\gamma_{V(s)} : t \longmapsto \Phi(t, s) = \exp_{c(s)}^\perp (tV(s))$$

is a geodesic, $\mathrm{II}\left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial t}\right) = 0$, and since $\frac{\partial \Phi}{\partial s}$ is parallel along $\gamma_{V(s)}$, $\mathrm{II}\left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial s}\right) = 0$. To determine $\mathrm{II}\left(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial s}\right)$, observe that the lift, $\left(\exp_N^\perp\right)^* \left(\frac{\partial \Phi}{\partial s}\right)$, of $\frac{\partial \Phi}{\partial s}$

via $\exp_N^{\perp}$, is a basic horizontal, geodesic field for the Riemannian submersion

$$\pi : (\nu(N), \left(\exp_N^{\perp}\right)^* (g)) \longrightarrow N.$$

Thus $\mathrm{II}\left(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial s}\right) = \nabla_{\frac{\partial \Phi}{\partial s}} \frac{\partial \Phi}{\partial s} \equiv 0$, and the image of $\Phi$ is totally geodesic. Since $\frac{\partial \Phi}{\partial s}$ is a parallel Jacobi field along $\gamma_{V(s)}$, the image of $\Phi$ is flat.

To prove Part 7, consider a $V \in \mathcal{V}$ along with orthonormal parallel fields $J_1, \ldots, J_{k-1}$ in $W$. Since $\sec(\gamma_v', J_i) \equiv 0$ and $\mathrm{Ric}_k \geq 0$, $\sec(\gamma_v', V) \geq 0$. Since $\Lambda(t) \equiv W(t) \oplus \mathcal{V}(t)$ is a parallel, orthogonal splitting, all curvatures of the form $\sec(\gamma_v', \cdot)$ are nonnegative.

Since $\mathrm{Ric}_k M \geq 0$, it follows from the Gauss Equation that to prove Part 8, it suffices to show that

$$\langle S(J), J \rangle \geq 0,$$

for all $J \in \Lambda_N$ and all $t \geq 0$. Since $\Lambda_N(t) \equiv W(t) \oplus \mathcal{V}(t)$ is a parallel, orthogonal splitting and $\langle S(J), J \rangle \equiv 0$ for all $J \in W$, it suffices to show that

$$\langle S(J), J \rangle \geq 0$$

for all $J \in \mathcal{V}$ and all $t \geq 0$. If not, then for some $t_0 > 0$ and some $J \in \mathcal{V}$, $\langle S(J), J \rangle < 0$. Set

$$U \equiv \{J, L_1, \ldots, L_{k-1}\},$$

where $L_1, \ldots, L_{k-1}$ are $(k-1)$–linearly independent fields of $W$. It follows that for some $c > t_0$,

$$Tr\left(S|_U\right)(t_0) < \frac{1}{t_0 - c} < 0,$$

and hence from Lemma 2.2 that $\Lambda_N$ has a singularity, which is contrary to our hypothesis that $N$ has infinite focal radius. □

In the case that $M$ is not simply connected, we have the following structure result.

**Corollary 3.2.** *Let $N$ be a closed submanifold in a compact nonnegatively curved manifold $M$. If $N$ has infinite focal radius, then the universal cover,*

*$\tilde{M}$ splits isometrically as*

$$\tilde{M} = \tilde{N}_0 \times \mathbb{R}^m$$

*where $\tilde{N}_0$ is compact and simply connected, and the universal cover $\tilde{N}$ of $N$ is isometrically embedded in $\tilde{M}$ as*

$$\tilde{N} = \tilde{N}_0 \times \mathbb{R}^l,$$

*where $\mathbb{R}^l$ is an affine subspace of $\mathbb{R}^m$.*

*Proof.* Let $\pi : \tilde{M} \longrightarrow M$ by the universal cover. By Theorem 9.1 in [2], $\tilde{M}$ splits isometrically as

