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A variant of the Mordell–Lang conjecture

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The Mordell–Lang conjecture (proven by Faltings, Vojta and McQuillan) states that the intersection of a subvariety V of a semiabelian variety G defined over an algebraically closed field $\Bbbk$ of characteristic 0 with a finite rank subgroup $\Gamma \leq G(\Bbbk)$ is a finite union of cosets of subgroups of Γ. We explore a variant of this conjecture when $G = \mathbb{G}_a \times A$ for an abelian variety A defined over $\Bbbk$.

1. Introduction

Throughout our paper, each subvariety is assumed to be closed. Unless otherwise noted, $\Bbbk$ will always denote an algebraically closed field of characteristic 0.

Faltings [Fal94] proved the Mordell–Lang conjecture, thus showing that any subvariety of an abelian variety A defined over $\Bbbk$ intersects a finitely generated subgroup $\Gamma \leq A(\Bbbk)$ in a finite union of cosets of subgroups of Γ. Vojta [Voj96] proved that Faltings' result holds when we replace the abelian variety A by extensions G of A by an algebraic torus, i.e., when G is a semiabelian variety. Then McQuillan [McQ95] extended further Vojta's theorem by proving that the conclusion holds when we replace Γ by any finite rank subgroup of a semiabelian variety. It is natural to ask whether a variant of the Mordell–Lang conjecture holds when we replace G by a more general commutative algebraic group, which is the extension of an abelian variety by some commutative linear algebraic group H. However, it is easy to see that if $\dim H \geq 2$ and H is not a torus (i.e., G is not a semiabelian variety), then there are examples when an irreducible subvariety $V \subseteq G$ meets a finitely generated subgroup of $G(\Bbbk)$ in a Zariski dense subset and moreover V is not a coset of an algebraic subgroup of G (as predicted by the variant of the Mordell–Lang conjecture).

Example 1.1. If $G = \mathbb{G}_a^2$, then the graph of any polynomial of degree larger than 1 with integer coefficients would contain infinitely many integral points, thus contradicting the corresponding Mordell–Lang principle for G.

Example 1.2. If $G = \mathbb{G}_a \times \mathbb{G}_m$, then the diagonal subvariety $\Delta \subseteq G$ contains infinitely many points of the subgroup Γ spanned by $(1,1)$ and $(0,2)$; more precisely, $(2^n, 2^n) \in \Delta$ for each $n \in \mathbb{N}$. However, Δ is not a coset of a subgroup of G.

Vojta [Voj96, page 134] noted the example, similar to our Example 1.1, of the subvariety of $\mathbb{G}_a^2$ defined by Pell's equation as evidence for his "doubt that this [Mordell–Lang] result can be generalized to a larger class of group varieties". We see, instead, that these examples suggest that the only commutative algebraic group G for which a variant of the Mordell–Lang conjecture might hold is an extension of an abelian variety by a single copy of $\mathbb{G}_a$. Assuming the Bombieri–Lang conjecture[1], we prove in Section 2 that a variant of the Mordell–Lang conjecture holds for an algebraic group isomorphic to a product of an abelian variety with the one-dimensional additive group. First, we state the aforementioned conjecture of Bombieri–Lang; for more details on this famous conjecture, see [BG06, Chapter 14]. We also note that the Bombieri–Lang conjecture is a special case of Vojta's conjectures which play a central role in arithmetic geometry (see [BG06, Conjecture 14.3.2 and its Remark 14.3.7]).

Conjecture 1.3 (Bombieri–Lang–Vojta). *Let X be a projective variety of general type defined over $\overline{\mathbb{Q}}$. Then for each number field $\mathbb{K}$, the set $X(\mathbb{K})$ is not Zariski dense in X.*

A smooth projective variety X is of *general type* if its canonical divisor K_X is big, i.e., $\kappa(X) := \kappa(X, K_X) = \dim X$; in general, a (possibly singular) variety is of *general type* if it admits a smooth model of general type. If $\dim X = 1$, then Conjecture 1.3 is equivalent to the well-known Mordell conjecture, proven by Faltings [Fal83]. We prove the following result.

