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Quantum Schur duality of affine type C with three parameters

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We establish a three-parameter Schur duality between the affine Hecke algebra of type C and a coideal subalgebra of quantum affine $\mathfrak{sl}_n$. At the equal parameter specializations, we obtain Schur dualities of types BCD.

1. Introduction

The classic Schur duality exhibits the fundamental interactions between representation theories of general linear Lie algebras and symmetric groups. A quantized Schur duality was obtained by Jimbo [13] between quantum groups and Hecke algebras of type A. There has also been various versions of affine type A Schur duality; cf. [5, 12].

In developing a Kazhdan-Lusztig theory of (super) type BCD, Bao and Wang [3] were led to a Schur type duality between Hecke algebra of type B and an $\imath$quantum group which is a coideal subalgebra of a quantum group of type A. There has been further development of such dualities which involve Hecke algebra of type B of unequal or two parameters; see [1, 4]. We recall a coideal subalgebra $\mathbb{U}^\imath$ of a quantum group $\mathbb{U}$ together form a quantum symmetric pair $(\mathbb{U}, \mathbb{U}^\imath)$; see [16, 17].

The goal of this paper is to formulate and establish a Schur $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n), \mathbb{H})$-duality on $\mathbb{V}^{\otimes d}$, for $n \geq 2d+2$ (and three additional variants). Here $\mathbb{V}$ is an infinite-dimensional vector space with a basis parametrized by $\mathbb{Z}$, $\mathbb{H}$ denotes the Hecke algebra of affine type C_d in three parameters, and $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$ is an affine $\imath$quantum group which is a coideal subalgebra of the affine type A quantum group $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$. The actions of $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$ and $\mathbb{H}$ on $\mathbb{V}^{\otimes d}$ are given by explicit formulas.

It is well known (cf. [15, 18, 19]) there is a 3-parameter Hecke algebra $\mathbb{H}$ of affine type C over $\mathbb{Q}(q, q_0, q_1)$ which specializes to all kinds of Hecke algebras of classical affine types. Remarkably, in the general theory of quantum symmetric pairs [16, 17], a coideal subalgebra of the quantum groups of affine type A allows different parameters. In our setting suitable choices of the parameters in the coideal subalgebras correspond to the 3 parameters of Hecke algebras of affine type C.

A geometric approach and a Hecke algebraic approach were systematically developed in [9, 10] (also see [2]) toward the realizations of coideal subalgebras of quantum groups of affine type A and constructions of their canonical bases. A Schur duality involving affine Hecke algebra of type C (of single parameter) was implicit in these papers and could be developed in those frameworks naturally. It is conceivable that there will be other type of Schur dualities (of single parameter) if one starts with different types of affine flag varieties or affine Hecke algebras, and it would take considerable work to set this up. Upon single parameter specializations, the 3-parameter duality here immediately leads to several dualities involving Hecke algebras of different affine types, which are expected to arise from geometric constructions using different types of flag varieties. In this way, the 3-parameter Schur duality in this paper could serve as a helpful guideline on the geometric and categorical realizations of various equal parameter Schur dualities of different types in the future.

In a very interesting work [6], Chen, Guay and Ma considered a duality which is reminiscent to our but different in several aspects. They considered 2-parameter (instead of 3-parameter here) affine Hecke algebras, and their formulation uses *finite-dimensional* tensor representations. The coideal algebra therein used a different definition via reflection equations, and it is not known (though is expected) if it is isomorphic to some suitable specialization of the one used in this paper. It is interesting and should be possible to adapt our work to study finite-dimensional representations of the multi-parameter coideal algebras as well.

The paper is organized as follows. In Section 2, we define an infinite-dimensional tensor module $\mathbb{V}^{\otimes d}$ for the affine Hecke algebra $\mathbb{H}$. The Schur $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n), \mathbb{H})$-duality is established in Section 3. In Section 4, inspired by the considerations in [9]–[10], we establish three additional variants of Schur duality: the $(\mathbb{U}^{\jmath\imath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}), \mathbb{H})$-duality, the $(\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}), \mathbb{H})$-duality, and the $(\mathbb{U}^{\imath\imath}(\widehat{\mathfrak{sl}}_\eta), \mathbb{H})$-duality. Here $\mathbb{U}^{\jmath\imath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})$, $\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})$, $\mathbb{U}^{\imath\imath}(\widehat{\mathfrak{sl}}_\eta)$ denote different coideal subalgebras in $\mathbb{U}(\widehat{\mathfrak{sl}}_{n-1}), \mathbb{U}(\widehat{\mathfrak{sl}}_{n-1}), \mathbb{U}(\widehat{\mathfrak{sl}}_{n-2})$, respectively.

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2. Quantum algebras and Hecke algebras

In this section we give a quick review on the quantum group of affine type A, its coideal subalgebra $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$, and the Hecke algebra $\mathbb{H}$ of affine type C. We formulate the actions of $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$ and $\mathbb{H}$ on the tensor space $\mathbb{V}^{\otimes d}$.

2.1. Quantum group of affine type A

Let q, q_0, q_1 be indeterminates, and denote by $\mathbb{F}$ the field

$$\mathbb{F} = \mathbb{Q}(q, q_0, q_1).$$

The quantum affine $\mathfrak{gl}_n$ is the associative algebra $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ over $\mathbb{F}$ generated by

$$\mathbf{E}_i, \mathbf{F}_i \quad (0 \leq i \leq n-1) \quad \mathbf{D}_a^{\pm 1} \quad (0 \leq a \leq n-1)$$

subject to the following relations for $0 \leq a, b \leq n-1$ and for $0 \leq i, j \leq n-1$:

1) q-Cartan relations:

$$\begin{aligned}
&\mathbf{D}_a\mathbf{D}_b = \mathbf{D}_b\mathbf{D}_a, \quad \mathbf{D}_a\mathbf{D}_a^{-1} = 1 = \mathbf{D}_a^{-1}\mathbf{D}_a,\\
&\mathbf{D}_a\mathbf{E}_j\mathbf{D}_a^{-1} = q^{\delta_{aj}-\delta_{a-1,j}}\mathbf{E}_j, \quad \mathbf{D}_a\mathbf{F}_j\mathbf{D}_a^{-1} = q^{-\delta_{aj}+\delta_{a-1,j}}\mathbf{F}_j,\\
&\mathbf{E}_i\mathbf{F}_j - \mathbf{F}_j\mathbf{E}_i = \delta_{ij}\frac{\mathbf{K}_i - \mathbf{K}_i^{-1}}{q-q^{-1}}.
\end{aligned}$$

(Here and below $\mathbf{K}_i := \mathbf{D}_i\mathbf{D}_{i+1}^{-1}$ and $\mathbf{D}_n = \mathbf{D}_0$.)

2) q-Serre relations:

$$\begin{aligned}
\mathbf{E}_i^2\mathbf{E}_j + \mathbf{E}_j\mathbf{E}_i^2 &= (q+q^{-1})\mathbf{E}_i\mathbf{E}_j\mathbf{E}_i, && \text{if } |i-j| \equiv 1,\\
\mathbf{F}_i^2\mathbf{F}_j + \mathbf{F}_j\mathbf{F}_i^2 &= (q+q^{-1})\mathbf{F}_i\mathbf{F}_j\mathbf{F}_i, && \text{if } |i-j| \equiv 1,\\
\mathbf{E}_i\mathbf{E}_j = \mathbf{E}_j\mathbf{E}_i, &\quad \mathbf{F}_i\mathbf{F}_j = \mathbf{F}_j\mathbf{F}_i, && \text{if } i \not\equiv j \pm 1,
\end{aligned}$$

where $i \equiv j$ means $i \equiv j \pmod n$. The quantum affine $\mathfrak{sl}_n$ is the $\mathbb{F}$-subalgebra $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ of $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ generated by $\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i^{\pm 1}$ $(0 \le i \le n-1)$.

Remark 2.1. The algebra $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ does not contain a "Heisenberg subalgebra" and it differs from $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ only on the finite Cartan subalgebra; it plays only an auxiliary role as it allows for simpler formulas. The algebra $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ has level 0 and is sometimes called the quantum loop algebra of $\mathfrak{sl}_n$.

The comultiplication Δ on $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ is given as follows:

$$\begin{aligned}
&\Delta(\mathbf{E}_i) = \mathbf{E}_i \otimes \mathbf{K}_i^{-1} + 1 \otimes \mathbf{E}_i,\\
&\Delta(\mathbf{F}_i) = \mathbf{F}_i \otimes 1 + \mathbf{K}_i \otimes \mathbf{F}_i, \quad \Delta(\mathbf{D}_a) = \mathbf{D}_a \otimes \mathbf{D}_a.
\end{aligned}$$

Let $\mathbb{V}$ be the $\mathbb{F}$-vector space with basis $\{v_j \mid j \in \mathbb{Z}\}$. It has a natural module structure over $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ (and hence over $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$) as follows:
(2.1.1)

$$\mathbf{E}_i v_{j+1} = \begin{cases} v_j & \text{if } j \equiv i;\\ 0 & \text{else,}\end{cases} \quad \mathbf{F}_i v_j = \begin{cases} v_{j+1} & \text{if } j \equiv i;\\ 0 & \text{else,}\end{cases} \quad \mathbf{D}_a v_j = \begin{cases} q v_j & \text{if } j \equiv a;\\ v_j & \text{else.}\end{cases}$$

2.2. An ıquantum group

From now on we take an integer $r \ge 1$, and let

$$n = 2r+2.$$

Let $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{gl}}_n)$ be the associative algebra over $\mathbb{F}$ generated by

$$\mathbf{e}_i, \mathbf{f}_i \quad (0 \le i \le r), \quad \mathbf{h}_a^{\pm 1} \quad (0 \le a \le r+1),$$

subject to the following relations (in which $\mathbf{k}_i := \mathbf{h}_i \mathbf{h}_{i+1}^{-1}$) for $0 \le a, b \le r+1, 0 \le i, j \le r$:

1) q-Cartan relations:

$$\mathbf{h}_a \mathbf{h}_b = \mathbf{h}_b \mathbf{h}_a, \quad \mathbf{h}_a \mathbf{h}_a^{-1} = 1 = \mathbf{h}_a^{-1} \mathbf{h}_a,$$
$$\mathbf{h}_a \mathbf{e}_j \mathbf{h}_a^{-1} = \begin{cases} q^{2\delta_{0j}} \mathbf{e}_j & \text{if } a = 0; \\ q^{-2\delta_{rj}} \mathbf{e}_j & \text{if } a = r+1; \\ q^{\delta_{aj} - \delta_{a-1,j}} \mathbf{e}_j & \text{otherwise,} \end{cases}$$
$$\mathbf{h}_a \mathbf{f}_j \mathbf{h}_a^{-1} = \begin{cases} q^{-2\delta_{0j}} \mathbf{f}_j & \text{if } a = 0; \\ q^{2\delta_{rj}} \mathbf{f}_j & \text{if } a = r+1; \\ q^{\delta_{a-1,j} - \delta_{aj}} \mathbf{f}_j & \text{otherwise,} \end{cases}$$
$$\mathbf{e}_i \mathbf{f}_j - \mathbf{f}_j \mathbf{e}_i = \delta_{i,j} \frac{\mathbf{k}_i - \mathbf{k}_i^{-1}}{q - q^{-1}} \quad (i,j) \neq (0,0), (r,r).$$

2) q-Serre relations:

$$\begin{aligned}
&\mathbf{e}_i^2 \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i^2 = (q+q^{-1}) \mathbf{e}_i \mathbf{e}_j \mathbf{e}_i, \ \mathbf{f}_i^2 \mathbf{f}_j + \mathbf{f}_j \mathbf{f}_i^2 = (q+q^{-1}) \mathbf{f}_i \mathbf{f}_j \mathbf{f}_i, \quad \text{if } |i-j| = 1, \\
&\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_j \mathbf{e}_i, \quad \mathbf{f}_i \mathbf{f}_j = \mathbf{f}_j \mathbf{f}_i, \qquad \text{if } i \neq j \pm 1, \\
&\mathbf{e}_r^2 \mathbf{f}_r + \mathbf{f}_r \mathbf{e}_r^2 = (q+q^{-1})(\mathbf{e}_r \mathbf{f}_r \mathbf{e}_r - q^2 q_0 q_1^{-1} \mathbf{e}_r \mathbf{k}_r - q^{-2} \mathbf{e}_r \mathbf{k}_r^{-1}), \\
&\mathbf{f}_r^2 \mathbf{e}_r + \mathbf{e}_r \mathbf{f}_r^2 = (q+q^{-1})(\mathbf{f}_r \mathbf{e}_r \mathbf{f}_r - q^2 q_0 q_1^{-1} \mathbf{k}_r \mathbf{f}_r - q^{-2} \mathbf{k}_r^{-1} \mathbf{f}_r), \\
&\mathbf{e}_0^2 \mathbf{f}_0 + \mathbf{f}_0 \mathbf{e}_0^2 = (q+q^{-1})(\mathbf{e}_0 \mathbf{f}_0 \mathbf{e}_0 - q_1 q \mathbf{e}_0 \mathbf{k}_0 - q_0^{-1} q^{-1} \mathbf{e}_0 \mathbf{k}_0^{-1}), \\
&\mathbf{f}_0^2 \mathbf{e}_0 + \mathbf{e}_0 \mathbf{f}_0^2 = (q+q^{-1})(\mathbf{f}_0 \mathbf{e}_0 \mathbf{f}_0 - q_1 q \mathbf{k}_0 \mathbf{f}_0 - q_0^{-1} q^{-1} \mathbf{k}_0^{-1} \mathbf{f}_0).
\end{aligned}$$

Let $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{gl}}_n)$ generated by $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i^{\pm}(0 \le i \le r)$. Sometimes, $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$ and $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{gl}}_n)$ are called $\imath$quantum groups.

