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On the characterising slopes of hyperbolic knots

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A slope p/q is a characterising slope for a knot K in S^3 if the oriented homeomorphism type of p/q-surgery on K determines K uniquely. We show that when K is a hyperbolic knot its set of characterising slopes contains all but finitely many slopes p/q with $q \geq 3$. We prove stronger results for hyperbolic L-space knots, showing that all but finitely many non-integer slopes are characterising. The proof is obtained by combining Lackenby's proof that for a hyperbolic knot any slope p/q with q sufficiently large is characterising with genus bounds derived from Heegaard Floer homology.

1. Introduction

Given a knot $K \subseteq S^3$, we say that $p/q \in \mathbb{Q}$ is a *characterising slope* for K if the oriented homeomorphism type of the manifold obtained by p/q-surgery on K determines K uniquely. That is p/q is characterising for K if $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some $K' \subseteq S^3$ implies that $K = K'$[1]. In general determining the set of characterising slopes for a given knot is challenging. It was a long-standing conjecture of Gordon, eventually proven by Kronheimer, Mrowka, Ozsváth and Szabó, that every slope is a characterising slope for the unknot [10]. Ozsváth and Szabó have also shown that every slope is a characterising slope for the trefoil and the figure-eight knot [16]. Since then, Ni and Zhang — who introduced the characterising slope terminology — studied characterising slopes for torus knots, showing that amongst slopes which are not negative integers all but finitely many are characterizing for $T_{5,2}$ and exhibiting infinitely many characterising slopes for each torus knot [15]. It was later shown that for a torus knot the set of non-integer non-characterizing slopes is finite [13]. More recently, Lackenby showed that every knot in S^3 has infinitely many characterising slopes [12].

[1]Throughout the paper, we use $Y' \cong Y$ to denote the existence of a orientation-preserving homeomorphism between Y and Y'.

Ni and Zhang asked whether a hyperbolic knot can have infinitely many non-characterising slopes. This question was resolved by Baker and Motegi who produced examples of knots (including hyperbolic knots) with infinitely many non-characterising slopes [2]. As their examples included only integer slopes as non-characterising slopes, one might wonder about the possibility of non-integer non-characterising slopes.

Question 1 (cf. Question 4.4 of [2]). *Does every hyperbolic knot have only finitely many non-integer non-characterising slopes?*

As evidence for a positive answer to this question Lackenby showed that for a hyperbolic knot p/q is characterising for K whenever q is sufficiently large [12, Theorem 1.2]. The purpose of this paper is to strengthen this result.

Theorem 2. *Let K be a hyperbolic knot in S^3. Then p/q is a characterizing slope for K provided that $|p| + |q|$ is sufficiently large and $q \geq 3$.*

If we add the condition that K is an L-space knot[2], then we can obtain stronger results, answering Question 1 affirmatively, as well as showing many integer slopes are characterising.

Theorem 3. *Let K be a hyperbolic L-space knot. Then p/q is a characterizing slope for K provided that $|p| + |q|$ is sufficiently large and p/q is not a negative integer.*

As far as the author is aware, this provides the only known examples of hyperbolic knots with integer characterising slopes other than the figure-eight knot. Since positive torus knots are known to have only finitely many non-characterising slopes amongst slopes which are not negative integers, this immediately yields the following corollary.

Corollary 4. *Let K be a non-satellite L-space knot. Then p/q is a characterizing slope for K provided that $|p| + |q|$ is sufficiently large and p/q is not a negative integer.*

In order to show that p/q is characterising for a hyperbolic knot K whenever q is sufficiently large, Lackenby shows that for any other hyperbolic knot K' with $S^3_{K'}(p/q) \cong S^3_K(p/q)$ the geometry of K' is sufficiently

[2]We say that K is an L-space knot if it admits *positive* L-space surgeries.

constrained to ensure that $K = K'$ for large q. Key to his argument is that the length of the slope of p/q (as measured on the boundary of a horoball neighbourhood of the cusp in the complement of K') is bounded below by an increasing function of $|q|$ which does not depend on K'. So in order to adapt Lackenby's approach to find characterising slopes for small q and large p, we need to bound the length of p/q below by an increasing function of $|p|$ which does not depend on K'. Such a lower bound is obtained by combining a result of Agol which allows us to bound the length of the longitude of a knot in terms of its genus [1, Theorem 5.1] with results derived from Heegaard Floer homology which constrain the genera of two knots with a common surgery. For Theorem 2 the Heegaard Floer input is hidden in the following theorem.

