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# Moduli spaces of anti-invariant vector bundles and twisted conformal blocks

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We prove a canonical identifications between the spaces of generalized theta functions on the moduli spaces of anti-invariant vector bundles in the ramified case and the conformal blocks associated to twisted Kac-Moody affine algebras. We also show a strange duality on level one in the unramified case, this gives the dimensions of the spaces of generalized theta functions of level one.

## 1. Introduction

Consider a smooth projective complex curve $X$ of genus $g_X \geqslant 2$ with an involution $\sigma$ and assume that the fixed locus of $\sigma$ is not empty and contains $2n$ points. Let $\pi : X \to Y = X/\sigma$ the associated double cover and denote $R$ the ramification locus and $B = \pi(R)$.

An anti-invariant vector bundle over $X$ is a vector bundle $E$ that has an isomorphism $\psi : \sigma^*E \to E^*$. If the this isomorphism verifies $\sigma^*\psi = {}^t\psi$ then $E$ is called $\sigma-$symmetric, and if it verifies $\sigma^*\psi = -{}^t\psi$ then it is called $\sigma-$alternating, in this case the rank has to be even. If $E$ is stable then $\psi$ is necessarily $\sigma-$symmetric or $\sigma-$alternating.

We constructed the moduli spaces of such vector bundles in [20], and in [19], we showed that the locus $\mathcal{SU}_X^{\sigma,+}(r)$ of stable $\sigma-$symmetric anti-invariant vector bundles with trivial determinant over $X$ is irreducible. We also proved that the locus $\mathcal{SU}_X^{\sigma,-}(r)$ (for even rank $r$) of stable $\sigma-$alternating vector bundles with trivial determinant has $2^{2n-1}$ connected components indexed by some types $\tau = (\tau_p)_{p\in R} \mod \pm 1$, where $\tau_p \in \{\pm 1\}$ is the *Pfaffian* of the $\sigma-$alternating isomorphism $\psi : \sigma^*E \to E^*$ over $p$.

The main topic of this paper is the study of generalized theta functions on the moduli stacks $\mathscr{SU}_X^{\sigma,+}(r)$ (resp. $\mathscr{SU}_X^{\sigma,\tau}(r)$) of $\sigma-$symmetric (resp. $\sigma-$alternating of type $\tau$) anti-invariant vector bundles $(E, \delta, \psi)$ over $X$, where $\delta$ is a trivialisation of $\det(E)$ and $\psi : \sigma^*E \cong E^*$ is $\sigma-$symmetric (resp. $\sigma-$alternating of type $\tau$) compatible with $\delta$. When the type is $\tau$ trivial, that's $\tau_i = +1$ for all $i$, the corresponding moduli stack is denoted simply by $\mathscr{SU}_X^{\sigma,-}(r)$.

These moduli stacks correspond to moduli stacks $\mathscr{M}_Y(\mathcal{G})$ of parahoric $\mathcal{G}-$torsors over the quotient curve $Y$, for some *twisted* parahoric Bruhat-Tits group schemes $\mathcal{G}$. Parahoric $\mathcal{G}-$torsors have attracted the attention of many mathematician recently (see [15], [10], [1], [2]), since they can be considered as a generalization of parabolic $G-$bundles. Their moduli spaces has been constructed in the untwisted case by Balaji and Seshadri [1]. For the twisted case, we have constructed their moduli spaces in type $A_n$ in [20].

The restriction of the determinant bundle $\mathcal{D}$ over the moduli stack of vector bundles of trivial determinant $\mathscr{SU}_X(r)$ to the moduli stack $\mathscr{SU}_X^{\sigma,-}(r)$ turns out to have a square root for each $\sigma-$invariant theta characteristic, we call them the *Pfaffian of cohomology line bundles*. However this is not true for the $\sigma-$symmetric case.

In the étale case, the two stacks are isomorphic (see [19]). In this case also, the determinant bundle has a square root. However, in this paper we stick to the ramified case.

Our main result is the identification of the space of global sections of the powers of a Pfaffian line bundle $\mathcal{P}$ and the determinant line bundle $\mathcal{D}$ (called space of generalized theta functions) with the *conformal blocks* $\mathcal{V}_{\sigma,\pm}(k)$ associated to the twisted universal central extension of $\mathfrak{sl}_r$ (see section 2 for a precise definition).

**Theorem 1.1.** *Let $\mathcal{P}$ be a Pfaffian line bundle over $\mathscr{SU}_X^{\sigma,-}(r)$ and $\mathcal{D}$ be the determinant line bundle over $\mathscr{SU}_X^{\sigma,+}$, and let $k \in \mathbb{N}$. Then we have canonical isomorphisms*

$$H^0(\mathscr{SU}_X^{\sigma,-}(r), \mathcal{P}^k) \cong \mathcal{V}_{\sigma,-}(k),$$
$$H^0(\mathscr{SU}_X^{\sigma,+}(r), \mathcal{D}^k) \cong \mathcal{V}_{\sigma,+}(k).$$

Using the results of Heinloth [10], describe the Picard group of this moduli stack. In fact it is infinite cyclic group whose generator is the Pfaffian bundle.

**Theorem 1.2.** *Let $\mathcal{P}$ be the Pfaffian bundle over $\mathscr{SU}_X^{\sigma,-}(r)$, then*

$$Pic(\mathscr{SU}_X^{\sigma,+}(r)) = \mathbb{Z}\mathcal{P}.$$

## 2. Preliminaries on twisted Kac-Moody algebras

In this first section, we recall briefly the construction of the twisted affine Kac-Moody Lie algebras and the attached conformal blocks. We use notations of [11]. The definition of twisted conformal blocks is adapted from [8], where a more general definition is given in the framework of vertex algebras.

Consider an outer automorphism $\tau$ of the Lie algebra $\mathfrak{sl}_r(\mathbb{C})$. It is an order two automorphism. The involution $\tau$ is extended to an automorphism of the affine Kac-Moody algebra $\widehat{\mathcal{L}}(\mathfrak{sl}_r) = \mathfrak{sl}_r(\mathscr{K}) \oplus \mathbb{C}K$, where $\mathscr{K} = \mathbb{C}((t))$ and $K$ a central element, by sending $x \otimes g(t)$ to $\tau(x) \otimes g(-t)$ and fixes the center. Then the fixed subalgebra of this involution, denoted by $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \tau)$, is an affine Lie algebra of type $A_l^{(2)}$ (after adding a scaling element $D$), where $l = \lfloor r/2 \rfloor$, and it is called *twisted* affine Lie algebra. Let $\mathring{\mathfrak{g}}$ be the finite simple Lie algebra of $\mathcal{L}(\mathfrak{sl}_r, \tau)$ (see [11, §6.3] for a precise definition). Then $\mathring{\mathfrak{g}}$ is of type $C_l$ if $r$ is odd, and is isomorphic to the fixed subalgebra $\mathfrak{sl}_r(\mathbb{C})^\tau$ if $r$ is even.

Since we will be interested mainly in the following two involutions

$$\sigma^+(a(t)) = -\,{}^t a(-t), \quad \sigma^-(a(t)) = -J_r\,{}^t a(-t) J_r^{-1},$$

where

$$J_r = \begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix},$$

we give an explicit constructions of $\mathcal{L}(\mathfrak{sl}_r, \sigma^\pm)$.

Let $M_{i,j}$ be the canonical basis of the vector space of square matrices of size $r$. Let $\mathfrak{h} \subset \mathfrak{sl}_r(\mathbb{C})$ be the Cartan subalgebra of diagonal matrices and let $\alpha'_1, \ldots, \alpha'_{r-1} \in \mathfrak{h}^*$ be the simple roots defined by $\alpha_i = M^*_{i,i} - M^*_{i+1,i+1}$. Denote by $E'_1, \ldots, E'_{r-1}$ and $F'_1, \ldots, F'_{r-1}$ the Chevalley generators of $\mathfrak{sl}_r(\mathbb{C})$: $E'_i = M_{i,i+1}$, $F'_i = -\,{}^t E'_i$. Let ${\alpha'_0}^\vee = M_{1,1} - M_{r,r}$, $E'_0 = M_{1,r}$ and $F'_0 = -\,{}^t E'_0$. Then the Chevalley generators of $\widehat{\mathcal{L}}(\mathfrak{sl}_r)$ are given by

$$e_0 = t \otimes E'_0, \quad f_0 = t^{-1} \otimes F'_0,$$

and for $i \in \{1, \ldots, r-1\}$

$$e_i = 1 \otimes E'_i, \quad f_i = 1 \otimes F'_i.$$

Recall the Lie bracket on $\hat{\mathcal{L}}(\mathfrak{sl}_r)$ is given by

$$[g(t), h(t)] = [g, h] \otimes P(t)Q(t) + (g, h)\mathrm{Res}\left(\frac{dP}{dt} Q\right) K,$$

where $g, h \in \mathfrak{sl}_r$, $P, Q \in \mathscr{K}$ and $( \, , \, )$ is the normalized Killing form on $\mathfrak{sl}_r$.

Moreover, by extending the linear forms $\alpha_i$ to $\mathfrak{h} \oplus \mathbb{C}K$ such that $\alpha_i(K) = 0$, then $\alpha_i$ are the simple roots of $\widehat{\mathcal{L}}(\mathfrak{sl}_r)$.

**Case** $\mathcal{L}(\mathfrak{sl}_r, \sigma^-)$**.** Let $r = 2l$. This is the algebra constructed in [11, Page 128]. We can assume, after conjugation, that $\sigma^-$ sends $E'_i$ to $E'_{r-i}$, $F'_i \to F'_{r-i}$ and $\alpha'_i \to \alpha'_{r-i}$. So let's define

- $\alpha_i^\vee = \alpha'^\vee_i + \alpha'^\vee_{r-i}$ $(1 \leqslant i \leqslant l-1)$, $\alpha_l^\vee = \alpha'^\vee_l$ and $\alpha_0^\vee = -2\alpha'^\vee_0 + \alpha'^\vee_1 + \alpha'^\vee_{r-1}$.
- $E_i = E'_i + E'_{r-i}$ $(1 \leqslant i \leqslant l-1)$, $E_l = E'_l$ and $E_0 = E'_{-\alpha'_0+\alpha'_{r-1}} - E'_{-\alpha'_0+\alpha'_1}$.
- $F_i = F'_i + F'_{r-i}$ $(1 \leqslant i \leqslant l-1)$, $F_l = F'_l$ and $F_0 = -E'_{\alpha'_0-\alpha'_{r-1}} + E'_{\alpha'_0-\alpha'_1}$.