Theorem 1.4. *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$ and let Γ be a finitely generated subgroup of $(\mathbb{G}_a \times A)(\overline{\mathbb{Q}})$. If Conjecture 1.3 holds, then for each subvariety $V \subseteq \mathbb{G}_a \times A$, the intersection $V(\overline{\mathbb{Q}}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.*

The problem of proving unconditionally the results of this paper in the most general case seems deeply linked with the conjecture of Vojta for a generically finite cover of an abelian variety. In turn, this seems to be out of reach of the present methods, unless one inserts additional assumptions.

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[1] Bombieri formulated this conjecture only in the case of surfaces.

Unconditionally, we can prove Theorem 1.4 when $\dim A = 1$ (see Theorem 1.5), and also when V is birational to a subvariety of an abelian variety (see Remark 2.2). In the case when V is birational to a subvariety of some abelian variety, the result follows as a consequence of Faltings' result [Fal91, Theorem 2] regarding the finiteness of the number of S-integral points on an abelian variety with respect to an ample divisor (see also Remark 2.2 and our proof of Theorem 1.4). In the case A is an elliptic curve, we can prove a more general result (inspired by the results of [CMZ13]), valid for any commutative algebraic group of dimension 2 that is an extension of an elliptic curve by a copy of the additive group.

Theorem 1.5. *Let $\mathbb{K}$ be a finitely generated field of characteristic 0, let $E/\mathbb{K}$ be an elliptic curve and let G be a commutative algebraic group which is an extension of E by $\mathbb{G}_a$, i.e., there is a short exact sequence of connected algebraic groups:*

$$0 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow E \longrightarrow 0.$$

Let Γ be a subgroup of $G(\mathbb{K})$ such that $\Gamma \cap \mathbb{G}_a$ is finitely generated, and let T be a subset of Γ. Then the Zariski closure of T is a finite union of translates of algebraic subgroups of G.

Assuming that A is an abelian variety of $\Bbbk/\overline{\mathbb{Q}}$-trace 0 (i.e., there is no nonconstant morphism between A and some abelian variety defined over $\overline{\mathbb{Q}}$), then we can prove unconditionally the conclusion from Theorem 1.4 even in the more general case when we intersect a subvariety $V \subseteq \mathbb{G}_a \times A$ with a finite rank subgroup.

Theorem 1.6. *Let $\Bbbk$ be an algebraically closed field of characteristic 0, let A be an abelian variety such that $\mathrm{Tr}_{\Bbbk/\overline{\mathbb{Q}}}(A) = 0$. Then each subvariety $V \subseteq \mathbb{G}_a \times A$ intersects a finite rank subgroup $\Gamma \leq (\mathbb{G}_a \times A)(\Bbbk)$ in a finite union of cosets of subgroups of Γ.*

We prove Theorem 1.6 in Section 3 using the ideas introduced by Hrushovski [Hru96] for his proof of the function field version of the Mordell–Lang conjecture. We note that Theorem 1.6 fails if the abelian variety were defined over a number field. Indeed, if A is an abelian variety whose $\mathbb{Q}$-rational points are Zariski dense, while $V \subseteq \mathbb{G}_a \times A$ is the graph of any non-constant rational function $f\colon A \longrightarrow \mathbb{P}^1$ defined over $\mathbb{Q}$, and $\Gamma := (\mathbb{G}_a \times A)(\mathbb{Q})$, then $V(\mathbb{Q}) \cap \Gamma$ is Zariski dense in V, even though V is not a coset of an algebraic subgroup of $\mathbb{G}_a \times A$.

2. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. We only need to prove the case when V is irreducible. Hence we may assume that V is integral. Next, it suffices to prove that if $V \cap \Gamma$ is Zariski dense in V then V is a translate of an algebraic subgroup of $\mathbb{G}_a \times A$. Let $\pi_2 \colon \mathbb{G}_a \times A \longrightarrow A$ be the usual projection morphism. Then by Faltings' theorem [Fal94], we may replace A by the Zariski closure of $\pi_2(V \cap \Gamma)$ which is a translate of an abelian subvariety of A, and assume that $\pi_2|_V \colon V \longrightarrow A$ is dominant. If $\pi_2|_V$ is of relative dimension 1, then $V = \mathbb{G}_a \times A$ and we are done. So we only need to consider the case that $\pi_2|_V$ is generically finite. Let $\pi_1 \colon \mathbb{G}_a \times A \longrightarrow \mathbb{G}_a$ be the usual projection morphism. If $\pi_1|_V \colon V \longrightarrow \mathbb{G}_a$ is not dominant, i.e., $\pi_1(V)$ is a point, then V is contained in a fibre $F \cong A$ of π_1. It follows that $V = F$ by the dimension reasoning and hence V is a translate of the abelian variety A. Hence, from now on, we may assume further that $\pi_1|_V \colon V \longrightarrow \mathbb{G}_a$ is dominant. We will prove that in this case, our hypotheses yield a contradiction.

Let $\mathbb{P}^1 \times A$ be the compactification of $\mathbb{G}_a \times A$ (such that $\mathbb{P}^1 - \mathbb{G}_a = \{\infty\}$) and $\overline{V}$ the Zariski closure of V in $\mathbb{P}^1 \times A$. Also, by a slight abuse of notation, we still denote by π_2 the projection morphism $\mathbb{P}^1 \times A \longrightarrow A$. Then by the Stein factorization, there exists a normal projective variety Y endowed with a birational morphism $\iota \colon \overline{V} \longrightarrow Y$ and with a finite morphism $f \colon Y \longrightarrow A$ such that $\pi_2|_{\overline{V}} = f \circ \iota$.

Let $\mathbb{K}$ be a number field such that V, $\overline{V}$, Y, ι and f are all defined over $\mathbb{K}$, and also $\Gamma \leq (\mathbb{G}_a \times A)(\mathbb{K})$. Since $V(\mathbb{K}) \cap \Gamma$ is Zariski dense in V, we conclude that $Y(\mathbb{K})$ is Zariski dense in Y.

Applying Kawamata's structure theorem [Kaw81, Theorem 13] to the finite morphism $f \colon Y \longrightarrow A$, there exists a finite étale cover $\phi \colon \widetilde{Y} \longrightarrow Y$ such that $\widetilde{Y}$ is isomorphic to the direct product $\widetilde{B} \times W$, where $\widetilde{B}$ is a finite étale cover of an abelian subvariety B of A and W is a normal projective variety of general type, i.e., $\kappa(W) = \dim W$. Since $Y(\mathbb{K})$ is Zariski dense in Y and $\phi \colon \widetilde{Y} \longrightarrow Y$ is étale, the Chevalley–Weil theorem (see [CZ17, p. 585] and [BG06, Theorem 10.3.11]) yields that there exists a finite extension $\mathbb{L}/\mathbb{K}$ such that $\widetilde{Y}(\mathbb{L})$ is Zariski dense in $\widetilde{Y}$. In particular, at the expense of replacing $\mathbb{L}$ by another finite extension, we obtain that $W(\mathbb{L})$ is Zariski dense in W, which contradicts Conjecture 1.3, if $\dim W > 0$.

So, from now on, we may assume that W is a point, which yields that $\widetilde{Y}$ and therefore Y itself is an abelian variety. Since Y is birational to V, composing this birational map with $\pi_1|_V$, we obtain a non-constant rational function $g \colon Y \longrightarrow \mathbb{P}^1$ (note that $\pi_1|_V \colon V \longrightarrow \mathbb{G}_a$ is dominant). Since Γ is a finitely generated subgroup of $(\mathbb{G}_a \times A)(\mathbb{K})$, we have that $\pi_1(\Gamma)$ is a set of S-integral points in $\mathbb{K}$ with respect to a suitable finite set S of places of $\mathbb{K}$.

