Math. Res. Lett.
Volume 26, Number 2, 421–445, 2019

Equidistribution of Neumann data mass on simplices and a simple inverse problem

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In this paper we study the behaviour of the Neumann data of Dirichlet eigenfunctions on simplices. We prove that the L^2 norm of the (semi-classical) Neumann data on each face is equal to $2/n$ times the $(n-1)$-dimensional volume of the face divided by the volume of the simplex. This is a generalization of [2] to higher dimensions. Again it is *not* an asymptotic, but an exact formula. The proof is by simple integrations by parts and linear algebra.

We also consider the following inverse problem: do the *norms* of the Neumann data on a simplex determine a constant coefficient elliptic operator? The answer is yes in dimension 2 and no in higher dimensions.

1. Introduction

In this paper we extend the results of [2] on triangles to simplices, which are the higher dimensional analogues of triangles. The proof has many similarities but involves more linear algebra and elementary geometry. We have chosen to separate the two proofs in order to make the paper about triangles simple and clean. We also have added to this paper some applications to rudimentary inverse problems.

Let $T \subset \mathbb{R}^n$ be an n-dimensional (non-degenerate) simplex with faces $G_0, \dots, G_n$. We consider the Dirichlet eigenfunction problem on T:

$$\begin{cases} -h^2 \Delta u = u \text{ in } T, \\ u|_{\partial T} = 0. \end{cases} \tag{1.1}$$

The semiclassical parameter $h > 0$ denotes the (inverse of) the eigenvalues hence takes values in a discrete set. We assume that the eigenfunctions are normalized: $\|u\|_{L^2(T)} = 1$. Our main result, similarly to in [2] is that the Neumann data on each face of the simplex is proportional to the volume of the face.

Theorem 1. *Let $T \subset \mathbb{R}^n$ be a non-degenerate simplex with faces $G_0, G_1, \ldots, G_n$ and suppose u solves* (1.1).

Then the (semi-classical) Neumann data on each of the the boundary faces satisfies

$$\int_{G_j} |h\partial_\omega u|^2 dS_j = \frac{2\, Vol_{n-1}(G_j)}{n\, Vol_n(T)}. \tag{1.2}$$

Here $h\partial_\omega$ is the semi-classical normal derivative on ∂T, dS_j is the surface measure on G_j, $Vol_n(T)$ is the volume of the simplex T, and $Vol_{n-1}(G_j)$ is the $(n-1)$-dimensional induced volume of G_j.

Remark 1.1. As in [2], we are calling this "equidistribution" of Neumann mass since it says that the Neumann data has mass proportional to the $(n-1)$-dimensional volume of the face to which it is restricted. Again it should be remarked that perhaps the most surprising part of this result is that it holds for the *entire sequence* of eigenfunctions, and is an exact formula, rather than an asymptotic.

The proportionality constant in (1.2) depends in a seemingly non-obvious way on the dimension n. However, it turns out this is the right dimensional constant in the case of the Cauchy data for quantum ergodic eigenfunctions restricted to a hypersurface, and indeed also for the boundary data quantum ergodic restriction theorems in the original studies [5, 7]. One of the author's motivations for the present paper was to see if one could isolate the mass of the Dirichlet vs. Neumann data of quantum ergodic eigenfunctions restricted to an interior simplex hypersurface in the Cauchy data restriction theorem in [4]. Unfortunately this does not help, and the present paper and [2] do not preclude the possibility of quantum ergodic eigenfunctions having $o(1)$ (in L^2) restrictions to the boundary of an interior simplex. See below for a brief history.

A statement such as Theorem 1 is false in general for other polygonal domains. It is clearly false in the case of a square, as discussed in [2], as well as for a rectangular parallelepiped in any dimension by looking at Fourier series.

Remark 1.2. From a big picture point of view, part of the method of proof is to study two equivalent problems: one is the study of the flat Laplacian on an arbitrary simplex, and the other is the study of an arbitrary constant coefficient elliptic operator on the standard simplex. Elementary linear algebra takes one problem to the other. Indeed, one could study an arbitrary constant coefficient elliptic operator on an arbitrary simplex using these methods as well. The duality of these two problems was what led to the statement of the inverse problem in Section 4.

1.1. Brief history

Previous results on restrictions to hypersurfaces primarily focused on upper bounds. Burq-Gérard-Tzvetkov [1] give an upper bound of the norm (squared) of the restrictions of eigenfunctions, of order $\mathcal{O}(h^{-1/2})$. In the author's paper with Hassell-Toth [3], an upper bound of $\mathcal{O}(1)$ was proved for (semi-classical) Neumann data restricted to arbitrary co-dimension 1 hypersurfaces in any dimension. Both of these estimates are shown to be sharp, so this gives a lower (and upper) bound for *some* eigenfunctions.

In the case of quantum ergodic eigenfunctions, a little more is known. Gérard-Leichtnam [5] and Hassell-Zelditch [7] give asymptotic formulae for (a density one subsequence of) the Neumann (respectively Dirichlet) boundary data of Dirichlet (respectively Neumann) quantum ergodic eigenfunctions. That means that there is a lower bound, and explicit local asymptotic formula in this special case, at least for most of the eigenfunctions. Similar statements were proved for interior hypersurfaces in [4, 8, 9]. However, for an interior hypersurface, it seems an intractible problem to separate the behaviour of the Dirichlet or Neumann data, or a sparse subsequence must be removed. This again gives lower bounds on the norms of the Dirichlet or Neumann data for *some* of the eigenfunctions.

2. The standard simplex in $\mathbb{R}^3$

In this section we prove the theorem for the standard simplex in dimension 3 as it is simple to see how the proof works in this case. In Section 3 we prove the general result.

Let $p_0 = (0,0,0)$, $p_1 = (1,0,0)$, $p_2 = (0,1,0)$, and $p_3 = (0,0,1)$. The standard simplex is given by all convex combinations of these vectors:

$$T = \left\{ \sum_{j=0}^{3} t_j p_j : \sum_{j=0}^{3} t_j = 1, \text{ and } t_j \geqslant 0 \right\}.$$

That is, T is the four sided solid with the p_j and 0 at the corners.

