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A flop formula for Donaldson-Thomas invariants

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Let X and X' be nonsingular projective 3-folds related by a flop of a disjoint union of (-2)-curves. We prove a flop formula relating the Donaldson-Thomas invariants of X to those of X', which implies some simple relations among BPS state counts. As an application, we show that if X satisfies the GW/DT correspondence for primary insertions and descendants of the point class, then so does X'. We also propose a conjectural flop formula for general flops.

1. Introduction

The Donaldson-Thomas theory of a nonsingular projective 3-fold X counts the number of stable sheaves on X [5, 32]. In particular, when considering ideal sheaves of curves, the theory gives virtual numbers of embedded curves in X. Another curve counting theory on X is the much studied Gromov-Witten theory, which essentially counts stable maps from curves with marked points to X. In [20, 21], Maulik, Nekrasov, Okounkov, and Pandharipande proposed a remarkable conjecture that the Gromov-Witten theory of X is equivalent to the Donaldson-Thomas theory of X in a subtle way. This suggests that many phenomenon in one theory have counterparts in the other theory.

The above mentioned curve counting theories are deformation invariant. A fundamental problem in Gromov-Witten theory is to investigate the transformation of Gromov-Witten invariants under birational surgeries [31]. The first breakthrough is the work of Li and Ruan [18], who showed that, for 3-folds, the primary Gromov-Witten theories are invariant under general flops. It is also important to study the effect of biraional surgeries on Donaldson-Thomas theory. The first progress in this direction is the work of Hu and Li [10], who used the degeneration formula to understand the change of Donaldson-Thomas invariants under blow-ups at points, and flops of a disjoint union of $(-1,-1)$ curves which are all numerally equivalent.

In this paper, we use the degeneration formula to prove a flop formula in Donaldson-Thomas theory for flops of a disjoint union (-2)-curves, and derive some interesting relations on BPS state counts. As an application, we give positive evidence for the conjectural GW/DT correspondence. Here an embedded curve in a 3-fold is a (-2)-curve [30] if it is a nonsingular rational curve with normal bundle of type $(-1,-1)$ or $(0,-2)$. Our flop formula generalizes the result of Hu and Li [10], since a $(-1,-1)$-curve is a (-2)-curve, and we do not assume that the curves are numerically equivalent.

Throughout this paper, let X and X' be nonsingular projective 3-folds over $\mathbb{C}$, which are related by a flop $f : X \dashrightarrow X'$ of some contraction [14]. Then f is a birational map, and it is biregular outside of a subvariety of codimension two in X, called the center of f. The center of f is a disjoint union of trees of rational curves, and it has a neighborhood with trivial canonial bundle. We have a natural isomorphism of groups

$$\mathscr{F} : H_2(X,\mathbb{Z}) \xrightarrow{\cong} H_2(X',\mathbb{Z}),$$

defined as follows. For any $\beta \in H_2(X,\mathbb{Z})$, we can choose a real 2-dimensional pseudo-submanifold Σ representing β in X, which lies in the complement of the center of f. Now $\mathscr{F}\beta$ is represented by $f(\Sigma)$ in X', which lies in the complement of the center of f^{-1}. Similarly, by considering Poincaré duals of classes of degree$\geqslant 3$, we also have an isomorphism

$$H^{\geqslant 3}(X,\mathbb{Q}) \to H^{\geqslant 3}(X',\mathbb{Q}),$$

which can be extended to an isomorphism of cohomology groups

$$H^*(X,\mathbb{Q}) \xrightarrow{\cong} H^*(X',\mathbb{Q}),$$

by requiring this isomorphism to preserve the Poincaré pairing. The isomorphism will also be denoted by $\mathscr{F}$ by abuse of notation. Let $Cen(f)$ be the subgroup of $H_2(X,\mathbb{Z})$ generated by the cycles of irreducible curves in the center of f. The main result of this paper is the following.

Proposition 1.1. *Let f be a flop of a disjoint union of (-2)-curves. Suppose that $\gamma_1,\ldots,\gamma_m \in H^*(X,\mathbb{Q})(m \geqslant 0)$ have supports away from the center* *of f, and $d_1, \ldots, d_m \in \mathbb{Z}_{\geqslant 0}$. Then we have*

$$\frac{\sum_{\beta \in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}(X; q| \prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i))_\beta}{\sum_{\beta \in Cen(f)} v^\beta Z'_{DT}(X; q|)_\beta} = \frac{\sum_{\beta \in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}(X'; q| \prod_{i=1}^m \tilde{\tau}_{d_i}(\mathscr{F}\gamma_i))_{\mathscr{F}\beta}}{\sum_{\beta \in Cen(f)} v^\beta Z'_{DT}(X'; q|)_{\mathscr{F}\beta}}, \tag{1}$$

$$Z'_{DT}(X; q|)_\beta = Z'_{DT}(X'; q|)_{-\mathscr{F}\beta}, \qquad \forall \beta \in Cen(f). \tag{2}$$

Remark 1.2. *We remark that we can choose the support of γ_i away from the center of f if $deg\gamma_i > 2$.*

We sketch the proof of Proposition 1.1, the detail of which will be given in Section 3. By a beautiful result of Reid [30], we can decompose the flop f of (-2)-curves into a sequence of blow-ups of (-2)-curves followed by a sequence of blow-downs. Since blow-ups can be described in terms of semi-stable degenerations, it follows that we can use the degeneration formula [19] and the absolute/relative correspondence [11, 23] to relate invariants of X to those of the blow-up of X (see (9)). Therefore, in principle, the Donaldson-Thomas invariants of X can be related to those of X'. Due to the denominators in (1), we need to understand the transformation of the invariants attached to classes in $Cen(f)$ under blow-ups. To this end, we give a detailed analysis of the change of effectiveness of classes in $Cen(f)$ under blow-up (see Lemma 3.1 and 3.2).

Proposition 1.1 relates the Donaldson-Thomas invariants of X to those of X' in a nontrivial way. In [9] and [13], we obtained some blow-up formulae for Gromov-Witten and stable pair theories which contain some extra factors, and we discovered that these formulae imply some interesting relation among BPS state counts. In this paper, we consider the change of BPS state counts of Donaldson-Thomas theory under flops. Proposition 1.1 implies the following simple flop formulae for BPS state counts.

Corollary 1.3. *Let f be a flop of a disjoint union of (-2)-curves. Suppose that $\gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Q})(m \geqslant 0)$, and $g \in \mathbb{Z}$. Then we have*

$$n^X_{g,\beta}(\gamma_1, \ldots, \gamma_m) = n^{X'}_{g,\mathscr{F}\beta}(\mathscr{F}\gamma^1, \ldots, \mathscr{F}\gamma_m), \quad \forall \beta \in H_2(X,\mathbb{Z}) \setminus Cen(f); \tag{3}$$

$$n^X_{g,\beta} = n^{X'}_{g,-\mathscr{F}\beta}, \quad \forall \beta \in Cen(f) \setminus \{0\}. \tag{4}$$

The Donaldson-Thomas theory of X counts embedded curves on X only in a virtual sense. A fundamental problem in the Donaldson-Thomas theory is to understand the hidden enumerative meanings of the invariants. It is conjectured that BPS state counts are enumerative. It is interesting to understand Corollary 1.3 from the point of view of enumerative geometry.