We let $D := g^*(\{\infty\})$ be the divisor of Y which is the pullback of the point at infinity for the inclusion $\mathbb{G}_a \subseteq \mathbb{P}^1$; note that D is a divisor because g is non-constant. We let $\tilde{\iota}\colon V \longrightarrow Y$ be the corresponding birational map and then for each point $x \in \tilde{\iota}(\Gamma \cap V)$ (note that $\Gamma \cap V$ is Zariski dense in V), we have that $g(x)$ is S-integral with respect to the divisor D of Y. If D is ample, then Faltings' theorem [Fal91, Theorem 2] yields a contradiction to the fact that there exist infinitely many such S-integral points (for a variant of Faltings' theorem in the context of semiabelian varieties, see [Voj99]). We show next that the general case reduces to this special case.

Assume now that D is not ample. Let C be the connected component of the stabilizer of D in Y. Then C is an abelian subvariety of Y of positive dimension. Let Z be a complement of C in Y, i.e., Z is a proper abelian subvariety of Y such that Y is isogenous to $C \times Z$. Therefore, without loss of generality, we may replace Y by $C \times Z$. We let $h := g|_Z$, which is still a non-constant rational function $Z \longrightarrow \mathbb{P}^1$ and moreover, $h^*(\{\infty\})$ is an ample divisor of Z. Another application of Faltings' [Fal91, Theorem 2] provides a contradiction, which finishes our proof of Theorem 1.4. □

Remark 2.1. Using the notation as in Theorem 1.4, if A is a *simple* abelian variety, then one does not need to use [Kaw81, Theorem 13] to finish the proof. Indeed, [Kaw81] was employed only in the case the finite morphism $f\colon Y \longrightarrow A$ is ramified, because in the case f is unramified, we immediately derive that Y must itself be an abelian variety and proceed as in the proof of Theorem 1.4 invoking only Faltings' theorem [Fal91] regarding the finiteness of the number of S-integral points on an abelian variety with respect to an ample divisor. Now, if f is ramified, the canonical divisor K_Y of Y is the ramification divisor of f, and moreover since f is a finite map, we obtain that $K_Y = f^*(D_f)$ for some effective divisor D_f of A. If A is simple, then each nontrivial effective divisor of it is ample and therefore we obtain that K_Y is big, which still allows us to apply the Bombieri–Lang–Vojta conjecture to obtain a contradiction.

Remark 2.2. As shown in our proof of Theorem 1.4, the only point in which we employed the validity of Conjecture 1.3 is for the case when the finite morphism $Y \longrightarrow A$ is ramified. In particular, this means that Theorem 1.4 holds unconditionally if the subvariety $V \subseteq \mathbb{G}_a \times A$ is birational to an abelian variety. Furthermore, if $V \subseteq \mathbb{G}_a \times A$ is birational to a subvariety Y of some arbitrary abelian variety B, then the assumption that V contains a Zariski dense set of $\mathbb{K}$-rational points yields that Y contains a Zariski dense set of rational points and therefore, Faltings' theorem [Fal94] yields that Y must be a coset of an abelian subvariety of B. So, V is birational to an abelian variety itself and we are done using Faltings' theorem [Fal91] regarding the finiteness of the S-integral points on an abelian variety.

For the general case of a non-split extension G of an arbitrary abelian variety A (defined over a field of characteristic 0) by a copy of the additive group, the corresponding variant of the Mordell–Lang conjecture is quite subtle. However, we can settle unconditionally the case when A is an elliptic curve.

Proof of Theorem 1.5. First, we observe that since Γ projects to $E(\mathbb{K})$, which is a finitely generated group (due to the classical Mordell–Weil theorem), we get that Γ must itself be finitely generated (since so is its intersection with $\mathbb{G}_a(\mathbb{K})$ by the assumption). Hence, our goal is to show that if $V \subseteq G$ is an irreducible curve with the property that $V \cap \Gamma$ is Zariski dense in V, then V must be a coset of a one-dimensional algebraic subgroup of G. We have two cases: either G is a split extension, or not.

Case 1. G is a split extension. So, at the expense of replacing G by an isogenous copy of it and also replace $\mathbb{k}$ by a finite extension, we may assume $G = \mathbb{G}_a \times E$.

