Local convexity of renormalized volume for rank-1 cusped manifolds

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We study the critical points of the renormalized volume for acylindrical geometrically finite hyperbolic 3-manifolds that include rank-1 cusps, and show that the renormalized volume is locally convex around these critical points. We give a modified definition of the renormalized volume that is additive under gluing, and study some local properties.

1. Introduction

Renormalized volume is a quantity that gives a notion of volume for hyperbolic manifolds which have infinite volume under the classical definition. Its study for convex co-compact hyperbolic 3-manifolds can be found in [KS08], while the geometrically finite case which includes rank 1-cusps has been developed in [GMR17].

The article is organized as follows: Sections (2) and (3) give a review of parametrization of hyperbolic metrics and renormalized volume for convex co-compact manifolds. Section (4) gives a way to complete a result of [KS08] using Ricci flow. Section (5) gives a proof for the convex co-compact case by showing that the Hessian of the renormalized volume is positive definite, while Section (6) shows how to extend this proof when rank-1 cusps are taken into account. Finally, in Section (7) we define the corrected renormalized volume, which is additive under gluing, and we prove convexity at one of its critical points.

Moroianu [Mor17] has proved independently that the Hessian is positive definite at the critical points for the convex co-compact case, by the use of minimal surfaces. Our method relies on computing the Hessian for quasi-Fuchsian manifolds, and then using the skinning map to compute it for acylindrical manifolds.

2. Local parametrization of hyperbolic metrics

Let R be a Riemann surface of genus g with n punctures, $\mathcal{T}$ its Teichmuller space, $\mathcal{B}$ its space of Beltrami differentials and Q its space of quadratic holomorphic differentials. There is a nice description of these two last spaces in terms of complex-valued functions in the following way (it can be found, for example, in [Gar87]):

Take the covering map $\mathbb{H}^2 \to R$ and denote by $\Gamma \subseteq \mathrm{PSL}(2, \mathbb{R})$ the group of deck transformations. Then elements of $\mathcal{B}$ are represented in $\mathbb{H}^2$ by measurable L^∞ functions μ such that

$$(1) \qquad \mu(A(z))\overline{A'(z)} = \mu(z)A'(z), \forall A \in \Gamma.$$

Similarly, elements of Q are represented in $\mathbb{H}^2$ by holomorphic functions ϕ such that

$$(2) \qquad \phi(A(z))A'(z)^2 = \phi(z), \forall A \in \Gamma$$

and $\iint_{R(\Gamma)} |\phi(z)| dx \wedge dy < \infty$, where $R(\Gamma)$ is any fundamental domain for Γ in $\mathbb{H}^2$. Observe that the hyperbolic metric on $\mathbb{H}^2$, $\frac{1}{y^2} dz d\bar{z}$, corresponds to $\rho(z) = \frac{1}{y^2}$ which has the property

$$(3) \qquad \rho(A(z))A'(z)\overline{A'(z)} = \rho(z), \forall A \in \mathrm{PSL}(2, \mathbb{R}),$$

and hence $\frac{\overline{\phi}}{2\rho}$ satisfies (1). Because $\iint_{R(\Gamma)} |\phi(z)| dx \wedge dy < \infty$, $\frac{\overline{\phi}}{2\rho}$ is in L^∞, and then we have a map $Q \to \mathcal{B}, \phi \mapsto \frac{\overline{\phi}}{2\rho}$.

Thanks to the theory of the Beltrami equation ([Ahl66], [AB60], [LV65]), we can find a solution of $f_{\bar{z}}(z) = \mu(z)f_z(z)$ for $\|\mu\|_\infty < 1$ by a unique quasiconformal self-mapping f^μ of $\mathbb{H}^2$ which extends continuously to $\partial\mathbb{H}^2$, fixes $0, 1, \infty$ and depends analytically on μ (f solves the Beltrami equation in the distributional sense). With this we have a map from the unit ball of $\mathcal{B}$ to $\mathcal{T}$, and by Teichmüller theory, the correspondence that sends $\phi \mapsto \frac{\overline{\phi}}{2\rho}$

and then to the solution of the Beltrami equation for $\frac{\bar{\phi}}{2\rho}$ defines a local homeomorphism between a neighbourhood of 0 in Q to a neighbourhood of $\mathcal{T}$.