The Chevalley generators of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ are given by

$$e_i = 1 \otimes E_i, \quad f_i = 1 \otimes F_i \quad \text{for } i = 1, \ldots, l.$$
$$e_0 = t \otimes E_0, \quad f_0 = t^{-1} \otimes F_0.$$

Consider the elements $\tilde{\alpha}_i^\vee = 2\alpha_i^\vee/(\alpha_i^\vee, \alpha_i^\vee) \in \mathfrak{h}$. Since the normalized bilinear form $( \, ; \, )$ is non-degenerate on $\mathfrak{h}$ it induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. So let $\tilde{\alpha}_i$ be the images of $\tilde{\alpha}_i^\vee$ under this bijection. Then the simple roots of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ are given by

$$\alpha_0 = \frac{1}{2} \otimes \tilde{\alpha}_0, \quad \alpha_i = 1 \otimes \tilde{\alpha}_i, \quad i = 1, \ldots, l.$$

The simple coroots are just $1 \otimes \alpha_i^\vee$, for $i = 1, \ldots, l$. We denote them again by $\alpha_i^\vee$. For $i = 0$ the simple coroot is $2K + 1 \otimes \alpha_0^\vee$. We denote it also by $\alpha_0^\vee$. In particular, the normalized bilinear form on $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ is given by

$$(P \otimes x; Q \otimes y) = \frac{1}{2}\mathrm{Res}(t^{-1}PQ)(x; y),$$

where $( \, , \, )$ is the normalized bilinear form on $\mathfrak{sl}_r(\mathbb{C})$. The $2-$cocycle on $\mathcal{L}(\mathfrak{sl}_r, \sigma^-)$ that defines $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ is given by

$$\psi(g(t), h(t)) = \frac{1}{2}\mathrm{Res}\left(\mathrm{Tr}\left(\frac{dg}{dt}h\right)\right).$$

**Case $\mathcal{L}(\mathfrak{sl}_r, \sigma^+)$.** We treat the case $r = 2l$ (the odd case is again treated in [11]). We can assume, after conjugation, that $\sigma^+$ sends $E'_i$ to $-E'_{r-i}$, $F'_i \to -F'_{r-i}$ and $\alpha'_i \to \alpha'_{r-i}$. So we define the following elements of $\mathfrak{sl}_{2l}$ :

- $\beta_i^\vee = \alpha'^\vee_{l-i} + \alpha'^\vee_{l+i}$ $(1 \leqslant i \leqslant l-1)$, $\beta_l^\vee = \alpha'^\vee_0$ and $\beta_0^\vee = 2\alpha'^\vee_l + \alpha'^\vee_{l-1} + \alpha'^\vee_{l+1}$.
- $E_i = E'_i - E'_{r-i}$ $(1 \leqslant i \leqslant l-1)$, $E_l = E'_0$ and $E_0 = E'_{\alpha'_l+\alpha'_{l+1}} - E'_{\alpha'_l+\alpha'_{l-1}}$.
- $F_i = F'_i - F'_{r-i}$ $(1 \leqslant i \leqslant l-1)$, $F_l = F'_0$ and $F_0 = -E'_{-\alpha'_l-\alpha'_{l+1}} + E'_{-\alpha'_l-\alpha'_{l-1}}$.

Remark that the affine node $\beta_0$ of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)$ is then the node $\alpha_l$ with the notation of Table Aff 2 of [11, Page 55]. Thus when deleting this node the remaining diagram is of type $D_l$.

As before, we define the Chevalley generators of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)$ by

$$e_i = 1 \otimes E_i, \quad f_i = 1 \otimes F_i \quad \text{for } i = 1, \dots, l.$$
$$e_0 = t \otimes E_0, \quad f_0 = t^{-1} \otimes F_0.$$

The simple coroots of the simple invariant Lie algebra ($= \mathfrak{so}_{2l}$) are given by

$$\tilde{\beta}_i^\vee = 2\beta_i^\vee / (\beta_i^\vee, \beta_i^\vee), \quad i = 0, \dots, l.$$

As above denote by $\tilde{\beta}_i$ the corresponding elements of $\mathfrak{h}^*$. Then the simple roots of $\widehat{\mathcal{L}}(\mathfrak{sl}_{2l}, \sigma^+)$ are given by

$$\beta_0 = 2K + 1 \otimes \tilde{\beta}_0, \quad \beta_i = 1 \otimes \tilde{\beta}_i, \quad i = 1, \dots, l.$$

From the construction of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)$, it is clear that the Coxeter coefficients and their duals in this case are taken in the inverse order. We recall the dual Coxeter coefficients of the twisted Kac-Moody algebras $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$ in the following table.

(1)

| | $a_0^\vee$ | $a_1^\vee$ | $a_2^\vee$ | $\cdots$ | $a_{l-1}^\vee$ | $a_l^\vee$ |
|---|---|---|---|---|---|---|
| $\widehat{\mathcal{L}}(\mathfrak{sl}_{2l}, \sigma^+)$ | 2 | 2 | 2 | $\cdots$ | 1 | 1 |
| $\widehat{\mathcal{L}}(\mathfrak{sl}_{2l}, \sigma^-)$ | 1 | 1 | 2 | $\cdots$ | 2 | 2 |
| $\widehat{\mathcal{L}}(\mathfrak{sl}_{2l+1}, \sigma^+)$ | 1 | 2 | 2 | $\cdots$ | 2 | 2 |

Dual Coxeter coefficients.

Now, when we add a scaling elements to the above algebras, i.e. derivations $D_\pm$ such that

$$[D_\pm, t^n \otimes x] = nt^n \otimes x,$$

then, by [11, Theorem 8.5], both Kac-Moody algebras $\hat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm) \oplus \mathbb{C}D_\pm$ are isomorphic to the Kac-Moody algebra $\mathfrak{g}(A)$, where $A$ is the affine generalized Cartan matrix of type $A^{(2)}_{r-1}$. In particular, we deduce an isomorphism

$$\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+) \oplus \mathbb{C}D_+ \cong \hat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-) \oplus \mathbb{C}D_-.$$

Moreover, the derivations $D_\pm$ induces a weight decomposition of the algebras $\mathcal{L}(\mathfrak{sl}_r, \sigma^\pm) \oplus \mathbb{C}D_\pm$. The main observation is that the above isomorphism does not respect the decompositions of these algebras in powers of $t$.
We will see in a moment that under the above isomorphism, the fundamental weight $\lambda_0^+$ of $\hat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+) \oplus \mathbb{C}D$ is sent to twice the fundamental weight $\lambda_0^-$.

### Twisted conformal blocks

Let $\lambda_0^\pm, \ldots, \lambda_l^\pm$ be the fundamental wights of the twisted affine Lie algebras $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$, i.e. $\lambda_i^\pm$ are linear forms on the Cartan subalgebras such that

$$\lambda_i^+(\beta_j) = \lambda_i^-(\alpha_j) = \delta_{ij}, \quad i, j = 0, \ldots, l.$$

Denote by $\mathring{\mathfrak{g}} \subset \mathcal{L}(\mathfrak{sl}_r, \sigma^\pm)$ the simple Lie algebra generated by $e_i$ and $f_i$ for $i = 1, \ldots, l$. Note that $\mathring{\mathfrak{g}}$ is of type $D_l$ in the case of $\sigma^+$ when $r$ is even, and it is of type $C_l$ otherwise. Moreover, we have the identifications (see [11, §12.4])

$$\lambda_i^\pm = \mathring{\lambda}_i + a_i^\vee \lambda_0^\pm, \quad i = 1, \ldots, l,$$

where $\mathring{\lambda}_i$ $(i = 1, \ldots, l)$ are the fundamental weights of $\mathring{\mathfrak{g}}$.

**Remark 2.1.** Remark that, for an even rank $r$, the weight $\lambda_0^+$ has level equals $a_0^\vee = 2$, while $\lambda_0^-$ has level $a_0^\vee = 1$ (see Table 1).

Denote by $\mathrm{P}^{\sigma,\pm}$ the set of dominant integral weights of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$. By [11, §12.4], one deduces a bijection between $\mathrm{P}^{\sigma,\pm}$ and the set

$$\tilde{\mathrm{P}}^{\sigma,\pm} = \{(\lambda, k) \mid \lambda \in \mathring{\mathrm{P}},\ \langle \lambda, \varrho \rangle \leqslant k\},$$

where $\mathring{\mathrm{P}}$ is the set of dominant weights of $\mathring{\mathfrak{g}}$, and $\varrho$ is the highest coroot of $\mathring{\mathfrak{g}}$ when $r$ is even, and $\varrho$ is twice the highest coroot of $\mathring{\mathfrak{g}}$ when $r$ is odd.

For $\mu^{\pm} \in \mathrm{P}^{\sigma,\pm}$, denote by $\mathcal{H}_{\mu^{\pm}}(k)$ the irreducible highest weight module of level $k$ of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^{\pm})$ of highest weight $\mu^{\pm}$. Let $\overrightarrow{\mu}^{\pm} = (\mu_1^{\pm}, \ldots, \mu_{2n}^{\pm})$ be a vector of elements of $\mathrm{P}^{\sigma,\pm}$ parameterized by the points of $R$, and define

$$\mathcal{H}_{\overrightarrow{\mu}^{\pm}}(k) = \mathcal{H}_{\mu_1^{\pm}}(k) \otimes \cdots \otimes \mathcal{H}_{\mu_{2n}^{\pm}}(k).$$

Finally, let $\mathscr{A}_R = H^0(X \smallsetminus R, \mathcal{O}_X)$. By considering the associated Laurent series at $p \in R$, we get an inclusion $\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^{\pm}} \subset \mathfrak{sl}_r(\mathscr{K}_p)^{\sigma^{\pm}}$. We can then define an action of $\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^{\pm}}$ on $\mathcal{H}_{\overrightarrow{\mu}^{\pm}}(k)$ as product of representations (i.e diagonal action). More explicitly, for $\alpha \in \mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^{\pm}}$ and $X = X_1 \otimes \cdots \otimes X_{2n}$, we have

$$\alpha \cdot X = \sum_i X_1 \otimes \cdots \otimes \alpha \cdot X_i \otimes \cdots \otimes X_{2n}.$$

**Definition 2.2.** The *conformal block* attached to the data $(X, \sigma, \overrightarrow{\mu}^{\pm}, \widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^{\pm}), k)$ is defined by

$$\mathcal{V}_{\sigma,\pm}(k) = \left[\left(\mathcal{H}_{\overrightarrow{\mu}^{\pm}}(k)\right)_{\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^{\pm}}}\right]^*,$$

where for a $\mathfrak{g}-$module $V$, we denote by $V_{\mathfrak{g}}$ the space of coinvariants of $V$, thus the largest quotient of $V$ on which $\mathfrak{g}$ acts trivially.