We adopt the following identification for all $i \in \mathbb{Z}$:

$$\mathbf{E}_i = \mathbf{E}_{i+n}, \quad \mathbf{F}_i = \mathbf{F}_{i+n}, \quad \mathbf{D}_i = \mathbf{D}_{i+n}, \quad \mathbf{K}_i = \mathbf{K}_{i+n}.$$

Proposition 2.2. *There are injective* $\mathbb{F}$*-algebra homomorphisms* $\jmath : \mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{gl}}_n) \to \mathbb{U}(\widehat{\mathfrak{gl}}_n)$ *and* $\jmath : \mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n) \to \mathbb{U}(\widehat{\mathfrak{sl}}_n)$ *defined by*

(2.2.1) $\mathbf{h}_a \mapsto \mathbf{D}_a\mathbf{D}_{-a}, \quad (0 \le a \le r+1)$

(2.2.2) $\mathbf{e}_i \mapsto \mathbf{E}_i + \mathbf{F}_{-i-1}\mathbf{K}_i^{-1}, \quad \mathbf{f}_i \mapsto \mathbf{E}_{-i-1} + \mathbf{F}_i\mathbf{K}_{-i-1}^{-1}, \quad (1 \le i \le r-1)$

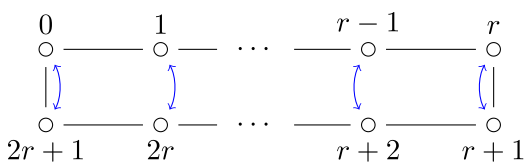
(2.2.3) $\mathbf{e}_0 \mapsto \mathbf{E}_0 + q_0^{-1}\mathbf{F}_{-1}\mathbf{K}_0^{-1}, \quad \mathbf{f}_0 \mapsto \mathbf{E}_{-1} + q_1q^{-1}\mathbf{F}_0\mathbf{K}_{-1}^{-1}.$

(2.2.4) $\mathbf{e}_r \mapsto \mathbf{E}_r + q^{-1}\mathbf{F}_{-r-1}\mathbf{K}_r^{-1}, \quad \mathbf{f}_r \mapsto \mathbf{E}_{-r-1} + q_0q_1^{-1}\mathbf{F}_r\mathbf{K}_{-r-1}^{-1},$

It follows that $\mathbf{k}_a \mapsto \mathbf{K}_a\mathbf{K}_{-a-1}^{-1}$, $(0 \le a \le r)$. It turns out $(\mathbb{U}(\widehat{\mathfrak{gl}}_n), \mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{gl}}_n))$ forms a quantum symmetric pair à la Letzter and Kolb.

Proof. Noting that the subalgebra of $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ generated by the right-hand sides of (2.2.1)–(2.2.4) is a quantum symmetric pair coideal subalgebra (in the sense of [16]) associated with the affine Dynkin diagram and involution below, the proposition follows from [16, Theorem 7.8]. □

Figure 1: Dynkin diagram of type $A_{2r+1}^{(1)}$ with involution of type $\jmath\jmath \equiv \mathfrak{c}$.



Combining (2.1.1) and (2.2.1)–(2.2.4), we obtain an explicit description of the $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{gl}}_n)$-action on $\mathbb{V}$ as below.

Lemma 2.3. *The vector space* $\mathbb{V}$ *admits a* $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{gl}}_n)$*-action as below. For* $0 \le a \le r+1$ *and for* $i \ne 0, r$,

$$\mathbf{h}_a(v_j) = \begin{cases} q^2 v_j & \text{if } a = 0, r+1; a \equiv j; \\ q v_j & \text{if } a \ne 0, r+1; \pm a \equiv j; \\ v_j & \text{otherwise,} \end{cases}$$

$$\mathbf{e}_i(v_j) = \begin{cases} v_{j-1} & \text{if } j \equiv i+1; \\ v_{j+1} & \text{if } -j \equiv i+1; \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{f}_i(v_j) = \begin{cases} v_{j+1} & \text{if } j \equiv i; \\ v_{j-1} & \text{if } -j \equiv i; \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathbf{e}_0(v_j) = \begin{cases} v_{j-1} & \text{if } j \equiv 1; \\ q_0^{-1} v_{j+1} & \text{if } j \equiv -1; \\ 0 & \text{otherwise;} \end{cases} \qquad \mathbf{f}_0(v_j) = \begin{cases} q_1 v_{j+1} + v_{j-1} & \text{if } j \equiv 0; \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathbf{e}_r(v_j) = \begin{cases} v_{j-1} + v_{j+1} & \text{if } j \equiv r+1; \\ 0 & \text{otherwise;} \end{cases} \qquad \mathbf{f}_r(v_j) = \begin{cases} q_0 q_1^{-1} v_{j+1} & \text{if } j \equiv r; \\ v_{j-1} & \text{if } j \equiv r+2; \\ 0 & \text{otherwise.} \end{cases}$$

2.3. Affine Hecke algebra in 3 parameters

Let W be the Weyl group of affine type C_d generated by $S = \{s_0, s_1, \dots, s_d\}$ with the affine Dynkin diagram

$$\begin{array}{ccccccc} \circ & \Longrightarrow & \circ & \text{——} \cdots \text{——} & \circ & \Longleftarrow & \circ \\ 0 & & 1 & & d-1 & & d \end{array}$$

Recall that $\mathbb{V}$ is the natural representation of $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ with $\mathbb{F}$-basis $\{v_i \mid i \in \mathbb{Z}\}$. The tensor space $\mathbb{V}^{\otimes d}$ then has an $\mathbb{F}$-basis $\{M_f \mid f \in \mathbb{Z}^d\}$, where

$$M_f = v_{f_1} \otimes \cdots \otimes v_{f_d} \in \mathbb{V}^{\otimes d} \quad \text{for} \quad f = (f_1, \dots, f_d) \in \mathbb{Z}^d.$$

The group W admits a natural right action on $\mathbb{Z}^d$. Precisely, for

$$f = (f_1, \dots, f_d) \in \mathbb{Z}^d,$$

we have

$$\text{(2.3.1)} \qquad f \cdot s_i = \begin{cases} (f_1, \dots, f_{i-1}, f_{i+1}, f_i, f_{i+2}, \dots, f_d) & \text{if } i \neq 0, d; \\ (-f_1, f_2, \dots, f_d) & \text{if } i = 0; \\ (f_1, f_2, \dots, f_{d-1}, n - f_d) & \text{if } i = d. \end{cases}$$

Let $\mathbb{H}$ be the affine Hecke algebra of type C_d with three parameters, that is, $\mathbb{H}$ is an $\mathbb{F}$-algebra generated by

$$T_i \quad (0 \le i \le d-1), \qquad X_a^{\pm 1} \quad (1 \le a \le d),$$

subject to the following relations, for $1 \le a, b \le d$ and for $0 \le i, j, k \le d-1$,

1) Toric relations:

$$X_a X_a^{-1} = 1 = X_a^{-1} X_a, \qquad X_a X_b = X_b X_a.$$

2) Hecke relations:

$$
\begin{aligned}
&(T_0 - q_0^{-1})(T_0 + q_1) = 0, \quad (T_i - q^{-1})(T_i + q) = 0 \quad (i \neq 0),\\
&T_k T_{k-1} T_k = T_{k-1} T_k T_{k-1} \quad (k \neq 0, 1),\\
&(T_0 T_1)^2 = (T_1 T_0)^2, \quad T_i T_j = T_j T_i \quad (|i-j| > 1).
\end{aligned}
\tag{2.3.2}
$$

3) Bernstein-Lusztig relations:

$$T_0 X_1^{-1} T_0 = q_0^{-1} q_1 X_1 + (q_0^{-1} q_1 - 1) T_0, \tag{2.3.3}$$

$$T_i X_i T_i = X_{i+1} \ (i \neq 0), \quad T_i X_j = X_j T_i \ (j \neq i, i+1). \tag{2.3.4}$$

We remark that the Hecke algebra $\mathbb{H}$ of affine type C_d in this paper can be matched with the version in [19, Appendix A] with the following parameter correspondence: our $q \leftrightarrow$ their p, our $q_0 \leftrightarrow$ their q_1, our $q_1 \leftrightarrow$ their q_0. Also see [15] in somewhat different notations.

The algebra $\mathbb{H}$ contains a subalgebra $\mathbb{H}_A$ generated by $T_1, \ldots, T_{d-1}$, $X_1^{\pm 1}, \ldots, X_d^{\pm 1}$, which is an affine Hecke algebra of type A.

We define $T_d \in \mathbb{H}$ by

$$T_d := q_0^{-1} X_d T_{d-1}^{-1} \cdots T_1^{-1} T_0^{-1} T_1^{-1} \cdots T_{d-1}^{-1}. \tag{2.3.5}$$

Lemma 2.4. *The element $T_d \in \mathbb{H}$ satisfies the following relations:*

1) $(T_d - q_1^{-1})(T_d + q_0^{-1}) = 0$.

2) $T_d T_i = T_i T_d$, for all $0 \leq i \leq d-2$.

3) $(T_{d-1} T_d)^2 = (T_d T_{d-1})^2$, if $d \geq 2$.

Proof. These relations are verified by direct computations. Here we only present proofs for (1) and (3) while leaving the verification of (2) to the reader.

Thanks to (2.3.4), we have

$$X_d T_{d-1}^{-1} \cdots T_1^{-1} T_0^{-1} = T_{d-1} \cdots T_1 (X_1 T_0^{-1}).$$

Hence

$$q_0 T_d = T_{d-1} \cdots T_1 (X_1 T_0^{-1}) T_1^{-1} \cdots T_{d-1}^{-1}. \tag{2.3.6}$$

It follows from (2.3.3) that $(X_1 T_0^{-1} - q_0 q_1^{-1})(X_1 T_0^{-1} + 1) = 0$, and thus $(q_0 T_d - q_0 q_1^{-1})(q_0 T_d + 1) = 0$ by (2.3.6). Part (1) follows.

We compute

$$\begin{aligned}(T_{d-1}^{-1}T_d^{-1})^2 &= q_0^2 T_{d-2}\cdots T_1T_0T_1\cdots T_{d-1}X_d^{-1}T_{d-2}\cdots T_1T_0T_1\cdots T_{d-1}X_d^{-1}\\ &= q_0^2 T_{d-2}\cdots T_1T_0T_1\cdots T_{d-1}T_{d-2}\cdots T_1T_0\\ &\quad \times T_1\ldots T_{d-2}T_{d-1}^{-1}X_{d-1}^{-1}X_d^{-1},\\ (T_d^{-1}T_{d-1}^{-1})^2 &= q_0^2 T_{d-1}\cdots T_1T_0T_1\cdots T_{d-1}X_d^{-1}\\ &\quad \times T_{d-2}\cdots T_1T_0T_1\cdots T_{d-1}X_d^{-1}T_{d-1}^{-1}\\ &= q_0^2 T_{d-1}\cdots T_1T_0T_1\cdots T_{d-1}T_{d-2}\cdots T_1T_0\\ &\quad \times T_1\cdots T_{d-2}T_{d-1}^{-1}X_{d-1}^{-1}X_d^{-1}T_{d-1}^{-1}\\ &= q_0^2 T_{d-1}\cdots T_1T_0T_1\cdots T_{d-1}T_{d-2}\cdots T_1T_0\\ &\quad \times T_1\cdots T_{d-2}T_{d-1}^{-2}X_{d-1}^{-1}X_d^{-1}.\end{aligned}$$

Thus, to show the identity in (3) it suffices to show that

$$\begin{aligned}&(T_{d-2}\cdots T_0\cdots T_{d-1}\cdots T_0\cdots T_{d-2})T_{d-1}\\ &= T_{d-1}(T_{d-2}\cdots T_0\cdots T_{d-1}\cdots T_0\cdots T_{d-2}).\end{aligned} \tag{2.3.7}$$

Let $\alpha_0 = -2\epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le d-1$. The highest root in the finite type C Weyl group is $\theta = \alpha_0 + 2\alpha_1 + ... + 2\alpha_{d-2} + \alpha_{d-1} = -\epsilon_{d-1} - \epsilon_d$ and hence

$$s_\theta s_{d-1} = s_{d-1}s_\theta. \tag{2.3.8}$$

Therefore, (2.3.7) follows by (2.3.8) and noting that

$$s_{d-2}\cdots s_0\cdots s_{d-1}\cdots s_0\cdots s_{d-2}$$

is a reduced expression of s_θ. □

It follows by (2.3.4)–(2.3.5) that the algebra $\mathbb{H}$ is generated by $T_0, T_1, \ldots, T_d$. For any $w \in W$ with a reduced form $w = s_{i_1}\cdots s_{i_l}$, set

$$T_w := T_{i_1}\cdots T_{i_l} \tag{2.3.9}$$

and

$$q_w := q_{s_{i_1}}\cdots q_{s_{i_l}} \quad \text{where} \quad q_{s_i} := \begin{cases} q_1 & \text{if } i = 0;\\ q & \text{if } i \neq 0, d;\\ q_0^{-1} & \text{if } i = d.\end{cases} \tag{2.3.10}$$

It follows by the braid relations in (2.3.2) and Lemma 2.4 that T_w is independent of the choice of the reduced form of w. Since the q_{s_i}'s satisfy the same braid relations, q_w is uniquely determined by w, too.