Now, the solution to the Beltrami equation for $\mu = \frac{\bar{\phi}}{2\rho}$, $(f^\mu)^*(\rho dz d\bar{z})$ is a local parametrization of hyperbolic metrics defined on the same space. We would like to compute the variation of this family of metrics at the origin. Because $f^{t\mu}$ satisfies the Beltrami equation for $t\mu$, we have

$$(4) \qquad (f^{t\mu})^*(\rho dz d\bar{z})(z) = \rho(f^{t\mu(z)})|f_z^{t\mu}|^2|dz + t\mu(z)d\bar{z}|^2.$$

Hence (here is implicit the analytic dependence of f^μ with respect to μ)

$$(5) \qquad \left.\frac{\partial}{\partial t}\right|_{t=0} (f^{t\mu})^*(\rho dz d\bar{z})(z) = \rho(z)\mu(z)d\bar{z}^2 + \rho\overline{\mu(z)}dz^2 + E dz d\bar{z},$$

where E groups the terms of the derivative that go together with $dz d\bar{z}$. We can then replace $\mu = \frac{\bar{\phi}}{2\rho}$ to obtain

$$(6) \qquad \left.\frac{\partial}{\partial t}\right|_{t=0} (f^{t\mu})^*(\rho dz d\bar{z})(z) = \mathrm{Re}(\phi dz^2) + E dz d\bar{z}.$$

Define RQ as the space of real parts of quadratic holomorphic differentials. Since taking the real part defines an isomorphism $Q \to RQ$, we have a local homeomorphism from a neighbourhood of 0 in RQ to a neighbourhood of $\mathcal{T}$. Moreover, if we define I_v to be the hyperbolic metric $(f^\mu)^*(\rho dz d\bar{z})$, where $\mu = \frac{\bar{\phi}}{2\rho}$ and $\mathrm{Re}(\phi dz^2) = v$, then (6) implies that at 0

$$(7) \qquad\qquad\qquad DI_v = v + E dz d\bar{z}.$$

Using the hyperbolic metric of R to define an inner product for tensors, DI_v projects orthogonally to v (because $dz^2, d\bar{z}^2$ are pointwise orthogonal to $dz d\bar{z}$). In particular, for $v, w \in RQ_c$ we have

$$(8) \qquad\qquad\qquad \langle DI_c(v), w \rangle = \langle v, w \rangle.$$

Observe finally that if we had chosen R with the same hyperbolic metric but with opposite orientation, the space RQ coincides with the one defined for the original Riemann surface structure.

3. Review of renormalized volume

Given a convex co-compact hyperbolic 3-manifold (M, g), Krasnov and Schlenker [KS08] defined its renormalized volume and calculated its first variation from the W-volume of a compact submanifold N ([KS08] Definition 3.1) as

$$(9) \qquad\qquad W(M, N) = V(N) - \frac{1}{4} \int_{\partial N} H \, da,$$

where da is the area form of the induced metric.

Expanding on the notation used in [KS08], denote by I the metric induced on ∂N, II its second fundamental form (so $\mathrm{II}(x, y) = \mathrm{I}(x, By)$, where B is the shape operator) and $\mathrm{III}(x, y) = \mathrm{I}(Bx, By)$ its third fundamental form.

If we further assume that N has convex boundary and that the normal exponential map (pointing towards the exterior of ∂N) defines a family of equidistant surfaces $\{S_r\}$ that exhaust the complement of N ($S_0 = \partial N$), then the W-volume of N_r (points on the interior of S_r) satisfies ([Sch13] Lemma 3.6)

$$(10) \qquad\qquad W(N_r) = W(N) - \pi r \chi(\partial N).$$

Also, as observed in ([Sch13] Definition 3.2, Proposition 3.3), $\mathrm{I}^* = 4 \lim_{r \to \infty} e^{-2r} \mathrm{I}_r$ (where I_r is the metric induced on S_r, which is identified with S by the normal exponential map) exists and lies in the conformal class of the boundary. The analogous re-scaled limits for II, III, B also exist and are denoted by $\mathrm{II}^*, \mathrm{III}^*, B^*$. The reason to multiply by 4 is so $\mathrm{I}^* = g|_N$ in the case when N is a totally geodesic surface.

For the case of convex co-compact manifolds, any metric at infinity that belongs to the conformal class given by the hyperbolic structure can be obtained as the rescaled limit of the induced metrics of some family of equidistant surfaces. Theorem 5.8 of [KS08] describes this by the use of Epstein surfaces (as stablished in [Eps84]), which in turn allows us to define

$$(11) \qquad\qquad W(M, h) = W(M, N_r) + \pi r \chi(\partial M),$$

where $\{N_r\}$ corresponds to the equidistant surfaces given by the Epstein surfaces of h (in other words, $h = \mathrm{I}^*$). Then $W(M, h)$ is well-defined as a consequence of (10).

We can finally define the renormalized volume of M as

$$(12) \qquad V_R(M) = W(M, h),$$

where h is the metric in the conformal class at infinity that has constant curvature -1.