$$\tilde{M} = M_0 \times \mathbb{R}^m$$

where $M_0$ is compact and simply connected. By Part 3 of Theorem D, $\pi : \tilde{M} \longrightarrow M$ factors through $\exp_N^{\perp} : \nu(N) \longrightarrow M$. That is we have a Riemannian cover $p : \tilde{M} \longrightarrow \nu(N)$ so that $\pi = \exp_N^{\perp} \circ p$. Since every normal vector to the zero section, $N_0$, in $\nu(N)$ exponentiates to a ray, every normal vector to $\pi^{-1}(N)$ exponentiates to a ray. Since $\tilde{M}$ is the metric product, $M_0 \times \mathbb{R}^m$, every normal vector to $\pi^{-1}(N)$ is tangent to an $\mathbb{R}^m$–factor. Thus every tangent space to $\pi^{-1}(N)$ has the form $TM_0 \times \mathbb{R}^l$, where $\mathbb{R}^l$ is an affine subspace of $\mathbb{R}^m$. Since $\pi^{-1}(N)$ is totally geodesic and without boundary it follows that $\pi^{-1}(N)$ is $M_0 \times \mathbb{R}^l$ where $\mathbb{R}^l$ is an affine subspace of $\mathbb{R}^m$. □

### 3.1. What can be done with just classical Riccati comparison?

Although weak versions of all of our results can be obtained using just classical Riccati comparison, to the best of our knowledge no theorem discussed here can be proven with out the Transverse Jacobi Equation. As a concrete example, we point out that classical comparison yields the following weak form of Theorem A.

**Weak Form of Theorem A:** *For $\kappa = -1, 0,$ or $1$, let $M$ be a complete Riemannian $n$–manifold with sectional curvature $\geq \kappa$, and let $N$ be any hypersurface of $M$. Then at every point of $N$ there is a* ***single*** *vector $v$ so that*

$$\mathrm{II}_N(v,v) \geq -\frac{|v|^2}{\cot(\mathrm{FocalRadius}(N))} \quad \text{if } \kappa = 1$$

$$\mathrm{II}_N(v,v) \geq -\frac{|v|^2}{\mathrm{FocalRadius}(N)} \quad \text{if } \kappa = 0$$

$$\mathrm{II}_N(v,v) \geq -\frac{|v|^2}{\coth(\mathrm{FocalRadius}(N))} \quad \text{if } \kappa = -1.$$

To clarify how classical comparison fails to yield Theorem A, we note that the sectional curvature version of Lemma 2.1 implies that if Inequality (7) fails for all 1–dimensional subspaces $H(t) \subset T_{\gamma(t)}M^{\perp}$ then Inequality (6) fails for all 1–dimensional subspaces $W_{t_0} \subset T_{\gamma(t_0)}M^{\perp}$. In contrast, the classical theorem of [8] only gives that Inequality (6) fails for some $W_{t_0} \subset T_{\gamma(t_0)}M^{\perp}$. Examples 2.37 and 2.38 in [18] show that there is no classical analog to Lemma 2.1, (also see the commentary after Lemma E in [18].)

## 4. Submanifold restrictions

The main step in the proof of Theorem B is to show that the intrinsic metrics on all of the submanifolds satisfy the hypothesis of Cheeger's Finiteness Theorem, [5].

**Lemma 4.1.** *Let $M$ be a compact Riemannian manifold. Given $D, r > 0$ let $\mathcal{S}$ be the class of closed Riemannian manifolds that can be isometrically embedded into $M$ with focal radius $\geq r$ and intrinsic diameter $\leq D$. Then there are positive numbers $K, v > 0$ so that for every $S \in \mathcal{S}$,*

$$|\sec_S| \leq K \text{ and } vol(S) > v.$$

*Proof.* The compactness of $M$ gives us ambient upper and lower curvature bounds. Combined with Theorem A, we get the existence of $K$.