As another application, we investigate the conjectural GW/DT correspondence. In the primary case, the correspondence is established for several classes of 3-folds, including toric 3-folds [22], and Calabi-Yau 3-folds which are complete intersections in products of projective spaces [28, 33]. However, in the descendent case, not much is known. The following result gives further positive evidence to the MNOP conjecture.

Corollary 1.4. *Let f be a flop of a disjoint union of (-2)-curves. Assume that X satisfies the GW/DT correspondence for primary insertions and descendants of the point class. Then so does X'.*

Remark 1.5. *Behrend [1] discovered that Donaldson-Thomas invariants of Calabi-Yau 3-folds are equal to weighted Euler characteristics of their moduli spaces. Based on this crucial observation, for general flops between Calabi-Yau 3-folds, Toda [34] used the wall crossing formula [12, 16] to obtain some flop formulae for Donaldson-Thomas type invarints (i.e. ordinary Euler characteristics), and Calabrese [3] used the method of Hall algebra identities and the integration morphism [16] to prove the flop formula (1) for Donaldson-Thomas invariants. Although we only consider flops of (-2)-curves in this paper, we do NOT assume that the target 3-folds are Calabi-Yau.*

Based on the established flop formulae of [3], [10] and ours, we propose the following conjecture.

Conjecture 1.6. *The formulae (1) and (2) hold for general flops.*

We expect that the degeneration formula will play a role in the proof of the conjecture. Note that an embedded nonsingular rational curve in a 3-fold is locally floppable only if it has normal bundle of type $(-1,-1), (0,-2)$ or $(1,-3)$ [17]. However, unlike the case of (-2)-curves, it is difficult to describe a general flop of $(1,-3)$-curves in terms of blow-ups and blow-downs (see [27] for some explicit examples).

Most of the results obtained in this paper also hold in the stable pair theory [29], since the behavior of stable pair invariants under degeneration is similar to that of Donaldson-Thomas theory. We also have corresponding corollaries on BPS state counts and GW/P correspondence, and conjectural flop formulae for general flops in the stable pair theory.

An outline of this paper is as follows. In Section 2, we review some basic materials on Donaldson-Thomas invariants. In Section 3, we recall Reid's result to decompose the flop under consideration into a sequence of blow-ups followed by a sequence of blow-downs, and use the degeneration formula to prove Proposition 1.1. In Section 4, we give a working definition of the BPS state counts for Donaldson-Thomas theory and prove Corollary 1.3. In Section 5, we review the conjectural GW/DT corespondence and prove Corollary 1.4.

2. Preliminaries on Donaldson-Thomas invariants

In this section, we briefly review some basic materials on Donaldson-Thomas invariants and fix notations. We refer readers to [5, 19–21, 32] for details.

Donaldson-Thomas theory is defined via integration over the moduli space of ideal sheaves of X. Here an ideal sheaf is a torsion-free sheaf of rank 1 with trivial determinant. Each ideal sheaf $\mathcal{I}$ determines a subscheme $Y \subset X$ via the exact sequence

$$0 \to \mathcal{I} \to \mathscr{O}_X \to \mathscr{O}_Y \to 0.$$

In this note, we will consider only the case $\dim Y \leqslant 1$. The one dimensional components of Y (weighted by their intrinsic multiplicities) determine an element,

$$[Y] \in H_2(X, \mathbb{Z}).$$

For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, let $I_n(X, \beta)$ be the moduli space of ideal sheaves $\mathcal{I}$ satisfying

$$\chi(\mathscr{O}_Y) = n, \quad [Y] = \beta,$$

where χ is the holomorphic Euler characteristic. From the deformation theory, $I_n(X, \beta)$ carries a virtual fundamental class of degree $\int_\beta c_1(X)$.

For $d \in \mathbb{Z}_{\geqslant 0}$ and $\gamma \in H^*(X, \mathbb{Q})$, the descendant insertion $\tilde{\tau}_d(\gamma)$ is defined as follows. Let

$$\begin{aligned} \pi_X : X \times I_n(X, \beta) &\to X, \\ \pi_I : X \times I_n(X, \beta) &\to I_n(X, \beta) \end{aligned}$$

be tautological projections. Let $\mathscr{I}$ be the universal sheaf over $X \times I_n(X,\beta)$. The operation

$$(-1)^{d+1}\pi_{I*}\left(\pi_X^*(\gamma)\cdot \mathrm{ch}_{2+d}(\mathscr{I})\cap \pi_I^*(\cdot)\right) : H_*(I_n(X,\beta),\mathbb{Q})\to H_*(I_n(X,\beta),\mathbb{Q})$$

is the action of $\tilde{\tau}_d(\gamma)$. The Donaldson-Thomas invariants with descendant insertions are defined as the virtual integration

$$\left\langle \prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\right\rangle_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} \prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i),$$

where $d_1,\dots,d_m \in \mathbb{Z}_{\geqslant 0}$, and $\gamma_1,\dots,\gamma_m \in H^*(X,\mathbb{Q})$. Here the integral is the push-forward to a point of the class

$$\tilde{\tau}_{d_i}(\gamma_i)\circ\cdots\tilde{\tau}_{d_m}(\gamma_m)([I_n(X,\beta)]^{vir}).$$

The partition function of the Donaldson-Thomas invariants is defined by

$$Z_{DT}\left(X;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\right)_\beta = \sum_{n\in\mathbb{Z}}\langle\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\rangle_{n,\beta}q^n,$$

and the reduced partition function is obtained by formally removing the degree zero contributions,

$$Z'_{DT}\left(X;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\right)_\beta = \frac{Z_{DT}\left(X;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\right)_\beta}{Z_{DT}\left(X;q\middle|\right)_0}.$$

Let $S\subset X$ be a nonsingular divisor. For $n\in\mathbb{Z}$ and nonzero $\beta\in H_2(X,\mathbb{Z})$ with $\int_\beta [S]\geqslant 0$, let $I_n(X/S,\beta)$ be the moduli space of relative ideal sheaves, which carries a virtual fundamental class of degree $\int_\beta c_1(X)$. We have the following natural morphism

$$\epsilon : I_n(X/S,\beta)\to \mathrm{Hilb}\left(S,\int_\beta [S]\right)$$

The pull-back of cohomology classes of $\mathrm{Hilb}(S,\int_\beta[S])$ gives relative insertions.

