Math. Res. Lett.
Volume 26, Number 6, 1789–1789, 2019

A counterexample to Batson's conjecture

Andrew Lobb

We give a counterexample to Batson's conjecture on the non-orientable smooth slice genera of torus knots.

Let $T_{p,q} \subset S^3$ be the (p,q) torus knot for $p > q \geq 2$, and let $D_{p,q}$ be the usual q-stranded braid closure diagram of $T_{p,q}$. Adding a blackboard-framed 1-handle between the first two strands of $D_{p,q}$ results in a simpler torus knot, whose usual braid closure diagram we then consider. Repeating this procedure eventually arrives at the unknot, which may be capped off in the 4-ball B^4 to give a surface $F_{p,q} \subset B^4$ with $\partial F_{p,q} = T_{p,q}$.

Batson conjectured [1] that $b_1(F_{p,q})$ is minimal among the first Betti numbers of non-orientable smooth surfaces in the 4-ball with boundary $T_{p,q}$. Van Cott and Jabuka have verified this conjecture in many cases [2].

Figure 1: The torus knot $T_{4,9}$ is shown on the left. In the middle we have added two 1-handles resulting in the unknot - this describes the surface $F_{4,9} \subset B^4$ which has $b_1(F_{4,9}) = 2$. On the right we show a way to add a single 1-handle to $T_{4,9}$ resulting in the smoothly slice knot 6_1. Capping off with a slicing disc gives a surface $\Sigma \subset B^4$ with $\partial\Sigma = T_{4,9}$ and $b_1(\Sigma) = 1$.

References

[1] J. Batson, *Nonorientable slice genus can be arbitrarily large*, Math. Res. Lett. **21** (2014), no. 3, 423–436.

[2] C. A. Van Cott and S. Jabuka, *On a nonorientable analogue of the Milnor conjecture*, `arXiv:1809.017793`, (2019).

Mathematical Sciences, Durham University, Durham, DH1 3LE, UK
E-mail address: `andrew.lobb@durham.ac.uk`

Received June 4, 2019