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# Lower bounds on the growth of Sobolev norms in some linear time dependent Schrödinger equations

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In this paper we consider linear, time dependent Schrödinger equations of the form $\mathrm{i}\partial_t \psi = K_0\psi + V(t)\psi$, where $K_0$ is a positive selfadjoint operator with discrete spectrum and whose spectral gaps are asymptotically constant.

We give a strategy to construct bounded perturbations $V(t)$ such that the Hamiltonian $K_0 + V(t)$ generates unbounded orbits. We apply our abstract construction to three cases: (i) the Harmonic oscillator on $\mathbb{R}$, (ii) the half-wave equation on $\mathbb{T}$ and (iii) the Dirac-Schrödinger equation on Zoll manifolds. In each case, $V(t)$ is a smooth and periodic in time pseudodifferential operator and the Schrödinger equation has solutions fulfilling the optimal lower bound estimate $\|\psi(t)\|_r \gtrsim |t|^r$ as $|t| \gg 1$.

## 1. Introduction

In this paper we study linear Schrödinger equations of the form

$$\mathrm{i}\partial_t \psi = K_0\psi + V(t)\psi \tag{1}$$

on a scale of Hilbert spaces $\mathcal{H}^r$. Here $K_0$ is a positive, selfadjoint operator with purely discrete spectrum, $V(t)$ is a time dependent self-adjoint perturbation, and the scale $\mathcal{H}^r$ is the one defined spectrally by $K_0$.

We develop an abstract technique to construct bounded operators $V(t)$ which are smooth and periodic in time, for which (1) has unbounded orbits. In particular we show that the norms of the solutions of (1) grow polynomially in time, despite every orbit of the unperturbed flow being bounded. The procedure that we develop is quite general, and it applies *in case $K_0$ has constant spectral gaps.*

In particular, we apply it successfully to three different models: (i) the Harmonic oscillator on $\mathbb{R}$, (ii) the half-wave equation on $\mathbb{T}$ and (iii) the Dirac-Schrödinger equation on Zoll manifolds. In each case, we construct $V(t)$ as a pseudodifferential operator of order 0, smooth and $2\pi$-periodic in time, so that the Hamiltonian $K_0 + V(t)$ has solutions $\psi(t)$ fulfilling

$$\|\psi(t)\|_r \geq C_r\, t^r, \quad \text{for } t \gg 1, \tag{2}$$

which display, therefore, growth of Sobolev norms with optimal lower bounds. Note that case (iii) differs from (i) and (ii), since the Dirac-Schrödinger operator on Zoll manifolds has only *asymptotically constant spectral gaps*; however, such operator is a smoothing perturbation of a $K_0$ with constant spectral gaps, and our method applies with just a minor modification. In particular, the difference between cases (i)-(ii) and (iii) is that in the former ones the perturbations $V(t)$ can be arbitrary small in size; on the contrary, in case (iii) $V(t)$ has to contain a not perturbative term to correct the spectral gaps.

The problem of constructing unbounded solutions in Schrödinger equations has recently attracted a lot of attention. However, even in the simpler case of *linear time dependent* equations there are not many results in the literature. Up to our knowledge, the only examples were given by Bourgain for a Klein-Gordon and Schrödinger equation on $\mathbb{T}$ [Bou99], by Delort for the Harmonic oscillator on $\mathbb{R}$ [Del14], and by Bambusi, Grébert, Robert and the author for the Harmonic oscillators on $\mathbb{R}^d$, $d \geq 1$ [BGMR18]. Strictly speaking, our result on the Harmonic oscillator is not new, but the proof is new, general, and so simple that we think it is of some interest.
Our construction is based on the idea of "reversing" a mechanism exploited by Graffi and Yajima [GY00] to prove existence of absolutely continuous spectrum, and it goes as follows. First it is enough to look for a time dependent perturbation $V(t)$ of the form

$$V_A(t) := e^{-\mathrm{i}tK_0}\, A\, e^{\mathrm{i}tK_0} \tag{3}$$

where $A$ is a *time independent operator whose flow* $e^{-\mathrm{i}tA}$ *generates unbounded orbits*. By this very definition, if $\psi(t)$ is a solution of (1) with $V(t) \equiv V_A(t)$, the change of coordinates $\psi(t) = e^{-\mathrm{i}tK_0}\varphi(t)$ preserves each spectral norm and conjugates (1) to $\mathrm{i}\dot{\varphi} = A\varphi$, which, by assumptions, has unbounded orbits. Secondly, in order to guarantee that $t \mapsto V_A(t)$ is smooth and periodic in time, it is enough to ask that $A$ and $K_0$ have pseudodifferential properties and that an Egorov-type theorem holds; it is here that we use (indirectly) that $K_0$ has constant spectral gaps, see Remark 2.1.

Therefore, all is left to do in applications is to find a time independent operator $A$ whose propagator generates unbounded orbits. This is a much simpler task, and the general philosophy is to look for pseudodifferential operators with absolutely continuous spectrum. While in the case of compact manifold this is trivial (any multiplication operator will work), in the case of the Harmonic oscillator it is less obvious how to proceed, since the only multiplication operators which are pseudodifferential are the polynomials (see Remark 3.4). So the strategy is the following: we define an operator with absolutely continuous spectrum by its action on the Hermite basis, and then we use Chodosh characterization [Cho11] to verify that it is pseudodifferential (Chodosh characterization allows to read pseudodifferential properties of an operator from its matrix representation). Actually it is enough to choose the discrete laplacian on the Hermite basis to complete this program.

The reason to look for operators with absolutely continuous spectrum is due to the Guarneri-Combes theorem [Gua89, Com93], which guarantees, for these operators, the existence of initial data $\psi$ for which the time-averaged Sobolev norms $\frac{1}{T}\int_0^T \|e^{-\mathrm{i}tA}\psi\|_r \, \mathrm{d}t$ grow in $T$. This is a slightly weaker statement than (2); however, in applications, one can typically prove the stronger estimate $\|e^{-\mathrm{i}tA}\psi\|_r \geq C_{r,\psi}\, t^r$ as $t \to \infty$.