Let us briefly recall Nakajima basis for the cohomology of Hilbert schemes of points of S. Let $\{\delta_i\}$ be a basis of $H^*(S,\mathbb{Q})$ with dual basis $\{\delta^i\}$. For any cohomology weighted partition η with respect to the basis $\{\delta_i\}$, Nakajima constructed a cohomology class $C_\eta \in H^*(\mathrm{Hilb}(S,|\eta|),\mathbb{Q})$. The Nakajima basis of $H^*(\mathrm{Hilb}(S,d),\mathbb{Q})$ is the set $\{C_\eta\}_{|\eta|=d}$. We refer readers to [24] for more details.

The partition function of the relative Donaldson-Thomas invariants is defined by

$$Z_{DT}\Big(X/S;q|\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)|\eta\Big)_\beta = \sum_{n\in\mathbb{Z}} q^n \int_{[I_n(X/S,\beta)]^{vir}} \prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\cdot \epsilon^* C_\eta,$$

and the reduced partition function is obtained by formally removing the degree zero contributions

$$Z'_{DT}\Big(X/S;q|\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)|\eta\Big)_\beta = \frac{Z_{DT}\Big(X/S;q|\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)|\eta\Big)_\beta}{Z_{DT}\Big(X/S;q||\Big)_0}$$

Let $\Delta \subset \mathbb{C}$ be the unit disc, and let $\pi : \chi \to \Delta$ be a nonsingular 4-fold over $\mathbb{D}$, such that $\chi_t = \pi^{-1}(t) \cong X$ for $t \neq 0$, and χ_0 is a union of two irreducible nonsingular projective 3-folds X_1 and X_2 intersecting transversally along a nonsingular projective surface S. (We can also consider the general case where the central fiber has several irreducible components, but we restrict ourselves to this simple case for simplicity of presentation.) Consider the natural inclusion maps

$$i_t : X = \chi_t \longrightarrow \chi, \qquad i_0 : \chi_0 \longrightarrow \chi,$$

and the gluing map

$$g = (j_1, j_2) : X_1 \coprod X_2 \longrightarrow \chi_0.$$

We have

$$H_2(X,\mathbb{Z}) \xrightarrow{i_{t*}} H_2(\chi,\mathbb{Z}) \xleftarrow{i_{0*}} H_2(\chi_0,\mathbb{Z}) \xleftarrow{g_*} H_2(X_1,\mathbb{Z}) \oplus H_2(X_2,\mathbb{Z}),$$

where i_{0*} is an isomorphism since there exists a deformation retract from χ to χ_0(see [4]). Also, in the next section, the family $\chi \to \Delta$ comes from a trivial family, and each $\gamma \in H^*(X,\mathbb{Q})$ has global liftings such that the restriction $\gamma(t)$ on χ_t is defined for all t.

The degeneration formula for the Donaldson-Thomas theory expresses the absolute invariants of X via the relative invariants of (X_1, S) and (X_2, S):

$$\begin{aligned} & Z'_{DT}\Big(X; q \Big| \prod_{i=1}^{m} \tilde{\tau}_{d_i}(\gamma_i)\Big)_{\beta} \\ &= \sum Z'_{DT}\Big(X_1/S; q \Big| \prod_{i\in P_1} \tilde{\tau}_{d_i}(j_1^*\gamma_i(0)) \Big| \eta\Big)_{\beta_1} \\ &\quad \times \frac{(-1)^{|\eta|-\ell(\eta)}\mathfrak{z}(\eta)}{q^{|\eta|}} \cdot Z'_{DT}\Big(X_2/S; q \Big| \prod_{i\in P_2} \tilde{\tau}_{d_i}(j_2^*\gamma_i(0)) \Big| \eta^\vee\Big)_{\beta_2}, \end{aligned}$$

where $\mathfrak{z}(\eta) = |\mathrm{Aut}(\eta)| \cdot \prod_{i=1}^{\ell(\eta)} \eta_i$, $\eta^\vee$ is defined by taking the Poincaré duals of the cohomology weights of η, and the sum is over cohomology weighted partitions η, degree splittings $i_{t*}\beta = i_{0*}(j_{1*}\beta_1 + j_{2*}\beta_2)$, and marking partitions $P_1 \coprod P_2 = \{1, \ldots, m\}$. In particular, if (η, β_1, β_2) has nontrivial contribution in the degeneration formula, then we have the following dimension constraint:

$$v\dim_{\mathbb{C}} P_n(X_1/S, \beta_1) + v\dim_{\mathbb{C}} P_n(X_2/S, \beta_2) = v\dim_{\mathbb{C}} P_n(X, \beta) + 2|\eta|.$$

3. Proof of main result

In this section, we give a detailed proof o Proposition 1.1. We first recall Reid's result to decompose a flop of a disjoint union of (-2)-curves into a sequence of blow-ups followed by a sequence of blow-downs, and then use the degeneration formula to prove our flop formula. We refer readers to [30] for explicit local description of the flop of a single (-2)-curve, and to [14, 15] for general materials on birational geometry of 3-folds.

Let $C_1, \ldots, C_l$ be the irreducible components of the center of f. We can contract these curves to obtain a contraction $\psi : X \to \bar{X}$, and then these curves generate an extremal face in $NE(X)$. The width of C_i in X is defined by Reid as follows [30]:

$$\begin{aligned} w_i := \mathit{width}(C_i \subset X) := \sup\{k \mid & \text{ there exists a scheme } \\ & S \cong C_i \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^k) \text{ such that } C_i \subset S \subset X\}. \end{aligned}$$

Since C_i is isolated, it follows that $1 \leqslant w_i < \infty$. Note that $\psi(C_i) \in \bar{X}$ is a hypersurface singularity given by

$$x^2 + y^2 + z^2 + t^{2w_i} = 0.$$

In particular, C_i is a $(-1,-1)$-curve if and only if $w_i = 1$.

Without loss of generality, assume that

$$w_1 \geqslant \cdots \geqslant w_l \geqslant 1.$$

Let $w = w_1$, and for $d = 1, \ldots, w$, set

$$k_d := \sup\{i | w_i \geqslant d\}.$$

Then

$$1 \leqslant k_w \leqslant \cdots \leqslant k_1 = l.$$

Write $X = X_0$ and $C_i = C_{0,i}$. Then proceeding inductively, we obtain a sequence of blow-ups:

$$X_w \xrightarrow{\phi_{w-1}} X_{w-1} \xrightarrow{\phi_{w-2}} \cdots \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0.$$

Here for $d = 0, 1, \ldots, w-2$, ϕ_d is the blow-up of X_d along the (-2)-curves $C_{d,1}, \ldots, C_{d,k_{d+1}}$. Let

$$E_{d+1,i} := \phi_d^{-1}(C_{d,i}) \cong \begin{cases} \mathbb{F}_2, & i = 1, \ldots, k_{d+2}, \\ \mathbb{F}_0, & i = k_{d+1} + 1, \ldots, k_{d+1}. \end{cases}$$