## 3. Loop groups and uniformization theorem

### 3.1. Bruhat-Tits parahoric $\mathcal{G}-$torsors

Let $\mathcal{G}$ be a smooth affine group scheme over $X$. $\mathcal{G}$ is said to be a parahoric Bruhat-Tits group scheme if there is a finite subset $R \subset X$ such that if $\mathcal{O}_x$ is the completion of the local ring at $x \in R$ then $\mathcal{G}_{\mathcal{O}_x}$ is a parahoric group scheme over $\mathrm{Spec}(\mathcal{O}_x)$ (in the sens of Bruhat-Tits, [6, Définition 5.2.6]) for each $x \in R$ and the fibers $\mathcal{G}_y$ is semisimple for all $y \in X \smallsetminus R$.
A class of examples of such group schemes is provided by the invariant Weil restriction. Given a Galois cover $\pi : X \to X/\Gamma$ of curves and a semisimple algebraic group $G$ over $X$ with an action of $\Gamma$ lifted from its action on $X$, then $\mathcal{G} = \pi_*^{\Gamma}(G)$ is a parahoric group scheme over $X/\Gamma$ (provided it is not empty). Moreover, it is shown in [1] that the stack of $\Gamma-$equivariant $G-$torsors over $X$ is in one to one correspondence with the stack of $\mathcal{G}-$torsors over $X/\Gamma$.
In our case, we have $\Gamma = \mathbb{Z}/2 = \langle \sigma \rangle$ and $G = \mathrm{SL}_r$. Consider the actions of $\sigma$

on $\mathrm{SL}_r$ given by

$$\sigma^+(g) = {}^t g^{-1}, \quad \sigma^- = J_r \sigma^+ J_r^{-1},$$

where

$$J_r = \begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix},$$

$I_{r/2}$ is the identity matrix of size $r/2$.

Let $\mathcal{G}$ and $\mathcal{H}$ be the invariant Weil restrictions of the constant group scheme $\mathbf{SL}_r = X \times \mathrm{SL}_r$ defined by

$$\mathcal{G} = (\pi_* \mathbf{SL}_r)^{\sigma^+}, \quad \mathcal{H} = (\pi_* \mathbf{SL}_r)^{\sigma^-}.$$

These two are smooth affine group schemes over $Y$ which are parahoric. Denote by $\mathscr{M}_Y(\mathcal{G})$ and $\mathscr{M}_Y(\mathcal{H})$ the stacks of $\mathcal{G}-$torsors and $\mathcal{H}-$torsors over $Y$. By [20, Proposition 2.4], we have isomorphisms

$$\mathscr{S}\mathscr{U}_X^{\sigma,+}(r) \cong \mathscr{M}_Y(\mathcal{G}), \quad \mathscr{S}\mathscr{U}_X^{\sigma,-}(r) \cong \mathscr{M}_Y(\mathcal{H}).$$

## 3.2. Uniformization theorem

For a ramification point $p \in X$, denote by $\mathscr{O}_p$ the completion of the local ring at $p$, $\mathscr{K}_p$ its fraction field and $\mathscr{V}_p$ a complementary vector subspace of $\mathscr{O}_p$ in $\mathscr{K}_p$. Let $\mathscr{S}\mathscr{U}_X(r)$ denote the moduli stack of rank $r$ vector bundles over $X$ with a trivialization of its determinant. Let's fix the canonical linearization on $\mathcal{O}_X$, so we identify $\sigma^*\mathcal{O}_X$ and $\mathcal{O}_X$. Moreover, since all the types are isomorphic, we assume hereafter that $\tau = (+1, \dots, +1) \mod \pm 1$ and denote the corresponding moduli stack by $\mathscr{S}\mathscr{U}_X^{\sigma,-}(r)$.
In [3], it is proved that

$$\mathscr{S}\mathscr{U}_X(r) \cong \mathbf{SL}_r(\mathscr{O}_p) \backslash \mathbf{SL}_r(\mathscr{K}_p) / \mathbf{SL}_r(\mathscr{A}_p),$$

where $\mathscr{A}_p = H^0(X - p, \mathcal{O}_X)$. Let $t$ be a local parameter at $p$, then $\mathscr{K}_p \cong \mathbb{C}((t))$, $\mathscr{O}_p \cong \mathbb{C}[[t]]$.

Consider the two involutions $\sigma^\pm$ on $\mathbf{SL}_r(\mathscr{K}_p)$ given by

$$g(t) \to \sigma^+(g(t)) = {}^t g(-t)^{-1},$$
$$g(t) \to \sigma^-(g(t)) = J_r \cdot {}^t g(-t)^{-1} \cdot J_r^{-1}.$$

Let $\mathscr{Q} = \mathbf{SL}_r(\mathscr{O}_p) \backslash \mathbf{SL}_r(\mathscr{K}_p)$. In [16], it is proved that

$$\mathscr{Q}^{\sigma^+} = \mathbf{SL}_r(\mathscr{O}_p)^{\sigma^+} \backslash \mathbf{SL}_r(\mathscr{K}_p)^{\sigma^+}.$$

Note that $\mathbf{SL}_r(\mathscr{O}_p)^{\sigma^+}$ is the maximal parahoric subgroup of $\mathbf{SL}_r(\mathscr{K}_p)^{\sigma^+}$ and, with the notations of loc. cit. this case corresponds to $I = \{0\}$. In fact, their involution is the conjugation of $\sigma^+$ by the anti-diagonal matrix $D_r$ with all entries equal 1. But this does not change much. Indeed, by taking a matrix $A$ such that $D_r = {}^tAA$ (such matrix can be constructed easily), then conjugation by $A$ realizes an isomorphism between $\mathbf{SL}_r(\mathscr{K})^{\sigma^+}$ and their invariant locus.
We denote in the sequel by $\mathscr{Q}^{\sigma,\pm}$ the quotient $\mathbf{SL}_r(\mathscr{O}_p)^{\sigma^\pm}\backslash\mathbf{SL}_r(\mathscr{K}_p)^{\sigma^\pm}$, for some $p \in R$.

**Theorem 3.1.** *We have an isomorphism of stacks*

$$\mathscr{SU}_X^{\sigma,\pm}(r) \cong \boldsymbol{SL}_r(\mathscr{O}_p)^{\sigma^\pm}\backslash\boldsymbol{SL}_r(\mathscr{K}_p)^{\sigma^\pm}/\boldsymbol{SL}_r(\mathscr{A}_p)^{\sigma^\pm}.$$

*Moreover, the projections $\mathscr{Q}^{\sigma,\pm} \to \mathscr{SU}_X^{\sigma,\pm}(r)$ are locally trivial for the fppf topology.*

*Proof.* Recall that we have isomorphisms

$$\mathscr{SU}_X^{\sigma,+}(r) \cong \mathscr{M}_Y(\mathcal{G}), \quad \mathscr{SU}_X^{\sigma,-}(r) \cong \mathscr{M}_Y(\mathcal{H}).$$

Now, using the main Theorem of [10], we deduce, for a ramification point $p \in X$ over a branch point $y \in Y$, that

$$\begin{aligned}\mathscr{M}_Y(\mathcal{G}) &\cong \mathcal{G}(\mathscr{O}_y)\backslash\mathcal{G}(\mathscr{K}_y)/H^0(Y \smallsetminus y, \mathcal{G}) \\ &\cong \mathbf{SL}_r(\mathscr{O}_p)^{\sigma^+}\backslash\mathbf{SL}_r(\mathscr{K}_p)^{\sigma^+}/\mathbf{SL}_r(\mathscr{A}_p)^{\sigma^+}, \\ \mathscr{M}_Y(\mathcal{H}) &\cong \mathcal{H}(\mathscr{O}_y)\backslash\mathcal{H}(\mathscr{K}_y)/H^0(Y \smallsetminus y, \mathcal{H}).\end{aligned}$$

And we have $H^0(Y \smallsetminus y, \mathcal{H}) \cong \mathbf{SL}_r(\mathscr{A}_p)^{\sigma^-}$. Thus

$$\mathscr{M}_Y(\mathcal{H}) \cong \mathbf{SL}_r(\mathscr{O}_p)^{\sigma^-}\backslash\mathbf{SL}_r(\mathscr{K}_p)^{\sigma^-}/\mathbf{SL}_r(\mathscr{A}_p)^{\sigma^-}.$$

□

### 3.3. The Grassmannian viewpoint

Note that $\mathscr{Q}^{\sigma,+}$ is an ind-variety, which is a direct limit of a system of projective varieties $(\mathscr{Q}_N^{\sigma,+})_{N\geqslant 0}$, the $\mathscr{Q}_N^{\sigma,+}$ are the quotients $(S^0)^{\sigma^+}\backslash(S^N)^{\sigma^+}$, where $S^N$ is the subscheme of $\mathbf{SL}_r(\mathscr{K})$ parameterizing matrices $A(t)$ such that $A(t)$ and $A(t)^{-1}$ have poles of order at most $N$. As we said above, since all the stacks $\mathscr{SU}_X^{\sigma,\tau}(r)$ are isomorphic, so for simplicity we assume that $\tau$ is the trivial type. So let's denote $\mathscr{Q}^{\sigma,-}$ the quotient $\mathbf{SL}_r(\mathscr{O}_p)^{\sigma^-}\backslash\mathbf{SL}_r(\mathscr{K}_p)^{\sigma^-}$. This is again an ind-variety direct limit of $(\mathscr{Q}_N^{\sigma,-})_{N\geqslant 0}$, the $\mathscr{Q}_N^{\sigma,-}$ are the quotients $(S^0)^{\sigma^-}\backslash(S^N)^{\sigma^-}$.

By [3, Proposition 2.4] , the varieties $\mathscr{Q}_N := S^0\backslash S^N$ are identified with subvarieties (with the same underlying topological spaces) of the Grassmannian $\mathrm{Gr}^t(rN, 2rN)$ of $t-$stable subspaces of dimension $rN$ of $F_N^r := t^{-N}\mathscr{O}^{\oplus r}/t^N\mathscr{O}^{\oplus r}$.

Consider the $\sigma-$Hermitian forms $\Psi_\pm : \mathscr{K}^r \times \mathscr{K}^r \longrightarrow \mathscr{K}$ defined by

$$\Psi_+(v, w) = {}^t v \cdot \sigma(w) = \sum_{i=1}^{r} v_i\sigma(w_i),$$
$$\Psi_-(v, w) = {}^t v \cdot J_r \cdot \sigma(w),$$

where $v = {}^t(v_1, \dots, v_i)$ and $w = {}^t(w_1, \dots, w_i)$ are in $\mathscr{K}^r$. Then the groups $\mathrm{SL}_r(\mathscr{K})^{\sigma^\pm}$ can be defined as the loci of matrices $A \in \mathrm{SL}_r(\mathscr{K})$ which are unitary with respect to the forms $\Psi_\pm$, i.e. $\Psi_\pm(A\cdot v, A\cdot w) = \Psi_\pm(v, w)$ for all $v, w \in \mathscr{K}^r$.