We use (x_1, x_2, x_3) as the standard rectangular coordinates in $\mathbb{R}^3$. Let F_1 denote the face in the (x_2, x_3) plane (where $x_1 = 0$), F_2 the face where $x_2 = 0$, F_3 the face where $x_3 = 0$, and F_4 the remaining face. Then the unit normals are $\nu_j = -e_j$, $j = 1, 2, 3$ and $\nu_4 = (3)^{-1/2}(1, 1, 1)$ respectively, where e_j are the standard basis vectors pointing in the direction of x_j respectively. Then the statement of the theorem involves the quantities $|\nu_j \cdot h\partial u|$ restricted to their respective faces.

Let us denote by X the vector field

$$X = (x_1 + m_1)\partial_{x_1} + (x_2 + m_2)\partial_{x_2} + (x_3 + m_3)\partial_{x_3},$$

where the m_js are parameters independent of x. A simple computation yields that $[-h^2\Delta - 1, X] = -2h^2\Delta$.

Remark 2.1. We are going to apply Green's formula on simplices to u and Xu. However, it should be noted that in order to apply Green's formula in this context, we need that both u and the *gradient* of u are H^2 functions. The referee has been kind enough to point out that this is non-trivial for a domain with Lipschitz boundary, but that simplices are convex, and much more is known about Sobolev regularity for elliptic equations on convex domains. In the appendix we very briefly summarize the results in [6] which allow us to conclude that Xu is in H^2.

Now the eigenfunction equation (1.1) tells us that

$$[-h^2\Delta - 1, X]u = -2h^2\Delta u = 2u.$$

As Remark 2.1 indicates, u and Xu are both H^2 functions, so we may apply Theorem 4 (Green's formula on simplices) to get

$$\begin{aligned}
&\int_T ([-h^2\Delta - 1, X]u)\bar{u}dV \\
&= \int_T ((-h^2\Delta - 1)Xu)\bar{u}dV \\
&= \int_{\partial T} (-h\partial_\nu hXu)\bar{u}dS + \int_{\partial T} (hXu)(h\partial_\nu \bar{u})dS,
\end{aligned}$$

or

$$\begin{aligned}
2 &= 2\int_T |u|^2 dV \\
&= -2\int_T (h^2\Delta u)\bar{u} dV \\
&= \int_{\partial T} (-h\partial_\nu hXu)\bar{u} dS + \int_{\partial T} (hXu)(h\partial_\nu \bar{u}) dS \qquad (2.1) \\
&= \int_{\partial T} (hXu)(h\partial_\nu \bar{u}) dS, \qquad (2.2)
\end{aligned}$$

since we have assumed Dirichlet boundary conditions.

Let us break the analysis into the four different faces. On F_1, we have

$$\begin{aligned}
&\int_{F_1} (hXu)(h\partial_\nu \bar{u}) dS \\
&\quad = \int_{F_1} (((x_1+m_1)h\partial_{x_1} + (x_2+m_2)h\partial_{x_2} + (x_3+m_3)h\partial_{x_3})u)\bar{u} dS_1 \\
&\quad = -m_1 \int_{F_1} |h\partial_{\nu_1} u|^2 dS_1,
\end{aligned}$$

since $h\partial_{x_1} = -h\partial_{\nu_1}$ and $h\partial_{x_j}$ is tangential when $j = 2, 3$. Similarly, for $j = 2, 3$ we have

$$\int_{F_j} (hXu)(h\partial_{\nu_j} \bar{u}) dS_j = -m_j \int_{F_j} |h\partial_{\nu_j}|^2 dS_j.$$

On F_4 we need to be a little bit more careful. The points on F_4 all satisfy $x_1 + x_2 + x_3 = 1$ since the normal is parallel to $(1, 1, 1)$. The normal derivative is $h\partial_{\nu_4} = 3^{-1/2}(h\partial_{x_1} + h\partial_{x_2} + h\partial_{x_3})$, and the tangent vectors are all linear combinations of $e_3 - e_1 = (-1, 0, 0)$ and $e_2 - e_1 = (-1, 1, 0)$, so that, acting on functions which vanish on F_4,

$$\partial_{x_j} = 3^{-1/2}\partial_{\nu_4}$$

for $j = 1, 2, 3$. Hence

$$\begin{aligned}
&\int_{F_4} (hXu)h\partial_{\nu_4}\bar{u}dS_4 \\
&= \int_{F_4} (((x_1+m_1)h\partial_{x_1} + (x_2+m_2)h\partial_{x_2} + (x_3+m_3)h\partial_{x_3})u)h\partial_{\nu_4}\bar{u}dS_4 \\
&= (3)^{-1/2}\int_{F_4} (((x_1+m_1)+(x_2+m_2)+(x_3+m_3))h\partial_{\nu_4}u)h\partial_{\nu_4}\bar{u}dS_4 \\
&= 3^{-1/2}(1+m_1+m_2+m_3)\int_{F_4} |h\partial_{\nu_4}u|^2 dS_4.
\end{aligned}$$

Summing up, we have

$$\begin{aligned}
(2.3)\quad 2 = &-m_1\int_{F_1}|h\partial_{\nu_1}u|^2dS_1 - m_2\int_{F_2}|h\partial_{\nu_2}u|^2dS_2 - m_3\int_{F_3}|h\partial_{\nu_3}u|^2dS_3 \\
&+ 3^{-1/2}(1+m_1+m_2+m_3)\int_{F_4}|h\partial_{\nu_4}u|^2dS_4.
\end{aligned}$$

Now if $m_j = 0$ for $j = 1, 2, 3$, using (2.3) we have

$$2 = (3)^{-1/2}\int_{F_4}|h\partial_{\nu_4}u|^2dS_4,$$

so that

$$\int_{F_4}|h\partial_{\nu_4}u|^2dS_4 = 3^{1/2}\cdot 2.$$

We know that $\mathrm{Vol}_3(T) = 1/3! = 1/6$. The cross product computes the area of the parallelogram, which is twice the area of the triangle, so that tells us that

$$\begin{aligned}
\mathrm{Vol}_2(F_4) &= |(-1,1,0)\times(-1,0,1)|/2 \\
&= \sqrt{3}/2.
\end{aligned}$$