As a further comment, it is interesting to compare the rate of growth (2) with the upper bounds proved in [MR17, BGMR17] for equations of the form (1). In particular, the results of [MR17] imply that for any $V(t)$ continuous in time (but otherwise arbitrary depending) and pseudodifferential of order $\rho \leq 1$, each solution of (1) fulfills

$$\|\psi(t)\|_r \leq C_{r,\rho} \langle t \rangle^{\frac{r}{1-\rho}}, \quad \forall t \in \mathbb{R}. \tag{4}$$

In this paper we will have $\rho = 0$, thus the solutions that we construct provide optimal lower bounds on the speed of growth.

The upper bound (4) can be improved adding the assumption that $V(t) \equiv V(\omega t)$ is quasiperiodic in time with a nonresonant frequency vector $\omega \in \mathbb{R}^n$. Indeed, in [BGMR17] it is proved that if $\omega$ fulfills the nonresonance condition

$$\begin{aligned} &\exists \gamma, \tau > 0 \ \text{ s.t. } \ \left|\ell + \omega \cdot k\right| \geq \frac{\gamma}{(1+|k|+|\ell|)^\tau}, \\ &\forall \ell \in \mathbb{Z}\setminus\{0\}, \ \forall k \in \mathbb{Z}^n \setminus \{0\}, \end{aligned} \tag{5}$$

then for any $r \geq 0$, any $\epsilon > 0$ arbitrary small, (4) improves to

$$\|\psi(t)\|_r \leq C_{r,\epsilon} \langle t \rangle^{\epsilon}, \quad \forall t \in \mathbb{R} \ ; \tag{6}$$

the last estimate means that the growth of norms, if happens, is subpolynomial in time. Note that (6) is not in contrast with the faster growth of the norms (2); indeed the spectral condition that we impose on $K_0$ (see (10)) implies that $V_A(t)$ defined in (3) is periodic in time with frequency $\omega = 1$, which is clearly resonant.

Finally, in some cases one can prove that the Sobolev norms of the solution stay uniformly bounded in time. This requires typically nonresonance conditions stronger than (5) and a smallness assumption on the size of the perturbation. In this case, one might try to prove a reducibility KAM theorem, conjugating $K_0 + V(t)$ to a new Hamiltonian which is time independent and commutes with $K_0$; as a consequence, one gets the upper bound

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_r \leq C_r. \tag{7}$$

Concerning the systems that we treat here, the reducibility scheme has been successfully implemented for the Harmonic oscillators on $\mathbb{R}^d$ [Com87, Wan08, GT11, GP16b, Bam18, BGMR18], wave equations on the torus [CY00, FHW14, Mon17] (these methods can be used to prove reducibility for the half-wave equation on $\mathbb{T}$), and Klein-Gordon equation on the sphere [GP16a]. In all cases the frequency $\omega$ must be chosen in a Cantor set of nonresonant vectors and the perturbation must be sufficiently small in size. We recall that also the perturbations constructed here (and which provoke growth of norms) can be arbitrary small in size; therefore the stability/instability of the system depends only on the resonance property of the frequency $\omega$.

Before closing this introduction, let us mention that, in case of *nonlinear* Schrödinger equations, the problem of constructing solutions with unbounded orbits is extremely difficult. A first breakthrough was achieved in [CKSTT], which constructs solutions of the cubic nonlinear Schrödinger equation on $\mathbb{T}^2$ whose Sobolev norms become arbitrary large (see also [Han14, Gua14, HP15, GHP16] for generalizations of this result); however the existence of unbounded orbits for this model is still open.
At the moment, existence of unbounded orbits has only been proved by Gérard and Grellier [GG17] for the cubic Szegő equation on $\mathbb{T}$, and by Hani, Pausader, Tzvetkov and Visciglia [HPTV15] for the cubic NLS on $\mathbb{R} \times \mathbb{T}^2$.

## 2. The abstract framework

We start with a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and a reference operator $K_0$, which we assume to be selfadjoint and positive, namely such that

$$\langle\psi; K_0\psi\rangle \geq c_{K_0}\|\psi\|_0^2, \quad \forall\psi \in \mathrm{Dom}(K_0^{1/2}), \quad c_{K_0} > 0,$$

and define as usual a scale of Hilbert spaces by $\mathcal{H}^r = \mathrm{Dom}(K_0^r)$ (the domain of the operator $K_0^r$) if $r \geq 0$, and $\mathcal{H}^r = (\mathcal{H}^{-r})'$ (the dual space) if $r < 0$. We endow $\mathcal{H}^r$ with the norm

$$\|\psi\|_r := \|K_0^r\psi\|_0,$$

where $\|\cdot\|_0$ is the norm of $\mathcal{H}^0 \equiv \mathcal{H}$.

By the very definition of $\mathcal{H}^r$, the unperturbed flow $e^{-\mathrm{i}tK_0}$ preserves each norm, $\|e^{-\mathrm{i}tK_0}\psi\|_r = \|\psi\|_r$ $\forall t \in \mathbb{R}$. Consequently, every orbit of the unperturbed equation is bounded.

For $\mathcal{X}, \mathcal{Y}$ Banach spaces, we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$; if $\mathcal{X} = \mathcal{Y}$, we simply write $\mathcal{L}(\mathcal{X})$.

We state now the abstract assumptions that we will verify in each model. The principal one regards the existence of a bounded, *time independent* operator with unbounded orbits in the scale $\mathcal{H}^r$:

**Assumption A:** There exists an operator $A \in \mathcal{L}(\mathcal{H}^r)$, an initial datum $\psi_0 \in \mathcal{H}^r$ and a real $\mu > 0$ such that the Schrödinger equation

$$\mathrm{i}\dot{\psi} = A\psi, \qquad \psi(0) = \psi_0 \tag{8}$$

has a solution $\psi(t) \in \mathcal{H}^r$ fulfilling, for some $C_r > 0$, the estimate

$$\|\psi(t)\|_r \geq C_r\, t^{\mu r}, \quad t \gg 1. \tag{9}$$

In Lemma 2.3 below we give sufficient conditions on the operator $A$ to obtain (9).

The second assumption is smoothness in time of the map $t \mapsto \mathrm{e}^{-\mathrm{i}tK_0} A \mathrm{e}^{\mathrm{i}tK_0}$:

**Assumption B:** The map $t \mapsto \mathrm{e}^{-\mathrm{i}tK_0}\, A\, \mathrm{e}^{\mathrm{i}tK_0} \in C^\infty(\mathbb{R}; \mathcal{L}(\mathcal{H}^r))$.