Krasnov-Schlenker [KS08] derived the variation formula of the W-volume in terms of the input at infinity (observe that because of the description by Epstein surfaces, I^* determines II^* and III^*) from the volume variation of Rivin-Schlenker [RS99]. Since for the rest of the paper we are going to write everything in terms of these limit tensors, let us omit the superscript *.

As we did in the previous section, let us fix $c \in \mathcal{T}(\partial M)$ and some metric I_c that represents it, so we can parametrize $\mathcal{T}(\partial M)$ by RQ_c. Then for $v \in RQ_c$ we have the variation of V_R at I_c ([KS08] Corollary 6.2, Lemma 8.5)

$$(13) \qquad DV_R(v) = -\frac{1}{4} \int_{\partial M} \langle D\mathrm{I}_c(v), \mathrm{II}_0 \rangle da,$$

where the metric between tensors and the area form da are defined from I_c and $\mathrm{II}_0 = \mathrm{II} - \frac{1}{2}\mathrm{I}$ is the traceless second fundamental form. This 2-form is (at each component of ∂M, after taking quotient by the action of $\pi_1(M)$) the negative of the real part of the Schwarzian derivative of the holomorphic map between one component of the region of discontinuity and a disk ([KS08] Lemma 8.3). In particular (as we stated in (8)) $\langle D\mathrm{I}_c(v), \mathrm{II}_0 \rangle = \langle v, \mathrm{II}_0 \rangle$ pointwise. Then if we take c to be a critical point (i.e. $DV_R(v) = 0$ at I_c for every $v \in RQ_c$) II_0 must vanish at every point. This in turn implies that the holomorphic map between a component of the region of discontinuity and a disk has Schwarzian derivative identically zero, which means that the components are disks and the boundary of the convex core is totally geodesic.

4. On the maximality of the W-volume among metrics of constant area

In Section 7 of [KS08], Krasnov and Schlenker study the variation of the W-volume among metrics of the same conformal class while keeping the area constant. One way of showing this is by observing that the Ricci flow is a gradient-like flow for the W-Volume. We include this argument due to the connection to Ricci flow, since earlier proofs of this fact appeared in [GMR17] (Proposition 7.1, which includes the cusped case) and previously in [GMS] (Proposition 3.11, convex co-compact case).

From [KS08] Corollary 6.2 we know that

$$(14) \qquad \delta W = -\frac{1}{4} \int_{\partial M} \delta H + \langle \delta \mathrm{I}, \mathbb{II}_0 \rangle da,$$

where we omit the superscript * and H denotes the mean curvature at infinity.

Now, since we are taking a conformal variation, $\delta \mathrm{I} = u\mathrm{I}$ for some function $u : \partial M \to \mathbb{R}$. Moreover, since the volume is preserved, $\int_{\partial M} u\, da = 0$.

Remember that $\mathbb{II}_0$ is the traceless part of the second fundamental form, so $\langle u\mathrm{I}, \mathbb{II}_0 \rangle$ vanishes pointwise. Also ([KS08] Remark 5.4) $H = -K$ and hence we have

$$(15) \qquad \delta W = \frac{1}{4} \int_{\partial M} \delta K\, da.$$

But by the Gauss-Bonnet formula $\int_{\partial M} \delta K\, da + \int_{\partial M} K \delta(da) = 0$ and the equality $\delta(da) = \frac{1}{2} \langle \delta \mathrm{I}, \mathrm{I} \rangle da = u\, da$, we can reduce it to

$$(16) \qquad \delta W = -\frac{1}{4} \int_{\partial M} K u\, da,$$

from where we can recover that $K = const.$ is the unique critical point. If two points had different values of K, we can take u supported around those points such that $\int_{\partial M} u\, da = 0$, but $-\int_{\partial M} K u\, da > 0$.

Note that

$$u = -K + \frac{2\pi \chi(\partial M)}{\mathrm{vol}(\partial M)}$$

has integral equal to 0 ($\int_{\partial M} K\, da = 2\pi \chi(\partial M)$), and by Hölder inequality

$$(17) \qquad \left(\int_{\partial M} K^2 da \right) \cdot (\mathrm{vol}(\partial M)) \geq \left(\int_{\partial M} K\, da \right)^2,$$

giving $-\frac{1}{4} \int_{\partial M} K u\, da \geq 0$. Hence the W-volume is no decreasing under the Ricci flow in two dimensions. It is known (see, for example, [IMS11]) that this flow converges to the metric of constant curvature, proving that this metric is a global maximum. Since $K = const.$ is the only critical point, the W-volume increases strictly under the flow, making this metric a strict global maximum.