It remains to derive a uniform lower volume bound for the $S \in \mathcal{S}$. To do this we use the first display formula on Page 1 of [20]:

$$\mathrm{vol}(M) \leq \mathrm{vol}(N) \cdot f_\delta(\mathrm{diam}(M), \Lambda).$$

Here $N$ is a compact, embedded submanifold of $M$, $\delta$ is a lower curvature bound for $M$, $\Lambda$ is an upper bound for the mean curvature of $N$, and the function $f_\delta$ is given explicitly on Page 453 of [20]. Theorem A gives us an upper bound for $\Lambda$ and hence a $C > 0$ so that

$$f_\delta(\mathrm{diam}(M), \Lambda) \leq C.$$

Setting $v = \frac{\mathrm{vol}(M)}{C}$ completes the proof. □

Recall that the Cheeger-Gromov compactness theorem states

**Theorem 4.2 (see [4, Theorem 3.6], [26, Theorem 11.3.6]).** *Given $0 < \beta < \alpha < 1$, $k, K \in \mathbb{R}$, $v, D > 0$, and $n \in \mathbb{N}$, let $\{M_i\}_{i=1}^{\infty}$ be a sequence of* *closed Riemannian manifolds with*

$$k \leq \sec M_i \leq K,\ \operatorname{vol}(M_i) \geq v,\ \textit{and}\ \operatorname{Diam}(M_i) \leq D.$$

*Then there is a $C^{1,\alpha}$–Riemannian manifold $M_\infty$ and a subsequence of $\{M_i\}_{i=1}^{\infty}$ that converges to $M_\infty$ in the $C^{1,\beta}$ topology.*

*Proof of Theorem B.* It follows from the previous Lemma 4.1 that the class $\mathcal{S}$ satisfies the hypotheses of Cheeger's finiteness theorem. So any sequence $\{S_i\} \subset \mathcal{S}$ has a subsequence (also called $\{S_i\}$) that converges in the $C^{1,\beta}$–topology to an abstract $C^{1,\alpha}$ Riemannian manifold $(S_\infty, g_\infty)$. Let $\varphi_i : S_\infty \longrightarrow S_i$ be diffeomorphisms so that $\varphi_i^*(g_i) \stackrel{C^{1,\beta}}{\longrightarrow} g_\infty$. Let $f_i : S_i \longrightarrow M$ be the sequence of inclusions of $S_i$ into $M$. Composing gives a sequence $f_i \circ \varphi_i : S_\infty \longrightarrow M$, that is uniformly bounded in the $C^{1,\beta}$–topology. From Arzela-Ascoli it follows that $\{f_i \circ \varphi_i\}_i$ subconverges in the $C^{1,\beta}$–topology to an isometric embedding $f_\infty : S_\infty \longrightarrow M$. □

**Example 4.3 (Theorem B is optimal).** The isometric embedding theorem of J. Nash says that for given $k$, there is some $n = n(k)$ such that any $k$-dimensional Riemannian manifold embeds isometrically in $\mathbb{R}^n$. Consider then any compact Riemannian manifold, and rescale its metric so that its diameter is bounded above by 1. If needed, rescale the metric in $\mathbb{R}^n$ so that the image of an isometric embedding $f : M \to \mathbb{R}^n$ is contained in the interior of some fundamental domain for the covering space $\pi : \mathbb{R}^n \to \mathbb{T}^n$. Taking the composition $\pi \circ f$, we get an isometric embedding of $M$ into $\mathbb{T}^n$ with intrinsic diameter bounded above; thus Theorem B is optimal in the sense that its conclusion is false if the hypothesis about the lower bound on the focal radii is removed.

To see that the hypothesis about the intrinsic diameters can not be removed, let $\lambda\mathbb{S}^1$ be the circle of radius $\lambda$. For each $k$-manifold $(M, g)$, choose a rational number $\lambda$ so that the image of the isometric embedding

$$j : \lambda M \hookrightarrow \lambda\mathbb{T}^n$$

has focal radius greater or equal than 1.