For $i = 1, \ldots, k_{d+2}$, $C_{d+1,i} \subset E_{d+1,i}$ is the unique nonsingular rational curve with negative self intersection number, which is also a (-2)-curve in X_{d+1} with

$$width(C_{d+1,i} \subset X_{d+1}) = w_i - d - 1, \quad d = 1, \ldots, w-2.$$

Moreover, ϕ_{w-1} is the blow-up of X_{w-1} along the $(-1,-1)$-curves $C_{w-1,1}, \ldots,$ C_{w-1,k_w}, and

$$E_{w,i} := \phi_{w-1}^{-1}(C_{w-1,i}) \cong \mathbb{F}_0, \quad i = 1, \ldots, k_w.$$

For $d = 1, \ldots, w-1$ and $i = 1, \ldots, k_{d+1}$, the strict transform of $E_{d,i}$ under ϕ_d, denoted by $\tilde{E}_{d,i}$, is isomorphic to $E_{d,i}$. Moreover, $\tilde{E}_{d,i} \cap E_{d+1,i}$ is a nonsingular rational curve, which has negative self intersection number on

$\tilde{E}_{d,i}$, and self intersection number 2 on $E_{d+1,i}$. In particular, $\tilde{E}_{w-1,i} \cap E_{w,i}$ is a $(1,1)$-curve on $E_{w,i} \cong \mathbb{F}_0$. Note that $\tilde{E}_{d,i}$ is not affected by blow-ups $\phi_{d+1}, \ldots, \phi_{w-1}$, and can be viewed as an embedded surface in X_w. For $d = 1, \ldots, w-1$ and $i = k_{d+1}+1, \ldots, k_d$, $E_{d,i}$ is not affected by blow-ups $\phi_d, \ldots, \phi_{w-1}$, and can be viewed as an embedded surface in X_w.

Write $E_{w,i} = E'_{w,i}$. Since each $E'_{w,i} \cong \mathbb{F}_0$ has a ruling not contracted by ϕ_{w-1}, it follows that we can blow down X_w along these rulings for all i simultaneously to obtain $\phi'_{w-1} : X_w \to X'_{w-1}$. Proceeding inductively, we also have a sequence of blow-downs:

$$X_w \xrightarrow{\phi'_{w-1}} X'_{w-1} \xrightarrow{\phi'_{w-2}} \cdots \xrightarrow{\phi'_1} X'_1 \xrightarrow{\phi'_0} X'_0.$$

For $d = 0, 1, \ldots, w-2$, let

$$C'_{w-1-d,i} := \phi'_{w-1-d}(E'_{w-d,i}), \quad i = 1, \ldots, k_{w-d}$$

and

$$E'_{w-1-d,i} = \begin{cases} \phi'_{w-1-d} \circ \cdots \circ \phi'_{w-1}(\tilde{E}_{w-1-d,i}) \cong \mathbb{F}_2, \\ \quad i = 1, \ldots, k_{w-d}, \\ \phi'_{w-1-d} \circ \cdots \circ \phi'_{w-1}(E_{w-1-d,i}) \cong \mathbb{F}_0 \\ \quad i = k_{w-d}+1, \ldots, k_{w-1-d}. \end{cases}$$

Then $C'_{w-1-d,i}$ is a (-2)-curve in X'_{w-1-d} with

$$width(C'_{w-1-d,i} \subset X'_{w-1-d}) = w_i - w + 1 + d,$$

and $C'_{w-1-d,i} \subset E'_{w-1-d,i}$ is the unique nonsingular rational curve with negative self intersection number. Since for $i = 1, \ldots, k_{w-d}$, each $E'_{w-1-d,i} \cong \mathbb{P}_{C'_{w-1-d,i}}(\mathscr{O} \oplus \mathscr{O}(-2))$ has a fiber ruling, and for $i = k_{w-d}+1, \ldots, k_{w-1-d}$, each $E_{w-1-d,i}$ has a ruling not contracted by ϕ_{w-1-d}, it follows that we can blow down X'_{w-1-d} along these rulings simultaneously to obtain $\phi'_{w-2-d} : X'_{w-1-d} \to X'_{w-2-d}$.

Now for $d = 0, 1, \ldots, w-1$, the birational map

$$f_d := \phi'_d \circ \cdots \circ \phi'_{w-1} \circ \phi_{w-1}^{-1} \circ \cdots \circ \phi_d^{-1} : X_d \dashrightarrow X'_d$$

is a flop of (-2)-curves $C_{d,1}, \ldots, C_{d,k_{d+1}}$, where $C_{d,i}$ is flopped to $C'_{d,i}$. In particular, we have $X' = X'_0$ and $f = f_0$.

Degenerate X along $C_1, \ldots, C_l$ simultaneously, and we have

$$
\begin{aligned}
&Z'_{DT}\left(X; q \Big| \prod_{i=1}^{m} \tilde{\tau}_{d_i}(\gamma_i)\right)_{\beta} \\
&= \sum Z'_{DT}\left(X_1/E_1; q \Big| \prod_{i=1}^{m} \tilde{\tau}_{d_i}(\phi_0^* \gamma_i) | \eta_1^{\vee}, \ldots, \eta_l^{\vee}\right)_{\tilde{\beta}} \\
&\quad \times \prod_{i=1}^{l} \frac{(-1)^{|\eta_i| - \ell(\eta_i)} \mathfrak{z}(\eta_i)}{q^{|\eta_i|}} Z'_{DT}(\mathbb{P}_i / D_i; q || \eta_i)_{\beta_i},
\end{aligned}
$$

where we have assumed that the support of γ_i is away from $\bigcup_{i=1}^{l} C_i$, and

$$
\begin{aligned}
E_1 &:= \bigcup_{i=1}^{l} E_{1,i}, \\
\mathbb{P}_i &:= \mathbb{P}_{C_i}(N_{C_i} \oplus \mathscr{O}_{C_i}), \quad (N_{C_i} \text{ is the normal bundle of } C_i \text{ in } X) \\
D_i &:= \mathbb{P}_{C_i}(N_{C_i} \oplus \{0\}).
\end{aligned}
$$