Theorem 5. *[13, Theorem 1.7] Let $K, K' \subseteq S^3$ be knots such that $S^3_K(p/q) \cong S^3_{K'}(p/q)$. If*

$$|p| \geq 12 + 4q^2 + 4qg(K) \quad \textit{and} \quad q \geq 3,$$

then $g(K) = g(K')$.

The stronger conclusions of Theorem 3 comes from a corresponding result for L-space knots.

Theorem 6. *[13, Theorem 1.8] Suppose that K is an L-space knot. If $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some $K' \subseteq S^3$ and either*

(i) $p \geq 12 + 4q^2 + 4qg(K)$ or

(ii) $p \leq -(12 + 4q^2 + 2qg(K))$ and $q \geq 2$

holds, then $g(K) = g(K')$ and K' is fibred.

If one wishes to prove that hyperbolic knots have only finitely many non-integer non-characterising slopes, then it suffices to prove an analogue of Theorem 5 that applies to half-integer surgeries. Theorem 5 is proven by calculating the Heegaard Floer homology of $S^3_K(p/q)$ and $S^3_{K'}(p/q)$ using the mapping cone formula and comparing the absolute gradings. Just as Theorem 5 can be extended to Theorem 6 for L-space knots, it is probable that this approach can yield results in the half-integer case for other knots with simple knot Floer homology. However, it seems unlikely to the author that an unconditional statement for half-integer surgeries can be achieved by this approach alone.

2. The proof

Given a 3-manifold M with a toroidal boundary component and a slope σ on this boundary component, we will use $M(\sigma)$ to denote the Dehn filling along σ. If K is a knot in S^3 we will use S^3_K to denote the knot exterior $S^3 \setminus \mathrm{int}(N(K))$. So p/q-surgery on K will be denoted by $S^3_K(p/q)$. A 3-manifold M is hyperbolic if its interior admits a complete finite-volume hyperbolic structure. A knot $K \subseteq S^3$ is hyperbolic when S^3_K is hyperbolic. Recall that Mostow rigidity guarantees that if two hyperbolic 3-manifolds are homeomorphic, then they are isometric. So we may assume that geometric features of a hyperbolic 3-manifold, such as the volume or the shortest closed geodesic, are preserved by homeomorpisms.

Given a slope σ on the boundary of a compact hyperbolic 3-manifold one can assign a length to σ by choosing a horoball neighbourhood N of the cusps of M. There is a natural Euclidean metric on ∂N and we say the length of σ is the length of the shortest curve on ∂N with the slope of σ. In general, the length of σ depends on the choice of horoball neighbourhood. However, if M has only a single cusp, then there is a unique choice of maximal horoball. Given a slope p/q for S^3_K, we will use $\ell_K(p/q)$ to denote the length of p/q with respect this maximal horoball neighbourhood.

2.1. Slope lengths

The first step is to verify the following proposition, which is a mild reformulation of [12, Theorem 3.1].

Lemma 7. *Let $K \subseteq S^3$ be a hyperbolic knot. There are constants C_1 and C_2 such that if K' is a hyperbolic knot with $S^3_K(p/q) \cong S^3_{K'}(p/q')$ for $\ell_{K'}(p/q') > C_1$ and $|p| + |q| > C_2$, then $K = K'$ and $q = q'$.*

Although a proof is provided for completeness, the reader should note that our proof is essentially the same as Lackenby's except with minor changes to emphasise the role of slope length. Three theorems from hyperbolic geometry are needed in the proof. They are taken largely unchanged from Section 2 of [12], where further discussion can be found. First, a precise version of Thurston's hyperbolic Dehn surgery theorem is required.