If V does not project dominantly onto one of the two factors of $\mathbb{G}_a \times E$, then we obtain the desired conclusion. So, assume the curve $V \subseteq \mathbb{G}_a \times E$ projects dominantly onto both factors of $\mathbb{G}_a \times E$; using the hypothesis that V contains infinitely many points of the subgroup $\Gamma \leq (\mathbb{G}_a \times E)(\mathbb{K})$, then we derive a contradiction. We observe that V must have positive genus since it projects dominantly onto the elliptic curve E; then we derive (similar to the proof of Theorem 1.4) a contradiction due to the finiteness of the number of S-integral points on a curve of positive genus.

Case 2. G is a non-split extension.

We settle this case using the fact that G does not contain complete curves (see [CMZ13] for this and other facts on such group extensions). So, if the curve $V \subseteq G$ contained infinitely many points from a finitely generated subgroup Γ of G, then these points would be S-integral with respect to the complement of V in the projective closure of it, where S is a finite set of places containing those of bad reduction either for G or for a set of generators for Γ. (See also [CMZ13] for an explicit projective embedding of G, obtained first by Serre, in a letter to Masser reproduced in the aforementioned paper.) But since V is affine, Siegel's Theorem would entail that V has genus 0; hence V could not dominate E and would be a translate of $\mathbb{G}_a$, as required. □

3. Proof of Theorem 1.6

We work with the differential algebraic methods of [Bui92, Hru96] and recommend the book [MTF06] for general background.

We begin by endowing $\Bbbk$ with a derivation $\partial\colon \Bbbk \to \Bbbk$ for which the field of constants $\Bbbk^{\partial} := \{x \in \Bbbk \ : \ \partial(x) = 0\}$ is the field $\overline{\mathbb{Q}}$ of algebraic numbers. Since the statement of Theorem 1.6 becomes only formally more difficult with $\Bbbk$ replaced by a larger field, we may, and do, replace $\Bbbk$ with its differential closure, which is still an algebraically closed differential field having field of constants equal to $\overline{\mathbb{Q}}$. For the sake of readability, we shall identify algebraic (and differential algebraic) varieties with their sets of $\Bbbk$-points.

By the usual reductions, we may assume that V is irreducible, contains the identity element of the group, and has a trivial stabilizer. We are charged with showing that if $V \cap \Gamma$ is Zariski dense in V (where $\Gamma \leq (\mathbb{G}_a \times A)(\Bbbk)$ is a group of finite rank), then V must consist of a single point.

We find a differential algebraic subgroup $\widetilde{\Gamma} \leq \mathbb{G}_a \times A$ for which $\Gamma \leq \widetilde{\Gamma}$ and $\widetilde{\Gamma}$ has finite Morley rank (see [Mar00, (5.1)]). Let Ξ be the image of $\widetilde{\Gamma}$ under the projection map $\mathbb{G}_a \times A \to A$ and let B be the connected component of the identity of the Zariski closure of Ξ. Since it is an algebraic subgroup of A, then B also has $\Bbbk/\overline{\mathbb{Q}}$-trace zero. Hence, by [HS, Proposition 2.6] (also proven in [PZ03]) the Manin kernel $B^\sharp$ of B is locally modular and is thus orthogonal to the field of constants. By [Pil96, Lemma 4.2], every Zariski dense differential algebraic subgroup of B contains $B^\sharp$; thus, $B^\sharp \leq \Xi$. Consider also the image Υ of $\widetilde{\Gamma}$ under the projection $\mathbb{G}_a \times A \to \mathbb{G}_a$. Since Υ is a finite Morley rank subgroup of the additive group, it is a finite dimensional vector space over the field of constants and is, therefore, fully orthogonal to $B^\sharp$. The group $\widetilde{\Gamma} \cap (\Upsilon \times B^\sharp)$ is a differential algebraic subgroup of $\Upsilon \times B^\sharp$ which projects onto $B^\sharp$. By the orthogonality of $B^\sharp$ and Υ, every differential algebraic subvariety is a union of products $T \times S$ where $S \subseteq B^\sharp$ and $T \subseteq \Upsilon$. It follows that $\{0\} \times B^\sharp \leq \widetilde{\Gamma}$.