Next use that, for the given $\lambda$, there is an isometric embedding

$$\iota : \lambda\mathbb{S}^1 \hookrightarrow \mathbb{T}^2,$$

and let

$$I : \lambda\mathbb{T}^n \to \mathbb{T}^{2n}$$

be the product embedding. The images of the composition $I \circ j : (M, g) \hookrightarrow \mathbb{T}^{2n}$ all have focal radius $\geq 1$. Thus Theorem B is false if the hypothesis about the upper bound on the diameter is removed.

### 4.1. Other finiteness statements

Various other finiteness theorems for submanifolds follow by combining the proof of Theorem B with existing results. For example, using the main theorem of [14] we have

**Theorem 4.4.** *Given $k \in \mathbb{R}$, $v, D > 0$, $n \in \mathbb{N}$, and $r > 0$, let $\mathcal{M}(k, v, n)$ be the class of closed Riemannian $n$-manifolds with sectional curvature $\geq k$ , volume $\geq v$, and diameter $\leq D$, and let $\mathcal{S}$ be the class of closed Riemannian manifolds that can be isometrically embedded into an element of $\mathcal{M}(k, v, n)$ so that the image has focal radius $\geq r$ and intrinsic diameter $\leq D$. Then $\mathcal{S}$ contains only finitely many homeomorphism types.*

## 5. Submetries and conjugate points

In this section we prove Theorem F. We start, in subsection 5.1 with a establishing some basic facts about holonomy for manifold submetries. We then prove Theorem F in subsection 5.2.

### 5.1. Submetries and holonomy

Throughout this section, we assume $M$ is an Alexandrov space with curvature bounded from below, $\pi : M \longrightarrow X$ is a submetry, and $\gamma : [0, b] \longrightarrow X$ is a geodesic.

The proof of Lemma 2.1 in [3] gives us the following.

**Proposition 5.1.** *1. Given any $y \in \pi^{-1}(\gamma(0))$, there is a lift of $\gamma$ starting at $y$.*
*2. If for some $\varepsilon > 0$, $\gamma$ extends as a geodesic to $[-\varepsilon, b]$, then the lift in Part 1 is unique.*

Part 2 allows us to define holonomy maps between the fibers of $\pi$ over the *interior* of $\gamma$ as follows.

**Definition 5.2.** Given any $s, t \in (0, b)$, we define the holonomy maps

$$H_{s,t} : \pi^{-1}(\gamma(s)) \longrightarrow \pi^{-1}(\gamma(t))$$

by

$$H_{s,t}(x) = \tilde{\gamma}_x(t),$$

where $\tilde{\gamma}_x$ is the unique lift of $\gamma$ so that $\tilde{\gamma}_x(s) = x$.

**Proposition 5.3.** *If $M$ is Riemannian and $\pi$ is a manifold submetry, then for all $s, t \in (0, b)$, $H_{s,t}$ is a $C^\infty$ diffeomorphism.*

*Proof.* Choose $\varepsilon_0 > 0$ so that $[s - \varepsilon_0, t + \varepsilon_0] \subset (0, b)$. By compactness we cover $[s, t]$ by finite number of open intervals of the form

$$(s_i - \iota_i, s_i + \iota_i),$$

were $\iota_i$ is one-fourth of the injectivity radius of $\pi^{-1}(\gamma(s_i))$, and

$$s - \varepsilon_0 = s_0 < s_1 < \cdots < s_m = t + \varepsilon_0.$$

Let $F_t^i$ be the flow of grad $\text{dist}_{\pi^{-1}(\gamma(s_i))}$. Then for $r_1, r_2 \in (s_i, s_{i+1} + \iota_{i+1})$, $H_{r_1,r_2}$ is the restriction of $F^i_{r_2 - r_1}$ to $\pi^{-1}(\gamma(r_1))$ and hence is a diffeomorphism onto its image $\pi^{-1}(\gamma(r_2))$. Since $H_{s,t}$ is the composition of a finite number of the diffeomorphisms $H_{r_1,r_2}$, it follows that $H_{s,t}$ is a diffeomorphism. □