Theorem 8. *[12, Theorem 2.1]. Let M be a compact orientable hyperbolic 3-manifold with toroidal boundary components $T_1, \dots, T_n$. Let $(\sigma^i_1, \dots, \sigma^i_n)$ be a sequence of slopes, where σ^i_j lies on T_j and $\sigma^i_j \neq \sigma^{i'}_j$ if $i \neq i'$. Then* *for all sufficiently large i, $M(\sigma_1^i, \ldots, \sigma_n^i)$ is hyperbolic and the cores of the filling solid tori are geodesics with lengths tending to zero as $i \to \infty$. Moreover there is $\varepsilon > 0$ independent of i such that all other primitive geodesics in $M(\sigma_1^i, \ldots, \sigma_n^i)$ have length at least ε. For any horoball neighbourhood N of the cusps of M, there is a horoball neighbourhood N_i of the cusps of $M(\sigma_1^i, \ldots, \sigma_n^i)$ such that the inclusion*

$$M \setminus N \to M(\sigma_1^i, \ldots, \sigma_n^i) \setminus N_i$$

is bilipschitz with constant tending to one as $i \to \infty$.

Secondly, we need to know that for an infinite collection of hyperbolic 3-manifolds of bounded volume, some subsequence of them can be obtained by Dehn filling on another hyperbolic manifold with more cusps. See [19] or [3, Theorem E.4.8].

Theorem 9. *[12, Theorem 2.2]. Let M_i be a sequence of distinct oriented hyperbolic 3-manifolds with volume bounded above by V. Then there is a hyperbolic 3-manifold M with volume at most V and toroidal boundary components $T_1, \ldots, T_n$ such that there is a subsequence of the M_i and a sequence of slopes $(\sigma_1^i, \ldots, \sigma_n^i)$ such that*

$$M_i = M(\sigma_1^i, \ldots, \sigma_n^i)$$

and $\sigma_j^i \neq \sigma_j^{i'}$ for $i \neq i'$ in the subsequence.

Finally we need explicit bounds on the volume of hyperbolic 3-manifolds in terms of the length of filling curves. The upper bound is due to Thurston [19] and the lower bound is due to Futer, Kalfagianni and Purcell [5, Theorem 1.1].

Theorem 10. *[12, Theorem 2.4] Let S_K^3 be the complement of a hyperbolic knot in S^3. If the slope p/q has length $\ell = \ell_K(p/q) > 2\pi$, then $S_K^3(p/q)$ is hyperbolic with volume satisfying*

$$\left(1 - \left(\frac{2\pi}{\ell}\right)^2\right)^{3/2} \operatorname{vol}(S_K^3) \leq \operatorname{vol}(S_K^3(p/q)) < \operatorname{vol}(S_K^3).$$

We are ready to proceed with the proof of Lemma 7.

Proof of Lemma 7. If the constants C_1 and C_2 do not exist, then there is a sequence of hyperbolic knots K_i with slopes p_i/q_i and p_i/q_i' such that

(a) $S^3_K(p_i/q_i) \cong S^3_{K_i}(p_i/q_i')$ for all i,

(b) $\ell_{K_i}(p_i'/q_i') \to \infty$ and $|p_i| + |q_i| \to \infty$ as $i \to \infty$, but

(c) for all i we have $K_i \neq K$ or $q_i' \neq q_i$.

We will show that such a sequence results in a contradiction.

First assume the sequence $S^3_{K_i}$ includes infinitely many distinct manifolds. By passing to a subsequence, we may assume that the $S^3_{K_i}$ are all distinct. Since $\ell_{K_i}(p_i/q_i')$ will exceed 4π for i large enough, Theorem 10 shows that $\left(\frac{3}{4}\right)^{3/2} \operatorname{vol}(S^3_{K_i}) \leq \operatorname{vol}(S^3_{K_i}(p_i/q_i'))$ for i sufficiently large. However $\operatorname{vol}(S^3_{K_i}(p_i/q_i')) = \operatorname{vol}(S^3_K(p_i/q_i))$ is bounded above by $\operatorname{vol}(S^3_K)$. This gives the upper bound $\operatorname{vol}(S^3_{K_i}) \leq (\frac{3}{4})^{3/2} \operatorname{vol}(S^3_K)$ for all sufficiently large i. Thus we see that there is some V such that $\operatorname{vol}(S^3_{K_i}) < V$ for all i.