Consider the forms $\tilde{\Psi}_\pm$ on $t^{-N}\mathscr{O}^r \subset \mathscr{K}^r$ defined as the composition

$$\tilde{\Psi}_\pm : t^{-N}\mathscr{O}^{\oplus r} \times t^{-N}\mathscr{O}^{\oplus r} \xrightarrow{\Psi_\pm} t^{-2N}\mathscr{O} \xrightarrow{\mathrm{Res}} \mathbb{C},$$

where $\mathrm{Res} : \mathscr{K} \to \mathbb{C}$ is the residue map. The forms $\tilde{\Psi}_\pm$ vanish on $t^N\mathscr{O}^{\oplus r} \subset t^{-N}\mathscr{O}^{\oplus r}$, hence they induce two forms, denoted again by $\tilde{\Psi}_\pm$, on $F_N^r$

$$\tilde{\Psi}_\pm : F_N^r \times F_N^r \longrightarrow \mathbb{C}.$$

**Lemma 3.2.** *$\tilde{\Psi}_+$ is an anti-symmetric non-degenerate bilinear form on $F_N^r$, while $\tilde{\Psi}_-$ is a symmetric non-degenerate bilinear form.*

*Proof.* Consider the canonical basis of the vector space $F_r$ given by the classes of $t^k$ for $k = -N, \dots, N-1$. It induces a canonical basis of $F_N^r$. Then for $v = (v_i)_i, w = (w_i)_i \in F_N^r$, the forms $\Psi_\pm$ are given explicitly in this basis by

$$\Psi_+(v, w) = \sum_{i=1}^{r}\sum_{j=-N}^{N-1} (-1)^{-j-1} a_j^i b_{-j-1}^i,$$
$$\Psi_-(v, w) = \sum_{i=1}^{r}\sum_{j=-N}^{N-1} (-1)^{-j-1+\varepsilon(i)} a_j^i b_{-j-1}^{r-i},$$

where $\varepsilon(i)$ equals 1 if $i \in \{1, \ldots, r/2\}$, and 0 otherwise, and $v_i = (a^i_j)$, $w_i = (b^i_j)$ are in $F_N$. From this the result follows easily. □

**Proposition 3.3.** *The spaces $\mathscr{Q}_N^{\sigma,\pm}$ are isomorphic to closed subvarieties (with the same underlying topological subspaces) of the isotropic Grassmannian $Gr^{t,\sigma}_{\pm}(rN, 2rN)$ which parameterizes $\tilde{\Psi}_\pm-$isotropic $t-$stable vector subspaces of $F_N^r$ of dimension $rN$.*

*Proof.* We prove it for the symmetric case, the other one follows similarly. The image of $\mathscr{O}^{\oplus r}$ in $F_N^r$ is $\tilde{\Psi}_+-$isotropic, hence, for every $A(t) \in (S^N)^{\sigma^+}$, the corresponding point in $\mathrm{Gr}^t(rN, 2rN)$ of the class of $A(t)$ in $\mathscr{Q}_N^{\sigma,+}$ is $\tilde{\Psi}_+-$isotropic. Thus it is in $\mathrm{Gr}^{t,\sigma}_+(rN, 2rN)$.

Conversely, assume that we have a point $W$ of the isotropic Grassmannian. Let $A(t) \in S^N$ be a representative of the corresponding class in $S^0\backslash S^N$. We have for every $v, w \in \mathscr{O}^r$, $\Psi_+(A \cdot v, A \cdot w) \in \mathscr{O}$, to see this assume that for some $v, w \in \mathscr{O}^r$, the coefficient of $t^{-k}$ of $\Psi_+(A \cdot v, A \cdot w)$ is nonzero $(k > 0)$, then one deduces that $\mathrm{Res}(\Psi_+(A \cdot (t^k v), A \cdot w)) = \tilde{\Psi}_+(A \cdot (t^k v), A \cdot w) \neq 0$ giving a contradiction. Now let $(e_i)_i$ be the canonical basis of the $\mathscr{K}-$vector space $\mathscr{K}^r$ and let $B(t) = (\Psi_+(A \cdot e_i, A \cdot e_j))_{i,j}$, we see that $B(t) \in \mathrm{SL}_r(\mathscr{O}) = S^0$, and we have by definition $B(t) = {}^tA(t)A(-t)$. In particular we see that $B(t) = {}^tB(-t)$, hence $B(t) = {}^tC(-t) \cdot C(t)$ for some $C(t) \in \mathrm{SL}_r(\mathscr{O})$, and $C(t)A(t)$ is also a representative of $W$ and it is of course in $(S^N)^{\sigma^+}$. In other words the corresponding point of $W$ in $S^0\backslash S^N$ is in $\mathscr{Q}_N^{\sigma,+}$. This proves the proposition. □

Consider the variety $\mathscr{Q}_N = S^0\backslash S^N$ which is as a topological space isomorphic to the Grassmannian $\mathrm{Gr}^t(rN, 2rN)$. Fix an identification of $\mathscr{Q}_N$ as subspace of the homogeneous space $\mathrm{SL}_{2rN}(\mathbb{C})/\mathrm{P}_N$, where $\mathrm{P}_N$ is the parabolic subgroup of $\mathrm{SL}_{2rN}$ of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $A, B$ and $C$ are square $rN \times rN$ matrices. Let $\mathcal{O}_{\mathscr{Q}_N}(1)$ be the line bundle attached to the character $\chi : \mathrm{P} \to \mathbb{C}^*$ which sends a matrix as above to $\det(A^{-1})$. It is well known that the Picard group of $\mathscr{Q}_N$ is infinite cyclic generated by $\mathcal{O}_{\mathscr{Q}_N}(1)$ (it is actually isomorphic to the character group of the maximal parabolic subgroup $P_N$).

**Proposition 3.4.** *The restriction of* $\mathcal{O}_{\mathcal{Q}_N}(1)$ *to* $\mathcal{Q}_N^{\sigma,-}$ *has a square root, which we denote by* $\mathcal{O}_{\mathcal{Q}_N^{\sigma,-}}(1)$. *It is in fact the generator of the Picard group of* $\mathcal{Q}_N^{\sigma,-}$.

*Proof.* By Proposition 3.3, the variety $\mathcal{Q}_N^{\sigma,-}$ is isomorphic to a subvariety of the classical Grassmannian $\mathrm{SO}_{2rN}(\mathbb{C})/\mathrm{P}'_N$, where $\mathrm{P}'_N = \mathrm{P}_N \cap \mathrm{SO}_{2rN}(\mathbb{C})$. The restriction of the character $\chi$ to $\mathrm{P}'_N$ is denoted again by $\chi$. Now, consider the universal cover of $\mathrm{SO}_{2rn}(\mathbb{C})$, which is the Spin group $\mathrm{Spin}_{2rN}(\mathbb{C})$. It is a double cover of $\mathrm{SO}_{2rN}(\mathbb{C})$. Let $\tilde{\mathrm{P}}_N \subset \mathrm{Spin}_{2rN}(\mathbb{C})$ the inverse image of $\mathrm{P}'_N$. Then, by [7, Chapter 5, Theorem 3.3.1], the lifting of $\chi$ to $\tilde{\mathrm{P}}_N$ has a square root which we denote by $\chi_-$. Since we have

$$\mathrm{Spin}_{2rn}(\mathbb{C})/\tilde{\mathrm{P}} \cong \mathrm{SO}_{2rn}/\mathrm{P}'_N,$$

we deduce that the line bundle over $\mathcal{Q}_N^{\sigma,-}$ attached to $\chi_-$ is the square root of the restriction of $\mathcal{O}_{\mathcal{Q}_N}(1)$.
The Picard group of $\mathrm{Spin}_{2rN}/\tilde{\mathrm{P}}_N$ is infinite cyclic isomorphic to the character group of $\tilde{\mathrm{P}}_N$, which is generated by $\chi_-$. This implies the second claim. □

**Proposition 3.5.** *The ind-varieties* $\mathcal{Q}^{\sigma,\pm}$ *are integral.*

*Proof.* We know already that $\mathcal{Q}^{\sigma,\pm}$ are connected, hence they are irreducible. Moreover, by [16, Theorem 0.2], the flag varieties $\mathcal{Q}^{\sigma,\pm}$ are reduced. □

### 3.4. Central extension

Consider the *canonical* central extension of $\mathbf{SL}_r(\mathscr{K})$ defined in [[5], Section 4]:

$$0 \to \mathbb{G}_m \to \widehat{\mathbf{SL}_r}(\mathscr{K}) \to \mathbf{SL}_r(\mathscr{K}) \to 0.$$

The actions of $\sigma^\pm$ lift to $\widehat{\mathbf{SL}_r}(\mathscr{K})$ giving a central extension of $\mathbf{SL}_r(\mathscr{K})^{\sigma^\pm}$

$$0 \to \mathbb{G}_m \to \widehat{\mathbf{SL}_r}(\mathscr{K})^{\sigma^\pm} \to \mathbf{SL}_r(\mathscr{K})^{\sigma^\pm} \to 0.$$

Indeed, let $R$ be a $\mathbb{C}-$algebra, for $\gamma \in \mathrm{SL}_r(R((t)))$ let

$$\gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$$

be its decomposition with respect to $R((t)) = \mathscr{V}_R \oplus R[[t]]$. Recall that $\mathscr{V}$ is a complementary vector subspace of $\mathscr{O}$ in $\mathscr{K}$.

By [3], an element of $\widehat{\mathrm{SL}}_r(R((t)))$ is given, locally on Spec$(R)$, by a pair $(\gamma, u)$ where $\gamma \in \mathrm{SL}_r(R((t)))$, $u \in \mathrm{Aut}(\mathscr{V}_R)$ such that $u \equiv a(\gamma) \mod \mathrm{End}^f(\mathscr{V}_R)$, where $\mathrm{End}^f(\mathscr{V}_R) \subset \mathrm{End}(\mathscr{V}_R)$ is the set of finite rank endomorphisms of $\mathscr{V}_R$. By [3] Proposition 4.3, the map $\gamma \longrightarrow \overline{a}(\gamma)$ is a group homomorphism from $\mathrm{SL}_r(R((t)))$ onto the group $\mathrm{Aut}^f(\mathscr{V}_R)$ of units of $\mathrm{End}(\mathscr{V}_R)/\mathrm{End}^f(\mathscr{V}_R)$. It follows that

$$\overline{a}(\gamma^{-1}) = \overline{a}(\gamma)^{-1},$$

hence

$$u^{-1} \equiv a(\gamma^{-1}) \mod \mathrm{End}^f(\mathscr{V}_R).$$

So, define the following actions on $\widehat{\mathbf{SL}_r}(\mathscr{K})$

$$\begin{aligned} \sigma^+ &: (\gamma, u) \longrightarrow ({}^t\gamma(-t)^{-1},\, {}^tu(-t)^{-1}),\\ \sigma^- &: (\gamma, u) \longrightarrow (J_r\, {}^t\gamma(-t)^{-1} J_r^{-1}, J_r\, {}^tu(-t)^{-1} J_r^{-1}). \end{aligned}$$

Clearly these are involutions which lift $\sigma^\pm$ on $\mathbf{SL}_r(\mathscr{K})$.