5. Local convexity at the geodesic class

In order to study local behavior around the critical points we want to compute the Hessian of V_R at these points. Let c be a critical point (i.e. $\mathbb{II}_0 \equiv 0$) and I_c a metric representing this class. For $v, w \in RQ_c$, we vary (6) with respect to w to obtain

$$(18) \qquad \mathrm{Hess}V_R(v, w) = -\frac{1}{4}\int_{\partial M} \langle D\mathrm{I}(v), D\mathbb{II}_0(w)\rangle da$$

$$+ \text{(terms obtained by varying the other tensors w.r.t. w)},$$

and since $\mathbb{II}_0$ vanishes identically, all the terms in parenthesis get canceled, so

$$(19) \qquad \mathrm{Hess}V_R(v, w) = -\frac{1}{4}\int_{\partial M} \langle D\mathrm{I}(v), D\mathbb{II}_0(w)\rangle da.$$

Let us first understand the quasi-Fuchsian case (here we are referring to hyperbolic structures on the product $S \times \mathbb{R}$, where S is a closed orientable surface of genus $g > 1$). In order to differentiate the two ends let us label them (as well as the tensors defined on each one) with $+$ and $-$.

Theorem 1. *Let M be a Fuchsian manifold (i.e. the conformal classes at infinty c_+, c_- both equal to say a conformal class c). Then the Hessian at M of the renormalized volume is positive definite in the orthogonal complement of the diagonal of $RQ_c \times RQ_c \approx T_c\mathcal{T}(\partial M) = T_{c^+}\mathcal{T}(\partial M^+) \times T_{c^-}\mathcal{T}(\partial M^-)$ (where we are using the parametrization by real parts of holomorphic quadratic differentials w.r.t. (c, c) and that the real parts of holomorphic quadratic differentials are the same for both orientations). Moreover, the Hessian agrees with the metric induced by I with a $\frac{1}{8}$ factor.*

Proof. Recall by (19) that since M is a critical point, then for $v = (v_+, v_-)$, $w = (w_+, w_-)$ tangent vectors at (c, c)

$$(20) \qquad 4\mathrm{Hess}V_R(v, w) = -\int_{\partial_+ M} \langle D\mathrm{I}^+(v), D\mathbb{II}_0^+(w)\rangle da$$

$$-\int_{\partial_- M} \langle D\mathrm{I}^-(v), D\mathbb{II}_0^-(w)\rangle da.$$

As mentioned in [KS08](Lemma 8.3) we know that

$$(21) \qquad \mathbb{II}_0^+(c_+, c) = -\mathrm{Re}(q_+(c_+, c)),$$

where $q_+(c_+, .) : \mathcal{T}_- \to Q_{c_+}$ is the Bers embedding. In particular $D\mathbb{II}_0^+(v, 0)$ lands in RQ_c, and because $D\mathbb{II}_0^+(v, v) = 0$, then the whole image of $D\mathbb{II}_0^+$ lands in RQ_c. Since an analogous argument works for $D\mathbb{II}_0^-$, we can then reduce to

$$(22) \quad 4\mathrm{Hess}V_R(v, w) = -\int_{\partial_+ M} \langle v_+, D\mathbb{II}_0^+(w)\rangle da - \int_{\partial_- M} \langle v_-, D\mathbb{II}_0^-(w)\rangle da$$
$$= -\langle v, D\mathbb{II}_0(w)\rangle_{L^2},$$

where $\langle \cdot, \cdot \rangle$ on forms denotes the L^2 scalar product defined by I_c.

Now $D\mathbb{II}_0$ is diagonalizable with orthogonal eigenvectors (is the expression of the Hessian in terms of the metric induced by $\mathrm{I}(c, c)$). We can exploit this fact in the following lemma.

Lemma 1. $D\mathbb{II}_0(v, -v) = -\frac{1}{2}(v, -v)$

Proof. We prove first the following claim.

Claim: Let M be a Fuchsian manifold with associated conformal class c. Then $D\mathbb{III}^+(v, 0)$ is orthogonal to RQ_c for all $v \in RQ_c$.

Recall from [KS08] (as stated in the comments of Definition 5.3) that almost-Fuchsian manifolds are quasi-Fuchsian manifolds with the principal curvatures at infinity between $-1, 1$, and for those manifolds $\mathbb{III}$ is conformal to I^-, so $D\mathbb{III}^+(v, 0)$ is $\mathrm{I}(c, c)$ multiplied by some function ($\mathbb{III}$ stays conformal to $\mathrm{I}(c, c)$ when we only varied c_+). To see that those tensors are orthogonal to RQ_c (note that for every tensor space we are taking the inner product induced by I_c and integration), recall that if we take conformal coordinates, elements of RQ_c are expressed in terms of dz^2 and $d\bar{z}^2$, where the metric is in terms of $dzd\bar{z}$, meaning that $D\mathbb{III}^+(v, 0)$ is even pointwise orthogonal to any element of RQ_c.