By Theorem 9 we can pass to a further subsequence and assume that there is a hyperbolic M of finite volume with toroidal boundary components $T_1, \dots, T_n, T_{n+1}$ and a sequence of slopes $(\sigma_1^i, \dots, \sigma_n^i)$ on $T_1, \dots, T_n$ such that $\sigma_j^i \neq \sigma_j^{i'}$ for $i \neq i'$ and $S^3_{K_i} \cong M(\sigma_1^i, \dots, \sigma_n^i, \star)$, where $\star$ denotes that we are leaving T_{n+1} unfilled. As the knot complements $S^3_{K_i}$ are distinct manifolds, we have $n \geq 1$. We may consider the slope p_i/q_i' as slope a σ^i on T_{n+1}. Thus, we get a sequence of slopes σ^i such that $S^3_{K_i}(p_i/q_i') \cong M(\sigma_1^i, \dots, \sigma_n^i, \sigma^i)$ for all i.

Let N be a horoball neighbourhood of the cusps of M. By Theorem 8, in each $S^3_{K_i}$ there is a horoball neighbourhood N_i of the cusp such that the inclusion $M \setminus N \to S^3_{K_i} \setminus N_i$ is bilipschitz with constant approaching one. Thus since $\ell_{K_i}(p_i/q_i') \to \infty$ the length of σ_i as measured in ∂N must also tend to infinity. In particular, by taking a further subsequence we can assume the slopes σ^i are distinct.

Thus Theorem 8 shows that the cores of the filling solid tori in $M(\sigma_1^i, \dots, \sigma_n^i, \sigma^i)$ are geodesics of length tending to zero. Thus for any $\varepsilon > 0$, $M(\sigma_1^i, \dots, \sigma_n^i, \sigma^i)$ contains at least $n + 1 \geq 2$ closed geodesics of length less than ε when i sufficiently large. However, Theorem 8 also shows that there is $\delta > 0$, such that for i sufficiently large the core of $S^3_K(p_i/q_i)$ is the only geodesic of length less than δ. This is clearly a contradiction.

Thus we can assume that $S^3_{K_i}$ include only finitely many distinct manifolds. By passing to a subsequence if necessary, we can further assume that there is some $K' \subseteq S^3$ such that $S^3_{K_i} \cong S^3_{K'}$ for all i.

We may assume that the homemorphisms $S^3_K(p_i/q_i) \to S^3_{K'}(p_i/q_i')$ map the shortest closed geodesic in $S^3_K(p_i/q_i)$ to shortest geodesic in $S^3_{K'}(p_i/q_i')$. However for i sufficiently large Theorem 8 shows this shortest geodesic is the core of the filling solid tori in both manifolds. Thus the homeomorphism restricts to give a homeomorphism of knot complements $S^3_{K'} \cong S^3_K$. By the knot complement theorem this shows that $K = K'$ and that the meridian of K is mapped to the meridian of K' [8]. Since the homeomorphism must also map null-homologous curves to null-homologous curves it must also preserve longitudes, showing that we also have $p_i/q_i' = p_i/q_i$. This contradicts the initial assumptions on the p_i/q_i' and K_i. □

The following lemma provides the slope length bounds required to apply Lemma 7.

Lemma 11. *Let $K \subseteq S^3$ be a hyperbolic knot of genus g. Then*

$$\ell_K(p/q) \geq \frac{\sqrt{3}\,|q|}{6} \quad \textit{and} \quad \ell_K(p/q) \geq \frac{\sqrt{3}\,|p|}{6(2g-1)}.$$

Proof. By using bounds on the area of a cusp, Cooper and Lackenby show that [4, Lemma 2.1]

$$\ell_K(\alpha) \geq \frac{\sqrt{3}\,\Delta(\alpha,\beta)}{\ell_K(\beta)},$$

where α and β are any two slopes on the boundary of K and $\Delta(\alpha,\beta)$ denotes their distance (cf. [1, Lemma 8.1]). Since $\Delta(1/0, p/q) = |q|$ and $\ell_K(1/0) \leq 6$ by the 6-theorem [1, 11], this gives the bound on $\ell_K(p/q)$ in terms of $|q|$. Since $\Delta(0/1, p/q) = |p|$ and $\ell_K(0/1) \leq 6(2g-1)$ by [1, Theorem 5.1], this also gives the bound on $\ell_K(p/q)$ in terms of $|p|$. □

2.2. Hyperbolic surgeries on satellite knots

We also need to understand when non-hyperbolic knots can have hyperbolic surgeries.