By dimension constraint, we find that $\eta_1 = \cdots = \eta_l = \emptyset$. So

$$\tilde{\beta} \cdot E_1 = \beta_i \cdot D_i = 0.$$

For $\tilde{\beta}$, note that ϕ_0 induces a natural injection via 'pull-back' of 2-cycles

$$\phi_1^! = PD_{X_1} \circ \phi_0^* \circ PD_X : H_2(X, \mathbb{Z}) \to H_2(X_1, \mathbb{Z}),$$

where the image of $\phi_0^!$ is the subset of $H_2(X_1, \mathbb{Z})$ consisting of 2-cycles having intersection number zero with E_1, and so we have $\tilde{\beta} \in Im\phi_0^!$. For β_i, note that

$$H_2(\mathbb{P}_i, \mathbb{Z}) = \mathbb{Z}[C_i] \oplus \mathbb{Z} f_i,$$

where we have used the identification $C_i \cong \mathbb{P}_{C_i}(\{0\} \oplus \mathscr{O}_{C_i})$, and f_i is the class of a line in the fiber of $\mathbb{P}_i$. So $\beta_i \cdot D_i = 0$ implies that $\beta_i \in \mathbb{Z}_{\geqslant 0}[C_i]$, since β_i is effective. Therefore

$$\begin{aligned}
&Z'_{DT}\Big(X;q|\prod_{i=1}^{m}\tilde{\tau}_{d_i}(\gamma_i)\Big)_\beta\\
&=\sum_{\substack{\beta'\in H_2(X,\mathbb{Z}),n_i\in\mathbb{Z}_{\geqslant 0}\\ \beta'+n_1[C_1]+\cdots n_l[C_l]=\beta}} Z'_{DT}\Big(X_1/E_1;q|\prod_{i=1}^{m}\tilde{\tau}_{d_i}(\phi_0^*\gamma_i)|\Big)_{\phi_0^!\beta'}\\
&\quad\times\prod_{i=1}^{l} Z'_{DT}(\mathbb{P}_i/D_i;q||)_{n_i[C_i]}.
\end{aligned}$$

In particular, since the irreducible curves in the center of f generate an extremal face in $NE(X)$, it follows that for $\beta\in Cen(f)$, we have

$$Z'_{DT}\Big(X;q|\Big)_\beta=\sum_{\substack{\beta'+\sum_{i=1}^{l} n_i[C_i]=\beta\\ \beta'\in Cen(f)}} Z'_{DT}\Big(X_1/E_1;q||\Big)_{\phi_0^!\beta'}\prod_{i=1}^{l} Z'_{DT}(\mathbb{P}_i/D_i;q||)_{n_i[C_i]}.$$

Therefore we have obtained the following:

$$\begin{aligned}
&\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\Big(X;q|\prod_{i=1}^{m}\tilde{\tau}_{d_i}(\gamma_i)\Big)_\beta\\
&=\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\Big(X_1/E_1;q|\prod_{i=1}^{m}\tilde{\tau}_{d_i}(\phi_0^*\gamma_i)|\Big)_{\phi_0^!\beta}\\
&\quad\times\prod_{i=1}^{l}\sum_{d\geqslant 0} v^{d[C_i]} Z'_{DT}(\mathbb{P}_i/D_i;q||)_{d[C_i]},
\end{aligned}$$

$$\begin{aligned}
\text{(5)}\qquad &\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X;q|)_\beta\\
&=\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}\Big(X_1/E_1;q||\Big)_{\phi_0^!\beta}\cdot\prod_{i=1}^{l}\sum_{d\geqslant 0} v^{d[C_i]} Z'_{DT}(\mathbb{P}_i/D_i;q||)_{d[C_i]},
\end{aligned}$$

which implies that

$$\frac{\sum\limits_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\Big(X;q\big|\prod\limits_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\Big)_\beta}{\sum\limits_{\beta\in Cen(f)} v^\beta Z'_{DT}(X;q|)_\beta} \tag{6}$$

$$= \frac{\sum\limits_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\Big(X_1/E_1;q\big|\prod\limits_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\big|\Big)_{\phi_0^!\beta}}{\sum\limits_{\beta\in Cen(f)} v^\beta Z'_{DT}\Big(X_1/E_1;q\big|\big|\Big)_{\phi_0^!\beta}}.$$

Now degenerate X_1 along $E_{1,1},\dots,E_{1,l}$ simultaneously, and we obtain

$$\begin{aligned}
&Z'_{DT}\Big(X_1;q\big|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\Big)_{\phi_1^!\beta}\\
&=\sum Z'_{DT}\Big(X_1/E_1;q\big|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\big|\eta_1^\vee,\dots,\eta_l^\vee\Big)_{\tilde{\beta}}\\
&\quad\times\prod_{i=1}^l \frac{(-1)^{|\eta_i|-\ell(\eta_i)}\mathfrak{z}(\eta_i)}{q^{|\eta_i|}} Z'_{DT}(\mathbb{P}_{1,i}/D_{1,i};q||\eta_i)_{\beta_i},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{P}_{1,i} &:= \mathbb{P}_{E_{1,i}}(N_{E_{1,i}}\oplus\mathscr{O}_{E_{1,i}}), \quad (N_{E_{1,i}} \text{ is the normal bundle of } E_{1,i} \text{ in } X_1)\\
D_{1,i} &:= \mathbb{P}_{E_{1,i}}(N_{E_{1,i}}\oplus\{0\}).
\end{aligned}$$

By dimension constraint, we find that $\eta_1=\cdots=\eta_l=\emptyset$. So we have

$$\begin{aligned}
&Z'_{DT}\Big(X_1;q\big|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\Big)_{\phi_1^!\beta}\\
&=\sum_{\substack{\beta'\in H_2(X,\mathbb{Z}),\beta_i\in H_2(\mathbb{P}_{1,i},\mathbb{Z})\\ \phi_0^!\beta'+(\pi_{1,1})_*\beta_1+\cdots+(\pi_{1,l})_*\beta_l=\phi_0^!\beta\\ \beta_i\cdot E_{1,i}=\beta_i\cdot D_i=0}} Z'_{DT}\Big(X_1/E_1;q\big|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\big|\Big)_{\phi_0^!\beta'}\\
&\quad\times\prod_{i=1}^l Z'_{DT}(\mathbb{P}_{1,i}/D_{1,i};q||)_{\beta_i},
\end{aligned}$$

where we have used the identification $E_{1,i} \cong \mathbb{P}_{E_{1,i}}(\{0\} \oplus \mathscr{O}_{E_{1,i}})$, and $\pi_{1,i}$ is the composition

$$\mathbb{P}_{1,i} \to E_{1,i} \hookrightarrow X_1.$$

In particular, since

$$\beta' + (\phi_0)_*(\pi_{1,1})_*\beta_1 + \cdots + (\phi_0)_*(\pi_{1,l})_*\beta_l = \beta,$$

it follows that for $\beta \in Cen(f)$, we have

$$Z'_{DT}\Big(X_1; q|\Big)_{\phi_0^!\beta} = \sum_{\substack{\beta' + (\phi_0)_*(\pi_{1,1})_*\beta_1 + \cdots + (\phi_0)_*(\pi_{1,l})_*\beta_l = \beta \\ \beta' \in Cen(f) \\ \beta_i \cdot E_{1,i} = \beta_i \cdot D_{1,i} = 0}} Z'_{DT}\Big(X_1/E_1; q||\Big)_{\phi_0^!\beta'} \times \prod_{i=1}^{l} Z'_{DT}(\mathbb{P}_{1,i}/D_{1,i}; q||)_{\beta_i},$$