Going back to the proof of the lemma, it follows from [KS08] (Definition 5.3) that the first variations at M satisfy

$$(23) \qquad\qquad \delta\mathbb{II}_0 = \delta\mathbb{II} - \frac{1}{2}\delta\mathrm{I} = \mathrm{I}(\delta B\cdot, \cdot)$$

$$(24) \qquad\qquad \delta\mathbb{III} = \frac{1}{4}\delta\mathrm{I} + \mathrm{I}(\delta B\cdot, \cdot),$$

hence

$$(25) \qquad\qquad D\mathbb{III} = D\mathbb{II}_0 + \frac{1}{4}D\mathrm{I}.$$

In particular, by the claim, $D\mathbb{II}_0(v, 0) = -\frac{1}{4}v$, and so

$$(26) \quad D\mathbb{II}_0^+(v, -v) = D\mathbb{II}_0^+(2v, 0) + D\mathbb{II}_0^+(-v, -v) = -\frac{1}{4}D\mathrm{I}^+(2v, 0) = -\frac{1}{2}v.$$

The lemma follows since $D\mathbb{II}_0$ is diagonalizable in the orthogonal complement of the diagonal (the diagonal is part of the 0-eigenspace). □

This lemma implies that the orthogonal complement is the $-\frac{1}{2}$-eigenspace for $D\mathbb{II}_0$, concluding the last part of the theorem. □

Now we will use this local behavior for quasi-Fuchsian manifolds to conclude our main result for acylindrical manifolds.

Theorem 2. *Let M be a compact acylindrical 3-manifold with hyperbolic interior such that $\partial M \neq \emptyset$. Then there is a unique critical point c for the renormalized volume of M, where c is the unique conformal class at the boundary that makes every component of the region of discontinuity a disk (a.k.a. the geodesic class). The Hessian at this critical point is positive definite.*

Proof. Since $DV_R(v) = -\frac{1}{4}\langle D\mathrm{I}_c(v), \mathbb{II}_0 \rangle = -\frac{1}{4}\langle v, \mathbb{II}_0 \rangle$ we have

$$(27) \qquad\qquad DV_R = 0 \Leftrightarrow \mathbb{II}_0 \equiv 0.$$

Now, $\mathbb{II}_0 \equiv 0$ at each boundary component corresponds to the nullity of the Schwarzian derivative between every component of the domain of discontinuity and a disk. This implies as we stated that every component of the region of discontinuity is a disk, and since our manifold is acylindrical, there is a unique such point in the Teichmüller space of the boundary. We can also observe that for this point the boundary of the convex core is completely geodesic.

To prove that it is a strict local minimum, it is sufficient to prove that the Hessian is positive definite at this point. Let $\partial M = S_1 \cup \ldots \cup S_n$, and $c = (c_1, \ldots, c_n)$ denote the geodesic class. To show that the Hessian is positive definite, take parametrization RQ for S_i based at c_i and use the same metric to compute variations of V_R for M and for both ends of quasi-Fuchsian manifolds $S_i \times \mathbb{R}$.

Recall that, since M is hyperbolic acylindrical, the subgroups associated to the components of ∂M are quasifuchsian. Hence we have a map from $\mathcal{T}(\partial M)$ (corresponding to hyperbolic metrics in M) to $\mathcal{T}(\partial M) \times \mathcal{T}(\overline{\partial M})$ (corresponding to the quasifuchsian subgroups). The first coordinate of this

map is the identity on $\mathcal{T}(\partial M)$, while the second coordinate $\sigma : \mathcal{T}(\partial M) \to \mathcal{T}(\overline{\partial M})$ is Thurston's *skinning map*. Observe then that I at S_i coincides with $\mathrm{I}^+(c_i, \sigma_i(c))$, where σ_i is the image of the skinning map σ corresponding to S_i. From the dependence of $\mathrm{I\!I}$ in terms of I (Remark 5.9 of [KS08]) we see that $\mathrm{I\!I}_0$ at S_i coincides with $\mathrm{I\!I}_0^+(c_i, \sigma_i(c))$, and hence $DI(v), D\mathrm{I\!I}_0(w)$ are equal to $DI^+(v_i, d\sigma_i(v)), D\mathrm{I\!I}_0^+(w_i, d\sigma_i(w))$ at S_i (note that here $d\sigma$ is written in terms of our charts given by RQ_c, so it is essentially the conjugation of the derivative of the skinning map, given our remark on how our charts behave with an orientation change). Hence

$$(28) \qquad 4\mathrm{Hess}V_R(v, w) = -\sum_{i=1}^{n} \langle v_i, D\mathrm{I\!I}_0^+(w_i, d\sigma_i(w)) \rangle$$

$$= \frac{1}{4} \sum_{i=1}^{n} \langle v_i, w_i - d\sigma_i(w) \rangle,$$

which is greater than zero for $v \neq 0$ according to the following result:

Theorem 3 (McMullen [McM90]). *Under the conditions above, $\|d\sigma\| < 1$, where the norm $\|d\sigma\|$ is calculated in terms of the Teichmüller metric.*

Indeed, (28) tells us that $d\sigma_c$ is diagonalizable, which together with McMullen's theorem implies that all eigenvalues are less than 1. Then $\mathrm{Hess}V_R$ is also diagonalizable with all eigenvalues positive. □

McMullen's result can be sketched as follows. The skinning map is holomorphic between Teichmüller spaces, in particular sending holomorphic disks to holomorphic disks. This implies that $\|d\sigma\| \leq 1$ with respect to the Kobayashi metric, which coincides with the Teichmüller metric. If $\|d\sigma\| = 1$ at a point, then σ would be an isometry on the extremal holomorphic disk, but an earlier result of Thurston states that σ is a strict contraction for the acylindrical case.

Corollary 1. *Let $c \in \mathcal{T}(\partial M)$ be as in the previous theorem. Then $d\sigma$ admits an orthonormal eigenbasis with respect to the L^2-norm on RQ_c, with all eigenvalues less than 1 in absolute value.*

Observe that if we take holomorphic quadratic differentials for the tangent space, we have the conclusion for $d\sigma$ after taking a complex conjugation.

This result was somehow expected thanks to a parallel between the skinning map and the Thurston map for postcritically finite rational maps, since

the Thurston map is contracting for non-Lattè maps. Moreover, if f is a postcritically finite quadratic polynomial, there is a uniform spectral gap for the derivative at its unique critical point (more precisely, all eigenvalues are greater than 1/8 in norm. For this refer to [BEK]). It is unknown to the author if there is an analog for hyperbolic 3-manifolds, namely a family of manifolds for which there is some uniform espectral gap.

It is worth mentioning that it is still open that the geodesic class is the absolute minimum of V_R (for acylindrical manifolds). It is also open that V_R is strictly positive for quasi-Fuchsian manifolds outside the Fuchsian locus, although this is known for almost-Fuchsian manifolds [CM].

6. Remarks for hyperbolic 3-manifolds with rank-1 cusps

Consider a quasi-Fuchsian manifold M with rank-1 cusps and a compact subset K of ∂M^+ (the top boundary component as we denoted in (5)). For a quasi-Fuchsian manifold sufficiently close to the Fuchsian locus, the equidistant foliation over K extends up to the 0-leaf and has principal curvatures between -1 and 1. As stated in the comments of [KS08] Definition 5.3, this foliation extends to the other end of M and III^+ is a metric conformal to I^- (all of this over K). Hence, at a Fuchsian manifold, $D\mathrm{III}^+(v,0)$ is a multiple of I^+ at any such compact K, and hence over all of ∂M^+.

On the other hand, all the formulas of [KS08] Theorem 5.8 apply for the rank 1-cusps case, so in particular we also have that II_0 is (at each component of ∂M after taking quotient by the action of $\pi_1(M)$) the negative of the real part of the Schwarzian derivative of the holomorphic map between one component of the region of discontinuity and a disk.

As we mentioned, the variation formula of the renormalized volume was proved in [GMR17]. The second variation at a critical point is well defined since for the quasi-Fuchsian case it is the inner product of quadratic holomorphic differentials, which are in L^2 with respect to the metric. The other terms that get canceled with II_0 in (18) do not overcome the exponential decay of the metric since each of them have at most polynomial growth. The general case will then be well defined since it is also an inner product of quadratic holomorphic differentials, thanks to the skinning map argument.

Finally, McMullen's result still applies for surfaces with punctures, so the proof of our theorem extends to the case of geometrically finite hyperbolic acylindrical manifolds with rank 1 cusps. We summarize this as follows.

Theorem 4. *Let M be a Fuchsian manifold (i.e. the conformal classes at infinty c_+, c_- coincide, say to a conformal class c) with possibly rank 1-cusps. Then the Hessian at M of the renormalized volume is positive definite in the orthogonal complement of the diagonal of $RQ_c \times RQ_c \approx T_c \mathcal{T}(\partial M) = T_{c^+} \mathcal{T}(\partial M^+) \times T_{c^-} \mathcal{T}(\partial M^-)$ (where we are using the parametrization by real parts of holomorphic quadratic differentials w.r.t. (c, c) and that the real parts of holomorphic quadratic differentials are the same for both orientations). Moreover, the Hessian agrees with the metric induced by I with a $\frac{1}{8}$ factor.*

Theorem 5. *Let M be a geometrically finite manifold with rank 1-cusps. Then there is a unique critical point c for the renormalized volume of M, where c is the unique conformal class at the boundary that makes every component of the region of discontinuity a disk(a.k.a. the geodesic class). The Hessian at this critical point is positive definite.*

We can also extend the Ricci flow argument for the maximality of the W-volume. We use the existence and convergence properties of the Ricci flow from the conditions stated in [JMS09]. In the language of [GMR17], let a cusp be parametrized as $(v, w) \in \left[0, \frac{1}{R}\right[\times \mathbb{R}/\frac{1}{2}\mathbb{Z}$, which under the notation of [JMS09] corresponds to (s, w), where $v = e^{-s}$.