2.4. A tensor module for Hecke algebra

We first recall a well-known action of the Hecke algebra $\mathbb{H}_A$ of affine type A on $\mathbb{V}^{\otimes d}$; see [14]. We introduce linear operators $z_1, \ldots, z_d$ which act on $\mathbb{V}^{\otimes d}$ (from the right) as below:

$$M_f z_i = M_{(f_1,\ldots,f_{i-1},f_i+n,f_{i+1},\ldots,f_d)}.$$

Since each $f_i \in \mathbb{Z}$ has a unique expression $f_i = \overline{f}_i + c_i n$, for some $c_i \in \mathbb{Z}$ such that $-r \le \overline{f}_i \le r+1$, each basis element M_f has a unique expression

$$M_f = M_{\overline{f}} Z_f, \quad Z_f = z_1^{c_1} \cdots z_d^{c_d}.$$

Recall the right W-action on $\mathbb{Z}^d$ in (2.3.1). Following [14, (32)], the action of $\mathbb{H}_A$ is given by, for $1 \le i \le d-1$ and $1 \le a \le d$,

$$M_f T_i = \begin{cases} M_{f\cdot s_i} + (q^{-1} - q) M_{\overline{f}} P_+^{(i)}(Z_f) & \text{if } \overline{f}_{i+1} > \overline{f}_i; \\ q^{-1} M_{\overline{f}} Z_{f\cdot s_i} + (q^{-1} - q) M_{\overline{f}} P_+^{(i)}(Z_f) & \text{if } \overline{f}_{i+1} = \overline{f}_i; \\ M_{f\cdot s_i} + (q^{-1} - q) M_{\overline{f}} P_-^{(i)}(Z_f) & \text{if } \overline{f}_{i+1} < \overline{f}_i, \end{cases} \tag{2.4.1}$$

$$M_f X_a = M_f z_a^{-1}. \tag{2.4.2}$$

Here $P_\pm^{(i)}$ are operators given by

$$P_-^{(i)}(Z_f) = \frac{z_{i+1}(Z_{f\cdot s_i}) - z_i Z_f}{z_{i+1} - z_i}, \quad P_+^{(i)}(Z_f) = \frac{z_i(Z_{f\cdot s_i} - Z_f)}{z_{i+1} - z_i}. \tag{2.4.3}$$

Now we shall enhance the action of $\mathbb{H}_A$ on $\mathbb{V}^{\otimes d}$ to an action of the Hecke algebra $\mathbb{H}$ of affine type C in 3 parameters. For convenience we denote the basis elements of $\mathbb{V}$ by

$$v_i z^j := v_{i+nj}, \quad (-r \le i \le r+1, j \in \mathbb{Z}).$$

Define

$$M_f T_0 = (v_{f_1} T_0) \otimes v_{f_2} \otimes \cdots \otimes v_{f_d}, \tag{2.4.4}$$

where $v_{f_1}T_0$ is given by (below we assume $f_1 = k + nj$, for $-r \le k \le r+1$):

$$
\begin{aligned}
&q_0^{-1}q_1 v_{-k}z^{-j} + (q_1 - q_0^{-1})\textstyle\sum_{l=1}^{j} v_k z^{j-2l} + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_k z^{j+1-2l} \\
&\qquad \text{if } k = r+1, j \ge 0; \\
&v_{-k}z^{-j} + (q_0^{-1} - q_1)\textstyle\sum_{l=1}^{-j} v_k z^{-j-2l} + (1 - q_0^{-1}q_1)\sum_{l=2}^{-j} v_k z^{-j+1-2l} \\
&\qquad \text{if } k = r+1, j < 0; \\
&v_{-k}z^{-j} + (q_1 - q_0^{-1})\textstyle\sum_{l=1}^{j} v_k z^{j-2l} + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_k z^{j+1-2l} \\
&\qquad \text{if } 0 < k \le r, j \ge 0; \\
&v_{-k}z^{-j} + (q_0^{-1} - q_1)\textstyle\sum_{l=1}^{-j} v_k z^{-j-2l} + (1 - q_0^{-1}q_1)\sum_{l=1}^{-j} v_k z^{-j+1-2l} \\
&\qquad \text{if } 0 < k \le r, j < 0; \\
&q_0^{-1}v_0 z^{-j} + (q_1 - q_0^{-1})\textstyle\sum_{l=1}^{j} v_0 z^{j-2l} + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_0 z^{j+1-2l} \\
&\qquad \text{if } k = 0, j \ge 0; \\
&q_1 v_0 z^{-j} + (q_0^{-1} - q_1)\textstyle\sum_{l=0}^{-j} v_0 z^{-j-2l} + (1 - q_0^{-1}q_1)\sum_{l=1}^{-j} v_0 z^{-j+1-2l} \\
&\qquad \text{if } k = 0, j < 0; \\
&q_0^{-1}q_1 v_{-k}z^{-j} + (q_1 - q_0^{-1})\textstyle\sum_{l=1}^{j-1} v_k z^{j-2l} + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_k z^{j+1-2l} \\
&\qquad \text{if } -r \le k < 0, j > 0; \\
&q_0^{-1}q_1 v_{-k}z^{-j} + (q_0^{-1} - q_1)\textstyle\sum_{l=0}^{-j} v_k z^{-j-2l} + (1 - q_0^{-1}q_1)\sum_{l=1}^{-j} v_k z^{-j+1-2l} \\
&\qquad \text{if } -r \le k < 0, j \le 0.
\end{aligned}
\tag{2.4.5}
$$

The formula (2.4.5) above is obtained as follows. We first define the action of T_0 on $\{v_k | -r \le k \le r+1\}$ in $\mathbb{V}$, and then extend the action to all the basis vectors by the relation (2.3.3).

Proposition 2.5. *The formulas in* (2.4.1)–(2.4.5) *define an action of* $\mathbb{H}$ *on* $\mathbb{V}^{\otimes d}$.

Proof. It suffices to check the Hecke relation (2.3.2) and the Bernstein-Lusztig relation (2.3.3) for T_0. This follows by a direct computation, and here we only present the borderline cases in (2.4.5). It is useful to give the following formulas for the borderline cases in (2.4.5), for $1 \le k \le r$,

$$\begin{aligned}
&v_{r+1}T_0 = q_0^{-1}q_1 v_{-r-1},\\
&v_{-r-1}T_0 = v_{r+1} + (q_0^{-1} - q_1)v_{-r-1},\\
&v_k T_0 = v_{-k},\\
&v_{k-n}T_0 = v_{n-k} + (q_0^{-1} - q_1)v_{k-n} + (1 - q_0^{-1}q_1)v_k,\\
&v_0 T_0 = q_0^{-1}v_0,\\
&v_{-n}T_0 = q_0^{-1}v_n + (q_0^{-1} - q_1)v_{-n} + (1 - q_0^{-1}q_1)v_0,\\
&v_{-k}T_0 = q_0^{-1}q_1 v_k + (q_0^{-1} - q_1)v_{-k},\\
&v_{n-k}T_0 = q_0^{-1}q_1 v_{k-n} + (q_0^{-1}q_1 - 1)v_{-k}.
\end{aligned}$$

We start with checking (2.3.2) for these cases as follows:

$$\begin{aligned}
v_{r+1}(T_0 - q_0^{-1})(T_0 + q_1) &= q_0^{-1}q_1(v_{r+1} + q_0^{-1}v_{-r-1})\\
&\quad - q_0^{-1}(q_0^{-1}q_1 v_{-r-1} + q_1 v_{r+1}) = 0,\\
v_{-r-1}(T_0 - q_0^{-1})(T_0 + q_1) &= q_0^{-1}q_1 v_{-r-1} + q_1 v_{r+1}\\
&\quad - q_1(v_{r+1} + q_0^{-1}v_{-r-1}) = 0,\\
v_k(T_0 - q_0^{-1})(T_0 + q_1) &= (q_0^{-1}q_1 v_k + q_0^{-1}v_{-k})\\
&\quad - q_0^{-1}(v_{-k} + q_1 v_k) = 0,\\
v_{-k}(T_0 - q_0^{-1})(T_0 + q_1) &= q_0^{-1}q_1(v_{-k} + q_1 v_k)\\
&\quad - q_1(q_0^{-1}q_1 v_k + q_0^{-1}v_{-k}) = 0,\\
v_0(T_0 - q_0^{-1})(T_0 + q_1) &= 0,\\
v_{-n}(T_0 - q_0^{-1})(T_0 + q_1) &= q_0^{-1}(q_1 v_{-n} + (q_0^{-1}q_1 - 1)v_0 + q_1 v_n)\\
&\quad + (1 - q_0^{-1}q_1)(q_0^{-1} + q_1)v_0\\
&\quad - q_1(q_0^{-1}v_n + (1 - q_0^{-1}q_1)v_0 + q_0^{-1}v_{-n}) = 0,\\
v_{k-n}(T_0 - q_0^{-1})(T_0 + q_1) &= q_0^{-1}q_1 v_{k-n} + (q_0^{-1}q_1 - 1)v_{-k} + q_1 v_{n-k}\\
&\quad - q_1(v_{n-k} + q_0^{-1}v_{k-n} + (1 - q_0^{-1}q_1)v_k)\\
&\quad + (1 - q_0^{-1}q_1)(v_{-k} + q_1 v_k) = 0,\\
v_{n-k}(T_0 - q_0^{-1})(T_0 + q_1) &= q_0^{-1}q_1(v_{n-k} + q_0^{-1}v_{k-n} + (1 - q_0^{-1}q_1)v_k)\\
&\quad + (q_0^{-1}q_1 - 1)(q_0^{-1}q_1 v_k + q_0^{-1}v_{-k})\\
&\quad - q_0^{-1}(q_0^{-1}q_1 v_{k-n} + (q_0^{-1}q_1 - 1)v_{-k} + q_1 v_{n-k}) = 0.
\end{aligned}$$

The Bernstein-Lusztig relation (2.3.3) for the extremal cases follow from the following computation:

$$\begin{aligned}
v_{r+1}T_0X_1^{-1}T_0 &= (q_0^{-1}q_1)^2 v_{-r-1},\\
v_{-r-1}T_0X_1^{-1}T_0 &= q_0^{-1}q_1 v_{-3(r+1)} + (q_0^{-1}q_1-1)(v_{r+1}+(q_0^{-1}-q_1)v_{-r-1}),\\
v_kT_0X_1^{-1}T_0 &= q_0^{-1}q_1v_{k-n} + (q_0^{-1}q_1-1)v_{-k},\\
v_{-k}T_0X_1^{-1}T_0 &= q_0^{-1}q_1v_{-k-n} + (q_0^{-1}q_1-1)(q_0^{-1}q_1v_k-(q_0^{-1}-q_1)v_{-k}),\\
v_0T_0X_1^{-1}T_0 &= q_0^{-1}(q_1v_{-n}+(q_0^{-1}q_1-1)v_0),\\
v_{-n}T_0X_1^{-1}T_0 &= q_0^{-1}q_1v_{-2n}\\
&\quad + (q_0^{-1}q_1-1)(q_0^{-1}v_n+(1-q_0^{-1}q_1)v_0+(q_0^{-1}-q_1)v_{-n}),\\
v_{k-n}T_0X_1^{-1}T_0 &= q_0^{-1}q_1v_{k-2n}\\
&\quad + (q_0^{-1}q_1-1)(v_{n-k}+(q_0^{-1}-q_1)v_{k-n}+(1-q_0^{-1}q_1)v_k),\\
v_{n-k}T_0X_1^{-1}T_0 &= q_0^{-1}q_1v_{-k} + (q_0^{-1}q_1-1)(q_0^{-1}q_1v_{k-n}+(q_0^{-1}q_1-1)v_{-k}).
\end{aligned}$$

The proposition is proved. □

The action of T_i $(0 \le i \le d-1)$ on the set $\{M_f \mid 0 \le f_1 \le f_2 \le \cdots \le f_d \le r+1\}$ behaves nicely as below:

(2.4.6) $$M_fT_i = \begin{cases} q^{-1}M_f & \text{if } 0 \le f_i = f_{i+1} \le r+1,\\ M_{f\cdot s_i} & \text{if } 0 \le f_i < f_{i+1} \le r+1, \end{cases} \qquad (i \neq 0)$$

(2.4.7) $$M_fT_0 = \begin{cases} q_0^{-1}M_f & \text{if } f_1 = 0,\\ M_{f\cdot s_0} & \text{if } 0 < f_1 < r+1;\\ q_0^{-1}q_1M_{f\cdot s_0} & \text{if } f_1 = r+1. \end{cases}$$

Combining (2.4.6)–(2.4.7) with (2.4.2), we obtain the following.

Corollary 2.6. *The tensor space $\mathbb{V}^{\otimes d}$ is generated by $\{M_f \mid 0 \le f_1 \le f_2 \le \cdots \le f_d \le r+1\}$ as an $\mathbb{H}$-module.*

3. Schur duality in three parameters

In this section, we establish the Schur $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n), \mathbb{H})$-duality on $\mathbb{V}^{\otimes d}$. To that end, we study the structures of the affine Schur algebra.

3.1. Affine Schur algebras

From now on, we fix

$$r, d \in \mathbb{Z} \quad \text{such that} \quad r \geq d \geq 1.$$

Recall $n = 2r + 2$. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Denote the set of (weak) compositions of d into $r + 2$ parts by

$$\Lambda_{n,d} := \left\{ \lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r+1}) \in \mathbb{N}^{r+2} \mid \sum_{i=0}^{r+1} \lambda_i = d \right\}. \tag{3.1.1}$$

For $\lambda \in \Lambda_{n,d}$, let W_λ be the parabolic (finite) subgroup of W generated by $S \backslash \{s_{\lambda_0}, s_{\lambda_{0,1}}, \dots, s_{\lambda_{0,r}}\}$, where $\lambda_{0,i} = \lambda_0 + \lambda_1 + \cdots + \lambda_i$ for $1 \leq i \leq r$; note $\lambda_{0,r} = d - \lambda_{r+1}$.

We note that the element

$$\omega := (0, \underbrace{1, \dots, 1}_{d}, \underbrace{0, \dots, 0}_{r-d}, 0) \in \Lambda_{n,d}$$

makes sense under the assumption $r \geq d$.

Recall T_w in (2.3.9) and q_w in (2.3.10). For any finite subset $X \subset W$ and for $\lambda \in \Lambda_{n,d}$, set

$$T_X := \sum_{w \in X} q_w^{-1} T_w \quad \text{and} \quad x_\lambda := T_{W_\lambda}.$$

Lemma 3.1. *For $\lambda \in \Lambda_{n,d}$ and for $i \in \{0, 1, \dots, d\} \setminus \{\lambda_0, \lambda_{0,1}, \dots, \lambda_{0,r}\}$, we have*

$$x_\lambda T_i = \begin{cases} q_0^{-1} x_\lambda & \text{if } i = 0; \\ q^{-1} x_\lambda & \text{if } i \neq 0, d; \\ q_1^{-1} x_\lambda & \text{if } i = d. \end{cases}$$

Proof. Let us write $x_\lambda = \sum_{\substack{w \in W_\lambda \\ ws_i < w}} q_w^{-1} (T_w + q_{s_i} T_{ws_i})$. Then

$$(T_w + q_{s_i} T_{ws_i}) T_i = T_{ws_i} (T_i + q_{s_i}) T_i = p_{s_i} T_{ws_i} (T_i + q_{s_i}) = p_{s_i} (T_w + q_{s_i} T_{ws_i}),$$

where

$$p_{s_i} := \begin{cases} q_0^{-1} & \text{if } i = 0; \\ q^{-1} & \text{if } i \neq 0, d; \\ q_1^{-1} & \text{if } i = d. \end{cases}$$

The lemma follows. □

The affine Schur algebra $\mathbb{S}^{\mathfrak{c}}_{n,d}$ of 3-parameter is defined as the following $\mathbb{F}$-algebra

$$\mathbb{S}^{\mathfrak{c}}_{n,d} := \mathrm{End}_{\mathbb{H}}(\oplus_{\lambda\in\Lambda_{n,d}} x_\lambda \mathbb{H}) = \bigoplus_{\lambda,\mu\in\Lambda_{n,d}} \mathrm{Hom}_{\mathbb{H}}(x_\mu\mathbb{H}, x_\lambda\mathbb{H}).$$

Denote by $\ell(g)$ the length of $g \in W$. Let

$$\mathscr{D}_\lambda := \big\{ g \in W \mid \ell(wg) = \ell(w) + \ell(g), \forall w \in W_\lambda \big\}.$$

Then $\mathscr{D}_\lambda$ (respectively, $\mathscr{D}_\lambda^{-1}$) is the set of minimal length right (respectively, left) coset representatives of W_λ in W. Denote by

$$\mathscr{D}_{\lambda\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$$

the set of minimal length double coset representatives for $W_\lambda\backslash W/W_\mu$.