Let $Y := \widetilde{\Gamma} \cap V$. Since $\Gamma \cap V$ is Zariski dense in V, we have that the differential algebraic variety Y is Zariski dense in V. Since V is irreducible as an algebraic variety, there exists some component X of Y which is Zariski dense in V. Translating, we may assume that X contains the identity element.

By [Hru96, Proposition 4.4] there is a differential algebraic groups $H \leq \widetilde{\Gamma}$ for which X is a union of cosets of the connected component of $(H \cap (\{0\} \times B^\sharp))$ and X is contained in $H + (\{0\} \times B^\sharp)$. The group $(H \cap (\{0\} \times B^\sharp))$ is contained in the stabilizer of X and the Zariski closure of this group is contained in the stabilizer of V. As we have reduced to the case that V has a trivial stabilizer, $H \cap (\{0\} \times B^\sharp)$ is itself trivial.

Using again that $B^\sharp$ is orthogonal to the field of constants, it follows that $H \leq \mathbb{G}_a \times \{0\}$. Indeed, as before we let Φ be the projection of H to A and Ψ be the projection of H to $\mathbb{G}_a$. As $\Psi \perp \Phi$ and $H \leq \Psi \times \Phi$, it must be that $H = \Psi \times \Phi$. By [Pil96, Lemma 4.2] again, Φ contains the Manin kernel $C^\sharp$ of the connected component C of its Zariski closure, and $C^\sharp \leq B^\sharp$. We know the group $(\{0\} \times B^\sharp) \cap H$ contains $\{0\} \times C^\sharp$ and is trivial. Hence $C^\sharp = \{0\}$ and therefore $C = \{0\}$ (note that $C^\sharp$ is Zariski dense in C); so, $H \leq \mathbb{G}_a \times \{0\}$.

Therefore, we know that $X \subseteq H + (\{0\} \times B^\sharp)$ and that the two groups in the sum are orthogonal. Hence, X may be expressed as $S + T$ with $S \subseteq H$ and $T \subseteq (\{0\} \times B^\sharp)$. Since $B^\sharp$ is locally modular, by [HP85] the set T is a translate of a subgroup. Using that X has a trivial stabilizer, it follows that T is a single point. Taking Zariski closures, we see that V is a translate of a subvariety of $\mathbb{G}_a \times \{0\}$. Because V has trivial stabilizer, this subvariety cannot be all of $\mathbb{G}_a \times \{0\}$. Thus, V is a single point as we needed to show. This concludes our proof of Theorem 1.6.

Remark 3.1. We expect that these differential algebraic techniques could be pushed to prove a relative Mordell–Lang theorem for general commutative algebraic groups. The statement we expect to be true is the following. Let G be a commutative algebraic group over the algebraically closed field $\Bbbk$ of characteristic zero. Let $\Gamma \leq G(\Bbbk)$ be a finite rank subgroup and let $V \subseteq G$ be an irreducible subvariety for which $\Gamma \cap V(\Bbbk)$ is Zariski dense in V. Then there should be an algebraic subgroup $H \leq G$ of G, an algebraic group J defined over $\overline{\mathbb{Q}}$, an algebraic variety $Y \subseteq J$ also defined over $\overline{\mathbb{Q}}$, a point $a \in G(\Bbbk)$, and a map of algebraic groups $h\colon H \to J_{\Bbbk}$ (where $J_{\Bbbk}$ is the base change of J from $\overline{\mathbb{Q}}$ to $\Bbbk$) so that $V = a + h^{-1}(Y_{\Bbbk})$. When G is an extension of an abelian variety of $\overline{\mathbb{Q}}$-trace zero by $\mathbb{G}_a$, then this conjecture would imply that V must be a translate of an algebraic subgroup of G. What is missing from the existing literature on the structure of differential algebraic groups is an appropriate analogue of the Socle Theorem ([Hru96, Propositions 4.3 and 4.4]) for groups with non-rigid semipluriminimal socles.

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