We need to show that $\psi \in C_r^\infty(\overline{\partial M}) \Rightarrow e^{2\phi} \in R + s^{-\mu} C^{2,\alpha}$ (where the first term describes the family of conformal factors for which the renormalized volume is defined in [GMR17], and the second term the sufficient conditions for running Ricci flow in the lines of [JMS09]) for some constants $\mu > 0, R$; where $v = e^{-s}$, $\psi(v, w) = \phi(s, w)$. $C_r^\infty(\overline{\partial M})$ denotes ([GMR17] 2.7) functions whose w-derivative vanishes with infinite order at $v = 0$. $C^{2,\alpha}$ denotes the usual Hölder space with respect to the hyperbolic metric on the cusp.

Since $\psi \in C_r^\infty(\overline{M})$, $\lim_{v \to 0} \psi(v, w)$ exists and does not depend on w, which we denote by R. Then we need to show that $a(s, w) = e^{2\phi(s,w)} - R$ belongs to $s^{-\mu} C^{2,\alpha}$. We will show that belongs to $s^{-\mu} C^3$ for $0 < \mu < 1$.

First observe that $b(v, w) = a(s, w)$ extends to $v = 0$ in C^∞ because ψ does. In particular,

$$(29) \qquad\qquad b_v(v, w) = 2\psi_v(v, w) e^{2\psi(v, w)}$$

also extends to $v = 0$, and $b_v(0, w)$ does not depend on w. There is a constant K such that $|b(v, w)| \leq Kv^\mu$, so $|a(s, w)| \leq Ke^{-s\mu} \leq K's^{-\mu}$, which gives $a \in s^{-\mu} C^0$.

Now $a_s(s,w) = 2\phi_s(s,w)e^{2\phi(s,w)}$, and we need to show that $|a_s(s,w)| \leq Ks^{-\mu}$. This follows from

$$(30) \qquad 2\phi_s(s,w)e^{2\phi(s,w)} = -2e^{-s}\psi_v(v,w)e^{2\psi(v,w)}$$

and the fact that $\psi_v(v,w)e^{2\psi(v,w)}$ extends to $v = 0$ not depending on w.

Now the pattern appears for higher y-derivatives of a. When we express the derivatives in v coordinates, the expressions gain a $v = e^{-s}$, so the decay follows since ψ extends C^∞ to $v = 0$ without w dependence.

For the case of taking a w-derivative, its norm in C^1 is $e^{2s}a_w$ (the metric to define $C^{k,\alpha}$ is $ds^2 + e^{-2s}dw^2$). We have

$$(31) \qquad e^{2s}2\phi_w(s,w)e^{2\phi(s,w)} = v^{-2}\psi_w(v,w)e^{2\psi(v,w)},$$

which is bounded by some power of v because ψ_w has infinite order 0 at $v = 0$. Similarly, we extend our argument to further w-derivatives. This shows that we can run Ricci flow with initial condition any metric considered by [GMR17].

Finally, to observe that the logarithm of the conformal factors along the Ricci flow still belong to $C_r^\infty(\overline{M})$, note that the curvature $e^{-2\psi}(-1 + \Delta\psi)$ is in $C_r^\infty(\overline{M})$. This makes the flow to preserve the set of metrics with such conformal factors, which in turn allows the argument to extend to this generality.

7. Corrected renormalized volume

Let M be a closed compact hyperbolic 3-manifold with an oriented incompressible surface Σ ($g > 2$) which divides M into two components M_1, M_2. We can consider now hyperbolic structures N_1, N_2 on the interiors of M_1, M_2 such that we have the inclusions $M_1 \subset N_1, M_2 \subset N_2$ by taking coverings with respect to $\pi_1(M_1), \pi_1(M_2)$, respectively. The fact that they glue along Σ tells us that the conformal classes at infinity of the open manifolds are each others skinning maps.

Now, for r sufficiently large, M_1 lives inside N_1^r, the r-leaf of the foliation defined by the renormalized volume, and hence

$$(32) \quad \mathrm{vol}(M_1) = V_R(M_1, c_1) + r\pi\chi(\Sigma) - \mathrm{vol}((\mathrm{int}N_1^r) \setminus M_1) + \frac{1}{4}\int_{N_1^r} H\,da,$$

where c_1 is the conformal class at infinity for $\mathrm{int}(M_1)$.