So we have obtained

$$\begin{aligned}
&\sum_{\beta \in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\Big(X_1; q|\prod_{i=1}^{m} \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\Big)_{\phi_0^!\beta} \\
&= \sum_{\beta \in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\Big(X_1/E_1; q|\prod_{i=1}^{m} \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)|\Big)_{\phi_0^!\beta} \\
&\quad \times \prod_{i=1}^{l} \sum_{d \geqslant 0} v^{d[C_i]} \sum_{\substack{\beta_i \in H_2(\mathbb{P}_{1,i},\mathbb{Z}) \\ \beta_i \cdot E_{1,i} = \beta_i \cdot D_{1,i} = 0 \\ (\phi_0)_*(\pi_{1,i})_*\beta_i = d[C_i]}} Z'_{DT}(\mathbb{P}_{1,i}/D_{1,i}; q||)_{\beta_i},
\end{aligned}$$

$$\sum_{\beta \in Cen(f)} v^\beta Z'_{DT}(X_1; q|)_{\phi_0^!\beta} = \sum_{\beta \in Cen(f)} v^\beta Z'_{DT}\Big(X_1/E_1; q||\Big)_{\phi_0^!\beta} \times \prod_{i=1}^{l} \sum_{d \geqslant 0} v^{d[C_i]} \sum_{\substack{\beta \in H_2(\mathbb{P}_{1,i},\mathbb{Z}) \\ \beta \cdot E_{1,i} = \beta \cdot D_{1,i} = 0 \\ (\phi_0)_*(\pi_{1,i})_*\beta = d[C_i]}} Z'_{DT}(\mathbb{P}_{1,i}/D_{1,i}; q||)_\beta, \tag{7}$$

which implies that

$$\frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\left(X_1;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\right)_{\phi_0^!\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X_1;q|)_{\phi_0^!\beta}} \tag{8}$$

$$= \frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\left(X_1/E_1;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\middle|\right)_{\phi_0^!\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}\left(X_1/E_1;q||\right)_{\phi_0^!\beta}}.$$

Then from (6) and (8), we have

$$\frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\left(X;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\gamma_i)\right)_{\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X;q|)_{\beta}} \tag{9}$$

$$= \frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\left(X_1;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i)\right)_{\phi_0^!\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X_1;q|)_{\phi_0^!\beta}}.$$

Using the identification $\mathscr{F} : H_2(X,\mathbb{Z}) \xrightarrow{\cong} H_2(X',\mathbb{Z})$, we also have

$$\frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\left(X';q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(\mathscr{F}\gamma_i)\right)_{\mathscr{F}\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X';q|)_{\mathscr{F}\beta}} \tag{10}$$

$$= \frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}\left(X'_1;q\middle|\prod_{i=1}^m \tilde{\tau}_{d_i}(((\phi'_0)^*\mathscr{F}\gamma_i)\right)_{((\phi'_0)^!\mathscr{F}\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X'_1;q|)_{(\phi'_0)^!\mathscr{F}\beta}}.$$

Now we use induction on $w = 1, 2, 3, \ldots$ to prove (1) in Proposition 1.1. For $w = 1$, we have the following observation.

Lemma 3.1. *For any nonzero $\beta \in Cen(f)$, $\phi_0^!\beta$ is not effective.*

Proof. Argue by contradiction, and then $\beta = (\phi_0)_*\phi_0^!\beta$ is also effective. We can write $\beta = \sum_{i=1}^{l} a_i[C_i]$ with $a_i \in \mathbb{Z}_{\geqslant 0}$. Note that $\mathscr{F}[C_i] = -[C_i']$, and then

$$(\phi_0')_*\phi_0^!\beta = \mathscr{F}\beta = -\sum_{i=1}^{l} a_i[C_i'].$$

Since $\sum_{i=1}^{l} a_i[C_i']$ is effective, it follows that $(\phi_0')_*\phi_0^!\beta$ is not effective, which implies that $\phi_0^!\beta$ is not effective. □

Therefore,

$$\sum_{\beta\in Cen(f)} v^{\beta} Z'_{DT}(X_1; q|)_{\phi_0^!\beta} = 1 \quad \text{and} \quad \sum_{\beta'\in Cen(f^{-1})} v^{\beta'} Z'_{DT}(X_1; q|)_{\phi_0^!\beta'} = 1.$$

Note that in (9) and (10), we have

$$\phi_0^*\gamma_i = (\phi_0')^*\mathscr{F}\gamma_i \text{ and } \phi_0^!\beta = (\phi_0')^!\mathscr{F}\beta.$$

So in the case $w = 1$, (1) follows from (9) and (10).

Assume that the case for $w = W \geqslant 1$ is proved. Then for $w = W + 1$, we have

$$\begin{aligned}&\frac{\sum\limits_{\beta_1\in H_2(X_1,\mathbb{Z})} v^{\beta_1} Z'_{DT}(X_1; q|\prod\limits_{i=1}^{m}\tilde{\tau}_{d_i}(\phi_0^*\gamma_i))_{\beta_1}}{\sum\limits_{\beta_1\in Cen(f_1)} v^{\beta_1} Z'_{DT}(X_1; q|)_{\beta_1}}\\
&= \frac{\sum\limits_{\beta_1\in H_2(X_1,\mathbb{Z})} v^{\beta_1} Z'_{DT}(X_1'; q|\prod\limits_{i=1}^{m}\tilde{\tau}_{d_i}(\mathscr{F}_1\phi_0^*\gamma_i))_{\mathscr{F}_1\beta_1}}{\sum\limits_{\beta_1\in Cen(f_1)} v^{\beta_1} Z'_{DT}(X_1'; q|)_{\mathscr{F}_1\beta_1}},\end{aligned}$$

where $\mathscr{F}_1$ is the correspondence on (co)homology groups induced by f_1. We have the following key observation.

Lemma 3.2. *Let $S =Span_{\mathbb{Z}}\{[C_1], \ldots, [C_{k_2}]\}$. For any $\beta \in Cen(f) \setminus S$, $\phi_0^!\beta$ is not effective.*

Proof. Without loss of generality, assume that

$$S \cap \{[C_{k_2+1}], \ldots, [C_l]\} = \{[C_{l'}], \ldots, [C_l]\}.$$

Argue by contradiction, and we can write $\phi_0^! \beta = \sum_{j=1}^{n} m_j [V_j]$, where $m_j \in \mathbb{Z}_{\geqslant 0}$, and $V_1, \ldots, V_n$ are mutually distinct irreducible curves in X_1. Since

$$\beta = (\phi_0)_* \phi_0^! \beta \in Cen(f) \setminus S,$$

and $[C_1], \ldots, [C_l]$ generate an extremal face in $NE(X)$, it follows that, for each j, V_j is mapped onto a point or some C_i. In the former case, V_j is a fiber of one irreducible component of E_1 and then $V_j \cdot E_1 < 0$. In the latter case, V_j is contained in $E_{1,i}$ and then $V_j \cdot E_{1,i} \leqslant 0$. Moreover, we can find some V_j which is contained in some $E_{1,i} \cong \mathbb{F}_0$ for $l' \leqslant i \leqslant l$, and then $V_j \cdot E_{1,i} < 0$. In sum, we have $\phi_0^! \beta \cdot E_1 < 0$, which is absurd. □