For $\lambda, \mu \in \Lambda_{n,d}$ and $g \in \mathscr{D}_{\lambda\mu}$, define $\phi^g_{\lambda,\mu} \in \mathbb{S}^{\mathfrak{c}}_{n,d}$ by

$$\phi^g_{\lambda,\mu}(x_\nu) = \delta_{\mu,\nu} T_{W_\lambda g W_\mu}, \quad \forall \nu \in \Lambda_{n,d}.$$

It is straightforward to show that $\{\phi^g_{\lambda,\mu} \mid \lambda,\mu \in \Lambda_{n,d}, g \in \mathscr{D}_{\lambda,\mu}\}$ form an $\mathbb{F}$-basis of $\mathbb{S}^{\mathfrak{c}}_{n,d}$ (cf., e.g., [7, 8, 10]).

Define the right $\mathbb{H}$-module

$$\mathbb{T}^{\mathfrak{c}}_{n,d} := \bigoplus_{\lambda\in\Lambda_{n,d}} x_\lambda \mathbb{H}.$$

Thanks to Corollary 2.6 and Lemma 3.1, we have the following.

Lemma 3.2. *There exists a unique $\mathbb{H}$-module isomorphism $\kappa : \mathbb{T}^{\mathfrak{c}}_{n,d} \longrightarrow \mathbb{V}^{\otimes d}$ which sends*

$$\begin{aligned} x_\lambda \mapsto M_\lambda &:= M_{(0^{\lambda_0},\ldots,r+1^{\lambda_{r+1}})} \\ &= v_0^{\otimes\lambda_0} \otimes \cdots \otimes v_{r+1}^{\otimes\lambda_{r+1}} \in \mathbb{V}^{\otimes d}, \quad \forall \lambda \in \Lambda_{n,d}. \end{aligned}$$

This induces an algebra isomorphism $\mathbb{S}^{\mathfrak{c}}_{n,d} \simeq \mathrm{End}_{\mathbb{H}}(\mathbb{V}^{\otimes d})$.

3.2. The $\jmath\jmath$-Schur duality

Proposition 3.3. *The actions of $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$ and $\mathbb{H}$ on $\mathbb{V}^{\otimes d}$ commute.*

Proof. It is known that the actions of $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ and $\mathbb{H}_A$ on $\mathbb{V}^{\otimes d}$ commute. It remains to check that the T_0-action commutes with the $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$-action, and it suffices to check the special case $d = 1$.

It follows from a direct computation (using Lemma 2.3 and (2.4.5)) that the T_0-action commutes with the actions of all generators of $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$. The calculation is simple except for $\mathbf{e}_0, \mathbf{e}_r$ and $\mathbf{f}_0, \mathbf{f}_r$, which are complicated but similar – here we only provide a verification for $(\mathbf{e}_0 v)T_0 = \mathbf{e}_0(vT_0)$ and $(\mathbf{e}_r v)T_0 = \mathbf{e}_r(vT_0)$ for $v = v_k z^j \in \mathbb{V}$.

(1) We claim that $(\mathbf{e}_0 v)T_0 = \mathbf{e}_0(vT_0)$.

Indeed, if $k \neq \pm 1$, then $(\mathbf{e}_0 v_k z^j)T_0 = 0 = \mathbf{e}_0(v_k z^j T_0)$. There are four cases remaining. If $k = 1$ and $j \geq 0$, we have

$$\begin{aligned}
(\mathbf{e}_0 v_1 z^j)T_0 &= q_0^{-1} v_0 z^{-j} + (q_1 - q_0^{-1}) \sum_{l=1}^{j} v_0 z^{j-2l} + (q_0^{-1} q_1 - 1) \sum_{l=1}^{j} v_0 z^{j+1-2l} \\
&= \mathbf{e}_0 \left(v_{-1} z^{-j} + (q_1 - q_0^{-1}) \sum_{l=1}^{j} v_1 z^{j-2l} + (q_0^{-1} q_1 - 1) \sum_{l=1}^{j} v_1 z^{j+1-2l} \right) \\
&= \mathbf{e}_0(v_1 z^j T_0).
\end{aligned}$$

If $k = 1$ and $j < 0$, we obtain

$$\begin{aligned}
(\mathbf{e}_0 v_1 z^j)T_0 &= q_1 v_0 z^{-j} + (q_0^{-1} - q_1) \sum_{l=0}^{-j} v_0 z^{-j-2l} + (1 - q_0^{-1} q_1) \sum_{l=1}^{-j} v_0 z^{-j+1-2l} \\
&= \mathbf{e}_0 \left(v_{-1} z^{-j} + (q_0^{-1} - q_1) \sum_{l=1}^{-j} v_1 z^{-j-2l} + (1 - q_0^{-1} q_1) \sum_{l=1}^{-j} v_1 z^{-j+1-2l} \right) \\
&= \mathbf{e}_0(v_1 z^j T_0).
\end{aligned}$$

For $k = -1$ and $j > 0$, we have

$$\begin{aligned}
&(\mathbf{e}_0 v_{-1} z^j)T_0 \\
&= q_0^{-2} v_0 z^{-j} + q_0^{-1}(q_1 - q_0^{-1}) \sum_{l=1}^{j} v_0 z^{j-2l} + q_0^{-1}(q_0^{-1} q_1 - 1) \sum_{l=1}^{j} v_0 z^{j+1-2l} \\
&= \mathbf{e}_0 \left(q_0^{-1} q_1 v_1 z^{-j} + (q_1 - q_0^{-1}) \sum_{l=1}^{j-1} v_{-1} z^{j-2l} + (q_0^{-1} q_1 - 1) \sum_{l=1}^{j} v_{-1} z^{j+1-2l} \right) \\
&= \mathbf{e}_0(v_{-1} z^j T_0).
\end{aligned}$$

Finally for $k = -1$ and $j \le 0$, we have

$$\begin{aligned}
&(\mathbf{e}_0 v_{-1} z^j)T_0 \\
&= q_0^{-1}\left(q_1 v_0 z^{-j} + (q_0^{-1} - q_1)\sum_{l=0}^{-j} v_0 z^{-j-2l} + (1 - q_0^{-1}q_1)\sum_{l=1}^{-j} v_0 z^{-j+1-2l}\right) \\
&= \mathbf{e}_0\left(q_0^{-1}q_1 v_1 z^{-j} + (q_0^{-1} - q_1)\sum_{l=0}^{-j} v_{-1} z^{-j-2l} + (1 - q_0^{-1}q_1)\sum_{l=1}^{-j} v_{-1} z^{-j+1-2l}\right) \\
&= \mathbf{e}_0(v_{-1} z^j T_0).
\end{aligned}$$

(2) We claim that $(\mathbf{e}_r v)T_0 = \mathbf{e}_r(vT_0)$.

Indeed, if $k \neq r+1$, then $(\mathbf{e}_r v_k z^j)T_0 = 0 = \mathbf{e}_r(v_k z^j T_0)$. There are two cases remaining. If $k = r+1$ and $j \ge 0$, we have

$$\begin{aligned}
(\mathbf{e}_r v_{r+1} z^j)T_0 &= (v_r z^j + v_{-r} z^{j+1})T_0 \\
&= v_{-r} z^{-j} + (q_1 - q_0^{-1})\sum_{l=1}^{j} v_r z^{j-2l} + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_r z^{j+1-2l} \\
&\quad + q_0^{-1}q_1 v_r z^{-j-1} + (q_1 - q_0^{-1})\sum_{l=1}^{j} v_{-r} z^{j+1-2l} \\
&\quad + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j+1} v_{-r} z^{j+2-2l} \\
&= q_0^{-1}q_1 v_{-r} z^{-j} + (q_1 - q_0^{-1})\sum_{l=1}^{j} v_r z^{j-2l} \\
&\quad + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_r z^{j+1-2l} + q_0^{-1}q_1 v_{-r-2} z^{-j} \\
&\quad + (q_1 - q_0^{-1})\sum_{l=1}^{j} v_{r+2} z^{j-2l} + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_{r+2} z^{j+1-2l} \\
&= \mathbf{e}_r\Bigg(q_0^{-1}q_1 v_{-r-1} z^{-j} + (q_1 - q_0^{-1})\sum_{l=1}^{j} v_{r+1} z^{j-2l} \\
&\qquad + (q_0^{-1}q_1 - 1)\sum_{l=1}^{j} v_{r+1} z^{j+1-2l}\Bigg) \\
&= \mathbf{e}_r(v_{r+1} z^j T_0).
\end{aligned}$$

If $k = r+1$ and $j < 0$, we have

$$
\begin{aligned}
&(\mathbf{e}_r v_{r+1} z^j) T_0 = (v_r z^j + v_{-r} z^{j+1}) T_0 \\
&= v_{-r} z^{-j} + (q_0^{-1} - q_1) \sum_{l=1}^{-j} v_r z^{-j-2l} + (1 - q_0^{-1} q_1) \sum_{l=1}^{-j} v_r z^{-j+1-2l} \\
&\quad + q_0^{-1} q_1 v_r z^{-j-1} + (q_0^{-1} - q_1) \sum_{l=0}^{-j-1} v_{-r} z^{-j-1-2l} \\
&\quad + (1 - q_0^{-1} q_1) \sum_{l=1}^{-j-1} v_{-r} z^{-j-2l} \\
&= v_{-r} z^{-j} + (q_0^{-1} - q_1) \sum_{l=1}^{-j} v_r z^{-j-2l} + (1 - q_0^{-1} q_1) \sum_{l=2}^{-j} v_r z^{-j+1-2l} \\
&\quad + v_{-r-2} z^{-j} + (q_0^{-1} - q_1) \sum_{l=1}^{-j} v_{r+2} z^{-j-2l} + (1 - q_0^{-1} q_1) \sum_{l=2}^{-j} v_{r+2} z^{-j+1-2l} \\
&= \mathbf{e}_r \Bigg(v_{-r-1} z^{-j} + (q_0^{-1} - q_1) \sum_{l=1}^{-j} v_{r+1} z^{-j-2l} \\
&\qquad + (1 - q_0^{-1} q_1) \sum_{l=2}^{-j} v_{r+1} z^{-j+1-2l} \Bigg) \\
&= \mathbf{e}_r (v_{r+1} z^j T_0).
\end{aligned}
$$

The proposition is proved. □

By Proposition 3.3 and the above identification $\mathbb{S}^{\mathfrak{c}}_{n,d} \simeq \mathrm{End}_{\mathbb{H}}(\mathbb{V}^{\otimes d})$, there exists an $\mathbb{F}$-algebra homomorphism

$$\Psi : \mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n) \longrightarrow \mathbb{S}^{\mathfrak{c}}_{n,d}.$$

The next lemma follows by a standard Vandermonde determinant type argument.

Lemma 3.4. *For each $\lambda \in \Lambda_{n,d}$, the element $\phi^e_{\lambda,\lambda} \in \mathbb{S}^{\mathfrak{c}}_{n,d}$ belongs to the subalgebra of $\mathbb{S}^{\mathfrak{c}}_{n,d}$ generated by $\Psi(\mathbf{h}_a^{\pm 1})$, $0 \leq a \leq r+1$.*

Here, we define two families of maps $\widetilde{e}_i,\ \widetilde{f}_i : \Lambda_{n,d} \to \Lambda_{n,d} \sqcup \{0\}$ (0 is a formal symbol) by

$$\widetilde{e}_i(\lambda) := \begin{cases} (\lambda_0, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1} - 1, \lambda_{i+2}, \dots, \lambda_{r+1}) & \text{if } \lambda_{i+1} > 0; \\ 0 & \text{if } \lambda_{i+1} = 0, \end{cases}$$
$$\widetilde{f}_i(\lambda) := \begin{cases} (\lambda_0, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1} + 1, \lambda_{i+2}, \dots, \lambda_{r+1}) & \text{if } \lambda_i > 0; \\ 0 & \text{if } \lambda_i = 0. \end{cases}$$

By convention, it is understood that $M_0 = 0$ and $\phi^g_{0,\mu} = 0 = \phi^g_{\lambda,0}$.