**Remark 5.4.** For $\gamma$ and $\tilde{\gamma}_x$ as above, we define the holonomy fields along $\tilde{\gamma}_x$ to be the Jacobi fields that correspond to variations by lifts of $\gamma$. If the Lagrangian subspace $\Lambda_{\pi^{-1}(\gamma(s))}$ has no singularities on $(s, t)$, that is, if the evaluation map $\mathcal{E}_u : \Lambda_{\pi^{-1}(\gamma(s))} \longrightarrow T_{\gamma(u)}M$ is one-to-one for all $u \in (s, t)$, it follows that a field $J \in \Lambda_{\pi^{-1}(\gamma(s))}$ is holonomy if $J(u) \in T\pi^{-1}(\gamma(u))$ for some $u \in (s, t)$.

### 5.2. Submetries and variational conjugate points

The following is the precise sense in which the term "conjugate point" is used in Theorem F.

**Definition 5.5.** (Variational Conjugate Point) Let $\gamma : [0, b] \longrightarrow X$ be a unit speed geodesic in a complete, locally compact length space $X$. We say that $\gamma(b)$ is variationally conjugate to $\gamma(0)$ along $\gamma$ if and only if for some $\varepsilon > 0$, there is a continuous map $V : [0,b] \times (-\varepsilon, \varepsilon) \longrightarrow X$ with the following properties.

1. For all $t \in (0,b)$,
$$\gamma(t) = V(t,0).$$
2. There is a $C > 0$ and a $t_0 \in (0,b)$ so that for all sufficiently small $s \neq 0$,
$$\mathrm{dist}(\gamma(t_0), V(t_0,s)) \geq Cs.$$
3. For each $s \in (-\varepsilon, \varepsilon)$,
$$t \mapsto V(t,s)$$
is a unit speed geodesic on $[0,b]$.
4. At the end points,
$$\mathrm{dist}(V(0,0), V(0,s)) \leq o(s) \text{ and}$$
$$\mathrm{dist}(V(b,0), V(b,s)) \leq o(s).$$

In the Riemannian case, this coincides with the usual definition of conjugacy, so it is not surprising that geodesics in Alexandrov spaces stop minimizing distance after variational conjugate points.

**Proposition 5.6.** *If $X$ is an Alexandrov space with curvature bounded from below and $\gamma(b)$ is variationally conjugate to $\gamma(0)$ along $\gamma$, then for all $\varepsilon > 0$, either $\gamma$ does not extend to $[0, b+\varepsilon]$ or $\gamma|_{[0,b+\varepsilon]}$ is not minimal.*

*Proof.* Suppose that $\gamma|_{[0,b+\eta]}$ is minimal and that $\eta$ is small enough so that $t_0 \in (\eta, b-\eta)$. Since the comparison angle $\tilde{\sphericalangle}(\gamma(0), \gamma(b), \gamma(b+\eta))$ is $\pi$, it follows that

$$\sphericalangle(\gamma(0), V(b,s), \gamma(b+\eta)) \geq \tilde{\sphericalangle}(\gamma(0), V(b,s), \gamma(b+\eta)) > \pi - o\left(\frac{s}{\eta}\right),$$
$$\sphericalangle(\gamma(t_0), V(b,s), \gamma(b+\eta)) \geq \tilde{\sphericalangle}(\gamma(t_0), V(b,s), \gamma(b+\eta)) > \pi - o\left(\frac{s}{\eta}\right).$$

The previous two inequalities, together with a hinge comparison argument in the *space of directions* of $X$ at $V(b,s)$, gives

$$\sphericalangle(\gamma(0), V(b,s), \gamma(t_0)) \leq o\left(\frac{s}{\eta}\right).$$

So by hinge comparison in $X$,

$$\operatorname{dist}\left(V\left(t_0,s\right),\gamma\left(t_0\right)\right)\leq o\left(\frac{s}{\eta}\right),$$

but this is contrary to Part 2 of the definition of variational conjugacy. □

**Lemma 5.7.** *Let $\pi : M \longrightarrow X$ be a manifold submetry. Let $\gamma : [0, b] \longrightarrow X$ be a geodesic, and let $\tilde{\gamma}$ be a horizontal lift of $\gamma$ that has its first focal point for $\pi^{-1}\left(\gamma\left(0\right)\right)$ at $b_0 \in (0, b)$. Then $\gamma$ has a variational conjugate point at $b_0$.*