Since $Cen(f_1) = \{\phi_0^! \beta : \beta \in S\}$, it follows that

$$\sum_{\beta_1 \in Cen(f_1)} v^{\beta_1} Z'_{DT}(X_1; q|)_{\beta_1} = \sum_{\beta \in Cen(f)} v^{\phi_0^! \beta} Z'_{DT}(X_1; q|)_{\phi_0^! \beta}.$$

Now we have

$$\begin{aligned}
\sum_{\beta_1 \in Cen(f_1)} v^{\mathscr{F}_1 \beta_1} Z'_{DT}(X'_1; q|)_{\mathscr{F}_1 \beta_1} &= \sum_{\beta'_1 \in Cen(f_1^{-1})} v^{\beta'_1} Z'_{DT}(X'_1; q|)_{\beta'_1} \\
&= \sum_{\beta' \in Cen(f^{-1})} v^{(\phi'_0)^! \beta'} Z'_{DT}(X'_1; q|)_{(\phi'_0)^! \beta'} \\
&= \sum_{\beta \in Cen(f)} v^{(\phi'_0)^! \mathscr{F} \beta} Z'_{DT}(X'_1; q|)_{(\phi'_0)^! \mathscr{F} \beta} \\
&= \sum_{\beta \in Cen(f)} v^{\mathscr{F}_1 \phi_0^! \beta} Z'_{DT}(X'_1; q|)_{(\phi'_0)^! \mathscr{F} \beta},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{\sum_{\beta_1 \in H_2(X_1, \mathbb{Z})} v^{\beta_1} Z'_{DT}(X_1; q| \prod_{i=1}^{m} \tilde{\tau}_{d_i}(\phi_0^* \gamma_i))_{\beta_1}}{\sum_{\beta \in Cen(f)} v^{\phi_0^! \beta} Z'_{DT}(X_1; q|)_{\phi_0^! \beta}} \\
= &\frac{\sum_{\beta_1 \in H_2(X_1, \mathbb{Z})} v^{\beta_1} Z'_{DT}(X'_1; q| \prod_{i=1}^{m} \tilde{\tau}_{d_i}(\mathscr{F}_1 \phi_0^* \gamma_i))_{\mathscr{F}_1 \beta_1}}{\sum_{\beta \in Cen(f)} v^{\phi_0^! \beta} Z'_{DT}(X'_1; q|)_{(\phi'_0)^! \mathscr{F} \beta}}.
\end{aligned}$$

Observe that we have the following decomposition

$$H_2(X_1,\mathbb{Z}) = \phi_0^! H_2(X,\mathbb{Z}) \oplus \mathbb{Z}f_{1,1} \oplus \cdots \oplus \mathbb{Z}f_{1,l},$$

where $f_{1,i}$ is the class of a fiber in $E_{1,i}$. So we obtain

$$\begin{aligned}(11)\qquad &\frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}(X_1;q|\prod_{i=1}^m \tilde{\tau}_{d_i}(\phi_0^*\gamma_i))_{\phi_0^!\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X_1;q|)_{\phi_0^!\beta}} \\ &= \frac{\sum_{\beta\in H_2(X,\mathbb{Z})} v^\beta Z'_{DT}(X'_1;q|\prod_{i=1}^m \tilde{\tau}_{d_i}(\mathscr{F}_1\phi_0^*\gamma_i))_{\mathscr{F}_1\phi_0^!\beta}}{\sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X'_1;q|)_{(\phi'_0)^!\mathscr{F}\beta}}.\end{aligned}$$

Note that

$$\mathscr{F}_1\phi_0^*\gamma_i = (\phi'_0)^!\mathscr{F}\gamma_i \text{ and } \mathscr{F}_1\phi_0^!\beta = (\phi'_0)^!\mathscr{F}\beta,$$

and we see that in the case $w = W+1$, (1) follows from (9), (10) and (11).

To prove (2), we have the following observation. Using the identification $-\mathscr{F} : H_2(X,\mathbb{Z}) \xrightarrow{\cong} H_2(X',\mathbb{Z})$, from (5), we have

$$\begin{aligned}(12)\qquad \sum_{\beta\in Cen(f)} v^{-\beta} Z'_{DT}(X';q|)_{\mathscr{F}\beta} &= \sum_{\beta\in Cen(f)} v^{-\beta} Z'_{DT}\Big(X'_1/E'_1;q||\Big)_{(\phi'_0)^!\mathscr{F}\beta} \\ &\times \prod_{i=1}^l \sum_{d\geqslant 0} v^{d[C_i]} Z'_{DT}(\mathbb{P}'_i/D'_i;q||)_{d[C'_i]},\end{aligned}$$

where

$$\begin{aligned}&E'_1 := \bigcup_{i=1}^l E'_{1,i},\\ &\mathbb{P}'_i := \mathbb{P}_{C'_i}(N_{C'_i}\oplus\mathscr{O}_{C'_i}),\quad (N_{C'_i} \text{ is the normal bundle of } C'_i \text{ in } X')\\ &D'_i := \mathbb{P}_{C'_i}(N_{C'_i}\oplus\{0\}),\\ &C'_i \cong \mathbb{P}_{C'_i}(\{0\}\oplus\mathscr{O}_{C'_i}).\end{aligned}$$

So from (5) and (12), (2) is equivalent to the following:

$$(13)\qquad \sum_{\beta\in Cen(f)} v^\beta Z'_{DT}\Big(X_1/E_1;q||\Big)_{\phi_0^!\beta} = \sum_{\beta\in Cen(f)} v^{-\beta} Z'_{DT}\Big(X'_1/E'_1;q||\Big)_{(\phi'_0)^!\mathscr{F}\beta}$$

Now we use induction on $w = 1, 2, 3, \dots$ to prove (2) (or (13)). For $w = 1$, Lemma 3.1 implies that both LHS and RHS of (13) are equal to 1. Assume that the case for $w = W \geqslant 1$ is proved. Then for $w = W + 1$, we have

$$\sum_{\beta_1 \in Cen(f_1)} v^{\beta_1} Z'_{DT}(X_1; q|)_{\beta_1} = \sum_{\beta_1 \in Cen(f_1)} v^{-\beta_1} Z'_{DT}(X'_1; q|)_{\mathscr{F}_1 \beta_1},$$

and by Lemma 3.2, this gives

$$\sum_{\beta \in Cen(f)} v^{\beta} Z'_{DT}(X_1; q|)_{\phi_0^! \beta} = \sum_{\beta \in Cen(f)} v^{-\beta} Z'_{DT}(X'_1; q|)_{\mathscr{F}_1 \phi_0^! \beta}, \tag{14}$$