Recall the comultiplication Δ of $\mathbb{U}(\widehat{\mathfrak{sl}}_n)$ from Section 2.1. Then, we have

$$\Delta^{(d-1)}(\mathbf{E}_i) = \sum_{k=0}^{d-1} 1^{\otimes k} \otimes \mathbf{E}_i \otimes (\mathbf{K}_i^{-1})^{\otimes d-k-1},$$
$$\Delta^{(d-1)}(\mathbf{F}_i) = \sum_{k=0}^{d-1} \mathbf{K}_i^{\otimes k} \otimes \mathbf{F}_i \otimes 1^{\otimes d-k-1}.$$

Lemma 3.5. *For $0 \leq i \leq r$, we have*

$$\Psi(\mathbf{e}_i) = \begin{cases} \sum_{\lambda \in \Lambda} q^{\lambda_{i+1}-1} \phi^e_{\widetilde{e}_i(\lambda),\lambda} & \text{if } i \neq r; \\ \sum_{\lambda \in \Lambda} q^{3(\lambda_{r+1}-1)} q_0 q_1^{-1} \phi^e_{\widetilde{e}_r(\lambda),\lambda} & \text{if } i = r, \end{cases}$$
$$\Psi(\mathbf{f}_i) = \begin{cases} \sum_{\lambda \in \Lambda} q^{\lambda_i - 1} \phi^e_{\widetilde{f}_i(\lambda),\lambda} & \text{if } i \neq 0, r; \\ \sum_{\lambda \in \Lambda} q_1 q^{2(\lambda_0 - 1)} \phi^e_{\widetilde{f}_0(\lambda),\lambda} & \text{if } i = 0; \\ \sum_{\lambda \in \Lambda} q_0 q_1^{-1} q^{\lambda_r - \lambda_{r+1} - 1} \phi^e_{\widetilde{f}_r(\lambda),\lambda} & \text{if } i = r. \end{cases}$$

Proof. The proof is by a direct computation. Below we present the details only for verifying the most complicated equation

$$\Psi(\mathbf{e}_r) = \sum_{\lambda \in \Lambda_{n,d}} q^{3(\lambda_{r+1}-1)} q_0 q_1^{-1} \phi^e_{\widetilde{e}_r(\lambda),\lambda}.$$

First, we compute $\Psi(\mathbf{e}_r)$. It suffices to compute $\Psi(\mathbf{e}_r)(M_\lambda)$ for all $\lambda \in \Lambda_{n,d}$. Since $\mathbf{e}_r = \mathbf{E}_r + q^{-1}\mathbf{F}_{-r-1}\mathbf{K}_r^{-1}$, we have

$$\begin{aligned}\Psi(\mathbf{e}_r)(M_\lambda) &= \Delta^{(d-1)}(\mathbf{E}_r)M_\lambda + q^{-1-\lambda_r+\lambda_{r+1}}\Delta^{(d-1)}(\mathbf{F}_{-r-1})M_\lambda \\ &= \sum_{k=1}^{\lambda_{r+1}} q^{\lambda_{r+1}-k} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},r,r+1^{\lambda_{r+1}-k})} \\ &\quad + q^{-1-\lambda_r+\lambda_{r+1}} \sum_{k=1}^{\lambda_{r+1}} q^{k-1} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},r+2,r+1^{\lambda_{r+1}-k})}.\end{aligned}$$

Next, we calculate $\phi^e_{\widetilde{e}_r(\lambda),\lambda}$. It suffices to compute $\phi^e_{\widetilde{e}_r(\lambda),\lambda}(M_\lambda)$. By the definition of $\phi^e_{\widetilde{e}_r(\lambda),\lambda}$, it follows that

$$\phi^e_{\widetilde{e}_r(\lambda),\lambda}(M_\lambda) = \sum_{w\in W_\lambda\cap\mathscr{D}_{\widetilde{e}_r(\lambda)}} q_w^{-1} M_{\widetilde{e}_r(\lambda)} T_w.$$

Note that

$$\begin{aligned}W_\lambda \cap \mathscr{D}_{\widetilde{e}_r(\lambda)} &= \{s_{\lambda_{0,r}+1}\cdots s_{\lambda_{0,r}+k-1}\}_{k=1}^{\lambda_{r+1}} \\ &\qquad \sqcup \{s_{\lambda_{0,r}+1}\cdots s_{d-1}s_d s_{d-1}\cdots s_{\lambda_{0,r}+k}\}_{k=1}^{\lambda_{r+1}}.\end{aligned}$$

Moreover, $s_{\lambda_{0,r}+1}\cdots s_{\lambda_{0,r}+k-1}$ and $s_{\lambda_{0,r}+1}\cdots s_{d-1}s_d s_{d-1}\cdots s_{\lambda_{0,r}+k}$ are reduced expressions for $1 \le k \le \lambda_{r+1}$. Hence, we have

$$\begin{aligned}\sum_{w\in W_\lambda\cap\mathscr{D}_{\widetilde{e}_r(\lambda)}} q_w^{-1} M_{\widetilde{e}_r(\lambda)} T_w &= \sum_{k=1}^{\lambda_{r+1}} q^{-k+1} M_{\widetilde{e}_r(\lambda)} T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+k-1} \\ &\quad + \sum_{k=1}^{\lambda_{r+1}} q^{-(2\lambda_{r+1}-k-1)} q_0 M_{\widetilde{e}_r(\lambda)} T_{\lambda_{0,r}+1}\cdots T_{d-1}T_d T_{d-1}\cdots T_{\lambda_{0,r}+k}.\end{aligned}$$

It is easily verified that

$$M_{\widetilde{e}_r(\lambda)} T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+k-1} = M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},r,r+1^{\lambda_{r+1}-k})}.$$

In order to compute the other terms, we first note that, from (2.3.6),

$$\begin{aligned}&T_{\lambda_{0,r}+1}\cdots T_{d-1}T_d T_{d-1}\cdots T_{\lambda_{0,r}+k} \\ &\quad = q_0^{-1} T_{\lambda_{0,r}+1}\cdots T_{d-1}(T_{d-1}\cdots T_1 X_1 T_0^{-1} T_1^{-1}\cdots T_{d-1}^{-1}) T_{d-1}\cdots T_{\lambda_{0,r}+k} \\ &\quad = q_0^{-1} T_{\lambda_{0,r}+1}\cdots T_{d-1} T_{d-1}\cdots T_1 X_1 T_0^{-1} T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1}.\end{aligned}$$

It then follows from (2.3.3) that

$$\begin{aligned}
&T_{\lambda_{0,r}+1}\cdots T_{d-1}T_dT_{d-1}\cdots T_{\lambda_{0,r}+k}\\
&= q_0^{-1}T_{\lambda_{0,r}+1}\cdots T_{d-1}T_{d-1}\cdots T_1q_0q_1^{-1}(T_0X_1^{-1}-(q_0^{-1}q_1-1))T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1}\\
&= q_1^{-1}T_{\lambda_{0,r}+1}\cdots T_{d-1}T_{d-1}\cdots T_1T_0X_1^{-1}T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1}\\
&\quad + (q_1^{-1}-q_0^{-1})T_{\lambda_{0,r}+1}\cdots T_{d-1}T_{d-1}\cdots T_{\lambda_{0,r}+k}.
\end{aligned}$$

Since for $a \le b$ we have $T_a\cdots T_bT_b\cdots T_a = 1 + (q^{-1}-q)\sum_{l=a}^{b} T_a\cdots T_l\cdots T_a$, we obtain

$$\begin{aligned}
&T_{\lambda_{0,r}+1}\cdots T_{d-1}T_dT_{d-1}\cdots T_{\lambda_{0,r}+k}\\
&= q_1^{-1}\Big(1+(q^{-1}-q)\sum_{l=1}^{\lambda_{r+1}-1} T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+l}\cdots T_{\lambda_{0,r}+1}\Big)\\
&\quad \times T_{\lambda_{0,r}}\cdots T_1T_0X_1^{-1}T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1}\\
&\quad + (q_1^{-1}-q_0^{-1})T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+k-1}\\
&\quad \times \Big(1+(q^{-1}-q)\sum_{l=k}^{\lambda_{r+1}-1} T_{\lambda_{0,r}+k}\cdots T_{\lambda_{0,r}+l}\cdots T_{\lambda_{0,r}+k}\Big)\\
&= q_1^{-1}T_{\lambda_{0,r}}\cdots T_1T_0X_1^{-1}T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1}\\
&\quad + q_1^{-1}(q^{-1}-q)\sum_{l=1}^{\lambda_{r+1}-1} T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+l-1}T_{\lambda_{0,r}+l}\\
&\quad \times T_{\lambda_{0,r}+l-1}\cdots T_1T_0X_1^{-1}T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1}\\
&\quad + (q_1^{-1}-q_0^{-1})T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+k-1}\\
&\quad + (q_1^{-1}-q_0^{-1})(q^{-1}-q)\\
&\quad \times \sum_{l=k}^{\lambda_{r+1}-1} T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+l-1}T_{\lambda_{0,r}+l}T_{\lambda_{0,r}+l-1}\cdots T_{\lambda_{0,r}+k}.
\end{aligned}$$

Therefore, we need to compute

(3.2.1) $$M_{\widetilde{e}_r(\lambda)}T_{\lambda_{0,r}}\cdots T_1T_0X_1^{-1}T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1},$$

(3.2.2) $$M_{\widetilde{e}_r(\lambda)}T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+l-1}T_{\lambda_{0,r}+l}T_{\lambda_{0,r}+l-1}\cdots T_{\lambda_{0,r}+k},$$

(3.2.3) $$\begin{aligned}&M_{\widetilde{e}_r(\lambda)}T_{\lambda_{0,r}+1}\cdots T_{\lambda_{0,r}+l-1}T_{\lambda_{0,r}+l}\\ &\qquad \times T_{\lambda_{0,r}+l-1}\cdots T_1T_0X_1^{-1}T_1^{-1}\cdots T_{\lambda_{0,r}+k-1}^{-1},\end{aligned}$$

which are given as below:

$$
\begin{aligned}
(3.2.1) &= M_{\widetilde{e}_r(\lambda)} T_{\lambda_{0,r}} \cdots T_1 T_0 T_1 \cdots T_{\lambda_{0,r}+k-1} X^{-1}_{\lambda_{0,r}+k} \\
&= q^{-\lambda_r} M_{0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},-r+n,r+1^{\lambda_{r+1}-k}}, \\
(3.2.2) &= M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l},r,r+1^{\lambda_{r+1}-l-1})} T_{\lambda_{0,r}+l-1} \cdots T_{\lambda_{0,r}+k} \\
&= q^{-(l-k)} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l},r,r+1^{\lambda_{r+1}-l-1})}, \\
(3.2.3) &= M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l},r,r+1^{\lambda_{r+1}-l-1})} T_{\lambda_{0,r}+l-1} \cdots T_0 X_1^{-1} T_1^{-1} \cdots T_{\lambda_{0,r}+k-1} \\
&= q^{-(l-1)} M_{(r+1,0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l-1},r,r+1^{\lambda_{r+1}-l-1})} T_0 X_1^{-1} T_1^{-1} \cdots T_{\lambda_{0,r}+k-1} \\
&= q^{-(l-1)} q_0^{-1} q_1 M_{(r+1-n,0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l-1},r,r+1^{\lambda_{r+1}-l-1})} X_1^{-1} T_1^{-1} \cdots T_{\lambda_{0,r}+k-1} \\
&= \begin{cases} q^{k-l-1} q_0^{-1} q_1 M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l-1},r,r+1^{\lambda_{r+1}-l})} & \text{if } 1 \le l \le k-1; \\ q^{k-l} q_0^{-1} q_1 M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l},r,r+1^{\lambda_{r+1}-l-1})} & \text{if } k \le l \le \lambda_{r+1}-1. \end{cases}
\end{aligned}
$$

Summarizing the calculations above, we have

$$
\begin{aligned}
\phi^{e}_{\widetilde{e}_r(\lambda),\lambda}(M_\lambda) &= \sum_{k=1}^{\lambda_{r+1}} q^{-k+1} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},r,r+1^{\lambda_{r+1}-k})} \\
&+ \sum_{k=1}^{\lambda_{r+1}} q^{-2\lambda_{r+1}-\lambda_r+k+1} q_0 q_1^{-1} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},-r+n,r+1^{\lambda_{r+1}-k})} \\
&+ \sum_{k=1}^{\lambda_{r+1}} q^{-2\lambda_{r+1}+k+1}(q^{-1}-q)\left(\sum_{l=1}^{k-1} q^{k-l-1} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l-1},r,r+1^{\lambda_{r+1}-l})} \right. \\
&\quad \left. + \sum_{l=k}^{\lambda_{r+1}-1} q^{k-l} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l},r,r+1^{\lambda_{r+1}-l-1})} \right) \\
&+ \sum_{k=1}^{\lambda_{r+1}} q^{-2\lambda_{r+1}+k+1}(q_0 q_1^{-1}-1) M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},r,r+1^{\lambda_{r+1}-k})} \\
&+ \sum_{k=1}^{\lambda_{r+1}} q^{-2\lambda_{r+1}+k+1}(q_0 q_1^{-1}-1)(q^{-1}-q) \\
&\times \sum_{l=k}^{\lambda_{r+1}-1} q^{-l+k} M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{l},r,r+1^{\lambda_{r+1}-l-1})}.
\end{aligned}
$$

The coefficient for $M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},-r+n,r+1^{\lambda_{r+1}-k})}$ is

$$
q^{-3(\lambda_{r+1}-1)} q_0 q_1^{-1} q^{-1-\lambda_r+\lambda_{r+1}+k-1},
$$

while that for $M_{(0^{\lambda_0},\dots,r^{\lambda_r},r+1^{k-1},r,r+1^{\lambda_{r+1}-k})}$ is

$$q^{-k+1}+q^{-2\lambda_{r+1}+k+1}(q_0q_1^{-1}-1)+\sum_{m=k+1}^{\lambda_{r+1}}q^{-2\lambda_{r+1}-k+2m}(q^{-1}-q)$$
$$+\sum_{m=1}^{k-1}q^{-2\lambda_{r+1}-k+2+2m}(q^{-1}-q)+\sum_{m=1}^{k-1}q^{-2\lambda_{r+1}-k+2+2m}(q^{-1}-q)(q_0q_1^{-1}-1).$$

One checks that the latter coincides with

$$q^{-2\lambda_{r+1}-k+3}q_0q_1^{-1}=q^{-3(\lambda_{r+1}-1)}q_0q_1^{-1}q^{\lambda_{r+1}-k}.$$

Thus, we obtain

$$\phi^e_{\widetilde{e}_r(\lambda),\lambda}(M_\lambda)=q^{-3(\lambda_{r+1}-1)}q_0q_1^{-1}\Psi(\mathbf{e}_r)(M_\lambda),$$

which proves the assertion. □

Proposition 3.6. *Assume $r\geq d$. Then the Schur algebra $\mathbb{S}^{\mathfrak{c}}_{n,d}$ is generated by $\Psi(\mathbf{e}_i)$, $\Psi(\mathbf{f}_i)$, $\Psi(\mathbf{h}_a^{\pm1})$ for $0\leq i\leq r$ and $0\leq a\leq r+1$.*

Proof. Let S denote the subalgebra of $\mathbb{S}^{\mathfrak{c}}_{n,d}$ generated by $\Psi(\mathbf{e}_i)$, $\Psi(\mathbf{f}_i)$, $\Psi(\mathbf{h}_a^{\pm1})$, for $0\leq i\leq r$, and for $0\leq a\leq r+1$. By Lemma 3.4 and 3.5, for each $\lambda\in\Lambda_{n,d}$ and $0\leq i\leq r$, we have $\phi^e_{\widetilde{e}_i(\lambda),\lambda}\in S$ and $\phi^e_{\widetilde{f}_i(\lambda),\lambda}\in S$.

Take $\lambda\in\Lambda_{n,d}$ arbitrarily. It is easy to check that there exists a sequence $(x_1,\dots,x_l)$ of $\widetilde{e}_i,\widetilde{f}_i$'s such that $\lambda=x_1\cdots x_l(\omega)$ and $\{e\}=W_{\lambda_0}\subset\cdots\subset W_{\lambda_l}=W_\lambda$, where $\lambda_0=\omega$ and $\lambda_i=x_i(\lambda_{i-1})$. Then, we have

$$\phi^e_{\lambda,\omega}=\phi^e_{\lambda_r,\lambda_{r-1}}\cdots\phi^e_{\lambda_1,\lambda_0}\in S.$$

By the same way, we obtain $\phi^e_{\omega,\lambda}\in S$.