Here $C^\infty(\mathbb{R}; \mathcal{L}(\mathcal{H}^r))$ is the class of smooth maps from $\mathbb{R}$ to $\mathcal{L}(\mathcal{H}^r)$.

In applications, Assumption B can be verified by requiring $A$ and $K_0$ to be pseudodifferential operators and $K_0$ to fulfill an Egorov-like theorem.

The last assumption concerns a spectral property of $K_0$:

**Assumption C:** $K_0$ has an entire discrete spectrum such that

$$\mathrm{spec}(K_0) \subseteq \mathbb{N} + \lambda \tag{10}$$

for some $\lambda \geq 0$.

Assumption C guarantees that $e^{\mathrm{i}2\pi K_0} = e^{\mathrm{i}2\pi\lambda}$. As a consequence, for any operator $V$, the map $t \mapsto e^{\mathrm{i}tK_0} V e^{-\mathrm{i}tK_0}$ is $2\pi$-periodic.

**Remark 2.1.** We do not require explicitly $K_0$ to have constant spectral gaps; however, in applications, the only operators that we could find that verify both Assumption B and C have constant spectral gaps.

It follows immediately this result.

**Theorem 2.2.** *Assume* A, B, C. *There exists $V_A(t) \in C^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H}^r))$ s.t. $K_0 + V_A(t)$ generates unbounded orbits. More precisely there exists a smooth solution $\psi(t)$ of* (1) *with $V(t) = V_A(t)$, such that $\psi(t)$ belongs to $\mathcal{H}^r$ $\forall t$ and fulfills* (9).

*Proof.* The proof is trivial. Define $V_A(t)$ as in (3). By Assumption B and C, it belongs to $C^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H}^r))$. The change of coordinates $\psi = e^{-\mathrm{i}tK_0}\varphi$ conjugates $\mathrm{i}\dot{\psi} = (K_0 + V_A(t))\psi$ to (8) and preserves the norm $\| \cdot \|_r$ for any time $t$. Then Assumption A implies the claim. $\square$

The main contribution of our paper is to show how to verify Assumption A in different settings and for many initial datum. Indeed, while on a compact manifold any $A$ multiplication operator will work (see Example 2.5 below for the case of the torus), for the Harmonic oscillator it is less trivial how to proceed. In the latter case the idea, that we already anticipated, is to define the operator by its action on the Hermite basis in such a way that it is easier to prove growth of the solution's norms, and then to use Chodosh characterization to verify its pseudodifferential properties. We stress that pseudodifferential properties are needed to verify Assumption B by invoking an Egorov-like theorem, and allow to obtain smoothness in time of the operator $V_A(t)$.

In order to verify that the flow of $A$ has unbounded paths in $\mathcal{H}^r$, $r > 0$, the following result might be useful. First remark that a necessary condition for (9) to be fulfilled is that $[K_0, A] \neq 0$. Then a sufficient condition is that only a finite number of iterated commutators of $A$ and $K_0$ are not zero. We define $\mathrm{ad}_A(B) := [A, B]$.

**Lemma 2.3.** *Assume that for some $N \in \mathbb{N}$ one has*

$$\mathrm{ad}_A^j(K_0) \neq 0, \quad \forall 1 \leq j \leq N, \quad \mathrm{ad}_A^{N+1}(K_0) = 0. \tag{11}$$

*Fix $r \in \mathbb{N}$ and choose $\psi_0 \in \mathcal{H}^r$ such that*

$$\left[\mathrm{ad}_A^N(K_0)\right]^r \psi_0 \neq 0. \tag{12}$$

*Then there exists $C(r, N, \psi_0) > 0$ such that the solution $\psi(t)$ of (9) with initial datum $\psi_0$ fulfills*

$$\|\psi(t)\|_r \geq C(r, N, \psi_0) \, \langle t \rangle^{rN}, \quad t \gg 1. \tag{13}$$

The proof of the Lemma is postponed in Appendix A.

**Remark 2.4.** Condition (11) can be replaced by $\mathrm{ad}_A^N(K_0') \neq 0$, $\mathrm{ad}_A^{N+1}(K_0') = 0$ where $K_0'$ is any operator defining norms equivalent to $\|\cdot\|_r$.

We conclude this section with an example of an operator $A$ with absolutely continuous spectrum which has unbounded orbits; such example will guide us in the applications:

**Example 2.5.** Let $H^r(\mathbb{T}) = \mathrm{Dom}((1 - \partial_{xx})^{r/2})$ be the classical Sobolev space on the one dimensional torus $\mathbb{T}$. Define $A = v(x)$ (multiplication operator), with $v(x) \in C^\infty(\mathbb{T}, \mathbb{R})$ and $\nabla v \not\equiv 0$. Then the equation

$$\mathrm{i}\dot{\varphi} = v(x)\varphi, \quad \text{with } \varphi(0) \in H^r(\mathbb{T}) \quad \text{and} \quad (\nabla v) \cdot \varphi(0) \not\equiv 0$$

has a solution $\varphi(t) \in H^r(\mathbb{T})$ fulfilling (9) with $\mu = 1$. This follows applying Lemma 2.3 and Remark 2.4 with $K_0' = \partial_x$ and noting that $[v(x), \partial_x] \neq 0$, $[v(x), [v(x), \partial_x]] = 0$.

More generally, any multiplication operator on a compact manifold will work as well.

## 3. Applications

In this section we show how to check the abstract assumptions in three different setups. In each case we construct a periodic in time perturbation which induces growth of norms. While Assumption B and C will be rather easy to check, the verification of Assumption A depends on the setup.

For $\Omega \subset \mathbb{R}^d$ and $\mathcal{F}$ a Fréchet space, we will denote by $C_b^m(\Omega, \mathcal{F})$ the space of $C^m$ maps $f : \Omega \ni x \mapsto f(x) \in \mathcal{F}$, such that, for every seminorm $\| \cdot \|_j$ of $\mathcal{F}$ one has

$$\sup_{x \in \Omega} \|\partial_x^\alpha f(x)\|_j < +\infty, \quad \forall \alpha \in \mathbb{N}^d \ : \ |\alpha| \leq m. \tag{14}$$

If (14) is true $\forall m$, we say $f \in C_b^\infty(\Omega, \mathcal{F})$.