*Proof.* Since $\tilde{\gamma}$ has its first focal point for $\pi^{-1}\left(\gamma\left(0\right)\right)$ at $b_0$, there is a variation

$$\tilde{V} : [0, b_0] \times (\varepsilon, \varepsilon) \longrightarrow M$$

of $\tilde{\gamma}$ by geodesics that leave $\pi^{-1}\left(x\right)$ orthogonally at time 0 with

$$\tilde{V}\left(0,s\right)\in\pi^{-1}\left(x\right),\quad \left.\frac{\partial}{\partial s}\tilde{V}\right|_{(b_0,0)}=0 \text{ and } \left.\frac{\partial}{\partial s}\tilde{V}\right|_{(t,0)}\neq 0 \tag{12}$$

for all $t \in (0, b_0)$. If $\frac{\partial}{\partial s}\tilde{V}\left(t,0\right)$ is vertical for all $t \in (0, b_0)$, then by Remark 5.4, $\tilde{V}$ is a holonomy field. In this event, since $b_0 \in (0, b)$, it follows from Proposition 5.3 that

$$\left.\frac{\partial}{\partial s}\tilde{V}\right|_{(b_0,0)}\neq 0,$$

which is contrary to the second equation in (12). So for some $t_0 \in (0, b_0)$,

$$\left.\frac{\partial}{\partial s}\tilde{V}\right|_{(t_0,0)} \text{ is not vertical.} \tag{13}$$

Projecting $\tilde{V}$ under $\pi$ produces a variation $V$ of $\gamma$ in $X$ by geodesics. It follows from (13) that for all sufficiently small $s \neq 0$, there is a $C > 0$ so that

$$\operatorname{dist}\left(\gamma\left(t_0\right),V\left(t_0,s\right)\right)\geq Cs.$$

Since

$$\tilde{V}\left(0,s\right)\in\pi^{-1}\left(x\right),$$
$$\operatorname{dist}\left(\tilde{V}\left(b_0,0\right),\tilde{V}\left(b_0,s\right)\right)\leq o\left(s\right),$$

and $\pi$ is distance nonincreasing,

$$V(0,0) = x \text{ and}$$
$$\operatorname{dist}(V(b_0,0), V(b_0,s)) \leq o(s).$$

Thus $\gamma$ has a variational conjugate point at time $b_0$. □

*Proof of Theorem F.* Suppose that

$$\pi : M \longrightarrow X$$

is a manifold submetry of a complete Riemannian $n$–manifold with $\mathrm{Ric}_k \geq k$ and that for some $x \in X$, $\dim \pi^{-1}(x) \geq k$. Let $\gamma$ be a geodesic of $X$ emanating from $x$. Suppose that $\gamma$ extends to an interval $I$ that properly contains $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $\gamma$ is defined on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and either extends past $\frac{\pi}{2}$ or extends past $-\frac{\pi}{2}$. Without loss of generality, assume that $\gamma$ extends past $\frac{\pi}{2}$. By Part 1 of Theorem 2.3, every horizontal lift of $\gamma$ has its first focal point for $\pi^{-1}(x)$ at some $t_0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If $\tilde{\gamma}$ is such a lift and $t_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then by Lemma 5.7, $\gamma(t_0)$ is variationally conjugate to $\gamma(0)$. If $t_0 = -\frac{\pi}{2}$, then for convenience, we reorient $\gamma$ so that it extends past $-\frac{\pi}{2}$ and has its first focal point at $\frac{\pi}{2}$. Applying Part 2 of Lemma 2.1 with $\kappa = 1$, $t_0 = 0$, $t_{\max} = \frac{\pi}{2}$, $\tilde{\lambda} = \cot\left(t + \frac{\pi}{2}\right)$, and