Next, for $0\leq i\leq d-1$, we have

$$\phi^e_{\omega,\widetilde{e}_i(\omega)}\cdot\phi^e_{\widetilde{e}_i(\omega),\omega}=\phi^e_{\omega,\omega}+q_{s_i}^{-1}\phi^{s_i}_{\omega,\omega}.$$

We also have

$$\phi^e_{\omega,\widetilde{f}_r(\omega)}\cdot\phi^e_{\widetilde{f}_r(\omega),\omega}=\phi^e_{\omega,\omega}+\phi^{s_d}_{\omega,\omega}.$$

These show that $\phi^{s_i}_{\omega,\omega}\in S$ for $0\leq i\leq d$. Since $\phi^e_{\omega,\omega}\mathbb{S}^{\mathfrak{c}}_{n,d}\phi^e_{\omega,\omega}$ is generated by $\phi^{s_i}_{\omega,\omega}\in S$, $0\leq i\leq r$, we have $\phi^e_{\omega,\omega}\mathbb{S}^{\mathfrak{c}}_{n,d}\phi^e_{\omega,\omega}\subset S$.

Finally, for each $\lambda, \mu \in \Lambda_{n,d}$, we have

$$\phi^e_{\lambda,\omega}\mathbb{S}^{\mathfrak{c}}_{n,d}\phi^e_{\omega,\mu} = \phi^e_{\lambda,\omega}\phi^e_{\omega,\omega}\mathbb{S}^{\mathfrak{c}}_{n,d}\phi^e_{\omega,\omega}\phi^e_{\omega,\mu} \subset S.$$

Since $\mathbb{S}^{\mathfrak{c}}_{n,d}$ is the direct sum of $\phi^e_{\lambda,\omega}\mathbb{S}^{\mathfrak{c}}_{n,d}\phi^e_{\omega,\mu}$, we conclude that $S = \mathbb{S}^{\mathfrak{c}}_{n,d}$. □

Theorem 3.7. *Suppose $r \geq d \geq 1$. We have the following Schur $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n), \mathbb{H})$-duality:*

$$\Psi(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)) \simeq \mathrm{End}_{\mathbb{H}}(\mathbb{V}^{\otimes d}),$$
$$\mathrm{End}_{\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)}(\mathbb{V}^{\otimes d}) \simeq \mathbb{H}^{op}.$$

(To be consistent with the variants in next section, we can refer to this as $\jmath$-Schur duality.)

Proof. It follows by Proposition 3.6 that $\Psi(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)) = \mathbb{S}^{\mathfrak{c}}_{n,d}$. Hence the first isomorphism follows by Lemma 3.2.

Since $\Psi(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)) = \mathbb{S}^{\mathfrak{c}}_{n,d}$ induces an isomorphism

$$\mathrm{End}_{\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)}(\mathbb{V}^{\otimes d}) \simeq \mathrm{End}_{\kappa\mathbb{S}^{\mathfrak{c}}_{n,d}\kappa^{-1}}(\mathbb{V}^{\otimes d}),$$

we have

$$\begin{aligned}\mathrm{End}_{\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)}(\mathbb{V}^{\otimes d}) &\simeq \mathrm{End}_{\kappa\mathbb{S}^{\mathfrak{c}}_{n,d}\kappa^{-1}}(\mathbb{V}^{\otimes d})\\ &\simeq \mathrm{End}_{\mathbb{S}^{\mathfrak{c}}_{n,d}}(\mathbb{T}^{\mathfrak{c}}_{n,d}) \simeq (\phi^e_{\omega,\omega}\mathbb{S}^{\mathfrak{c}}_{n,d}\phi^e_{\omega,\omega})^{op} \simeq \mathbb{H}^{op}.\end{aligned}$$

The second isomorphism follows. □

3.3. Specializations

When specializing $\mathbb{H}$ to the single parameter case by letting $q_0 = 1$ and $q_1 = q^2$, we obtain the affine Hecke algebra of type C over $\mathbb{Q}(q)$, denoted here by $\mathbf{H}_{C_d}$. This is the Hecke algebra appearing in [9]-[10].

When specializing $\mathbb{H}$ to $q_0 = q_1$, we obtain the extended affine Hecke algebra of type B over $\mathbb{Q}(q, q_1)$ in 2 parameters q, q_1. When specializing $\mathbb{H}$ to the single parameter case by letting $q_0 = q_1 = q$, we obtain the extended affine Hecke algebra of type B over $\mathbb{Q}(q)$, denoted here by $\mathbf{H}_{B_d}$.

When specializing $\mathbb{H}$ to $q_0 = q_1 = 1$, we obtain the extended affine Hecke algebra of type D over $\mathbb{Q}(q)$, denoted here by $\mathbf{H}_{D_d}$.

Specializing our main Theorem 3.7 on the 3-parameter Schur duality to 2-parameter or 1-parameter cases, we obtain several versions of dualities, each of which is meaningful in its own way. In this sense, the duality in Theorem 3.7 is a master duality which unifies dualities of different types (among which the 1-parameter dualities should admit geometric interpretations using different types of flags).

The framework in [9] provides a geometric setting for the $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)|_{q_0=1,q_1=q^2}, \mathbf{H}_{C_d})$-duality on $\mathbb{V}|^{\otimes d}_{q_0=1,q_1=q^2}$. Both $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)|_{q_0=1,q_1=q^2}$ and $\mathbf{H}_{C_d}$ are geometrically realized; while not discussed explicitly therein, $\mathbb{V}|^{\otimes d}_{q_0=1,q_1=q^2}$ can also be geometrically realized in terms of varieties of pairs of an "n-step" partial flag and a complete flag.

Remark 3.8. Our work can lead to several interesting future projects, which are highly nontrivial to carry out. One bonus of carrying out these geometric constructions will be the positivity of the resulting $\imath$canonical bases.

1) A geometric setting in flag variety of affine type B similar to [9] for the $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)|_{q_0=q_1=q}, \mathbf{H}_{B_d})$-duality on $\mathbb{V}|^{\otimes d}_{q_0=q_1=q}$ is expected.
2) A geometric setting in flag variety of affine type D similar to [9] for the $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)|_{q_0=q_1=1}, \mathbf{H}_{D_d})$-duality on $\mathbb{V}|^{\otimes d}_{q_0=q_1=1}$ is expected. The finite type version of this duality would be a modification of the construction in [11].
3) The algebraic construction in [10] is expected to generalize to the 3-parameter case or various 2-parameter or equal parameter specializations.
4) Classify the finite-dimensional irreducible $\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n)$-modules.
5) All remarks in §3.3 here are valid for the variants of Schur $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n), \mathbb{H})$-duality considered in Section 4 below.

4. Variants of Schur dualities

Motivated by [9]–[10], we formulate in this section several variants of the Schur $(\mathbb{U}^{\mathfrak{c}}(\widehat{\mathfrak{sl}}_n), \mathbb{H})$-duality in Theorem 3.7. We continue to assume $r \geq d \geq 1$. Furthermore we set

$$\mathfrak{n} = n - 1 = 2r + 1, \qquad \eta = n - 2 = 2r.$$

4.1. The $\jmath$-Schur duality

Let $'\mathbb{V}_{\mathfrak{n}}$ be the $\mathbb{F}$-subspace of $\mathbb{V}$ spanned by v_i, for $i \in \mathbb{Z}$ such that $i \not\equiv r+1 \pmod{n}$. Note that $'\mathbb{V}_{\mathfrak{n}}$ is naturally an $\mathbb{H}$-submodule of $\mathbb{V}$, and moreover, it is a direct sum of permutation modules.

We consider an isomorphic copy of $\mathbb{U}(\widehat{\mathfrak{gl}}_{\mathfrak{n}})$ (with a different indexing set for generators), denoted by $\mathbb{U}('\widehat{\mathfrak{gl}}_{\mathfrak{n}})$. The algebra $\mathbb{U}('\widehat{\mathfrak{gl}}_{\mathfrak{n}})$ is generated by $\mathbf{E}_i, \mathbf{F}_i$ $(i \in [0, n-1]\backslash\{r+1\}), \mathbf{D}_a^{\pm 1}$ $(a \in [0, n-1]\backslash\{r+1\})$; here we regard indices $r, r+2$ adjacent. Denote by $\mathbb{U}('\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ the subalgebra of $\mathbb{U}('\widehat{\mathfrak{gl}}_{\mathfrak{n}})$ generated by $\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i$ $(i \in [0, n-1]\backslash\{r+1\})$, where $\mathbf{K}_r = \mathbf{D}_r\mathbf{D}_{r+2}^{-1}, \mathbf{K}_i = \mathbf{D}_i\mathbf{D}_{i+1}^{-1}$ $(i \neq r)$. Then $'\mathbb{V}_{\mathfrak{n}}$ is a natural representation of $\mathbb{U}('\widehat{\mathfrak{gl}}_{\mathfrak{n}})$, with the action given by: for $i \in [0, n-1]\backslash\{r+1\}, a \in [0, n]\backslash\{r+1\}$,

$$\mathbf{E}_i v_{j+1} = \begin{cases} v_j & \text{if } j \equiv i \neq r; \\ v_{j-1} & \text{if } j-1 \equiv i = r; \\ 0 & \text{else,} \end{cases} \tag{4.1.1}$$

$$\mathbf{F}_i v_j = \begin{cases} v_{j+1} & \text{if } j \equiv i \neq r; \\ v_{j+2} & \text{if } j \equiv i = r; \\ 0 & \text{else,} \end{cases} \qquad \mathbf{D}_a v_j = \begin{cases} q v_j & \text{if } j \equiv a; \\ v_j & \text{else.} \end{cases}$$

Then $\mathbb{U}('\widehat{\mathfrak{gl}}_{\mathfrak{n}})$ and $\mathbb{U}('\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ act on $'\mathbb{V}_{\mathfrak{n}}^{\otimes d}$ via iterated comultiplication.

For $i, j \in [0, r]$, we denote the Cartan integers by

$$\mathsf{c}_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}.$$

Define $\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ (cf. [9, Chapter 7]) to be the $\mathbb{F}$-algebra generated by $\mathbf{e}_i, \mathbf{f}_i$, and $\mathbf{k}_i^{\pm 1}$ $(0 \leq i \leq r-1)$ and $\mathbf{t}_r$, subject to the following relations: for all $0 \leq i, j \leq r-1$,

$$\begin{aligned}
&\mathbf{k}_0(\mathbf{k}_1^2 \cdots \mathbf{k}_{r-1}^2) = q^{-1}, \quad \mathbf{k}_i\mathbf{k}_i^{-1} = 1, \quad \mathbf{k}_i\mathbf{k}_j = \mathbf{k}_j\mathbf{k}_i, \quad \mathbf{k}_i\mathbf{t}_r = \mathbf{t}_r\mathbf{k}_i, \\
&\mathbf{k}_i\mathbf{e}_j\mathbf{k}_i^{-1} = q^{\mathsf{c}_{ij}+\delta_{i,0}\delta_{j,0}}\mathbf{e}_j, \quad \mathbf{k}_i\mathbf{f}_j\mathbf{k}_i^{-1} = q^{-\mathsf{c}_{ij}-\delta_{i,0}\delta_{j,0}}\mathbf{f}_j, \\
&\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i, \quad \mathbf{f}_i\mathbf{f}_j = \mathbf{f}_j\mathbf{f}_i, \quad \forall |i-j| > 1, \\
&\mathbf{e}_i\mathbf{f}_j - \mathbf{f}_j\mathbf{e}_i = \delta_{ij}\frac{\mathbf{k}_i - \mathbf{k}_i^{-1}}{q - q^{-1}}, \quad \forall (i,j) \neq (0,0), \\
&\mathbf{e}_i\mathbf{t}_r = \mathbf{t}_r\mathbf{e}_i, \quad \mathbf{f}_i\mathbf{t}_r = \mathbf{t}_r\mathbf{f}_i, \quad \forall i \leq r-2,
\end{aligned}$$

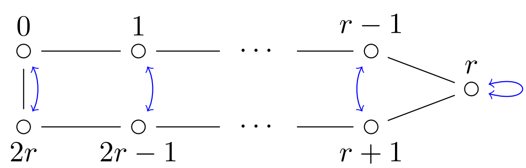
$$\mathbf{e}_{r-1}^2\mathbf{t}_r + \mathbf{t}_r\mathbf{e}_{r-1}^2 = (q+q^{-1})\mathbf{e}_{r-1}\mathbf{t}_r\mathbf{e}_{r-1},$$
$$\mathbf{f}_{r-1}^2\mathbf{t}_r + \mathbf{t}_r\mathbf{f}_{r-1}^2 = (q+q^{-1})\mathbf{f}_{r-1}\mathbf{t}_r\mathbf{f}_{r-1},$$
$$\mathbf{t}_r^2\mathbf{e}_{r-1} + \mathbf{e}_{r-1}\mathbf{t}_r^2 = (q+q^{-1})\mathbf{t}_r\mathbf{e}_{r-1}\mathbf{t}_r + q_0q_1^{-1}\mathbf{e}_{r-1},$$
$$\mathbf{t}_r^2\mathbf{f}_{r-1} + \mathbf{f}_{r-1}\mathbf{t}_r^2 = (q+q^{-1})\mathbf{t}_r\mathbf{f}_{r-1}\mathbf{t}_r + q_0q_1^{-1}\mathbf{f}_{r-1},$$
$$\mathbf{e}_i^2\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i^2 = (q+q^{-1})\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i, \quad \mathbf{f}_i^2\mathbf{f}_j + \mathbf{f}_j\mathbf{f}_i^2 = (q+q^{-1})\mathbf{f}_i\mathbf{f}_j\mathbf{f}_i, \quad \forall |i-j| = 1,$$
$$\mathbf{e}_0^2\mathbf{f}_0 + \mathbf{f}_0\mathbf{e}_0^2 = (q+q^{-1})\big(\mathbf{e}_0\mathbf{f}_0\mathbf{e}_0 - q_1q\mathbf{e}_0\mathbf{k}_0 - q_0^{-1}q^{-1}\mathbf{e}_0\mathbf{k}_0^{-1}\big),$$
$$\mathbf{f}_0^2\mathbf{e}_0 + \mathbf{e}_0\mathbf{f}_0^2 = (q+q^{-1})\big(\mathbf{f}_0\mathbf{e}_0\mathbf{f}_0 - qq_1\mathbf{k}_0\mathbf{f}_0 - q_0^{-1}q^{-1}\mathbf{k}_0^{-1}\mathbf{f}_0\big).$$

Proposition 4.1. *There is an injective $\mathbb{F}$-algebra homomorphism $\jmath\imath$: $\mathbb{U}^{\jmath\imath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}) \to \mathbb{U}({}'\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ such that*

$$\begin{aligned}
&\mathbf{k}_a \mapsto \mathbf{K}_a\mathbf{K}_{-a-1}^{-1}, \quad (0 \le a \le r-1)\\
&\mathbf{e}_i \mapsto \mathbf{E}_i + \mathbf{F}_{-i-1}\mathbf{K}_i^{-1}, \quad \mathbf{f}_i \mapsto \mathbf{E}_{-i-1} + \mathbf{F}_i\mathbf{K}_{-i-1}^{-1}, \quad (1 \le i \le r-1)\\
&\mathbf{e}_0 \mapsto \mathbf{E}_0 + q_0^{-1}\mathbf{F}_{-1}\mathbf{K}_0^{-1}, \quad \mathbf{f}_0 \mapsto \mathbf{E}_{-1} + q_1q^{-1}\mathbf{F}_0\mathbf{K}_{-1}^{-1},\\
(4.1.2)\quad &\mathbf{t}_r \mapsto \mathbf{E}_r + qq_0q_1^{-1}\mathbf{F}_r\mathbf{K}_r^{-1} + (1-q_0q_1^{-1})/(q-q^{-1})\mathbf{K}_r^{-1}.
\end{aligned}$$

Proof. The proof is similar to the proof for Proposition 2.2. The subalgebra here is a quantum symmetric pair coideal subalgebra associated with the Dynkin diagram and involution below, and the proposition follows from [16, Theorem 7.8]. □

Figure 2: Dynkin diagram of type $A_{2r}^{(1)}$ with involution of type $\jmath\imath$.