Lemma 12. *Suppose that K' is a satellite knot with $S^3_{K'}(p/q)$ hyperbolic. Then there is a hyperbolic knot K'' with $S^3_{K'}(p/q) \cong S^3_{K''}(p/q')$ for some $q' > q$. Moreover if $q \geq 2$ or K' is fibred, then $g(K'') \leq g(K')$.*

Proof. Let T be an incompressible torus in $S^3 \setminus K'$. We can consider K' as a knot in the solid torus V bounded by T. Thus we can consider K' as a satellite with companion given by the core K'' of V. By choosing T to ensure that $S^3 \setminus K''$ contains no further incompressible tori, we can assume that K'' is not a satellite knot. Hence by the work of Thurston K'' is a torus knot or a hyperbolic knot [18]. Since $S^3_{K'}(p/q)$ is hyperbolic, it is atoroidal and irreducible. Consequently the Dehn filling when considered as a surgery on

V must produce a $S^1 \times D^2$. However, Gabai has classified knots in $S^1 \times D^2$ with non-trivial $S^1 \times D^2$ surgeries, showing that K' is either a torus knot or a 1-bridge braid in V [6]. Moreover since $S^1 \times D^2$ fillings on 1-bridge braids only occur for integer surgery slopes, K' is a cable of K'' unless $q = 1$. In either event, we have that

$$S^3_{K'}(p/q) \cong S^3_{K''}(p/q'),$$

for some q'. However, it is known that $q' = qw^2$, where $w > 1$ is the winding number of K' in V [7, Lemma 3.3]. Surgery on a torus knot results in either a Seifert fibered space or a reducible manifold [14]. Since neither a Seifert fibered space nor a reducible manifold can be a hyperbolic manifold, it follows that K'' cannot be a torus knot. Therefore K'' is hyperbolic as required.

If K' is fibred, then K'' must also be fibred [9], thus the inequality $g(K'') \leq g(K')$, follows by considering the degrees of their Alexander polynomials. If $q \geq 2$, then K' is a cable of K''. It is known that $g(K') \geq g(K'')$ in this case [17]. $\square$

2.3. Proof of Theorem 2 and Theorem 3

Let K be a hyperbolic knot. Suppose that $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some K' and some p/q and that one of the following two conditions hold:

1) $q \geq 3$ or
2) either K is an L-space knot and $q \geq 2$ or $q = 1$ and $p > 0$.

Assume also that $|p| + |q|$ is large enough to guarantee $S^3_{p/q}(K)$ is hyperbolic. Since torus knots have no hyperbolic surgeries, Thurston's work shows that K' is either a hyperbolic knot or a satellite knot [18]. If K' is a satellite, then Lemma 12 shows there is a hyperbolic knot K'' with $S^3_{K''}(p/q') \cong S^3_K(p/q)$ for some $q' > q$ and $g(K'') \leq g(K')$. In either event, there is a hyperbolic knot L with $g(L) \leq g(K')$ such that $S^3_L(p/q') \cong S^3_K(p/q)$ for $q' \geq q$. Thus to show that p/q is a characterising slope for K it suffices to show that the only possibility is $q = q'$ and $L = K$.

Lemma 11 shows that $\ell_L(p/q') \geq \frac{\sqrt{3}|q|}{6}$. So Lemma 7 shows that there is C such that $q > C$ implies that $K = L$ and $q' = q$. This implies that p/q is a characterising slope if $q > C$. Thus we assume from now on that $q \leq C$. However by Theorem 5 and Theorem 6, if we assume further that $|p| \geq 12 + 4C^2 + 4Cg(K)$, then $g(K) = g(K') \geq g(L)$. Therefore for $|p|$ large enough Lemma 11 shows that $\ell_L(p/q') \geq \frac{\sqrt{3}\,|p|}{6(2g(K)-1)}$. Hence by taking $|p|$ even larger if necessary, Lemma 7 applies again to show that $q = q'$ and $K = L$, as required. This concludes the proof and the paper.

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