Hence

$$\begin{aligned}\int_{F_4} |h\partial_{\nu_4} u|^2 dS_4 &= 2\cdot 3^{1/2} \\ &= 4(\sqrt{3}/2) \\ &= (2/3)\left(\frac{2\cdot 3^{1/2}}{1/6}\right) \\ &= \frac{2\mathrm{Vol}_2(F_4)}{n\mathrm{Vol}_3(T)}.\end{aligned}$$

For $j = 1, 2, 3$ we have

$$\mathrm{Vol}_2(F_j) = 1/2.$$

Differentiating (2.3) with respect to m_j, we have

$$0 = -\int_{F_j} |h\partial_{\nu_j} u|^2 dS_j + (3)^{-1/2}\int_{F_4} |h\partial_{\nu_4} u|^2 dS_4,$$

or

$$\begin{aligned}2 &= \int_{F_j} |h\partial_{\nu_j} u|^2 dS_j \\ &= \left(\frac{2}{3}\right)\left(\frac{1/2}{1/6}\right) \\ &= \left(\frac{2}{3}\right)\frac{\mathrm{Vol}_2(F_j)}{\mathrm{Vol}_3(T)}.\end{aligned}$$

This proves the theorem for the standard simplex in dimension 3.

3. Proof of Theorem 1

Let $p_1, \ldots, p_n$ be independent vectors in $\mathbb{R}^n$, and let $p_0 = (0, \ldots, 0)$ denote the origin. Then

$$T = \left\{\sum_0^n t_j p_j : \sum t_j = 1 \text{ and } t_j \geqslant 0\right\} \tag{3.1}$$

is a simplex. If $p_j = e_j$ (standard rectangular basis vectors) for each j, then we say T is the standard simplex and denote it by T_0.

Since the p_js are independent, the matrix

$$A = \begin{bmatrix} | & | & \cdots & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & \cdots & | \end{bmatrix}$$

is invertible. Let $B = A^{-1}$, and for $x \in \mathbb{R}^n$ set

$$y = Bx.$$

This transformation simply takes the simplex T to the standard simplex T_0. Indeed, if $x = p_j$, then $Bx = e_j$. Hence

$$T_0 = \left\{ \sum t_j B p_j, \sum t_j = 1, \, t_j \geqslant 0 \forall j \right\}.$$

We pause briefly to point out that this change of variables induces a volume element, so that

$$\det(A) = n! \mathrm{Vol}(T).$$

This is easily seen using the volume of the standard simplex is $1/n!$ and the Jacobian for a change of volume integral is $\det(A)$.

We lift the transformation to $T^*\mathbb{R}^n$: for $\xi \in \mathbb{R}^n$, let $\eta = (B^{-1})^T \xi$. Then since the symbol of the Laplacian in $\mathbb{R}^n$ is $\xi_1^2 + \cdots + \xi_n^2$, the symbol for the Laplacian in our new coordinates is

$$\xi^T \xi = \eta^T B B^T \eta.$$

Set $\Gamma = BB^T$ and

$$-h^2 \widetilde{\Delta} = -\sum \Gamma_{ij} \partial_{y_i} \partial_{y_j},$$

the Laplacian in the y coordinates on the standard simplex T_0.

For the eigenfunctions u on T, let $v(y) = u(Ay)$ be the eigenfunctions in the y coordinates. Since $-h^2 \widetilde{\Delta}$ is constant coefficient, the same commutator argument can be used here. Indeed, let

$$Y = \sum (y_j + m_j) \partial_{y_j},$$

and a simple calculation gives

$$[-h^2 \widetilde{\Delta} - 1, Y] = -2h^2 \widetilde{\Delta}.$$

Following the recipe in Section 2 and Remark 2.1, we have using $-h^2\widetilde{\Delta}v = v$ and Green's formula on simplices

$$\begin{aligned}
2\int_{T_0} |v|^2 dy &= -2\int_{T_0} (h^2\widetilde{\Delta}v)\bar{v}dy \\
&= \int_{T_0} ([-h^2\widetilde{\Delta} - 1, Y]v)\bar{v}dy \\
&= \int_{T_0} ((-h^2\widetilde{\Delta} - 1)Yv)\bar{v}dy \\
&= \int_{T_0} ((-(h\partial)^T BB^T h\partial - 1)Yv)\bar{v}dy \\
&= \int_{T_0} (BB^T h\partial Yv) \cdot (h\partial\bar{v})dy - \int_{T_0} (Yv)\bar{v}dy \\
&\quad + \int_{\partial T_0} (-\nu^T BB^T h\partial(hYv))\bar{v}dy \\
&= \int_{T_0} (BB^T h\partial Yv) \cdot (h\partial\bar{v})dy - \int_{T_0} (Yv)\bar{v}dy
\end{aligned}$$

since we have assumed Dirichlet boundary condtions. Here ν denotes the unit outward normal and dS denotes the induced surface measure. Continuing,

$$\begin{aligned}
2\int_{T_0} |v|^2 dy &= \int_{T_0} (BB^T h\partial Yv) \cdot (h\partial\bar{v})dy - \int_{T_0} (Yv)\bar{v}dy \\
&= \int_{T_0} (Yv)(-h\partial^T BB^T h\partial\bar{v})dy - \int_{T_0} (Yv)\bar{v}dy \\
&\quad + \int_{\partial T_0} (hYv)(\nu^T BB^T h\partial\bar{v})dS \\
&= \int_{\partial T_0} (hYv)(\nu^T BB^T h\partial\bar{v})dS \qquad (3.2)
\end{aligned}$$

since $\widetilde{\Delta}\bar{v} = \bar{v}$.