Recalling $\Lambda_{n,d}$ from (3.1.1), we define

$$\Lambda^{\jmath\imath}_{\mathfrak{n},d} = \{\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r+1}) \in \Lambda_{n,d} \mid \lambda_{r+1} = 0\}.$$

Note that $\omega \in \Lambda^{\jmath\imath}_{\mathfrak{n},d}$. Define the right $\mathbb{H}$-module

$$\mathbb{T}^{\jmath\imath}_{\mathfrak{n},d} = \bigoplus_{\lambda \in \Lambda^{\jmath\imath}_{\mathfrak{n},d}} x_\lambda \mathbb{H}.$$

Following [10], we define the $\jmath$-variant of the Schur algebra $\mathbb{S}^{\mathfrak{c}}_{n,d}$ as follows:

$$\mathbb{S}^{\jmath}_{\mathfrak{n},d} = \mathrm{End}_{\mathbb{H}}(\oplus_{\lambda\in\Lambda^{\jmath}_{\mathfrak{n},d}} x_\lambda \mathbb{H}) = \bigoplus_{\lambda,\mu\in\Lambda^{\jmath}_{\mathfrak{n},d}} \mathrm{Hom}_{\mathbb{H}}(x_\mu\mathbb{H}, x_\lambda\mathbb{H}).$$

It is routine to show that $\{\phi^g_{\lambda,\mu} \mid \lambda,\mu\in\Lambda^{\jmath}_{\mathfrak{n},d}, g\in\mathscr{D}_{\lambda,\mu}\}$ form an $\mathbb{F}$-basis of $\mathbb{S}^{\jmath}_{\mathfrak{n},d}$.

The following is a variant of Lemma 3.2.

Lemma 4.2. *We have an isomorphism of* $\mathbb{H}$*-modules:* $\mathbb{T}^{\jmath}_{\mathfrak{n},d} \cong {}'\mathbb{V}_{\mathfrak{n}}^{\otimes d}$.

Note $\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ acts on ${}'\mathbb{V}_{\mathfrak{n}}^{\otimes d}$ via the embedding $\jmath : \mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}) \to \mathbb{U}({}'\widehat{\mathfrak{sl}}_{\mathfrak{n}})$; we denote this action by $\Psi^{\jmath}$.

Theorem 4.3. *We have the following Schur* $(\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}), \mathbb{H})$*-duality:*

$$\Psi^{\jmath}(\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})) \simeq \mathrm{End}_{\mathbb{H}}({}'\mathbb{V}_{\mathfrak{n}}^{\otimes d}),$$
$$\mathrm{End}_{\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})}({}'\mathbb{V}_{\mathfrak{n}}^{\otimes d}) \simeq \mathbb{H}^{op}.$$

Proof. We first check that the actions of $\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ and $\mathbb{H}$ on ${}'\mathbb{V}_{\mathfrak{n}}^{\otimes d}$ commute. As seen in Proposition 3.3, it remains to verify that the T_0-action commutes with the $\mathbf{t}_r$-action on ${}'\mathbb{V}_{\mathfrak{n}}$. For the unique expression $f = k + jn$ such that $-r \le k \le r$, we combine (4.1.2) and (4.1.1) and then obtain

$$\mathbf{t}_r v_f = \frac{1-q_0q_1^{-1}}{q-q^{-1}} v_f \quad (f \not\equiv r, r+2),$$
$$\mathbf{t}_r v_r z^j = q_0q_1^{-1} v_{-r} z^{j+1} + \frac{1-q_0q_1^{-1}}{q-q^{-1}} q^{-1} v_r z^j,$$
$$\mathbf{t}_r v_{-r} z^j = v_r z^{j-1} + \frac{1-q_0q_1^{-1}}{q-q^{-1}} q v_{-r} z^j.$$

Note that the $\mathbf{t}_r$-action is a scalar multiplication for $f \not\equiv r, r+2$ and hence it commutes with the T_0-action (2.4.5). For $k = r, j \ge 0$ we have

$$
\begin{aligned}
(\mathbf{t}_r v_r z^j)T_0 = q_0 q_1^{-1}\Bigg(& q_0^{-1} q_1 v_r z^{-j-1} + (q_1 - q_0^{-1}) \sum_{l=1}^{j} v_{-r} z^{j-2l+1} \\
& + (q_0^{-1} q_1 - 1) \sum_{l=1}^{j+1} v_{-r} z^{j-2l+2} \Bigg) \\
& + \frac{1 - q_0 q_1^{-1}}{q - q^{-1}} q^{-1} \Bigg(v_{-r} z^{-j} + (q_1 - q_0^{-1}) \sum_{l=1}^{j} v_r z^{j-2l} \\
& + (q_0^{-1} q_1 - 1) \sum_{l=1}^{j} v_r z^{j-2l+1} \Bigg), \\
\mathbf{t}_r(v_r z^j T_0) = v_r z^{-j-1} & + \frac{1 - q_0 q_1^{-1}}{q - q^{-1}} q v_{-r} z^{-j} \\
& + (q_1 - q_0^{-1}) \sum_{l=1}^{j} \left(q_0 q_1^{-1} v_{-r} z^{j-2l+1} + \frac{1 - q_0 q_1^{-1}}{q - q^{-1}} q^{-1} v_r z^{j-2l} \right) \\
& + (q_0^{-1} q_1 - 1) \sum_{l=1}^{j} \left(q_0 q_1^{-1} v_{-r} z^{j-2l+2} + \frac{1 - q_0 q_1^{-1}}{q - q^{-1}} q^{-1} v_r z^{j-2l+1} \right).
\end{aligned}
$$

They are indeed equal. The rest can be checked similarly and the commutivity follows.

For the first isomorphism, it suffices to show that

$$
\Psi^{\jmath\imath}(\mathbb{U}^{\jmath\imath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})) \simeq \mathbb{S}^{\jmath\imath}_{\mathfrak{n},d}, \tag{4.1.3}
$$

which follows from a variant of Proposition 3.6 as below. Let $'S$ be the subalgebra of $\mathbb{S}^{\jmath\imath}_{\mathfrak{n},d}$ generated by $\Psi^{\jmath\imath}(\mathbf{e}_i), \Psi^{\jmath\imath}(\mathbf{f}_i), \Psi^{\jmath\imath}(\mathbf{k}_i)$ and $\Psi^{\jmath\imath}(\mathbf{t}_r)$ for $0 \leq i \leq r-1$. Similar to the proof of Proposition 3.6, one can show that $'S$ contains the elements $\phi^e_{\omega,\lambda}, \phi^e_{\lambda,\omega}$ for all $\lambda \in \Lambda^{\jmath\imath}_{\mathfrak{n},d}$ and the elements $\phi^{s_i}_{\omega,\omega}$ for $0 \leq i \leq d-1$. The only difference here is that $\phi^{s_d}_{\omega,\omega} \in {}'S$ follows from

$$
\Psi^{\jmath\imath}(\mathbf{t}_r) \in \sum_{\lambda \in \Lambda^{\jmath\imath}_{\mathfrak{n},d}} \mathbb{F}\phi^{s_d}_{\lambda,\lambda}.
$$

For the second isomorphism, note that (4.1.3) together with Lemma 4.2 induce an isomorphism $\mathrm{End}_{\mathbb{U}^{\jmath\imath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})}({}'\mathbb{V}_{\mathfrak{n}}^{\otimes d}) \simeq \mathrm{End}_{\mathbb{S}^{\jmath\imath}_{\mathfrak{n},d}}(\mathbb{T}^{\jmath\imath}_{\mathfrak{n},d})$. The theorem now follows since

$$
\mathrm{End}_{\mathbb{S}^{\jmath\imath}_{\mathfrak{n},d}}(\mathbb{T}^{\jmath\imath}_{\mathfrak{n},d}) \simeq (\phi^e_{\omega,\omega}\mathbb{S}^{\jmath\imath}_{\mathfrak{n},d}\phi^e_{\omega,\omega})^{op} \simeq \mathbb{H}^{op}.
$$

□

4.2. The $\imath\jmath$-Schur duality

Let $\mathbb{V}_\mathfrak{n}$ be the $\mathbb{F}$-subspace of $\mathbb{V}$ spanned by v_i, for $i \in \mathbb{Z}$ such that $i \not\equiv 0 \pmod n$. Note that $\mathbb{V}_\mathfrak{n}$ is naturally an $\mathbb{H}$-submodule of $\mathbb{V}$, and moreover, it is a direct sum of permutation modules.

Recall $\mathbb{U}(\widehat{\mathfrak{gl}}_\mathfrak{n})$ is generated by $\mathbf{E}_i, \mathbf{F}_i (0 \le i \le \mathfrak{n}-1), \mathbf{D}_a^{\pm 1} (1 \le a \le \mathfrak{n})$. Denote by $\mathbb{U}(\widehat{\mathfrak{sl}}_\mathfrak{n})$ the subalgebra of $\mathbb{U}(\widehat{\mathfrak{gl}}_\mathfrak{n})$ generated by $\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i$ $(i \in [0, \mathfrak{n}-1])$, where $\mathbf{K}_0 = \mathbf{D}_{n-1}\mathbf{D}_1^{-1}, \mathbf{K}_i = \mathbf{D}_i \mathbf{D}_{i+1}^{-1}$ $(i \neq 0)$. Then $\mathbb{V}_\mathfrak{n}$ is a natural representation of $\mathbb{U}(\widehat{\mathfrak{gl}}_\mathfrak{n})$, with the action given by

$$\mathbf{E}_i v_{j+1} = \begin{cases} v_j & \text{if } j \equiv i \neq 0; \\ v_{j-1} & \text{if } j \equiv i = 0; \\ 0 & \text{else,} \end{cases}$$

$$\mathbf{F}_i v_j = \begin{cases} v_{j+1} & \text{if } j \equiv i \neq 0; \\ v_{j+2} & \text{if } j+1 \equiv i = 0; \\ 0 & \text{else,} \end{cases} \qquad \mathbf{D}_a v_j = \begin{cases} q v_j & \text{if } j \equiv a; \\ v_j & \text{else.} \end{cases}$$

Then $\mathbb{U}(\widehat{\mathfrak{gl}}_\mathfrak{n})$ and $\mathbb{U}(\widehat{\mathfrak{sl}}_\mathfrak{n})$ act on $\mathbb{V}_\mathfrak{n}^{\otimes d}$ via iterated comultiplication.

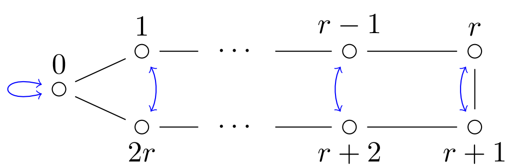
Define $\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_\mathfrak{n})$ to be an $\mathbb{F}$-algebra generated by $\mathbf{t}_0, \mathbf{e}_i, \mathbf{f}_i$, and $\mathbf{k}_i^{\pm 1}$ $(1 \le i \le r)$. We will not write down all its relations explicitly, as there is a $\mathbb{Q}(q)$-algebra isomorphism $\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_\mathfrak{n}) \to \mathbb{U}^{\jmath\imath}(\widehat{\mathfrak{sl}}_\mathfrak{n})$, which sends $q_0 \mapsto q_1, q_1 \mapsto q_0, \mathbf{t}_0 \mapsto \mathbf{t}_r, \mathbf{e}_i \mapsto \mathbf{e}_{r-i}, \mathbf{f}_i \mapsto \mathbf{f}_{r-i}, \mathbf{k}_i \mapsto \mathbf{k}_{r-i}$, for $1 \le i \le r$. In particular, the Serre relations for $\mathbf{t}_0$ are as follows:

$$\mathbf{t}_0^2 \mathbf{e}_1 + \mathbf{e}_1 \mathbf{t}_0^2 = (q+q^{-1}) \mathbf{t}_0 \mathbf{e}_1 \mathbf{t}_0 + q_0^{-1} q_1 \mathbf{e}_1,$$
$$\mathbf{t}_0^2 \mathbf{f}_1 + \mathbf{f}_1 \mathbf{t}_0^2 = (q+q^{-1}) \mathbf{t}_0 \mathbf{f}_1 \mathbf{t}_0 + q_0^{-1} q_1 \mathbf{f}_1.$$

We refer to §4.1 for the rest of relations of $\mathbb{U}^{\jmath\imath}(\widehat{\mathfrak{sl}}_\mathfrak{n})$.

The following proposition is a variant of Proposition 4.1 associated to the Dynkin diagram below in Figure 3.

Figure 3: Dynkin diagram of type $A_{2r}^{(1)}$ with involution of type $\imath\jmath$.