The Lie algebra attached to $\widehat{\mathbf{SL}}_r(\mathscr{K})$ is given by the central extension

$$0 \to \mathbb{C} \to \widehat{\mathfrak{sl}_r}(\mathscr{K}) \to \mathfrak{sl}_r(\mathscr{K}) \to 0. \tag{2}$$

It is in fact isomorphic to the affine Lie algebra $\hat{\mathcal{L}}(\mathfrak{sl}_r) = \mathfrak{sl}_r(\mathscr{K}) \oplus \mathbb{C}$, with the Lie algebra structure given by

$$[(\alpha, u), (\beta, v)] = \left([\alpha;\beta], \mathrm{Res}(\mathrm{Tr}(\frac{d\alpha}{dt}\beta))\right),$$

where Res stands for the *residue*. By pulling back the exact sequence (2) via the inclusions $\mathfrak{sl}_r(\mathscr{K})^{\sigma^\pm} \hookrightarrow \mathfrak{sl}_r(\mathscr{K})$ we get the central extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathfrak{sl}_r}(\mathscr{K}) & \longrightarrow & \mathfrak{sl}_r(\mathscr{K}) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathfrak{sl}_r}(\mathscr{K})^{\sigma^\pm} & \longrightarrow & \mathfrak{sl}_r(\mathscr{K})^{\sigma^\pm} & \longrightarrow & 0, \end{array}$$

where $\sigma^\pm$ act on $\widehat{\mathfrak{sl}_r}(\mathscr{K})$ by their actions on the first summand (which are given in Lemma 3.6 below). These are (after adding scaling elements) affine Kac-Moody Lie algebras of twisted type $A^{(2)}_{r-1}$. They are in fact the Lie algebras of the twisted groups $\widehat{\mathbf{SL}}_r(\mathscr{K})^{\sigma^\pm}$.

**Lemma 3.6.** *The Lie algebras associated to* $\widehat{\boldsymbol{SL}}_r(\mathscr{K})^{\sigma^\pm}$ *are the twisted affine Lie algebras of* $\mathfrak{sl}_r(\mathscr{K})$ *given by*

$$\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm) = \mathfrak{sl}_r(\mathscr{K})^{\sigma^\pm} \oplus \mathbb{C},$$

*where the actions of* $\sigma^\pm$ *on* $\mathfrak{sl}_r(\mathscr{K})$ *are given by*

$$\sigma^+(g(t)) = -\,{}^t g(-t), \quad \sigma^-(g(t)) = -J_r\,{}^t g(-t) J_r^{-1}.$$

*Proof.* The proof is straightforward, we just remark that

$${}^t(I_r + \varepsilon\alpha)^{-1} = I_r - \varepsilon\,{}^t\alpha,$$

where $\varepsilon^2 = 0$. □

## 4. Determinant and Pfaffian line bundles

Let $T$ be a locally noetherian $\mathbb{C}-$scheme. Denote by $p_1$ and $p_2$ the projection maps from $X \times T$ to $X$ and $T$ respectively. Let $\mathscr{E}$ be a vector bundle over $X \times T$. The derived direct image complex $Rp_{2*}(\mathscr{E})$ is represented by a complex of vector bundles $0 \to F_0 \to F_1 \to 0$. The line bundle $\mathcal{D}_{\mathscr{E}} := \det(F_0)^{-1} \otimes \det(F_1)$ over $T$ is independent of the choice of the representing complex and is called the *determinant of cohomology* of $\mathscr{E}$. The determinant of the universal family $\mathscr{L}$ over $X \times \mathscr{SU}_X(r)$ is called the *determinant bundle* over $\mathscr{SU}_X(r)$.

### 4.1. Ramified case

Assume that $\pi : X \to Y$ is ramified along a divisor $R$ and let

$$\mathrm{P}^- = \mathrm{Nm}^{-1}(K_Y\Delta).$$

**Proposition 4.1.** *Let* $(\mathscr{E}, \psi)$ *be a family of* $\sigma-$*alternating vector bundles over* $X$ *parameterized by* $T$*, with a* $\sigma-$*alternating non-degenerate form* $\psi : \sigma^*\mathscr{E} \longrightarrow \mathscr{E}^*$*. For any* $L \in P^-$ *let* $\mathscr{E}_L = \mathscr{E} \otimes p_1^* L$*. Then the determinant of cohomology line bundle* $\mathcal{D}_{\mathscr{E}_L}$ *admits a square root* $\mathcal{P}_{\mathscr{E}_L}$ *which we call* Pfaffian of cohomology *line bundle.*

*Proof.* Consider the family $\pi_*\mathscr{E}_L$ over $Y$. It is equipped with a non-degenerated quadratic form with values in $K_Y$. Indeed, by the projection formula, $\psi$ induces an isomorphism

$$\begin{aligned}\pi_*\mathscr{E}_L &\cong \pi_*(\sigma^*\mathscr{E}_L)\\ &\cong \pi_*(\mathscr{E}_L^*(q_1^{-1}(R)))\otimes q_1^*K_Y\\ &\cong (\pi_*\mathscr{E}_L)^*\otimes q_1^*K_Y,\end{aligned}$$

where the last isomorphism is the relative duality (see [9, Ex. III.6.10]) and $q_1 : Y\times T\to Y$ is the first projection. In fact the associated bilinear form is given by the composition

$$\pi_*\mathscr{E}_L\otimes\pi_*\mathscr{E}_L\longrightarrow\pi_*(p_1^*K_X)=q_1^*(K_Y\otimes\Delta)\oplus q_1^*K_Y\longrightarrow q_1^*K_Y.$$

Since we project on the $-1$ eigenspace of the linearization on $K_X$ (recall that $\pi_*K_X=K_Y\Delta\oplus K_Y$) and because $\psi$ is $\sigma-$alternating, we deduce that this bilinear form is symmetric.

We can apply now [13, Proposition 7.9] to get a square root of $\mathcal{D}_{\pi_*\mathscr{E}_L}$. To finish the proof we just have to remark that

$$\mathcal{D}_{\mathscr{E}_L}=\mathcal{D}_{\pi_*\mathscr{E}_L}.$$

□

In particular, if we consider the universal family over $X\times\mathscr{SU}_X^{\sigma,-}(r)$, we get, for each $L\in\mathrm{P}^-$, a Pfaffian of cohomology line bundle $\mathcal{P}_L$ over $\mathscr{SU}_X^{\sigma,-}(r)$.

On the other hand, consider the character $\chi:\widehat{\mathbf{SL}_r}(\mathscr{O})\to\mathbb{G}_m$ which is just the second projection (recall that $\widehat{\mathbf{SL}_r}(\mathscr{O})$ splits). More precisely, a point of $\widehat{\mathbf{SL}_r}(\mathscr{O})$ can be represented locally on $\mathrm{Spec}(R)$ by a pair $(\gamma,u)$, for $\gamma\in\mathrm{SL}_r(R[[t]])$ and $u$ an automorphism of $\mathscr{V}_R$ such that $a(\gamma)\equiv u \mod \mathrm{End}^f(\mathscr{V}_R)$. So $\chi$ sends this point to $\det(a(\gamma)^{-1}u)$. To this character one may associate a line bundle $\mathcal{L}_\chi$ over $\mathscr{Q}$ (see [3, §3]). Moreover $\mathcal{L}_\chi$ is isomorphic to the pullback of the determinant bundle.

**Lemma 4.2.** *The restriction of the character $\chi$ to $\widehat{\boldsymbol{SL}_r}(\mathscr{O})^{\sigma^-}$ has a square root which we denote by $\chi_-$.*

*Proof.* With the notations of the proof of Proposition 3.4, one can see that $\mathbf{SL}_r(\mathscr{O})^{\sigma^-}$ is the direct limit of the parabolic subgroups $\mathrm{P}'_N$. So just take the direct limit in Proposition 3.4. □

Let $\mathcal{L}_-$ be the line bundle over $\mathscr{Q}^{\sigma,-}$ defined by the character $\chi_-$ and denote by $q:\mathscr{Q}^{\sigma,-}\longrightarrow\mathscr{SU}_X^{\sigma,-}(r)$ the quotient maps (there should be no confusion about which map is considered).

Denote by $\mathcal{D}$ the determinant of cohomology line bundle over $\mathscr{SU}_X^{\sigma,-}(r)$ and by $\mathcal{L}$ its pullback to $\mathscr{Q}^{\sigma,-}$.

**Theorem 4.3.** *Let $r \geqslant 3$. Then the isomorphism class of the Pfaffian bundles $\mathcal{P}_L$ is independent of $L$, so we just denote it by $\mathcal{P}$, and its pullback $q^*\mathcal{P}$ is isomorphic to $\mathcal{L}_{\chi_-}$. Moreover we have*

$$Pic(\mathscr{SU}_X^{\sigma,+}(r)) = \mathbb{Z}\mathcal{D}, \quad Pic(\mathscr{SU}_{X,0}^{\sigma,-}(r)) = \mathbb{Z}\mathcal{P}.$$

*Proof.* Since $\mathrm{P}^-$ is connected and the group $Pic(\mathscr{SU}_X^{\sigma,-}(r))$ is discrete, we deduce that the map $L \to \mathcal{P}_L$ is constant. Hence all $\mathcal{P}_L$ are isomorphic. Now, by [20, Proposition 2.3], we have

$$0 \to \mathrm{Sym}_r^0(\mathbb{C}) \xrightarrow{i} \mathcal{G}_p \to \mathrm{SO}_r(\mathbb{C}) \to 0.$$

Even though that this extension does not split, $\mathrm{SO}_r$ can be embedded in $\mathcal{G}_p$ via the map $g \to g + \varepsilon \times 0$. Let $X^*(\mathcal{G}_p)$ is the character group of $\mathcal{G}_p$ and $\lambda \in X^*(\mathcal{G}_p)$. Since $\mathrm{SO}_r(\mathbb{C})$ is semi-simple, the restriction of $\lambda$ to $\mathrm{SO}_r(\mathbb{C})$ is trivial. Moreover, the restriction of $\chi$ to $i(\mathrm{Sym}_r^0(\mathbb{C}))$ is trivial too because $\mathrm{Sym}_r^0(\mathbb{C})$ is a unipotent.

Now, by [10, Theorem 3], we have

$$0 \to \Pi_{p\in B} X^*(\mathcal{G}_p) \to \mathrm{Pic}(\mathscr{M}_Y(\mathcal{G})) \to \mathbb{Z} \to 0,$$

it follows that $\mathrm{Pic}(\mathscr{SU}_X^{\sigma,+}(r)) \cong \mathbb{Z}$. It is well known that the pullback of the determinant line bundle over $\mathscr{SU}_X(r)$ to $\mathscr{Q}$ is $\mathcal{L}_\chi$ (see for example [13]). Furthermore, $\mathcal{D}$ has no square root. Indeed, by [19, Theorem 4.16], the stack $\mathscr{SU}_X^{\sigma,+}(r)$ is dominated by the intersection of two Prym varieties $\tilde{\mathrm{P}} \cap \tilde{\mathrm{Q}}$ which is not principally polarized. Hence if $\mathcal{D}$ has a square root, it follows that the polarization of $\tilde{\mathrm{P}} \cap \tilde{\mathrm{Q}}$ has also a square root, which is not true. Thus $\mathcal{D}$ is a generator of $\mathrm{Pic}(\mathscr{SU}_X^{\sigma,+}(r))$.