We have changed variables to be on T_0 in order to make sure the normal vectors are easy to compute. For T_0, let F_j be the side where $y_j = 0$, $1 \leqslant j \leqslant n$, and F_0 the remaining face. Then for $1 \leqslant j \leqslant n$, we have the outgoing normal vectors to F_j $\nu_j = -e_j$, where the e_j are the standard basis vectors. For F_0, we have transformed to T_0 so that

$$\nu_0 = n^{-1/2}(1, \ldots, 1).$$

Then the unit normal derivatives are

$$h\partial_{\nu_j} = -h\partial_{y_j}$$

for $1 \leqslant j \leqslant n$ and

$$h\partial_{\nu_0} = n^{-1/2}(h\partial_{x_1} + \cdots + h\partial_{x_n}).$$

We are assuming Dirichlet boundary conditions, so all of the tangential derivatives of v vanish. That is, for $1 \leqslant j \leqslant n$,

$$h\partial_\ell v = 0,$$

except for $\ell = j$. We also have using symmetry that on F_0,

$$h\partial_{\nu_0} v = n^{1/2} h\partial_{y_j} v$$

for every $1 \leqslant j \leqslant n$. We recall again that $y_j = 0$ on F_j for $1 \leqslant j \leqslant n$ and on F_0 we have $y_1 + y_2 + \cdots + y_n = 1$.

Plugging these observations in to (3.2), we have

$$\begin{aligned}
2\int_{T_0} |v|^2 dy &= \int_{\partial T_0} (hYv)(\nu^T BB^T h\partial\bar{v}) dS \\
&= \sum_{j=1}^n \int_{F_j} \left(\left(\sum_\ell (m_\ell + y_\ell) h\partial_{y_\ell} \right) v \right) (\nu_j^T BB^T h\partial\bar{v}) dS_j \\
&\quad + \int_{F_0} \left(\left(\sum_\ell (m_\ell + y_\ell) h\partial_{y_\ell} \right) v \right) ({\nu_0}^T BB^T h\partial\bar{v}) dS_0 \\
&= \sum_{j=1}^n \int_{F_j} (m_j h\partial_{y_j} v)(h\partial_{\nu_j} \bar{v}) dS_j \\
&\quad + \int_{F_0} \left(\sum_1^n (n^{-1/2}(y_j + m_j)) h\partial_{\nu_0} v \right) (\nu_0^T BB^T h\partial\bar{v}) dS_0 \\
&= \sum_{j=1}^n \int_{F_j} (-m_j h\partial_{\nu_j} v)(\nu_j^T BB^T h\partial\bar{v}) dS_j \\
&\quad + \int_{F_0} n^{-1/2}(1 + m_1 + \cdots + m_n)((h\partial_{\nu_0})v)(\nu_0^T BB^T h\partial\bar{v}) dS_0 \\
&= \sum_{j=1}^n (-m_j) I_j + n^{-1/2}(1 + m_1 + \cdots + m_n) I_0, \qquad (3.3)
\end{aligned}$$

where for each $0 \leqslant j \leqslant n$

$$I_j = \int_{F_j} (h\partial_{\nu_j} v)(\nu_j^T BB^T h\partial \bar{v}) dS_j.$$

Let us now compute the I_js. Using equation (3.3), setting $m_j = 0$ for all $1 \leqslant j \leqslant n$, we have

$$I_0 = 2n^{1/2} \int_{T_0} |v|^2 dy.$$

Differentiating equation (3.3) with respect to m_j yields for $1 \leqslant j \leqslant n$

$$I_j = n^{-1/2} I_0 = 2 \int_{T_0} |v|^2 dy.$$

Now we must compute the I_j in terms of the corresponding integrals on the original simplex T. We first observe that, since for $1 \leqslant j \leqslant n$ we have $F_j \subset \{y_j = 0\}$, changing variables on one of the boundary integrals induces the area of the $(n-1)$-dimensional parallelepiped spanned by $p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n$. Denote this parallelepiped Γ_j, and observe that

$$\mathrm{Vol}_{n-1}\Gamma_j = (n-1)!\mathrm{Vol}_{n-1}(G_j),$$

where G_j is the $(n-1)$-dimensional simplex spanned by $p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n$.

For F_0, our area element is $n^{1/2} dy$, so changing variables in the integral over F_0 induces the area of the parallelepiped spanned by $p_1, p_2 - p_1, p_3 - p_1, \ldots, p_n - p_1$ divided by $n^{1/2}$. Denote this parallelepiped by Γ_0, and again we have

$$\mathrm{Vol}_{n-1}(\Gamma_0) = (n-1)!\mathrm{Vol}_{n-1}(G_0).$$

We now need to compute the integrand inside of each I_j in terms of the corresponding normal derivatives on G_j of u.

We first observe that on F_j, for $1 \leqslant j \leqslant n$, $h\partial_{y_\ell} \bar{v} = 0$ for $\ell \neq j$, so that the semiclassical gradient can be written

$$h\partial_y v|_{F_j} = e_j h\partial_{y_j} v|_{F_j} = \nu_j h\partial_{\nu_j} v|_{F_j}.$$

Similarly, for $j = 0$, we have on F_0

$$
\begin{aligned}
h\partial v &= \begin{bmatrix} h\partial_{y_1} \\ \vdots \\ h\partial_{y_n} \end{bmatrix} v \\
&= n^{-1/2} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} h\partial_{\nu_0} v \\
&= \nu_0 h\partial_{\nu_0} v.
\end{aligned}
$$

Now for each j, let ω_j be the unit outward normal on G_j. We know for each j on the face G_j

$$
\begin{aligned}
h\partial_{\omega_j} u|_{G_j} &= \omega_j^T h\partial_x u|_{G_j} \\
&= \omega_j^T B^T h\partial_y v|_{F_j} \\
&= (B\omega_j)^T h\partial_y v|_{F_j} \\
&= (B\omega_j)^T \nu_j h\partial_{\nu_j} v|_{F_j} \\
&= (\omega_j^T B^T \nu_j) h\partial_{\nu_j} v|_{F_j},
\end{aligned}
$$

so that

$$
h\partial_{\nu_j} v|_{F_j} = (\omega_j^T B^T \nu_j)^{-1} h\partial_{\omega_j} u|_{G_j}
$$

written in the y and x coordinates respectively.