### 3.1. Harmonic oscillator on $\mathbb{R}$

Consider the Schrödinger equation

$$\mathrm{i}\dot{\psi} = \frac{1}{2}\left(-\partial_x^2 + x^2\right)\psi + V(t, x, D_x)\psi, \quad x \in \mathbb{R}. \tag{15}$$

Here $K_0 := \frac{1}{2}\left(-\partial_x^2 + x^2\right)$ is the Harmonic oscillator, the scale of Hilbert spaces is defined as usual by $\mathcal{H}^r = \mathrm{Dom}\,(K_0^r)$, and the base space $(\mathcal{H}^0, \langle \cdot, \cdot \rangle)$ is $L^2(\mathbb{R}, \mathbb{C})$ with its standard scalar product. The perturbation $V$ is chosen as the Weyl quantization of a symbol belonging to the following class:

**Definition 3.1.** A function $f$ is a symbol of order $\rho \in \mathbb{R}$ if $f \in C^\infty(\mathbb{R}_x \times \mathbb{R}_\xi, \mathbb{R})$ and $\forall \alpha, \beta \in \mathbb{N}$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha\, \partial_\xi^\beta f(x, \xi)| \leq C_{\alpha,\beta}\ (1 + |x|^2 + |\xi|^2)^{\rho - \frac{|\beta| + |\alpha|}{2}}.$$

We will write $f \in S_{\mathrm{ho}}^\rho$.

We endow $S_{\mathrm{ho}}^\rho$ with the family of seminorms

$$\wp_j^\rho(f) := \sum_{|\alpha| + |\beta| \leq j} \ \sup_{(x,\xi) \in \mathbb{R}^2} \frac{\left|\partial_x^\alpha\, \partial_\xi^\beta f(x, \xi)\right|}{[1 + |x|^2 + |\xi|^2]^{\rho - \frac{\beta + \alpha}{2}}}, \quad j \in \mathbb{N} \cup \{0\}.$$

Such seminorms turn $S_{\mathrm{ho}}^\rho$ into a Fréchet space. If a symbol $f$ depends on additional parameters (e.g. it is time dependent), we ask that all the seminorms are uniform w.r.t. such parameters.
To a symbol $f \in S_{\mathrm{ho}}^\rho$ we associate the operator $f(x, D_x)$ by standard Weyl quantization

$$\Big(f(x,D_x)\psi\Big)(x) := \frac{1}{2\pi}\iint_{y,\xi\in\mathbb{R}} \mathrm{e}^{\mathrm{i}(x-y)\xi} f\left(\frac{x+y}{2},\xi\right)\psi(y)\,\mathrm{d}y\mathrm{d}\xi.$$

**Definition 3.2.** We say that $F \in \mathcal{A}_\rho$ if it is a pseudodifferential operator with symbol of class $S^\rho_{\rm ho}$, i.e., if there exists $f \in S^\rho_{\rm ho}$ such that $F = f(x, D_x)$.

**Remark 3.3.** The harmonic oscillator $K_0$ has symbol given by $x^2 + \xi^2$; by our definition $K_0 \in \mathcal{A}_1$.

**Remark 3.4.** It follows from Definition 3.1 that symbols depending only on one variable are polynomials. Therefore the only multiplication operators which are pseudodifferential are the quantizations of polynomial functions in $x$.

As usual we give $\mathcal{A}_\rho$ a Fréchet structure by endowing it with the seminorms of the symbols.

Our first application is to construct a time dependent pseudodifferential operator of order 0 which provokes growth of Sobolev norms. In such a way we obtain an alternative, simpler construction of the result of Delort [Del14].

The main problem is to verify Assumption A. We do this by defining $A$ by its action on the Hermite functions $(\mathbf{e}_n)_{n\in\mathbb{N}_0}$ (which are eigenvectors of the Harmonic oscillator and form a basis of $L^2(\mathbb{R})$). We take $\delta \neq 0$ arbitrary and define

$$A\mathbf{e}_0 := \delta\mathbf{e}_1, \quad A\mathbf{e}_n := \delta\left(\mathbf{e}_{n+1} + \mathbf{e}_{n-1}\right) \quad \text{for } n \geq 1. \tag{16}$$

The action of $A$ is extended on all $\mathcal{H}^r$ by linearity, giving

$$A\psi = \delta\sum_{n\geq 0}(\psi_{n-1} + \psi_{n+1})\mathbf{e}_n,$$

where we defined $\psi_n = \langle\psi, \mathbf{e}_n\rangle$ for $n \geq 0$ and $\psi_{-1} = 0$. Clearly $A \in \mathcal{L}(\mathcal{H}^r)$ $\forall r \geq 0$.

**Remark 3.5.** In the basis of Hermite functions, the operator $A$ is the discrete Laplacian on the half line $\ell^2(\mathbb{N}_0)$ with Dirichlet boundary conditions, hence it has absolutely continuous spectrum.

**Remark 3.6.** The Fourier transform $(\psi_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n\geq 0}\psi_n \sin((n+1)y)$ maps $\mathcal{H}^r$ in $H^r(\mathbb{T})$ and conjugates $A$ to the multiplication operator by $\delta\cos(x)$; therefore we are in the framework of Example 2.5.

We prove now that $A$ fulfills Assumption A and it is a pseudodifferential operator of order 0. First we show that its propagator has unbounded paths in $\mathcal{H}^r$, $r > 0$.

**Lemma 3.7.** *Let $A$ be defined in* (16) *and consider the equation* $\mathrm{i}\dot{\psi} = A\psi$. *For any $r \in \mathbb{N}$, any nonzero $\psi_0 \in \mathcal{H}^r$, there exists a constant $C_{r,\psi_0} > 0$ such that $\|e^{-\mathrm{i}tA}\psi_0\|_r \geq C_{r,\psi_0} \langle t \rangle^r$ for $t \gg 1$.*

*Proof.* Remark 3.6 gives essentially a proof of the statement. Alternatively, one can also apply Lemma 2.3 as follows. Let $\mathbf{e}_n$ be the $n^{th}$ Hermite function; then a direct computation using (16) and $K_0\mathbf{e}_n = (n + \frac{1}{2})\mathbf{e}_n$ shows that

$$[A, K_0]\mathbf{e}_n = \delta(\mathbf{e}_{n-1} - \mathbf{e}_{n+1}), \quad [A, [A, K_0]]\mathbf{e}_n = 0, \quad \forall n \in \mathbb{N}_0.$$

Then we apply Lemma 2.3 with $N = 1$. Condition (12) is fulfilled for any nontrivial $\psi_0 \in \mathcal{H}^r$. $\square$

To prove that $A \in \mathcal{A}_0$ we use Chodosh characterization [Cho11], which allows to read the pseudodifferential properties of an operator by its matrix coefficients on the basis of Hermite functions. Any linear self-adjoint operator $A : \mathcal{H}^r \to \mathcal{H}^{r'}$ is completely determined by its *matrix*

$$M^{(A)} \colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}, \quad (m, n) \to \langle A\mathbf{e}_m, \mathbf{e}_n \rangle . \tag{17}$$

Define the discrete difference operator $\triangle$ on a function $M \colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ by

$$(\triangle M)(m, n) := M(m + 1, n + 1) - M(m, n),$$

and its powers $\triangle^\gamma$ by $\triangle$ applied $\gamma$-times.