Proposition 4.4. *There is an injective* $\mathbb{F}$*-algebra homomorphism* $\jmath : \mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}) \to \mathbb{U}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ *such that*

$$\begin{aligned}
&\mathbf{k}_a \mapsto \mathbf{K}_a\mathbf{K}_{-a-1}^{-1}, \quad (1 \le a \le r)\\
&\mathbf{e}_i \mapsto \mathbf{E}_i + \mathbf{F}_{-i-1}\mathbf{K}_i^{-1}, \quad \mathbf{f}_i \mapsto \mathbf{E}_{-i-1} + \mathbf{F}_i\mathbf{K}_{-i-1}^{-1}, \quad (1 \le i \le r-1)\\
&\mathbf{e}_r \mapsto \mathbf{E}_r + q^{-1}\mathbf{F}_{-r-1}\mathbf{K}_r^{-1}, \quad \mathbf{f}_r \mapsto \mathbf{E}_{-r-1} + q_0 q_1^{-1}\mathbf{F}_r\mathbf{K}_{-r-1}^{-1},\\
&\mathbf{t}_0 \mapsto \mathbf{E}_0 + qq_0^{-1}q_1\mathbf{F}_0\mathbf{K}_0^{-1} + (q_1 - q_0^{-1})/(q-q^{-1})\mathbf{K}_0^{-1}.
\end{aligned}$$

Recalling $\Lambda_{n,d}$ from (3.1.1), we define

$$\Lambda^{\imath\jmath}_{\mathfrak{n},d} = \{\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r+1}) \in \Lambda_{n,d} \mid \lambda_0 = 0\},$$

and define the right $\mathbb{H}$-module

$$\mathbb{T}^{\imath\jmath}_{\mathfrak{n},d} = \bigoplus_{\lambda\in\Lambda^{\imath\jmath}_{\mathfrak{n},d}} x_\lambda\mathbb{H}.$$

Following [10], we define the $\jmath$ variant of the Schur algebra $\mathbb{S}^{\mathfrak{c}}_{n,d}$ as follows:

$$\mathbb{S}^{\imath\jmath}_{\mathfrak{n},d} = \mathrm{End}_{\mathbb{H}}(\oplus_{\lambda\in\Lambda^{\imath\jmath}_{\mathfrak{n},d}} x_\lambda\mathbb{H}) = \bigoplus_{\lambda,\mu\in\Lambda^{\imath\jmath}_{\mathfrak{n},d}} \mathrm{Hom}_{\mathbb{H}}(x_\mu\mathbb{H}, x_\lambda\mathbb{H}).$$

The following is a variant of Lemma 3.2.

Lemma 4.5. *We have an isomorphism of* $\mathbb{H}$*-modules:* $\mathbb{T}^{\imath\jmath}_{\mathfrak{n},d} \cong \mathbb{V}_{\mathfrak{n}}^{\otimes d}$.

Note $\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})$ acts on $\mathbb{V}_{\mathfrak{n}}^{\otimes d}$ via the embedding $\jmath : \mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}) \to \mathbb{U}(\widehat{\mathfrak{sl}}_n)$; we denote this action by $\Psi^{\imath\jmath}$. In particular, we give the $\mathbf{t}_0$-action on $\mathbb{V}_{\mathfrak{n}}$ for record in the following: for $f \not\equiv 0$, we have

$$\mathbf{t}_0 v_f = \begin{cases} v_{f-2} + \frac{q_1 - q_0^{-1}}{q-q^{-1}} q v_f & \text{if } f \equiv 1;\\ q_0^{-1}q_1 v_{f+2} + \frac{q_1 - q_0^{-1}}{q-q^{-1}} q^{-1} v_f & \text{if } f \equiv -1;\\ \frac{q_1 - q_0^{-1}}{q-q^{-1}} v_f & \text{otherwise.}\end{cases}$$

The following is a variant of Theorem 4.3, and can be proved similarly.

Theorem 4.6. *We have the following Schur* $(\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}}), \mathbb{H})$*-duality:*

$$\Psi^{\imath\jmath}(\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})) \simeq \mathrm{End}_{\mathbb{H}}(\mathbb{V}_{\mathfrak{n}}^{\otimes d}),$$
$$\mathrm{End}_{\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_{\mathfrak{n}})}(\mathbb{V}_{\mathfrak{n}}^{\otimes d}) \simeq \mathbb{H}^{op}.$$

Remark 4.7. Starting with a natural $\mathbb{U}(\widehat{\mathfrak{sl}}_\mathfrak{n})$-module $\mathbb{V}_{\mathfrak{n},\frac{1}{2}}$ with a basis parametrized by $\frac{1}{2}+\mathbb{Z}$ of periodicity $\mathfrak{n}$, we can reformulate the Schur $(\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_\mathfrak{n}), \mathbb{H})$-duality in Theorem 4.3 on $\mathbb{V}_{\mathfrak{n},\frac{1}{2}}^{\otimes d}$ accordingly. Similarly, starting with a natural $\mathbb{U}(\widehat{\mathfrak{sl}}_\mathfrak{n})$-module $\mathbb{V}_{\mathfrak{n},0}$ with a basis parametrized by $\mathbb{Z}$ of periodicity $\mathfrak{n}$, we can reformulate the Schur $(\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_\mathfrak{n}), \mathbb{H})$-duality in Theorem 4.6 on $\mathbb{V}_{\mathfrak{n},0}^{\otimes d}$ accordingly.

4.3. The u-Schur duality

We shall assume $r \geq 2$ in this subsection. Let $\mathbb{V}_\eta$ be the $\mathbb{F}$-subspace of $\mathbb{V}$ spanned by v_i, for $i \in \mathbb{Z}$ such that $i \not\equiv 0 \pmod n$ and $i \not\equiv r+1 \pmod n$. Note $\mathbb{V}_\eta = \mathbb{V}_\mathfrak{n} \cap {}'\mathbb{V}_\mathfrak{n}$ is naturally an $\mathbb{H}$-submodule of $\mathbb{V}$, and moreover, it is a direct sum of permutation modules.

We consider the $\mathbb{F}$-algebra $\mathbb{U}(\widehat{\mathfrak{gl}}_\eta)$ (with an unusual indexing set of generators). The algebra $\mathbb{U}(\widehat{\mathfrak{gl}}_\eta)$ is generated by $\mathbf{E}_i, \mathbf{F}_i$ ($i \in [0, \mathfrak{n}-1]\backslash\{r+1\}$), $\mathbf{D}_a^{\pm 1}$ ($a \in [1, \mathfrak{n}]\backslash\{r+1\}$); here we regard indices $r, r+2$ adjacent. Denote by $\mathbb{U}(\widehat{\mathfrak{sl}}_\eta)$ the subalgebra of $\mathbb{U}(\widehat{\mathfrak{gl}}_\eta)$ generated by $\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i$ ($i \in [0, \mathfrak{n}-1]\backslash\{r+1\}$), where $\mathbf{K}_0 = \mathbf{D}_{-1}\mathbf{D}_1^{-1}, \mathbf{K}_r = \mathbf{D}_r\mathbf{D}_{r+2}^{-1}, \mathbf{K}_i = \mathbf{D}_i\mathbf{D}_{i+1}^{-1}$ ($i \neq 0, r$). Then $\mathbb{V}_\eta$ is a natural representation of $\mathbb{U}(\widehat{\mathfrak{gl}}_\eta)$, with the action given by: for $i \in [0, \mathfrak{n}-1]\backslash\{r+1\}, a \in [0, \mathfrak{n}]\backslash\{r+1\}$,

$$\mathbf{E}_i v_{j+1} = \begin{cases} v_j & \text{if } j \equiv i \neq 0, r; \\ v_{j-1} & \text{if } j \equiv i = 0; \\ v_{j-1} & \text{if } j-1 \equiv i = r; \\ 0 & \text{else,} \end{cases}$$

$$\mathbf{F}_i v_j = \begin{cases} v_{j+1} & \text{if } j \equiv i \neq 0, r; \\ v_{j+2} & \text{if } j+1 \equiv i = 0; \\ v_{j+2} & \text{if } j \equiv i = r; \\ 0 & \text{else,} \end{cases} \qquad \mathbf{D}_a v_j = \begin{cases} q v_j & \text{if } j \equiv a; \\ v_j & \text{else.} \end{cases}$$

Then $\mathbb{U}(\widehat{\mathfrak{gl}}_\eta)$ acts on $\mathbb{V}_\eta^{\otimes d}$ via iterated comultiplication.

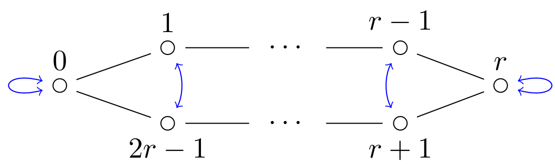
Define $\mathbb{U}^u(\widehat{\mathfrak{sl}}_\eta)$ to be the $\mathbb{F}$-algebra generated by $\mathbf{t}_0, \mathbf{t}_r, \mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i^{\pm 1}$ ($1 \leq i \leq r-1$), subject to the relation $\mathbf{t}_0\mathbf{t}_r = \mathbf{t}_r\mathbf{t}_0$, and other defining relations which can be found in the algebras $\mathbb{U}^{\jmath}(\widehat{\mathfrak{sl}}_\mathfrak{n})$ and $\mathbb{U}^{\imath\jmath}(\widehat{\mathfrak{sl}}_\mathfrak{n})$. (The relations would be different in case $r = 1$ as $\mathbf{t}_0$ and $\mathbf{t}_r$ no longer commute.)

Proposition 4.8. *There is an injective* $\mathbb{F}$*-algebra homomorphism* $\imath\imath$: $\mathbb{U}^{\imath\imath}(\widehat{\mathfrak{sl}}_\eta) \to \mathbb{U}(\widehat{\mathfrak{sl}}_\eta)$ *defined by*

$$\begin{aligned}
&\mathbf{k}_i \mapsto \mathbf{K}_i\mathbf{K}_{-i-1}^{-1},\\
&\mathbf{e}_i \mapsto \mathbf{E}_i + \mathbf{F}_{-i-1}\mathbf{K}_i^{-1}, \quad \mathbf{f}_i \mapsto \mathbf{E}_{-i-1} + \mathbf{F}_i\mathbf{K}_{-i-1}^{-1}, \quad (1 \le i \le r-1)\\
&\mathbf{t}_0 \mapsto \mathbf{E}_0 + qq_0^{-1}q_1\mathbf{F}_0\mathbf{K}_0^{-1} + (q_1 - q_0^{-1})/(q-q^{-1})\mathbf{K}_0^{-1},\\
&\mathbf{t}_r \mapsto \mathbf{E}_r + qq_0q_1^{-1}\mathbf{F}_r\mathbf{K}_r^{-1} + (1 - q_0q_1^{-1})/(q-q^{-1})\mathbf{K}_r^{-1}.
\end{aligned}$$

Proof. The proof is similar to the proof for Proposition 2.2. The subalgebra here is a quantum symmetric pair coideal subalgebra associated with the Dynkin diagram and involution below, and the proposition follows from [16, Theorem 7.8]. □

Figure 4: Dynkin diagram of type $A_{2r-1}^{(1)}$ with involution of type $\imath\imath$.



Recalling $\Lambda_{n,d}$ from (3.1.1), we define

$$\Lambda^{\imath\imath}_{\eta,d} = \{\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r+1}) \in \Lambda_{n,d} \mid \lambda_0 = \lambda_{r+1} = 0\} \ (= \Lambda^{\jmath\imath}_{\mathfrak{n},d} \cap \Lambda^{\imath\jmath}_{\mathfrak{n},d}).$$

Define the right $\mathbb{H}$-module

$$\mathbb{T}^{\imath\imath}_{\eta,d} = \bigoplus_{\lambda \in \Lambda^{\imath\imath}_{\eta,d}} x_\lambda \mathbb{H}.$$

Following [10], we define the $\imath\imath$-variant of the Schur algebra $\mathbb{S}^{\mathfrak{c}}_{n,d}$ as follows:

$$\mathbb{S}^{\imath\imath}_{\eta,d} = \mathrm{End}_{\mathbb{H}}(\oplus_{\lambda \in \Lambda^{\imath\imath}_{\eta,d}} x_\lambda \mathbb{H}) = \bigoplus_{\lambda,\mu \in \Lambda^{\imath\imath}_{\eta,d}} \mathrm{Hom}_{\mathbb{H}}(x_\mu \mathbb{H}, x_\lambda \mathbb{H}).$$

The following is a variant of Lemma 3.2.

Lemma 4.9. *We have an isomorphism of* $\mathbb{H}$*-modules:* $\mathbb{T}^{\imath\imath}_{\eta,d} \cong \mathbb{V}_\eta^{\otimes d}$.

Note $\mathbb{U}^{\imath}(\widehat{\mathfrak{sl}}_\eta)$ acts on $\mathbb{V}_\eta^{\otimes d}$ via the embedding $\imath : \mathbb{U}^{\imath}(\widehat{\mathfrak{sl}}_\eta) \to \mathbb{U}(\widehat{\mathfrak{sl}}_\eta)$; we denote this action by $\Psi^{\imath}$. The following theorem can be established similarly as for Theorem 4.3.

Theorem 4.10. *Let $r \geq d \geq 2$. We have the following Schur $(\mathbb{U}^{\imath}(\widehat{\mathfrak{sl}}_\eta), \mathbb{H})$-duality:*

$$\Psi^{\imath}(\mathbb{U}^{\imath}(\widehat{\mathfrak{sl}}_\eta)) \simeq \mathrm{End}_{\mathbb{H}}(\mathbb{V}_\eta^{\otimes d}),$$
$$\mathrm{End}_{\mathbb{U}^{\imath}(\widehat{\mathfrak{sl}}_\eta)}(\mathbb{V}_\eta^{\otimes d}) \simeq \mathbb{H}^{op}.$$

Proof. The proof is similar to the previous counterparts except that we need the following formulas for the images of $\mathbf{t}_0, \mathbf{t}_r$ which follow from a direct computation:

$$\Psi^{\imath}(\mathbf{t}_0) \in \sum_{\lambda \in \Lambda^{\imath}_{\eta,d}} \mathbb{F}\phi^{s_0}_{\lambda,\lambda}, \quad \Psi^{\imath}(\mathbf{t}_r) \in \sum_{\lambda \in \Lambda^{\imath}_{\eta,d}} \mathbb{F}\phi^{s_d}_{\lambda,\lambda}.$$

The theorem is proved. □

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