On the other hand, we have $h\partial_x = B^T h\partial_y$, so that

$$
\nu_j^T B h\partial_x u = \nu_j^T B B^T h\partial_y v.
$$

The left hand side is zero except for the projection on to the ω_j, so that on each G_j we have

$$
\begin{aligned}
\nu_j^T B h\partial_x u &= (\nu_j^T B\omega_j)\omega_j^T h\partial_x u \\
&= (\omega_j^T B^T \nu_j) h\partial_{\omega_j} u. \qquad (3.4)
\end{aligned}
$$

Hence

$$
\nu_j^T B B^T h\partial_y v|_{F_j} = (\omega_j^T B^T \nu_j) h\partial_{\omega_j} u|_{G_j}.
$$

Plugging these observations in to the formulae for the I_j, we get for $1 \leqslant j \leqslant n$

$$I_j = \int_{F_j} (h\partial_{\nu_j} v)(\nu_j^T BB^T h\partial\bar{v}) dS_j$$
$$= \frac{1}{(n-1)!\mathrm{Vol}_{n-1}(G_j)} \int_{G_j} \left((\omega_j^T B^T \nu_j)^{-1} h\partial_{\omega_j} u|_{G_j}\right) \left((\omega_j^T B^T \nu_j) h\partial_{\omega_j} \bar{u}|_{G_j}\right) d\widetilde{S}_j$$
$$= \frac{1}{(n-1)!\mathrm{Vol}_{n-1}(G_j)} \int_{G_j} |h\partial_{\omega_j} u|^2 d\widetilde{S}_j,$$

where $d\widetilde{S}_j$ is the induced surface measure on G_j.

On the other hand, for I_0, we have

$$I_0 = \int_{F_0} (h\partial_{\nu_0} v)(\nu_0^T BB^T h\partial\bar{v}) dS_0$$
$$= \frac{n^{1/2}}{(n-1)!\mathrm{Vol}_{n-1}(G_0)} \int_{G_0} |\partial_{\omega_0} u|^2 d\widetilde{S}_0,$$

where $d\widetilde{S}_0$ is the induced surface measure on G_0.

We recall that

$$\int_{T_0} |v|^2 dy = \frac{1}{n!\mathrm{Vol}_n(T)},$$

so that rearranging we have for each $1 \leqslant j \leqslant n$

$$I_j = \frac{2}{n!\mathrm{Vol}_n(T)},$$

and

$$I_0 = n^{1/2} \frac{2}{n!\mathrm{Vol}_n(T)}.$$

Rearranging, we have for $0 \leqslant j \leqslant n$

$$\int_{G_j} |h\partial_{\omega_j} u|^2 d\widetilde{S}_j$$
$$= \frac{2(n-1)!\mathrm{Vol}_{n-1}(G_j)}{n!\mathrm{Vol}_n(T)}$$
$$= \frac{2\mathrm{Vol}_{n-1}(G_j)}{n\mathrm{Vol}_n(T)},$$

which completes the proof of Theorem 1.

4. A simple inverse problem

The proof of Theorem 1 suggests a further question: If u solves a constant coefficient eigenfunction equation, does the Neumann data determine the coefficients? In fact, in this paper, we only have information about the norms of the Neumann data, so we cannot fully answer this question using only this very elementary information. In fact, in the general case, the answer is that the norms of the Neumann data do not determine the coefficients (see Subsection 4.1 below). However, in dimension 2 the norms *do* determine the coefficients. We will return to this question after a few easier results.

This question is, of course intimately related to posing the standard Laplacian eigenfunction problem on a different simplex. Let us pose it as such in dimension 2. Let $T \subset \mathbb{R}^2$ be a triangle with sides a, b, c, with the convention that the length of the sides are a, b, c respectively. Suppose u solves

$$\begin{cases} (-h^2\Delta - 1)u = 0 \text{ on } T, \\ u|_{\partial T} = 0, \\ \|u\|_{L^2(T)}. \end{cases} \tag{4.1}$$

We have the following Theorem.

Theorem 2. *Suppose u solves* (4.1), *and suppose $N_a = \int_a |h\partial_\nu u|^2 dS$ and similarly for N_b and N_c. Then the three quantities N_a, N_b, N_c uniquely determine the triangle T (up to reflection).*

This theorem seems obvious, but in the formulae for the N_a, N_b, and N_c, there is both the length of the side *and* the area of the triangle. The proof is by scaling.

Proof. Suppose we have another triangle T_1 with the same Neumann data norms. Let a_1, b_1, c_1 denote the three sides of T_1, again with the convention that a_1, b_1, c_1 denote also the length of the sides. We know that the Neumann data relates the lengths of the sides to the area of the triangle. We have

$$N_a = \frac{a}{\text{Area}(T)},$$

and similarly for b, c. On the other hand, we also have

$$N_a = \frac{a_1}{\text{Area}(T_1)},$$

and similarly for b_1, c_1. Equating these quantities, we have

$$\frac{a}{a_1} = \frac{\text{Area}(T)}{\text{Area}(T_1)},$$

and similarly

$$\frac{b}{b_1} = \frac{c}{c_1} = \frac{\text{Area}(T)}{\text{Area}(T_1)}.$$

This means that the side lengths of T_1 are all scalar multiples of the corresponding sides on T with the same scalar. Hence T_1 is similar to T. Let

$$\lambda = \frac{\text{Area}(T)}{\text{Area}(T_1)}.$$

On the one hand, this implies that

$$\text{Area}(T) = \lambda \text{Area}(T_1). \tag{4.2}$$

On the other hand, we have

$$a = \lambda a_1 \tag{4.3}$$

and similarly

$$b = \lambda b_1, \quad c = \lambda c_1. \tag{4.4}$$

As the lengths scale linearly, the area scales quadratically. That is, (4.3) and (4.4) imply that

$$\text{Area}(T) = \lambda^2 \text{Area}(T_1).$$

Hence combining with (4.2), we have $\lambda^2 = \lambda$, so that $\lambda = 1$. This means precisely that $T = T_1$ (up to reflection). □

We now consider the question of determining the coefficients of a constant coefficient elliptic operator on the standard 2-simplex. Let B be a non-degenerate 2×2 matrix, and let $\Gamma = BB^T$. Consider $P = -\Gamma_{ij} h\partial_{x_i} h\partial_{x_j}$ be the associated positive definite elliptic operator. Our next result is that the semi-classical Neumann data uniquely determines the operator P. Interestingly, this does not determine the matrix B (see Remark 4.5).