**Definition 3.8.** A symmetric function $M \colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ will be said to be a *symbol matrix of order $\rho$* if for any $\gamma, N \in \mathbb{N}$, there exists $C_{\gamma,N} > 0$ such that

$$\left|(\triangle^\gamma M)(m, n)\right| \leq C_{\gamma,N} \frac{(1 + m + n)^{\rho - |\gamma|}}{(1 + |m - n|)^N}.$$

The connection between pseudodifferential operators of order $\rho$ and symbol matrices of order $\rho$ is given by Chodosh's characterization [Cho11]:

**Theorem 3.9 (Chodosh's characterization).** *A selfadjoint operator $A$ belongs to $\mathcal{A}_\rho$ if and only if its matrix $M^{(A)}$ (as defined in* (17)*) is a symbol matrix of order $\rho$.*

We can now prove:

**Lemma 3.10.** *The operator $A$ defined in* (16) *belongs to $\mathcal{A}_0$.*

*Proof.* By formula (16), the matrix of $A$ is given by

$$M^{(A)}(m,n) := \langle A\mathbf{e}_m, \mathbf{e}_n \rangle = \delta_{n+1,m} + \delta_{n-1,m}, \quad n, m \in \mathbb{N}_0.$$

It is a trivial computation to verify that $M^{(A)}$ is a symbol matrix of order 0, hence by Theorem 3.9 it is a pseudodifferential operator in $\mathcal{A}_0$. □

We are now able to recover Delort theorem [Del14], for any initial datum.

**Theorem 3.11.** *There exists a time periodic pseudodifferential operator of order 0, $V \in C_b^\infty(\mathbb{T}, \mathcal{A}_0)$, such that the following holds true. For any $r \in \mathbb{N}$, for any initial datum $\psi_0 \in \mathcal{H}^r$, there exists a constant $C_{r,\psi_0} > 0$ such that the solution $\psi(t)$ of* (15) *with initial datum $\psi_0$ belongs to $\mathcal{H}^r$ for any $t \geq 0$ and fulfills $\|\psi(t)\|_r \geq C_{r,\psi_0} \langle t \rangle^r$ for any $t > 0$ large enough.*

*Proof.* We verify that Assumptions A, B, and C are met.
Assumption A: It follows by Lemma 3.7 with $\mu = 1$ and any initial datum.
Assumption B: By Lemma 3.10 one has $A \in \mathcal{A}_0$. By Egorov theorem for the Harmonic oscillator [Hör79] (and using the periodicity of the flow of $K_0$) the map $t \mapsto e^{-\mathrm{i}tK_0} A e^{\mathrm{i}tK_0} \equiv V_A(t) \in C_b^\infty(\mathbb{T}, \mathcal{A}_0)$. This can be seen e.g. by remarking that the symbol of $V_A(t)$ is $a \circ \phi_{ho}^t$, where $a \in S^0_{\mathrm{ho}}$ is the symbol of $A$ and $\phi_{ho}^t$ is the time $t$ flow of the harmonic oscillator; explicitly

$$\left(a \circ \phi_{ho}^t\right)(x,\xi) = a(x\cos t + \xi \sin t, -x \sin t + \xi \cos t). \tag{18}$$

Assumption C: It follows from $\sigma(K_0) = \{n + \frac{1}{2}\}_{n \geq 0}$.

□

We conclude with a list of remarks.

**Remark 3.12.** The parameter $\delta$ in (16) can be arbitrary small; therefore, also the perturbation $V_A(t)$ can be arbitrary small in size.

**Remark 3.13.** Consider (15) with $V_A(\omega t, x, D_x)$, $\omega \in \mathbb{R}$, so that the perturbation is now periodic in time with frequency $\omega$. Then it is proved in [Bam18] that for any $\delta$ sufficiently small, there exists a Cantor set $\mathcal{O}_\delta \subset [0,1]$ such that if $\omega \in \mathcal{O}_\delta$, each solution of (15) fulfills (7). It is clear therefore that the growth of Sobolev norms depends on resonance properties of the frequency $\omega$.

**Remark 3.14.** One has $V_A \in C^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H}^r))$ for any $r \geq 0$. Indeed by Calderon-Vaillancourt theorem for the class $\mathcal{A}_\rho$ (see e.g. [Rob87]), for any $r \in \mathbb{N}$, there exists $N \in \mathbb{N}$ and $C_{r,N} > 0$ such that

$$\sup_{t\in\mathbb{T}} \|\partial_t^\ell V_A(t)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_{r,N}\, \wp_N^\rho \left(\partial_t^\ell a(t,x,\xi)\right) < \infty, \quad \forall \ell \in \mathbb{N}_0,$$

where $a(t,x,\xi) \equiv (a \circ \phi_{ho}^t)(x,\xi)$ is the symbol of $V_A(t)$, see (18), and belongs to $C_b^\infty(\mathbb{R}, S^0_{\rm ho})$.

### 3.2. Half-wave equation on $\mathbb{T}$

The half-wave equation on $\mathbb{T}$ is given by

$$\mathrm{i}\dot{\psi} = |D_x|\psi + V(t,x,D_x)\psi, \quad x \in \mathbb{T}. \tag{19}$$

Here $|D_x|\psi$ is the Fourier multiplier defined by

$$|D_x|\psi := \sum_{j\in\mathbb{Z}} |j|\, \psi_j\, e^{\mathrm{i}jx},$$

where $\psi_j := \int_{\mathbb{T}} \psi(x) e^{-\mathrm{i}jx} \mathrm{d}x$ is the $j^{th}$ Fourier coefficient.