$$W_{t_0} = \left\{ J | J(0) \in T_{\tilde{\gamma}(0)}\pi^{-1}(x) \text{ and } J'(0) = \mathrm{S}_{\tilde{\gamma}'(0)} J(0) \right\},$$

we see that $W_{t_0}$ is spanned by Jacobi fields of the form $\sin\left(t + \frac{\pi}{2}\right) E$, where $E$ is a parallel field. In particular, $\mathrm{S}_{\tilde{\gamma}'(0)} \equiv 0$. So we can apply Part 1 of Lemma 2.1 and conclude that $\tilde{\gamma}$ also has a focal point at $s_0 \in \left[-\frac{\pi}{2}, 0\right)$. Since $\gamma$ extends past $-\frac{\pi}{2}$, by Theorem 5.7, $\gamma(s_0)$ is variationally conjugate to $\gamma(0)$.

If all geodesics emanating from $x \in X$ extend to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and are free of variational conjugate points on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then by Lemma 5.7, $\pi^{-1}(x)$ has focal radius $\geq \frac{\pi}{2}$. So if $\dim \pi^{-1}(x) \geq k$, then by Part 3 of Theorem 2.3, the universal cover of $M$ is isometric to the sphere or a to projective space with the standard metrics, and $\pi^{-1}(x)$ is totally geodesic in $M$.

To prove Part 3 of Theorem F, suppose that $p, q \in X$ are at maximal distance $> \frac{\pi}{2}$, and $\dim \pi^{-1}(p) \geq k$. Since $M$ is a compact Riemannian manifold and

$$\pi : M \longrightarrow X$$

is a submetry, $X$ is an Alexandrov space with some lower curvature bound. Since $p$ and $q$ are at maximal distance, $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are at maximal distance. It follows that for any $\tilde{p} \in \pi^{-1}(p)$,

$$\Uparrow_{\tilde{p}}^{\pi^{-1}(q)} \equiv \left\{ \tilde{v} \in \nu_{\tilde{p}}^{1}\left(\pi^{-1}(p)\right) \;\middle|\; \gamma_{\tilde{v}} \text{ is a segment from } \tilde{p} \text{ to } \pi^{-1}(q) \right\}$$

is a $\frac{\pi}{2}$–net in $\nu_{\tilde{p}}^{1}\left(\pi^{-1}(p)\right)$. Let $W$ be any $k$–dimensional subspace of $T_{\tilde{p}}\pi^{-1}(p)$. For $\tilde{v} \in \nu_{\tilde{p}}^{1}\left(\pi^{-1}(p)\right)$ and $\{E_i\}_{i=1}^{k}$ an orthonormal basis for $W$,

$$\mathrm{Trace}\left(S_{\tilde{v}}|_W\right) = \left\langle -\sum_{i=1}^{k} \mathrm{II}\left(E_i, E_i\right), \tilde{v} \right\rangle.$$

Since $\Uparrow_{\tilde{p}}^{\pi^{-1}(q)}$ is a $\frac{\pi}{2}$–net in $\nu_{\tilde{p}}^{1}\left(\pi^{-1}(p)\right)$, it follows that for some $\tilde{v} \in \Uparrow_{\tilde{p}}^{\pi^{-1}(q)}$,

$$\mathrm{Trace}\left(S_{\tilde{v}}|_W\right) \leq 0. \tag{14}$$

Let $\Lambda_{\pi^{-1}(p)}$ be the Lagrangian family of Jacobi fields along $\gamma_{\tilde{v}}$ that correspond to variations by geodesics that leave $\pi^{-1}(p)$ orthogonally at time 0. Then Inequality (14) combined with Lemma 2.1 gives us that $\gamma_{\tilde{v}}$ has a focal point in $\left[0, \frac{\pi}{2}\right]$. As before, it follows that either $\pi \circ \gamma_{\tilde{v}}$ does not extend to an interval that properly contains $\left[0, \frac{\pi}{2}\right]$, or $\pi \circ \gamma_{\tilde{v}}$ has a variational conjugate point in $\left[0, \frac{\pi}{2}\right]$. Since $\pi \circ \gamma_{\tilde{v}}$ is a minimal geodesic from $p$ to $q$ and $\mathrm{dist}(p, q) > \frac{\pi}{2}$, the former case is excluded. The latter case implies, via Proposition 5.6, that for all $\varepsilon > 0$, $\pi \circ \gamma_{\tilde{v}}|_{\left[0, \frac{\pi}{2}+\varepsilon\right]}$ is not minimal, so it is also contrary to our hypothesis that $\mathrm{dist}(p, q) > \frac{\pi}{2}$. □