Theorem 3. *Let B be a non-degenerate 2×2 matrix and $\Gamma = BB^T$. Let $P = -\Gamma_{ij} h\partial_{x_i} h\partial_{x_j}$. Let T_0 be the standard triangle in $\mathbb{R}^2$ generated by the vectors $(1,0)$ and $(0,1)$. Suppose u solves the eigenfunction problem*

$$\begin{cases} Pu = u \text{ in } T_0, \\ u|_{\partial T_0} = 0, \\ \|u\|_{L^2(T_0)} = 1. \end{cases}$$

Let F_1 and F_2 denote the sides of length 1 and F_0 the hypotenuse of length $\sqrt{2}$. Then the norms

$$\|h\partial_\nu u\|^2_{L^2(F_1)},\ \|h\partial_\nu u\|^2_{L^2(F_2)},\ \text{and } \|h\partial_\nu u\|^2_{L^2(F_0)}$$

uniquely determine Γ.

Remark 4.1. We pause to remark that in the statement of the theorem is buried a rather astounding fact: the norms of the (semi-classical) Neumann data of *any single* eigenfunction determine Γ. Of course this requires some knowledge also about the spectrum. In other words, if one eigenvalue and corresponding eigenfunction's Neumann mass is known, then Γ is uniquely determined.

Remark 4.2. It is also very interesting that the proof in fact computes the entries of Γ explicitly in terms of the Neumann data norms. Indeed, if we label

$$J_1 = \|h\partial_\nu u\|^2_{L^2(F_1)},\ J_2 = \|h\partial_\nu u\|^2_{L^2(F_2)},$$

and

$$J_0 = \|h\partial_\nu u\|^2_{L^2(F_0)},$$

and we write $\Gamma = (\Gamma)_{jk}$, we have

$$\Gamma_{11} = \frac{2}{J_1},$$

$$\Gamma_{22} = \frac{2}{J_2},$$

and

$$\Gamma_{12} = \Gamma_{21} = \frac{2\sqrt{2}}{J_0} - \frac{1}{J_1} - \frac{1}{J_2}.$$

In particular, if $J_1 = J_2 = 2$ and $J_0 = 2\sqrt{2}$, we have $\Gamma = I$ as expected (since each J_j is twice the length of the sides, which is the length of the side divided by the area of the triangle).

First we write a Lemma giving yet another way of computing the Neumann data mass. We state this Lemma in any dimension.

Lemma 4.3. *Let B be a non-degenerate $n \times n$ matrix, $\Gamma = BB^T$, and*

$$P = -\Gamma_{ij} h\partial_{x_i} h\partial_{x_j}.$$

Let T_0 be the standard simplex in $\mathbb{R}^n$ with faces $F_0, F_1, \ldots, F_n$ in the notation of earlier in this paper. Suppose u solves the eigenfunction problem

$$\begin{cases} Pu = u \text{ in } T_0, \\ u|_{\partial T_0} = 0, \\ \|u\|_{L^2(T_0)} = 1. \end{cases} \tag{4.5}$$

Then on each face F_j, $0 \leqslant j \leqslant n$, we have

$$\int_{F_j} (h\partial_{\nu_j} u)(\nu_j^T BB^T h\partial_x \bar{u}) dS_j = |B^T \nu_j|^2 \int_{F_j} |h\partial_{\nu_j} u|^2 dS_j,$$

where dS_j is the induced surface measure on F_j as usual.

Remark 4.4. Note that this is a different way of computing this quantity than in (3.4). Indeed, the rest of this section is based on the following idea: The proof of Theorem 1 involves $n+1$ normal derivative quantities in dimension n, and Lemma 4.3 shows how the Neumann data mass determines the quantities $|B^T \nu_j|^2$. In two dimensions, there are three such quantities for the three different faces of the simplex. As BB^T is symmetric, there are only three elements to determine in dimension 2. In higher dimensions, there are $n(n+1)/2$ elements to determine in BB^T, and $n(n+1)/2 > n+1$ for $n > 2$. Hence in the case of $n > 2$ we do not expect such a result to hold true. In Subsection 4.1 below, an explicit example is given.

Proof. We observe that on F_j, for $1 \leqslant j \leqslant n$, $h\partial_{x_\ell} \bar{u} = 0$ for $\ell \neq j$, so that

$$h\partial_x u = e_j h\partial_{x_j} u = \nu_j h\partial_{\nu_j} u.$$

Similarly, for $j = 0$, we have on F_0

$$\begin{aligned} h\partial u &= \begin{bmatrix} h\partial_{y_1} \\ \vdots \\ h\partial_{y_n} \end{bmatrix} u \\ &= n^{-1/2} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} h\partial_{\nu_0} u \\ &= \nu_0 h\partial_{\nu_0} u. \end{aligned}$$

Then on each face F_j with normal ν_j, we have

$$\begin{aligned} \nu_j^T BB^T h\partial\bar{u} &= \nu_j^T BB^T \nu_j h\partial_{\nu_j}\bar{u} \\ &= (B^T\nu_j)^T(B^T\nu_j)h\partial_{\nu_j}\bar{u} \\ &= |B^T\nu_j|^2 h\partial_{\nu_j}\bar{u}. \end{aligned}$$

Hence on each face F_j, we have

$$\int_{F_j} (h\partial_{\nu_j} u)(\nu_j^T BB^T h\partial\bar{u}) dS_j = |B^T\nu_j|^2 \int_{F_j} |h\partial_{\nu_j} u|^2 dS_j.$$

This completes the proof. □

Proof of Theorem 3. The proof proceeds by using an eigenvector diagonalization argument. It is interesting that, although the argument uses the *existence* of eigenvalues/vectors of Γ, we do not need to know them.

Let v_1, v_2 be orthonormal eigenvectors for Γ. Since $\Gamma = BB^T$ is positive definite, write λ_1^2, λ_2^2 for the eigenvalues of Γ so that $\Gamma v_j = \lambda_j^2 v_j$ for $j = 1, 2$. Let

$$L = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix},$$

so that (since L is orthogonal),

$$L^T \Gamma L = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}.$$

Let us denote

$$G = L \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

so that $GG^T = \Gamma$.