With the same method we deduce that $\mathcal{P}$ has no root, this implies that that $\mathrm{Pic}(\mathscr{SU}_X^{\sigma,-}(r)) \cong \mathbb{Z}\mathcal{P}$. □

**Remark 4.4.** The case $r = 2$ is special. In this case, the moduli stack $\mathscr{SU}_X^{\sigma,+}(2)$ is connected and $\mathscr{SU}_X^{\sigma,-}(2)$ has $2^{2n-1}$ connected component. It is pointed out in [19] that these moduli stacks can be identified with some moduli of parabolic rank 2 bundles on $Y$. Hence one can deduce their Picard groups using [13].

### 4.2. Unramified case

Assume here that $\pi : X \to Y$ is étale. Let $\mathrm{P}^+ = \mathrm{P}^{ev}_+ \cup \mathrm{P}^{od}_+ = \mathrm{Nm}^{-1}(K_Y)$ and $\mathrm{P}^- = P^a_- \cup P^b_- = \mathrm{Nm}^{-1}(K_Y\Delta)$. Note the subscript in the $+$ case is canonical and it corresponds to the parity of $h^0$ of the line bundles. Then we have the following

**Proposition 4.5.** *Let $(\mathscr{E}^{\pm}, \psi)$ be a family of $\sigma-$symmetric (resp. $\sigma-$alternating) vector bundles over $X$ parameterized by $T$, with a $\sigma-$symmetric (resp. $\sigma-$alternating) non-degenerated form $\psi : \sigma^*\mathscr{E}^{\pm} \longrightarrow \mathscr{E}^{\pm *}$. For any $L \in P^+$ (resp. $L \in P^-$) let $\mathscr{E}^{\pm}_L = \mathscr{E}^{\pm} \otimes p_1^* L$. Then the determinant of cohomology line bundle $\mathcal{D}_{\mathscr{E}^{\pm}_L}$ admits a square root $\mathcal{P}^{\pm}_{\mathscr{E}^{\pm}_L}$ which we call* Pfaffian of cohomology *line bundle.*

*Proof.* The proof is similar to that of Proposition 4.1. We note just that when $L \in \mathrm{P}^+$, the norm map induces a quadratic form on $\pi_* L$, and when it is in $\mathrm{P}^-$, the induced form is alternating. □

Let $\mathscr{U}^{\pm}$ be universal families on $\mathscr{S}\mathscr{U}^{\sigma,+}_X(r)$ and $\mathscr{S}\mathscr{U}^{\sigma,-}_X(2r)$, then for any $L \in P^{\pm}$, we have a Pfaffian of cohomology line bundle $\mathcal{P}^{\pm}_L := \mathrm{Pf}(\mathscr{U}^{\pm}_L)$. Moreover, since $\mathrm{Pic}(\mathscr{S}\mathscr{U}^{\sigma,\pm}_X(r))$ are discrete groups (this can be deduced from the uniformization theorem for example), we deduce that there is at most two isomorphism classes of $\mathcal{P}^{\pm}_L$ parametrized by the connected components of $\mathrm{P}^{\pm}$. So we denote them by $\mathcal{P}^{\pm}_{ev}$ and $\mathcal{P}^{\pm}_{od}$.

## 5. Generalized theta functions and conformal blocks

Assume in this section that the cover $\pi : X \to Y$ is ramified. We have formulated the uniformization theorem over a single ramification point. However we can use a bunch of points to uniformize our moduli stack. If we consider all the ramification points $R$, then we get the following

$$\mathscr{S}\mathscr{U}^{\sigma,\pm}_X(r) \cong \prod_{p\in R} \mathscr{Q}^{\sigma,\pm}_p / \mathbf{SL}_r(\mathscr{A}_R)^{\sigma^{\pm}},$$

where $\mathscr{Q}^{\sigma,\pm}_p = \mathbf{SL}_r(\mathscr{O}_p)^{\sigma^{\pm}} \backslash \mathbf{SL}_r(\mathscr{K}_p)^{\sigma^{\pm}}$, and $\mathscr{A}_R = H^0(X \smallsetminus R, \mathcal{O}_X)$. Of course all $\mathscr{Q}^{\sigma,\pm}_p$ are isomorphic, but we emphasise on the fixed points.
Roughly speaking, this isomorphism can be seen as follows: choose a formal neighborhood $D_p$ of each $p \in R$. Then giving a $\sigma-$symmetric vector bundle $(E, \psi)$ of trivial determinant, we choose a $\sigma-$invariant local trivializations

$\varphi_p$ near each $p$ and a $\sigma-$invariant trivialization $\varphi_0$ on $X \smallsetminus R$. Then the corresponding point of the right hand side is just the class of $(\varphi_p \circ \varphi_0^{-1})_{p\in R}$. Conversely, giving a class of functions $(f_p)_{p\in R}$ of the RHS, we can construct a $\sigma-$symmetric vector bundle by gluing the trivial bundles on $D_p$ and $X \smallsetminus R$ using the functions $f_p$.

We have seen that the line bundle $\mathcal{L}_-$ over $\mathscr{Q}_p^{\sigma,-}$ is isomorphic to $q^*\mathcal{P}$ and that the line bundle $\mathcal{L}$ over $\mathscr{Q}_p^{\sigma,+}$ is isomorphic to $q^*\mathcal{D}$. For $x \in R$, let $q_x : \prod_{p\in R} \mathscr{Q}_p^{\sigma,\pm} \to \mathscr{Q}_x^{\sigma,\pm}$ be the canonical projection. We define the line bundles

$$\mathscr{L}_- = \bigotimes_{p\in R} q_p^*\mathcal{L}_- \quad \text{and} \quad \mathscr{L} = \bigotimes_{p\in R} q_p^*\mathcal{L}$$

over $\prod_{p\in R} \mathscr{Q}_p^{\sigma,-}$ and $\prod_{p\in R} \mathscr{Q}_p^{\sigma,+}$ respectively. One can see that $\mathscr{L}_-$ and $\mathscr{L}$ are in fact the pullback via the projections $\prod_{p\in R} \mathscr{Q}_p^{\sigma,\pm} \to \mathscr{S}\mathscr{U}_X^{\sigma,\pm}(r)$ of the line bundles $\mathcal{P}$ and $\mathcal{D}$ respectively. In particular, both of these line bundles have canonical $\mathbf{SL}(\mathscr{A}_R)^{\sigma^\pm}-$linearizations. In fact these are the only ones due to the following

**Proposition 5.1.** *$\boldsymbol{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$ are integral and they have only the trivial character.*

*Proof.* The proof is inspired from [13].

Using the local triviality of the projection $\prod_{p\in R} \mathscr{Q}_p^{\sigma,\pm} \to \mathscr{S}\mathscr{U}_X^{\sigma,\pm}(r)$ and Proposition 3.5 we deduce that $\mathbf{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$ are reduced.

Now, since connected ind-groups are irreducible, it is sufficient to prove that $\mathbf{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$ is connected. For a points $p_1, \dots, p_k \in X \smallsetminus R$ we denote by $R_i = R \cup \{p_1, \sigma(p_1), \dots, p_i, \sigma(p_i)\}$. We claim the following

**Claim.** *We have an isomorphism*

$$\boldsymbol{SL}_r(\mathscr{A}_{R_i})^{\sigma^\pm} / \boldsymbol{SL}_r(\mathscr{A}_{R_{i-1}})^{\sigma^\pm} \cong (\mathscr{Q}_{p_i} \times \mathscr{Q}_{\sigma(p_i)})^{\sigma^\pm},$$

*where the action of $\sigma^\pm$ on the right hand side is given by*

$$\sigma^\pm(f,g) = (\sigma^\pm(g), \sigma^\pm(f)).$$

*Proof.* We have a canonical map $\mathbf{SL}_r(\mathscr{A}_{R_i})^{\sigma^\pm} \to (\mathscr{Q}_{p_i} \times \mathscr{Q}_{\sigma(p_i)})^{\sigma^\pm}$ which is clearly trivial on $\mathbf{SL}_r(\mathscr{A}_{R_{i-1}})^{\sigma^\pm}$. Hence we deduce a map

$$\mathbf{SL}_r(\mathscr{A}_{R_i})^{\sigma^\pm} / \mathbf{SL}_r(\mathscr{A}_{R_{i-1}})^{\sigma^\pm} \to (\mathscr{Q}_{p_i} \times \mathscr{Q}_{\sigma(p_i)})^{\sigma^\pm}$$

which is actually injective. Now, by considering the uniformization over the two points $\{p_i, \sigma(p_i)\}$, we get

$$\mathscr{SU}_X^{\sigma,\pm}(r) \cong (\mathscr{Q}_{p_i} \times \mathscr{Q}_{\sigma(p_i)})^{\sigma^\pm}/\mathbf{SL}_r(\mathscr{A}_{\{p_i,\sigma(p_i)\}})^{\sigma^\pm}.$$

Hence, for an $\mathbb{C}-$algebra $S$, giving a point of $(\mathscr{Q}_{p_i} \times \mathscr{Q}_{\sigma(p_i)})^{\sigma^\pm}(S)$ is the same as giving an anti-invariant ($\sigma-$symmetric or $\sigma-$alternating following $\pm$) vector bundle $E$ over $X_S$ and a $\sigma^\pm-$invariant trivialization $\delta : E|_{X_S^*} \to X_S^* \times \mathbb{C}^r$, where $X_S^* = X_S \smallsetminus \{p_i, \sigma(p_i)\}$. For an $S-$algebra $S'$, let $T(S')$ be the space of $\sigma^\pm-$invariant trivializations of $E_{S'}$ over $X_{S,i-1} = X_S \smallsetminus R_{i-1}$. Then $\mathbf{SL}_r(\mathscr{A}_{R_{i-1}})^{\sigma^\pm}$ acts on $T$, and in fact it is a torsor under that group. Moreover $\delta$ induces a map $\tilde{\delta} : T \to \mathbf{SL}_r(\mathscr{A}_{R_i})^{\sigma^\pm}$ by sending a trivialization $\phi$ to $\phi \circ \delta^{-1}$. Associating to $(E, \delta)$ the map $\tilde{\delta}$ gives an inverse to the above inclusion. □

It is clear to see that $(\mathscr{Q}_{p_i} \times \mathscr{Q}_{\sigma(p_i)})^{\sigma^\pm} \cong \mathscr{Q}_{p_i} = \mathbf{SL}_r(\mathscr{O}_{p_i})\backslash\mathbf{SL}_(\mathscr{K}_{p_i})$ which is simply connected. So using the homotopy exact sequence, we deduce that

$$\pi_0(\mathbf{SL}_r(\mathscr{A}_{R_i})) = \pi_0(\mathbf{SL}_r(\mathscr{A}_{R_{i-1}})).$$

Now let $g \in \mathrm{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$ and consider $g$ as an element of $\mathrm{SL}_r(\mathcal{K})^{\sigma^\pm}$, where $\mathcal{K}$ is the function field of $X$. By [18] (see also [15] Section 4), we know that the special unitary groups are simply connected and quasi-split. Steinberg ([17]) has showed the Kneser-Tits property for quasi-split simply connected groups over any field (Recall that this property means that these groups are generated by the unipotent radicals of their standard parabolic subgroups). So applying that to $\mathrm{SL}_r(\mathcal{K})^{\sigma^\pm}$, we can assume that $g = \prod_i \exp(N_i)$, where $N_i$ are nilpotent elements of $\mathfrak{sl}_r(\mathcal{K})^{\sigma^\pm}$. Let $\{p_1, \dots, p_k\}$ be the set of poles of the $N_i$'s. For $t \in \mathbb{A}^1$, we let $g_t = \prod_i \exp(tN_i)$. Then for any $t \in \mathbb{A}^1$ we see that $g_t \in \mathrm{SL}_r(\mathscr{A}_{R_k})^{\sigma^\pm}$ and $t \to g_t$ is a path in $\mathrm{SL}_r(\mathscr{A}_{R_k})^{\sigma^\pm}$ that relates $g$ to the identity. Hence $\mathbf{SL}_r(\mathscr{A}_{R_k})^{\sigma^\pm}$ is connected. So the same is true for $\mathbf{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$ by what we have shown above.