We recall now the class of global pseudodifferential operators on $\mathbb{T}$ (see e.g. [SW87] or the monograph [SV02]). For a function $f : \mathbb{T}\times\mathbb{Z} \to \mathbb{R}$, define the difference operator $\triangle f(x,j) := f(x,j+1) - f(x,j)$. We define the class of symbols as follows.

**Definition 3.15.** A function $f : \mathbb{T}\times\mathbb{Z} \to \mathbb{R}$, will be called a symbol of order $\rho \in \mathbb{R}$ if $x \mapsto f(x,j)$ is smooth for any $j \in \mathbb{Z}$ and $\forall \alpha, \beta \in \mathbb{N}$, there exists $C_{\alpha,\beta} > 0$ s.t.

$$\left|\partial_x^\alpha \triangle^\beta f(x,j)\right| \leq C_{\alpha,\beta}\, \langle j\rangle^{\rho-\beta}.$$

We will write $f \in S^\rho_{\rm to}$.

Again we endow $S^\rho_{\rm to}$ with the family of seminorms

$$\wp_\ell^\rho(f) := \sum_{\alpha+\beta\leq\ell} \sup_{(x,j)\in\mathbb{T}\times\mathbb{Z}} \langle j\rangle^{-\rho+\beta} \left|\partial_x^\alpha \triangle_\xi^\beta f(x,j)\right|, \quad \ell \in \mathbb{N}_0.$$

If a symbol $f$ depends on additional parameters (e.g. it is time dependent), we ask that the constants $C_{\alpha,\beta}$ are uniform w.r.t. such parameters.

To a symbol $f \in S^{\rho}_{\mathrm{to}}$ we associate the operator $f(x, D_x)$ by standard quantization:

$$\Big(f(x, D_x)\psi\Big)(x) := \sum_{j\in\mathbb{Z}} f(x, j)\, \psi_j\, e^{\mathrm{i}jx}.$$

Then we have the following definition.

**Definition 3.16.** We say that $F \in \mathcal{A}_\rho$ if it is a pseudodifferential operator with symbol of class $S^{\rho}_{\mathrm{to}}$, i.e., if there exist $f \in S^{\rho}_{\mathrm{to}}$ such that $F = f(x, D_x)$.

**Remark 3.17.** The operator $|D_x|$ has symbol given by $|j|$; therefore $|D_x| \in \mathcal{A}_1$.

**Remark 3.18.** The class of pseudodifferential operators defined globally on the torus through Fourier analysis coincides with the usual definition of pseudodifferential operators on a compact manifold, see e.g. [SV02] or [SW87] for a proof.

As usual we give $\mathcal{A}_\rho$ a Fréchet structure by endowing it with the seminorms of the symbols. In this case we have:

**Theorem 3.19.** *Consider the half-wave equation* (19)*. There exist a pseudodifferential operator of order* 0*,* $V \in C^\infty(\mathbb{T}, \mathcal{A}_0)$*, and for any* $r \in \mathbb{N}$ *a constant* $C_r > 0$ *and a solution* $\psi(t)$ *of* (19) *such that, for any* $t \geq 0$*,* $\psi(t) \in H^r(\mathbb{T})$ *and satisfies* $\|\psi(t)\|_{H^r(\mathbb{T})} \geq C_r\, t^r$ *for* $t$ *large enough.*

*Proof.* First we show how to put ourselves in the setting of the abstract problem. Define

$$K_0 := |D_x| + \lambda, \quad \lambda > 0.$$

The space $\mathcal{H}^r := \mathrm{Dom}(K_0^r)$ coincides with $H^r(\mathbb{T})$, with equivalent norms. We take perturbations of the form $V(t, x, D_x) = \lambda\left(1 + e^{-\mathrm{i}tK_0} A e^{\mathrm{i}tK_0}\right)$, so that (19) becomes

$$\mathrm{i}\dot{\psi} = K_0\psi + V_A(t, x, D_x)\psi, \quad V_A(t, x, D_x) := \lambda\, e^{-\mathrm{i}tK_0} A e^{\mathrm{i}tK_0},$$

and we are back to the abstract setting. We need only to verity Assumptions A, B, C.

Assumption A: Take $A$ and the initial datum $\psi_0$ as in Example 2.5.

Assumption B: One has $A \in \mathcal{A}_0$ and $K_0 \in \mathcal{A}_1$. By a classical result of Hörmander [Hör85] (see also [DG75]), $t \mapsto e^{-\mathrm{i}tK_0} A e^{\mathrm{i}tK_0} \in C^\infty(\mathbb{R}, \mathcal{A}_0)$; actually, being periodic in time, it belongs to $C^\infty(\mathbb{T}, \mathcal{A}_0)$.

Assumption C: Trivial, since $\sigma(K_0) = \{j + \lambda\}_{j\in\mathbb{N}_0}$. $\square$

**Remark 3.20.** The parameter $\lambda$ can be arbitrary small; therefore also in this case $V(t,x,D_x)$ can be arbitrary small in size.

**Remark 3.21.** By Calderon-Vaillancourt theorem,

$$V(t,x,D_x) \in C^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H}^r)) \quad \forall r \in \mathbb{N}.$$

### 3.3. Schrödinger-Dirac equation on Zoll manifolds

Consider the Schrödinger-Dirac equation on a Zoll manifold $M$ (e.g., $M$ can be a $n$-dimensional sphere)

$$\mathrm{i}\dot{\psi} = \sqrt{-\Delta_g + m^2}\,\psi + V(t,x,D_x)\psi, \quad x \in M\ ; \tag{20}$$

here $m \neq 0$ is a real number and $-\Delta_g$ is the positive Laplace-Beltrami operator on $M$. Let $H^r(M) = \mathrm{Dom}\left((1-\Delta_g)^{r/2}\right)$, $r \geq 0$, the usual scale of Sobolev spaces on $M$. Finally we denote by $S^\rho_{\mathrm{cl}}$ the space of classical real valued symbols of order $\rho \in \mathbb{R}$ on the cotangent $T^*(M)$ of $M$ (see Hörmander [Hör85] for more details).

**Definition 3.22.** We say that $F \in \mathcal{A}_\rho$ if it is a pseudodifferential operator (in the sense of Hörmander [Hör85]) with symbol of class $S^\rho_{\mathrm{cl}}$.