We now change variables using the matrix G. Let T_1 denote the triangle spanned by the new coordinates v_1, v_2. Rescaling in each variable $v_j \mapsto \lambda_j^{-1} v_j$ gives a new triangle T. Let $w(x) = u(Gx)$, so that

$$\int_T |w|^2 dV = \int_T |u(Gx)|^2 dV = |G|^{-1} \int_{T_0} |u|^2 dV = \frac{1}{\lambda_1 \lambda_2}. \tag{4.6}$$

We also have $-h^2 \Delta w = w$ on T_0, so we can use the same commutator argument as above to compute the mass of the Neumann data. For $j = 1, 2$, let

$$I_j = \int_{\lambda_j^{-1} v_j} |h \partial_\nu w|^2 dS,$$

and $I_0 = \int_H |h\partial_\nu w|^2 dS$ be the Neumann mass of the function w on the legs spanned by the $\lambda_j^{-1} v_j$ and the hypotenuse H. Using Theorem 1 and (4.6), we have for $j = 1, 2$

$$\begin{aligned} I_j &= \left(\frac{1}{\lambda_1 \lambda_2} \right) \left(\frac{\text{length of } \lambda_j^{-1} v_j}{\text{area}(T)} \right) \\ &= \left(\frac{1}{\lambda_1 \lambda_2} \right) \left(\frac{\lambda_j^{-1}}{(\lambda_1^{-1} \lambda_2^{-1}/2)} \right) \\ &= \frac{2}{\lambda_j}. \end{aligned}$$

Further,

$$I_0 = 2(\lambda_1^{-2} + \lambda_2^{-2})^{1/2}.$$

For $j = 0, 1, 2$, let

$$J_j = \int_{F_j} |h \partial_\nu u|^2 dS$$

be the Neumann mass of the original eigenfunction u on the faces of T_0. These are the quantities we are assuming we know.

Using Lemma 4.3, that means

$$\begin{aligned} J_j &= \frac{1}{|G^T\nu|^2}\int_{F_j}(h\partial_\nu u)(\nu^T G G^T h\nabla\bar{u})dS \\ &= \left(\frac{1}{|G^T\nu|^2}\right)\left(\frac{\text{length of } F_j}{\text{length of } \lambda_j^{-1}v_j}\right) I_j \\ &= \left(\frac{1}{|G^T\nu|^2}\right)\lambda_j\left(\frac{2}{\lambda_j}\right) \\ &= \frac{2}{|G^T\nu|^2} \qquad (4.7) \end{aligned}$$

for $j = 1, 2$, and

$$J_0 = \frac{2\sqrt{2}}{|G^T\nu|^2}.$$

We pause momentarily to recall that the normal vectors ν in the above expressions are the normals to the original faces F_j, $j = 0, 1, 2$ on the standard triangle T_0.

Recall that $\nu_1 = (-1, 0)$, $\nu_2 = (0, -1)$ and $\nu_0 = (\sqrt{2})^{-1}(1, 1)$, which will help us determine the matrix Γ.

Write

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}.$$

As Γ is symmetric, we have $\Gamma_{12} = \Gamma_{21}$, so we only need to determine the three numbers Γ_{11}, Γ_{12}, and Γ_{22}.

The quantities we need to examine are all of the form $|G^T\nu_j|^2$, which we rewrite:

$$\begin{aligned} |G^T\nu_j|^2 &= (G^T\nu_j)^T(G^T\nu_j) \\ &= \nu_j^T G G^T \nu_j \\ &= \nu_j^T \Gamma \nu_j. \end{aligned}$$

Plugging in the ν_j, $j = 0, 1, 2$, we have:

$$\begin{aligned} \nu_1^T\Gamma\nu_1 &= (-1, 0)\Gamma\begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ &= \Gamma_{11}, \end{aligned}$$

and similarly

$$\nu_2^T\Gamma\nu_2 = \Gamma_{22}.$$

For ν_0, we get information about the off diagonal terms as well:

$$\begin{aligned}\nu_0^T \Gamma \nu_0 &= \frac{1}{2}(1,1)\Gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2}(1,1) \begin{pmatrix} \Gamma_{11} + \Gamma_{12} \\ \Gamma_{21} + \Gamma_{22} \end{pmatrix} \\ &= \frac{1}{2}(\Gamma_{11} + \Gamma_{12} + \Gamma_{21} + \Gamma_{22}) \\ &= \frac{1}{2}(\Gamma_{11} + 2\Gamma_{12} + \Gamma_{22}) \qquad (4.8)\end{aligned}$$

again due to Γ being symmetric.

Returning now to (4.7), we have for $j = 1, 2$

$$\begin{aligned}J_j &= \frac{2}{|G^T \nu_j|^2} \\ &= \frac{2}{\Gamma_{jj}}.\end{aligned}$$

Hence

$$\Gamma_{11} = \frac{2}{J_1}$$

and similarly for Γ_{22}. For Γ_{12}, we appeal to equation (4.8) to get

$$\begin{aligned}J_0 &= \frac{2\sqrt{2}}{|G^T \nu_0|^2} \\ &= \frac{2\sqrt{2}}{\frac{1}{2}(\Gamma_{11} + 2\Gamma_{12} + \Gamma_{22})} \\ &= \frac{4\sqrt{2}}{\Gamma_{11} + 2\Gamma_{12} + \Gamma_{22}}.\end{aligned}$$

Rearranging, we have

$$\Gamma_{11} + 2\Gamma_{12} + \Gamma_{22} = \frac{4\sqrt{2}}{J_0},$$

so that solving for Γ_{12}, we have

$$\Gamma_{12} = \frac{2\sqrt{2}}{J_0} - \frac{1}{2}(\Gamma_{11} + \Gamma_{22}).$$

Plugging in the known values of Γ_{11} and Γ_{22}, we have

$$\Gamma_{12} = \frac{2\sqrt{2}}{J_0} - \frac{1}{2}\left(\frac{2}{J_1} + \frac{2}{J_2}\right) = \frac{2\sqrt{2}}{J_0} - \frac{1}{J_1} - \frac{1}{J_2}.$$

This gives the Γ_{jk} in terms of the known quantities J_1, J_2, and J_0, completing the proof. □

Remark 4.5. It is interesting to note that the proof of Theorem 3 does not uniquely determine the matrix B, due to rotational invariance. Indeed, if

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a = c = d = 2^{-1/2}$ and $b = -2^{-1/2}$, then we still have $a^2 + c^2 = b^2 + d^2 = 1$, and $ac + bd = 0$. Note, however, that $BB^T = I$ in this case as well.