We can take r even larger so that we have a similar statement for M_2. Observe that the regions $(\mathrm{int}N_1^r) \setminus M_1, (\mathrm{int}N_2^r) \setminus M_2$ glue along Σ and live inside the quasi-Fuchsian manifold obtained by gluing $N_1 \setminus M_1, N_2 \setminus M_2$ along Σ. In particular the quasi-Fuchsian manifold $(N_1 \setminus M_1) \cup (N_2 \setminus M_2)$ has (c_1, c_2) as a conformal class at infinity and N_1^r, N_2^r as the r-leaves for the renormalized volume, from where we conclude

$$(33) \qquad \mathrm{vol}((\mathrm{int}N_1^r) \setminus M_1 \cup (\mathrm{int}N_2^r) \setminus M_2)$$

$$= V_R(\Sigma \times [0,1], c_1, c_2) + \frac{1}{4} \int_{N_1^r} H da$$

$$+ \frac{1}{4} \int_{N_2^r} H da + 2r\pi\chi(\Sigma).$$

This gives the following proposition.

Proposition 1. *For M as above,*

$$\mathrm{vol}(M) = V_R(M_1, c_1) + V_R(M_2, c_2) - V_R(\Sigma \times [0,1], c_1, c_2).$$

In order to use this proposition to find a lower bound for the volume of M (and since c_1, c_2 are related by the skinning map), we want to show that the minimum of the corrected renormalized volume (defined as follows) is attained at the geodesic class.

Definition 1. *Let M be a compact acylindrical 3-manifold with hyperbolic interior, connected boundary of genus $g > 1$, and let $c \in \mathcal{T}(\partial M)$ the element that defines the conformal boundary at infinity. Then the corrected renormalized volume is defined as*

$$(34) \qquad \overline{V}_R(M) = V_R(M, c) - \frac{1}{2}V_R(\partial M \times [0,1], (c, \sigma(c))).$$

Observe that proposition (1) implies that, under cutting, the volume of M is equal to the sum of the corrected renormalized volumes of its parts. It is straightforward to extend this to the case when Σ has multiple components, and if M is open without cusps, we can replace $\mathrm{vol}(M)$ by $\overline{V}_R(M)$ in Proposition (1). Then if we consider the corrected renormalized volume to be an extension of the volume for closed hyperbolic manifolds, we obtain as a corollary of proposition 1:

Corollary 2. *$\overline{V}_R$ is additive under cutting.*

Now, from the first variation of V_R at $\mathrm{I} = \mathrm{I}^+$, we have

$$(35) \qquad 8D\overline{V}_R(v) = -\int_{\partial M^+} \langle D\mathrm{I}^+(v, d\sigma(v)), \mathrm{II}_0^+(c, \sigma(c))\rangle da^+$$
$$+ \int_{\partial M^-} \langle D\mathrm{I}^-(v, d\sigma(v)), \mathrm{II}_0^-(c, \sigma(c))\rangle da^-,$$

where da^+, da^- are the volume forms for $\mathrm{I}^+, \mathrm{I}^-$, respectively.

Observe that if we take c to be the geodesic class, $\mathrm{II}_0^+, \mathrm{II}_0^-$ are both zero, and hence c is also a critical point for the corrected renormalized volume. Moreover, the Hessian is expressed as

$$(36) \qquad 8\mathrm{Hess}\overline{V}_R(v, w) = -\langle D\mathrm{I}^+(v, d\sigma(v)), D\mathrm{II}_0^+(w, \sigma(w))\rangle\rangle$$
$$+ \langle D\mathrm{I}^-(v, d\sigma(v)), D\mathrm{II}_0^-(w, \sigma(w))\rangle,$$

which by Lemma 1 is equal to

$$(37) \qquad 8\mathrm{Hess}\overline{V}_R(v, w) = -\left\langle v, -\frac{1}{4}(w - d\sigma(w)) \right\rangle$$
$$+ \left\langle d\sigma(v), \frac{1}{4}(w - d\sigma(w)) \right\rangle.$$

Then

$$(38) \qquad 32\mathrm{Hess}\overline{V}_R(v, w) = \langle v + d\sigma(v), w - d\sigma(w)\rangle,$$

and since all eigenvalues of $d\sigma$ are between -1 and 1, we obtain the following result.

Theorem 6. *Let M be a compact acylindrical 3-manifold with hyperbolic interior, $\partial M \neq \emptyset$ without cusps, and $c \in \mathcal{T}(\partial M)$ be the geodesic class. Then c is a local minimum for the corrected renormalized volume of M.*

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