**Remark 3.23.** The operator $\sqrt{-\Delta_g + m^2}$ belongs to $\mathcal{A}_1$ [Hör85].

**Remark 3.24.** By [CdV79], there exist $c_0, c_1 > 0$ such that

$$\sigma\left(\sqrt{-\Delta_g + m^2}\right) \subseteq \bigcup_{j \geq 0} \left[j + c_0 - \frac{c_1}{j},\ j + c_0 + \frac{c_1}{j}\right],$$

so in this case the spectral gaps are only *asymptotically* constant.

We have the following

**Theorem 3.25.** *Consider the Schrödinger-Dirac equation* (20)*. There exists a pseudodifferential operator of order* 0*,* $V \in C^\infty(\mathbb{T}, \mathcal{A}_0)$*, and for any* $r \in \mathbb{N}$ *a constant* $C_r > 0$ *and a solution* $\psi(t)$ *of* (20) *fulfilling* $\psi(t) \in H^r(M)$ *for any time* $t$*, and*

$$\|\psi(t)\|_{H^r(M)} \geq C_r\, t^r, \quad t \gg 1.$$

*Proof.* To begin with we show how to put ourselves in the abstract setup. So first we define the operator $K_0$. This is achieved by exploiting the spectral properties of the operator $-\Delta_g$. Applying Theorem 1 of Colin de Verdière [CdV79], there exists a pseudodifferential operator $Q$ of order $-1$, commuting with $-\Delta_g$, such that $\mathrm{Spec}[\sqrt{-\Delta_g + m^2} + Q] \subseteq \mathbb{N} + \lambda$ with some $\lambda > 0$. So we define

$$K_0 := \sqrt{-\Delta_g + m^2} + Q \in \mathcal{A}_1. \tag{21}$$

Since $Q \in \mathcal{A}_{-1}$, the space $\mathcal{H}^r := \mathrm{Dom}(K_0^r)$, $r \geq 0$, coincides with the classical Sobolev space $H^r(M)$ and one has the equivalence of norms

$$c_r \, \|\psi\|_{H^r(M)} \leq \|\psi\|_r \leq \widetilde{c}_r \, \|\psi\|_{H^r(M)}, \quad \forall r \in \mathbb{R}.$$

We take the perturbation of the form $V(t, x, D_x) = Q + e^{-\mathrm{i}tK_0} A e^{\mathrm{i}tK_0}$, so that (20) becomes

$$\mathrm{i}\dot{\psi} = K_0\psi + V_A(t, x, D_x), \quad V_A(t, x, D_x) = e^{-\mathrm{i}tK_0} A e^{\mathrm{i}tK_0}$$

and we are back to the abstract setting. We need only to verify Assumptions A, B, C.

Assumption A: It follows by a trivial variant of Example 2.5. Choose any non-constant $v(x) \in C^\infty(M, \mathbb{R})$, define $A$ as the multiplication operator by $v(x)$, and take an initial datum $\psi_0 \in \mathcal{H}^r$ fulfilling $(\nabla_g v(x))\psi_0 \not\equiv 0$. For example, $v(x)$ can be any non-costant eigenfunction of $-\Delta_g$. Then the Schrödinger equation (8) has orbits fulfilling (9) with $\mu = 1$ (it is enough to apply Lemma 2.3 and Remark 2.4 using $[\nabla_g, v(x)] \neq 0$, $[[\nabla_g, v(x)], v(x)] = 0$).

Assumption B: One has $A \in \mathcal{A}_0$ and $K_0 \in \mathcal{A}_1$. Then $e^{-\mathrm{i}tK_0} A e^{\mathrm{i}tK_0} \in C^\infty(\mathbb{R}, \mathcal{A}_0)$ by a classical result of Hörmander [Hör85].

Assumption C: True by construction. □

**Remark 3.26.** In this case, the perturbation $V(t, x, D_x)$ cannot be chosen arbitrary small in size, since we have to add the smoothing operator $Q$ to correct the spectral gaps.

**Remark 3.27.** One could also choose $V(t, x, D_x)$ as

$$e^{-\mathrm{i}t\sqrt{-\Delta_g + m^2}} \, \epsilon A \, e^{\mathrm{i}t\sqrt{-\Delta_g + m^2}};$$

in such a way the perturbation is arbitrary small in size and fulfills Assumption C (again by [Hör85]), but it is not periodic in time.

## Appendix A. Proof of Lemma 2.3

Since the linear operator $\mathrm{ad}_A$ fulfills Leibniz rule, for any $M, r \in \mathbb{N}$ one has the identity

(A.1)
$$\mathrm{ad}_A^M(K_0^{2r}) = \sum_{k_1+\ldots+k_{2r}=M} \binom{M}{k_1 \cdots k_{2r}} \mathrm{ad}_A^{k_1}(K_0)\,\mathrm{ad}_A^{k_2}(K_0) \cdots \mathrm{ad}_A^{k_{2r}}(K_0)$$

If $M \geq 2rN+1$ then in (A.1) at least one index $k_j$ is greater equal $N+1$, so by assumption (11) the whole expression is zero. By the same argument, if $M = 2rN$ then the only term not null in (A.1) is the one with $k_j = N \ \forall j$, which is $\left[\mathrm{ad}_A^N(K_0)\right]^{2r}$.

Consider now the solution $\psi(t) \equiv e^{-\mathrm{i}tA}\psi_0$ of equation (8). Since $A$ is self-adjoint,

$$\|\psi(t)\|_r^2 \equiv \left\langle e^{\mathrm{i}tA}\, K_0^{2r}\, e^{-\mathrm{i}tA}\psi_0, \psi_0 \right\rangle ,$$

where we used $\|\psi\|_r^2 \equiv \left\langle K_0^{2r}\psi, \psi\right\rangle$. Now we use the Lie formula $e^{\mathrm{i}tA}\, B\, e^{-\mathrm{i}tA} \equiv \sum_{j\geq 0} \frac{(\mathrm{i}t)^j}{j!}\mathrm{ad}_A^j(B)$, assumption (11) and our previous considerations to obtain

$$e^{\mathrm{i}tA}\, K_0^{2r}\, e^{-\mathrm{i}tA} = \sum_{M=0}^{2rN} \frac{(\mathrm{i}t)^M}{M!}\,\mathrm{ad}_A^M(K_0^{2r}) \equiv \frac{(\mathrm{i}t)^{2rN}}{(2rN)!}\left[\mathrm{ad}_A^N(K_0)\right]^{2r} + O(t^{2rN-1}) \ ;$$

provided $\left[\mathrm{ad}_A^N(K_0)\right]^r \psi_0 \neq 0$, it follows that

$$\liminf_{t\to+\infty} \frac{\|\psi(t)\|_r^2}{|t^{2rN}|} \geq \frac{1}{(2rN)!}\,\left|\left\langle \left[\mathrm{ad}_A^N(K_0)\right]^{2r}\psi_0, \psi_0\right\rangle\right| > 0. \tag{A.2}$$

In particular there exists a constant $C(r, N, \psi_0) > 0$ such that (13) holds true.