Now let $\lambda$ be a character of $\mathrm{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$; seeing $\lambda$ as a function, we consider its derivative at the identity which turns out to be a Lie algebras morphism from $\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^\pm}$ to the trivial algebra $\mathbb{C}$. However, the affine algebra $\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^\pm}$ equals the direct sum of two commutator subalgebras. Indeed, the algebra $\mathfrak{sl}_r(\mathscr{A}_R)$ equals to its commutator, and we have eigenspace decomposition with respect to $\sigma^\pm$

$$\mathfrak{sl}_r(\mathscr{A}_R) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1,$$

it follows

$$[\mathfrak{g}_{-1} \oplus \mathfrak{g}_1, \mathfrak{g}_{-1} \oplus \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus [\mathfrak{g}_1, \mathfrak{g}_{-1}] \oplus [\mathfrak{g}_1, \mathfrak{g}_1].$$

Hence $\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^\pm} = \mathfrak{g}_1 = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \oplus [\mathfrak{g}_1, \mathfrak{g}_1]$. So the derivative of $\lambda$ at the identity is zero. Since $\lambda$ is a group homomorphism, its derivative is identically zero everywhere. Since $\mathbf{SL}_r(\mathscr{A}_X)^{\sigma^\pm}$ is integral, we can write it as limit of integral varieties $V_n$ and for $n$ large $1 \in V_n$, so $\lambda|_{V_n} = 1$, hence $\lambda = 1$. □

Fix an integer $k > 0$. For any dominant weight $\lambda^\pm \in \mathrm{P}^{\sigma,\pm}$, there is a line bundle $\mathscr{L}(\lambda^\pm)$ over $\mathscr{Q}^{\sigma,\pm}$ associated to the principal $\mathbf{SL}_r(\mathscr{O})^{\sigma^\pm}-$bundle:

$$\mathbf{SL}_r(\mathscr{K})^{\sigma^\pm} \longrightarrow \mathscr{Q}^{\sigma,\pm},$$

defined using the character $e^{-\lambda^\pm}$ on $\mathbf{SL}_r(\mathscr{O})^{\sigma^\pm}$. Further, it is shown in [12] that the space of global sections of powers of $\mathscr{L}(\lambda)$ is isomorphic to the dual of the irreducible highest integrable representation of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$ associated to $\lambda^\pm$.
We are mainly interested in the case where $\lambda^\pm = \lambda_0^\pm$. Denote by $\mathcal{H}_\pm(k)$ the highest weight representation of level $k$ of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$ associated to the weight $\lambda_0^\pm$. It is called the basic representation of level $k$. So the above result of [12] (see also [14]) can be formulated as follows

**Theorem 5.2 (Kumar, Mathieu).**

1) *The space $H^0(\mathscr{Q}^{\sigma,-}, q^*\mathcal{P}^k)$ is canonically isomorphic, as $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)-$module, to the dual of the basic representation $\mathcal{H}_-(k)$.*
2) *The space $H^0(\mathscr{Q}^{\sigma,+}, q^*\mathcal{D}^k)$ is canonically isomorphic, as $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)-$module, to the dual of the basic representation $\mathcal{H}_+(k)$.*

Note that by Remark 2.1, when $r$ is even, the weight $\lambda_0^+$ has level 2 while $\lambda_0^-$ is of level 1. This explains why we have to take the determinant line bundle in $\sigma^+$ case and the Pfaffian line bundle in $\sigma^-$ case.

The point that should be stressed here is that in [12], Kumar has defined the ind-group $\mathbf{SL}_r(\mathscr{O})\backslash\mathbf{SL}_r(\mathscr{K})$ using representation theory of Kac-Moody algebras. It is shown in [3] that this construction coincides with the usual functorial definition. Moreover, Pappas and Rapoport have claimed in [15] (page 3) that the constructions of Kumar coincide with their definitions of the Schubert varieties. In particular, we deduce in our special case that the ind-variety structure on the twisted flag varieties $\mathscr{Q}^{\sigma,\pm}$ are the same as those defined by Kumar.

Conjecture 3.7 of Pappas and Rapoport ([15]) relates the space of sections of a line bundle on the moduli stack of parahoric Bruhat-Tits torsors with the space of invariants sections of the pullback of that line bundle to the flag variety. The following proposition is a special case of this conjecture. The key point in its proof is the integrality of the ind-group $\mathbf{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$ (Proposition 5.1).

**Proposition 5.3.** *We have isomorphisms*

$$H^0(\mathscr{SU}_X^{\sigma,-}(r), \mathcal{P}^k) \cong \left(\prod_{p\in R} H^0(\mathscr{Q}_p^{\sigma,-}, \mathcal{L}_-^k)\right)^{\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^-}},$$

$$H^0(\mathscr{SU}_X^{\sigma,+}(r), \mathcal{D}^k) \cong \left(\prod_{p\in R} H^0(\mathscr{Q}_p^{\sigma,+}, \mathcal{L}^k)\right)^{\mathfrak{sl}_r(\mathscr{A}_R)^{\sigma^+}}.$$

*Proof.* Since $\mathbf{SL}_r(\mathscr{A}_R)^{\sigma^\pm}$ and $\mathscr{Q}^{\sigma^\pm}$ are integral, the result follows, using the Künneth formula, from [5, Proposition 7.4]. □

Now, Proposition 5.3 and Theorem 5.2 imply our main result

**Theorem 5.4.** *Let $k \in \mathbb{N}$, we have*

1) *The space of global sections $H^0(\mathscr{SU}_X^{\sigma,-}(r), \mathcal{P}^k)$ is canonically isomorphic to the conformal block space $\mathcal{V}_{\sigma,-}(k)$.*
2) *The space of global sections $H^0(\mathscr{SU}_X^{\sigma,+}(r), \mathcal{D}^k)$ is canonically isomorphic to the conformal block space $\mathcal{V}_{\sigma,+}(k)$.*

## 6. Strange duality at level one

In this section, we show a strange duality at level one for the moduli spaces $\mathcal{SU}_X^{\sigma,+}(r)$ of $\sigma-$symmetric vector bundles in the unramified case. Since $\mathcal{U}_X^{\sigma,-}(r) \cong \mathcal{U}_X^{\sigma,+}(r)$, similar results hold for the $\sigma-$alternating case.

So assume that $\pi : X \to Y$ is étale. Let $\Delta \in J_Y[2]$ the line bundle associated to this cover. We use notation from subsection 4.2. In particular, since we will deal just with the $\sigma-$symmetric case, we shall denote simply by $\mathrm{P}^{ev}$ the space $\mathrm{P}_+^{ev}$ and by $\mathcal{P}_{ev}$ the isomorphism class of the Pfaffian bundles $\mathcal{P}_L$, for $L \in \mathrm{P}^{ev}$.

The moduli stack $\mathscr{U}_X^{\sigma,+}(r)$ of $\sigma-$symmetric bundles has two connected components distinguished by the parity of $h^0(E \otimes L)$, for fixed $L \in \mathrm{P}^{ev}$. The even connected component is denoted $\mathscr{U}_{X,0}^{\sigma,+}(r)$. The moduli $\mathscr{S}\mathscr{U}_X^{\sigma,+}(r)$ are connected. The associated moduli spaces has been constructed in [20]. Here we consider the moduli spaces $\mathcal{U}_X^{\sigma,+}(r)$ (resp. $\mathcal{SU}_X^{\sigma,+}(r)$) of *stable* $\sigma-$symmetric vector bundles (resp. with trivial determinant).

**Lemma 6.1.** *The Pfaffian line bundle $\mathcal{P}_{ev}$ over $\mathscr{U}_{X,0}^{\sigma,+}(r)$ descends to the moduli space $\mathcal{U}_{X,0}^{\sigma,+}(r)$.*

*Proof.* The moduli space $\mathcal{U}_X^{\sigma,+}(r)$ is constructed using GIT as a $\mathrm{SL}(H)-$quotient of a parameter scheme $\mathrm{Quot}^\sigma(\mathbb{C})$, where $H = \mathbb{C}^m$ for some $m$ (see [20]). Let $L \in \mathrm{P}^{ev}$ and $a = (E, q, \overline{\psi})$ be a point of $\mathrm{Quot}^\sigma(\mathbb{C})$. Since $E$ is stable, the stabilizer of $a$ under the action of $\mathrm{SL}(H)$ is just $\{\pm 1\}$. The action of this stabilizer on $(\mathcal{P}_L)_a$ is by definition multiplication by $g^{h^1(E\otimes L)}$, for $g \in \{\pm 1\}$. Since $\mathcal{U}_{X,0}^{\sigma,+}(r)$ is connected, we have

$$h^1(E \otimes L) = \begin{cases} 1 & \text{if } r \equiv 1 \mod 2 \text{ and } L \in \mathrm{P}^{od} \\ 0 & \text{otherwise.} \end{cases}$$

This can be shown using Hitchin system (cf. [19, Theorem 4.12]). Since $L$ is even, it follows that $-1$ acts trivially on $(\mathcal{P}_L)_a$, for any $a$. Using Kempf's Lemma we deduce the result. □

Now we show the existence of the Pfaffian divisor. Let $\mathcal{U}_X(r, 0)$ be the moduli space of rank $r$ and degree 0 stable vector bundles over $X$, and let $\Theta_L$ be the divisor in $\mathcal{U}_X(r, 0)$ supported on vector bundles $E$ such that $E \otimes L$ has a non-zero global section, where $L \in \mathrm{P}^{ev}$ is fixed.