**Remark.** By Theorem 1 of [27], $X$ needs not have positive Ricci curvature, even when $\pi$ is a Riemannian submersion. So neither the first nor third conclusion of Theorem F follow from the Bonnet-Myers Theorem.

**Remark.** There are also various notions of conjugacy in length spaces proposed by Shankar and Sormani in [30]. Our variational notion is more readily adaptable to the situation of Theorem F than are any of those in [30]. All of the definitions have the common feature that $\gamma$ stops minimizing after a conjugate point.

**Remark.** By results in [12] and [34], the possibilities for $\pi$ in Part 2 of Theorem F can be listed, if $\pi$ is a Riemannian submersion. More generally, Riemannian foliations of round spheres are classified if they are either nonsingular ([21]) or they are singular and have fiber dimension $\leq 3$ ([29]). However, the singular Riemannian foliations of round spheres have not been classified, and there is an abundance of examples ([28]).

The version of Theorem F when $k = 1$ yields, via a different proof, the inequality statements of Chen and Grove in Theorems A and B in [6], with the additional information about the behavior of geodesics from conclusion 1. In particular, if $\pi$ is a Riemannian submersion, it follows that the conjugate radius of $X$ is $\leq \frac{\pi}{2}$. For a Riemannian submersion $\pi : M^{n+k} \longrightarrow B^n$ with the sectional curvature of $M \geq 1$, Theorem A of [10] gives that

$$\pi \frac{n-1}{k+n-1} \geq \operatorname{conj}(B).$$

In particular, the conjugate radius of $B$ is $\leq \frac{\pi}{2}$ if $k \geq n-1$. (Cf also Corollary 1.2 of [10].)

Theorem F has the following consequence for cohomogeneity one actions.

**Corollary 5.8.** *Let $M$ be a complete Riemannian manifold $M$ with* $\mathrm{Ric}_k \geq k$. *If $G \times M \to M$ is an isometric, cohomogeneity one action with a singular orbit of dimension $\geq k$, then the following hold.*
*1. The diameter of $M/G$ is $\leq \frac{\pi}{2}$.*
*2. If the diameter of $M/G$ is $\frac{\pi}{2}$, then the universal cover of $M$ is isometric to the sphere or a projective space with the standard metrics, and the singular orbits are totally geodesic in $M$.*

Part 3 of Theorem F has the following corollary.

**Corollary 5.9.** *Let $M$ be a complete Riemannian manifold $M$ with* $\mathrm{Ric}_k \geq k$. *If $G \times M \to M$ is an isometric group action, the diameter of $M/G$ is $> \frac{\pi}{2}$, and $x$ is a point that realizes the diameter of $M/G$, then* $\dim \pi^{-1}(x) \leq k-1$.

*In particular, if $G \times M \longrightarrow M$ is as above and is also a cohomogeneity one action, then both singular orbits have dimension $\leq k-1$.*

**Remark.** The sectional curvature case of Corollary 5.9 can be inferred from Corollary 2.7 of [6].

Examples $D$ and $E$ of [18] show that the hypothesis about the dimensions of the submanifolds can not be removed from Theorem F.

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Department of Mathematics, Universidad Autónoma de Madrid
and ICMAT CSIC-UAM-UCM-UC3M, 28049 Madrid, Spain
*E-mail address*: `luis.guijarro@uam.es`

Department of Mathematics, University of California
Riverside, CA 92521, USA
*E-mail address*: `fred@math.ucr.edu`

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