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## References


[Bam18] D. Bambusi, *Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations I*, Trans. Amer. Math. Soc. **370** (2018), no. 3, 1823–1865.

[BGMR17] D. Bambusi, B. Grébert, A. Maspero, and D. Robert, *Growth of Sobolev norms for abstract linear Schrödinger equations*, J. Eur. Math. Soc. (JEMS), in press, (2017).

[BGMR18] D. Bambusi, B. Grébert, A. Maspero, and D. Robert, *Reducibility of the quantum harmonic oscillator in d-dimensions with polynomial time-dependent perturbation*, Anal. PDE **11** (2018), no. 3, 775–799.

[Bou99] J. Bourgain, *Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential*, Comm. Math. Phys. **204** (1999), no. 1, 207–247.

[CdV79] Y. Colin de Verdière, *Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques*, Comment. Math. Helv. **54** (1979), no. 3, 508–522.

[Cho11] O. Chodosh, *Infinite matrix representations of isotropic pseudodifferential operators*, Methods Appl. Anal. **18** (2011), no. 4, 351–371.

[CKSTT] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Invent. Math. **181** (2010), no. 1, 39–113.

[Com87] M. Combescure, *The quantum stability problem for time-periodic perturbations of the harmonic oscillator*, Ann. Inst. H. Poincaré Phys. Théor **47** (1987), no. 1, 63–83.

[Com93] J. Combes, *Connections between quantum dynamics and spectral properties of time-evolution operators*, Diff. Eq. Appl. Math. Phys. **192** (1993), 59–68.

[CY00] L. Chierchia and J. You, *KAM tori for 1D nonlinear wave equations with periodic boundary conditions*, Comm. Math. Phys. **211** (2000), no. 2, 497–525.

[Del14] J.-M. Delort, *Growth of Sobolev norms for solutions of time dependent Schrödinger operators with harmonic oscillator potential*, Comm. Partial Differential Equations **39** (2014), no. 1, 1–33.

[DG75] J. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29(1):39–79, 1975.

[FHW14] D. Fang, Z. Han, and W.-M. Wang, *Bounded Sobolev norms for Klein-Gordon equations under non-resonant perturbation*, J. Math. Phys. **55** (2014), no. 12, 121503.

[GG17] P. Gérard and S. Grellier, *The cubic Szegő equation and Hankel operators*, Astérisque (2017), no. 389, vi+112.

[GHP16] M. Guardia, E. Haus, and M. Procesi, *Growth of Sobolev norms for the analytic NLS on* $\mathbb{T}^2$, Adv. Math. (2016), no. 301, 615–692.

[GP16a] B. Grébert and E. Paturel, *KAM for the Klein Gordon equation on* $\mathbb{S}^d$. Boll. Unione Mat. Ital. **9** (2016), no. 2, 237–288.

[GP16b] B. Grébert and E. Paturel, *On reducibility of quantum Harmonic Oscillator on* $\mathbb{R}^d$ *with quasiperiodic in time potential*, `arXiv:1603.07455`, (2016).

[GT11] B. Grébert and L. Thomann, *KAM for the quantum harmonic oscillator*, Comm. Math. Phys. **307** (2011), no. 2, 383–427.

[Gua89] I. Guarneri, *Spectral properties of quantum diffusion on discrete lattices*, Europhysics Letters **10** (1989), no. 2, 95.

[Gua14] M. Guardia, *Growth of Sobolev norms in the cubic nonlinear Schrödinger equation with a convolution potential*, Comm. Math. Phys. **329** (2014), no. 1, 405–434.

[GY00] S. Graffi and K. Yajima, *Absolute continuity of the Floquet spectrum for a nonlinearly forced harmonic oscillator*, Comm. Math. Phys. **215** (2000), no. 2, 245–250.

[Han14] Z. Hani, *Long-time instability and unbounded Sobolev orbits for some periodic nonlinear Schrödinger equations*, Arch. Ration. Mech. Anal. **211** (2014), no. 3, 929–964.

[Hör85] L. Hörmander, The Analysis of Linear Partial Differential Operators IV, Grundlehren der Mathematischen Wissenschaften **256**, Springer-Verlag, (1985).

[HP15] E. Haus and M. Procesi, *Growth of Sobolev norms for the quintic NLS on* $\mathbb{T}^2$, Anal. PDE **8** (2015), no. 4, 883–922.

[HPTV15] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia, *Modified scattering for the cubic Schrödinger equation on product spaces and applications*, Forum Math. Pi **3** (2015), e4, 63.

[Hör79] L. Hörmander, *Weyl Calculus of pseudodifferential operators*, CPAM **32** (1979), no. 3, 359–443.

[Mon17] R. Montalto, *A reducibility result for a class of linear wave equations on* $\mathbb{T}^d$, IMRN, rnx167, `DOI:10.1093/imrn/rnx167`, (2017).

[MR17] A. Maspero and D. Robert, *On time dependent Schrödinger equations: Global well-posedness and growth of Sobolev norms*, Journal of Functional Analysis **273** (2017), no. 2, 721–781.

[Rob87] D. Robert, Autour de l'Approximation Semi-Classique, PM **68**, Birkhäuser, (1987).

[SV02] J. Saranen and G. Vainikko, *Periodic integral and pseudodifferential equations with numerical approximation*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, (2002).

[SW87] J. Saranen and W. Wendland, *The Fourier series representation of pseudodifferential operators on closed curves*, Complex Var. Theory Appl. **8** (1987), no. 1-2, 55–64.

[Wan08] W.-M. Wang, *Pure point spectrum of the Floquet Hamiltonian for the quantum harmonic oscillator under time quasi-periodic perturbations*, Comm. Math. Phys. **277** (2008), no. 2, 459–496.

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