Let us recall from [19] some basic results about the Hitchin system in this case. The Hitchin morphism on $\mathcal{U}_X(r, 0)$ induces a fibration

$$\mathscr{H} : T^*\mathcal{U}_X^{\sigma,+}(r) \longrightarrow W^{\sigma,+},$$

where $W^{\sigma,+} = \bigoplus H^0(K_X^i)_+$. For general $s \in W^{\sigma,+}$ the associated spectral curve $q : \tilde{X}_s \to X$ is smooth and it has a fixed point free involution $\tilde{\sigma}$ that lifts $\sigma$. Moreover, the quotient curve $\tilde{Y}_s = \tilde{X}_s/\tilde{\sigma}$ is a smooth spectral curve over $Y$ with spectral data in $K_Y \otimes \Delta$. Let $S$ be the ramification divisor of $\tilde{Y}_s \to Y$. Then the fiber $\mathrm{Nm}^{-1}_{\tilde{X}/\tilde{Y}_s}(\mathcal{O}(S))$ has two connected components $\tilde{\mathrm{P}}^{ev} \cup \tilde{\mathrm{P}}^{od}$, distinguished by the parity of $h^0(- \otimes q^*L)$ (+ for even), where $L$

is in $\mathrm{P}^{ev}$. Now by [19, Theorem 4.17] the push-forward map

$$q_* : \tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}} \dashrightarrow \mathcal{SU}_X^{\sigma,+}(r)$$

is dominant, where $\tilde{\mathrm{Q}} = \mathrm{Nm}^{-1}_{\tilde{X}_s/X}(\delta)$, $\delta = \det(q_*\mathcal{O}_{\tilde{X}_s})^{-1}$.

**Lemma 6.2.** *Let $L \in P^{ev}$. The restrictions of the divisor $\Theta_L \subset \mathcal{U}_X(r,0)$ to $\mathcal{U}_{X,0}^{\sigma,+}(r)$ and $\mathcal{SU}_X^{\sigma,+}(r)$ are again divisors. The associated reduced divisors are denoted $\Xi_L$.*

*Proof.* It is enough to produce a semistable $\sigma-$symmetric vector bundle with trivial determinant that does not belong to $\Theta_L$. Let $\Xi_L$ be the (principal) polarization on $\mathrm{P} := Prym(X \to Y)$. Then the linear system $|r\Xi_L|$ is base point free for any $r \geqslant 2$ and the group $\mathrm{P}[r]$ acts irreducibly on it. Hence let $\alpha \in \mathrm{P}[r]$ such that $\mathcal{O}_X \notin T_\alpha^*\Xi_L = \Xi_{\alpha\otimes L}$. In other words $h^0(\alpha \otimes L) = 0$. Define $E := \alpha^{\oplus r}$. It is obviously a semistable $\sigma-$symmetric vector bundle with trivial determinant, hence it belongs to the closures of $\mathcal{SU}_X^{\sigma,\pm}(r)$ and $\mathcal{U}_{X,0}^{\sigma,+}(r)$ and it is not in the restriction of the divisor $\Theta_L$. $\square$

Note that the other connected component $\mathcal{U}_{X,1}^{\sigma,+}(r)$ is entirely included in $\Theta_L$ for any $L \in \mathrm{P}^{ev}$. For the moduli of $\sigma-$alternating bundles, the same happens, i.e. the restriction of $\Theta_L$ to $\mathcal{U}_{X,0}^{\sigma,-}(r)$ is again a divisor and $\mathcal{U}_{X,1}^{\sigma,-}(r) \subset \Theta_L$.

**Lemma 6.3.** *We have $dim(H^0(\mathcal{U}_{X,0}^{\sigma,+}(r), \mathcal{P}_{ev})) = 1$.*

*Proof.* Let $q : \tilde{X}_s \to X$ be a smooth spectral curve over $X$ attached to a general $s \in W^{\sigma,+}$ (see [19] for more details and notations). First, for some positive integer $m$, the pullback of the determinant bundle via $q_* : J^m_{\tilde{X}_s} \to \mathcal{U}_X(r,0)$ is the line bundle $\mathcal{O}(\Theta_{q^*\kappa})$ attached to the Riemann theta divisor $\Theta_{q^*\kappa}$ over $J^m_{\tilde{X}_s}$ (modulo a translation). Let $\mathcal{S} \subset \tilde{\mathrm{P}}^{ev}$ be the locus of line bundles $L$ such that $q_*L$ is stable. The codimension of the complement of $\mathcal{S}$ is at least 2. Indeed, it is clear that this codimension is at least 1. Now let $\tilde{\mathcal{S}}$ be the locus of line bundle such that $q_*L$ is semi-stable. Then using the same argument as in [4, Proposition 5.1], we get that codimension of the complement of $\tilde{\mathcal{S}}$ is at least 2. Since the codimension of $\mathcal{S} \subset \tilde{\mathcal{S}}$ is at least 1, one get that the codimension of the complement of $\mathcal{S}$ is at least 2.

Since $q_* : \mathcal{S} \longrightarrow \mathcal{U}_{X,0}^{\sigma,+}(r)$ is dominant, we get an injection

$$H^0(\mathcal{U}_{X,0}^{\sigma,+}(r), \mathcal{P}_L) \hookrightarrow H^0(\tilde{\mathrm{P}}^{ev}, \tilde{\mathcal{L}}),$$

where $\tilde{\mathcal{L}}$ is the principle polarization on $\tilde{\mathrm{P}}^{ev}$. So $h^0(\mathcal{U}_{X,0}^{\sigma,+}(r), \mathcal{P}_L)$ is at most 1.

Now by Lemma 6.2, there is an effective divisor $\Xi_L$ such that $2\Xi_L = \Theta_L|_{\mathcal{U}_{X,0}^{\sigma,+}(r)}$. In particular, $\mathcal{P}_L$ has a non trivial global section. □

Denote by $\mathcal{L}$ and $\tilde{\mathcal{L}}$ line bundles defining principal polarizations on $\mathrm{P}^{ev}$ and $\tilde{\mathrm{P}}^{ev}$ respectively. The restriction of $\mathcal{P}_{ev}$ to $\mathcal{SU}_X^{\sigma,+}(r)$ is denoted again by $\mathcal{P}_{ev}$.

**Theorem 6.4.** *We have an isomorphism*

$$H^0(P^{ev}, \mathcal{L}^r)^* \cong H^0(\mathcal{SU}_X^{\sigma,+}(r), \mathcal{P}_{ev}).$$

*In particular we deduce*

$$dim(H^0(\mathcal{SU}_X^{\sigma,+}(r), \mathcal{P}_{ev})) = r^{g_Y - 1}.$$

*Proof.* Consider the following commutative diagram

$$\begin{array}{ccc} \tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}} \times \mathrm{P}^{ev} & \longrightarrow & \tilde{\mathrm{P}}^{ev} \\ \downarrow & & \downarrow \\ \mathcal{SU}_X^{\sigma,+}(r) \times \mathrm{P}^{ev} & \longrightarrow & \mathcal{U}_{X,0}^{\sigma,+}(r, K_X), \end{array}$$

where $\mathcal{U}_{X,0}^{\sigma,+}(r, K_X) = \{E \otimes L \mid E \in \mathcal{U}_{X,0}^{\sigma,+}(r)\}$, it has a canonical Pfaffian line bundle $\mathcal{P}$. Using [4, Theorem 3], we deduce that the pullback of the line bundle $\mathcal{P}$ to $\mathcal{SU}_X^{\sigma,+}(r) \times \mathrm{P}^{ev}$ is isomorphic to $p_1^*\mathcal{P}_{ev} \otimes p_2^*\mathcal{L}^r$.
Now the rational map $\tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}} \longrightarrow \mathcal{SU}_X^{\sigma,+}(r)$ is dominant ([19, Theorem 4.16]). It follows, by the same argument used in the proof above, that the map

$$H^0(\mathcal{SU}_X^{\sigma,+}(r), \mathcal{P}_{ev}) \to H^0(\tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}}, \tilde{\mathcal{L}})$$

is injective, where here we denote abusively by $\tilde{\mathcal{L}}$ the restriction of $\tilde{\mathcal{L}}$ to $\tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}} \subset \tilde{\mathrm{P}}^{ev}$. Since the two subvarieties $\mathrm{P}^{ev}$ and $\tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}}$ are (torsors under) complementary pair inside $\tilde{\mathrm{P}}^{ev}$, we obtain, using [4, Proposition 2.4], an isomorphism

$$H^0(\tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}}, \tilde{\mathcal{L}}) \cong H^0(\mathrm{P}^{ev}, \mathcal{L}^r)^*.$$

Hence we deduce an injective map

$$H^0(\mathcal{SU}_X^{\sigma,+}(r), \mathcal{P}_{ev}) \hookrightarrow H^0(\mathrm{P}^{ev}, \mathcal{L}^r)^*.$$

Moreover the group $\mathrm{P}[r]$ acts on $\mathbb{P}H^0(\mathrm{P}^{ev}, \mathcal{L}^r)^*$ as well as on $\mathcal{SU}_X^{\sigma,+}(r)$, hence it acts also on the linear system $\mathbb{P}H^0(\mathcal{SU}_X^{\sigma,+}(r), \mathcal{P}_{ev})$. Since the projective representation $\mathbb{P}H^0(\mathrm{P}^{ev}, \mathcal{L}^r)^*$ is irreducible, the map $H^0(\mathcal{SU}_X^{\sigma,+}(r), \mathcal{P}_{ev}) \hookrightarrow H^0(\mathrm{P}^{ev}, \mathcal{L}^r)^*$, which is equivariant for these actions, is necessarily an isomorphism. □

Note that we have a map

$$\rho : \mathcal{SU}_X^{\sigma,+}(r) \dashrightarrow |\mathcal{L}^r| = \mathbb{P}H^0(\mathrm{P}^{ev}, \mathcal{L}^r),$$

given by

$$E \longrightarrow \rho(E) = \mathrm{divPf}(\pi_*(p_2 E \otimes \mathscr{K})),$$

where $\mathscr{K}$ is the normalized Poincaré bundle over $\mathrm{P}^{ev}$ such that $\sigma^*\mathscr{K} \simeq \mathscr{K}^{-1} \otimes p_2^* K_X$. Note that the family $\pi_*(p_2^* E \otimes \mathscr{K})$ has a non-degenerated quadratic form with values in $K_Y$. Note that $\rho^*\mathcal{O}(1) \cong \mathcal{P}_{ev}$. So this map and the duality of complimentary pairs induce the following commutative diagram

$$\begin{array}{ccc} H^0(\mathrm{P}^{ev}, \mathcal{L}^r)^* & \longrightarrow & H^0(\mathcal{SU}_X^{\sigma,+}(r), \mathcal{P}_{ev}) \\ & \searrow & \downarrow \\ & & H^0(\tilde{\mathrm{P}}^{ev} \cap \tilde{\mathrm{Q}}, \tilde{\mathcal{L}}), \end{array}$$

where all maps in the above diagram are isomorphisms. In other words, the isomorphism of Theorem 6.4 is exactly $\rho^*$.


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