4.1. Dimension 3: an example

The result in Theorem 3 is false in higher dimensions, even for small perturbations of I. Let T_0 be the standard simplex in $\mathbb{R}^3$, B be a 3×3 nondegenerate matrix, $\Gamma = BB^T$, and $P = -\Gamma_{ij} h\partial_{x_i} h\partial_{x_j}$. Suppose u solves the eigenfunction problem (4.5). Lemma 4.3 still applies, with $\nu_j = -e_j$ for $1 \leqslant j \leqslant 3$ and $\nu_0 = 3^{-1/2}(1,1,1)$. For $0 < \epsilon < 1$, define the matrix B by

$$B^T = \begin{pmatrix} a & 0 & 0 \\ d & (1-\epsilon^2)^{1/2} & \epsilon \\ \epsilon & \epsilon & (1-\epsilon^2)^{1/2} \end{pmatrix},$$

where

$$d = \frac{-3\epsilon(1-\epsilon^2)^{1/2} - \epsilon^2}{(1-\epsilon^2)^{1/2} + \epsilon}$$

and

$$a = (1 - d^2 - \epsilon^2)^{1/2}.$$

Observe that $B = I + \mathcal{O}(\epsilon)$ and satisfies

$$|B^T e_1|^2 = a^2 + d^2 + \epsilon^2 = 1,$$

$$|B^T e_2|^2 = (1-\epsilon^2) + \epsilon^2 = 1,$$

$$|B^T e_3|^2 = \epsilon^2 + (1-\epsilon^2) = 1,$$

and

$$\begin{aligned}|B^T(1,1,1)^T|^2 &= a^2 + (d + (1-\epsilon^2)^{1/2} + \epsilon)^2 + (2\epsilon + (1-\epsilon^2)^{1/2})^2 \\ &= a^2 + d^2 + (1-\epsilon^2) + \epsilon^2 + 2d(1-\epsilon^2)^{1/2} + 2d\epsilon \\ &\quad + 2\epsilon(1-\epsilon^2)^{1/2} + 4\epsilon^2 + (1-\epsilon^2) + 4\epsilon(1-\epsilon^2)^{1/2} \\ &= (1-\epsilon^2) + (1-\epsilon^2) + \epsilon^2 + 2d((1-\epsilon^2)^{1/2} + \epsilon) \\ &\quad + 2\epsilon(1-\epsilon^2)^{1/2} + 4\epsilon^2 + (1-\epsilon^2) + 4\epsilon(1-\epsilon^2)^{1/2} \\ &= 2 - \epsilon^2 - 6\epsilon(1-\epsilon^2)^{1/2} - 2\epsilon^2 + 6\epsilon(1-\epsilon^2)^{1/2} + 1 + 3\epsilon^2 \\ &= 3.\end{aligned}$$

These are the same values one gets from $B = I = \Gamma$, however $BB^T \neq I$, so these 4 numbers do not determine Γ.

Appendix A. Green's formula on simplices

As pointed out by the referee, a simplex $T \subset \mathbb{R}^n$ is a Lipschitz domain, so Green's formula requires a little bit of justification. Naturally we expect eigenfunctions to be well behaved enough for Green's formula to apply. We recall here how this is done in the book of Grisvard [6]. We continue to let $u = u(h)$ denote the sequence of normalized Dirichlet eigenfunctions on T, satisfying (1.1).

Standard elliptic regularity theory guarantees that the eigenfunctions on T are $\mathcal{C}^\infty$ on the interior, however we need more control over Sobolev estimates in order to apply Green's formula. Using that the simplex T is convex, we can apply [6, Theorem 3.1.3.1] on a priori elliptic estimates for convex domains, which tells us that an eigenfunction is in any (semiclassical) H^m space with the implicit constants in the estimates independent of m. Then we may apply [6, Theorem 1.4.4.6] to conclude that the derivative is bounded from $H^m(T)$ to $H^{m-1}(T)$, $m \geqslant 1$. From this we deduce that $Yu \in H^2(T)$, where $Y = \sum(y_j + m_j)\partial_{y_j}$ is the perturbed radial vector field used in (3.2). We can then use [6, Lemma 1.5.3.2], which is Green's formula for Lipschitz domains and H^2 functions. For convenience, we state it here for the specific case of the standard simplex and a constant coefficient elliptic operator as used in Section 3.

Theorem 4 (Green's formula for simplices). *Let K be an $n \times n$ real symmetric positive definite matrix, and let $P = -\sum_{ij} K_{ij}\partial_{x_i}\partial_{x_j}$ be the associated elliptic operator. Let T_0 be the standard simplex in $\mathbb{R}^n$ with faces*

$F_0, F_1, \ldots, F_n$. Then for $f, g \in H^2(T_0)$, we have

$$\int_{T_0} (Pf)\bar{g} dV = \int_{T_0} f(P\bar{g}) dV - \sum_{j=0}^{n} \int_{F_j} (\nu_j^T K \partial f)\bar{g} ds_j + \sum_{j=0}^{n} \int_{F_j} f(\nu_j^T K \partial \bar{g}) ds_j,$$

where ν_j denotes the outward unit normal vector on F_j, ds_j the induced surface measure on F_j, and ∂ the n-dimensional gradient.

Acknowledgements

The work in this paper is supported in part by NSF grant DMS-1500812. The author is grateful to the anonymous referee for helpful suggestions and comments. The author would also like to thank Luc Hillairet for many interesting discussions about the topics in this paper.

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Received August 7, 2017