Note that using the identification $-\mathscr{F} : H_2(X, \mathbb{Z}) \xrightarrow{\cong} H_2(X', \mathbb{Z})$, (7) gives

$$\begin{aligned} &\sum_{\beta \in Cen(f)} v^{-\beta} Z'_{DT}(X'_1; q|)_{(\phi'_0)^! \mathscr{F} \beta} \\ &= \sum_{\beta \in Cen(f)} v^{-\beta} Z'_{DT}\Big(X'_1/E'_1; q||\Big)_{(\phi'_0)^! \mathscr{F} \beta} \\ &\times \prod_{i=1}^{l} \sum_{d \geqslant 0} v^{d[C_i]} \sum_{\substack{\beta \in H_2(\mathbb{P}'_{1,i}, \mathbb{Z}) \\ \beta \cdot E'_{1,i} = \beta \cdot D'_{1,i} = 0 \\ (\phi'_0)_* (\pi'_{1,i})_* \beta = d[C'_i]}} Z'_{DT}(\mathbb{P}'_{1,i}/D'_{1,i}; q||)_{\beta}, \end{aligned} \tag{15}$$

where

$\mathbb{P}'_{1,i} := \mathbb{P}_{E'_{1,i}}(N_{E'_{1,i}} \oplus \mathscr{O}_{E'_{1,i}})$, ($N_{E'_{1,i}}$ is the normal bundle of $E'_{1,i}$ in X'_1)
$D'_{1,i} := \mathbb{P}_{E'_{1,i}}(N_{E'_{1,i}} \oplus \{0\})$,
$E'_{1,i} \cong \mathbb{P}_{E'_{1,i}}(\{0\} \oplus \mathscr{O}_{E'_{1,i}})$,

and $\pi'_{1,i}$ is the composition $\mathbb{P}'_{1,i} \to E'_{1,i} \hookrightarrow X'_1$. Note that

$$\mathscr{F}_1 \phi_0^! \beta = (\phi'_0)^! \mathscr{F} \beta,$$

and we see that in the case $w = W + 1$, (13) follows from (7), (14) and (15).

4. BPS state counts

BPS state counts were first introduced in the Gromov-Witten theory. In a study of Type IIA string theory via M-theory, Gopakumar and Vafa defined

BPS state counts on Calabi-Yau 3-folds [7, 8]. Motivated by the Calabi-Yau case together with the degenerate contribution computation, Pandharipande defined BPS state counts for arbitrary 3-folds [25]. We refer interested readers to [26] for a precise description of the working definition of BPS state counts of Gromov-Witten theory of X.

Now we give a working definition of BPS state counts of Donaldson-Thomas theory. Let $\{T_i\}_{0\leqslant i\leqslant N}$ be a basis of $H^*(X,\mathbb{Q})$, and we define the BPS state counts of Donaldson-Thomas theory by the following identity:

$$\sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_\beta c_1(X)=0}} v^\beta Z'_{DT}(X;q|)_\beta + \sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_\beta c_1(X)>0}} v^\beta \sum_{e_0,\ldots,e_N\in\mathbb{Z}_{\geqslant 0}} Z'_{DT}(X;q|\prod_{i=0}^N \tilde{\tau}_0(T_i)^{e_i})_\beta \prod_{i=0}^N \frac{t_i^{e_i}}{e_1!}$$

$$= \exp\Bigg\{ \sum_{\substack{\beta\in H_2(X,\mathbb{Z})\setminus\{0\}\\ \int_\beta c_1(X)=0}} v^\beta \sum_{g\in\mathbb{Z}} \sum_{r\in div(\beta)} n^X_{g,\frac{\beta}{r}} \cdot \frac{(-1)^{g-1}}{r}\Big[(-q)^r - 2 + (-q)^{-r}\Big]^{g-1}$$

$$+ \sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_\beta c_1(X)>0}} v^\beta \sum_{g\in\mathbb{Z}} \sum_{e_0,\ldots,e_N\in\mathbb{Z}_{\geqslant 0}} n^X_{g,\beta}\Big(\prod_{i=0}^N T_i^{e_i}\Big)$$

$$\times \prod_{i=0}^N \frac{t_i^{e_i}}{e_1!} \cdot (-1)^{g-1}\Big[(-q) - 2 + (-q)^{-1}\Big]^{g-1}(1+q)^{\int_\beta c_1(X)}\Bigg\}.$$

Since by Lemma 3.1 in [6], the full primary Donaldson-Thomas theory is determined by those invariants with primary insertions (if any) of degree> 2, it follows that the BPS state counts vanish if insertions of degree< 2 appear, and they satisfy the divisor equation.

Note that the Donaldson-Thomas theory counts curves only in a virtual sense. However, it is expected that BPS state counts are enumerative. More precisely, assume that $\gamma_1,\ldots,\gamma_m$ are integral, and let $X_i\subset X$ be a subvariety which is the Poincaé dual of γ_i in general position. Then $n^X_{g,\beta}(\gamma_1,\ldots,\gamma_m)$ is expected to the number of irreducible embedded curves in X of geometric genus g, with homology class β and intersecting with all X_i's.

To prove Corollary 1.3, we only need to consider insertions of degree> 2. Without loss of generality, let $\{T_i\}_{0\leqslant i\leqslant L}$ be a basis of $H^{>2}(X,\mathbb{Q})$. Since $[C_1],\ldots,[C_l]$ generate an extremal face in $NE(X)$, it follows that

$$(16)\qquad \sum_{\beta\in Cen(f)} v^\beta Z'_{DT}(X;q|)_\beta$$

$$= \exp\Bigg\{ \sum_{\beta\in Cen(f)} v^\beta \sum_{g\in\mathbb{Z}} \sum_{r\in div(\beta)} n^X_{g,\frac{\beta}{r}} \cdot \frac{(-1)^{g-1}}{r}\Big[(-q)^r-2+(-q)^{-r}\Big]^{g-1}\Bigg\},$$

and then

(17)

$$\frac{1}{\sum\limits_{\beta\in Cen(f)} v^{\beta} Z'_{DT}(X;q|)_{\beta}}\Bigg(\sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_\beta c_1(X)=0}} v^{\beta} Z'_{DT}(X;q|)_{\beta}$$

$$+\sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_\beta c_1(X)>0}} v^{\beta} \sum_{e_0,\ldots,e_L\in\mathbb{Z}_{\geqslant 0}} Z'_{DT}(X;q|\prod_{i=0}^{L}\tilde{\tau}_0(T_i)^{e_i})_{\beta}\prod_{i=0}^{L}\frac{t_i^{e_i}}{e_1!}\Bigg)$$

$$=\exp\Bigg\{\sum_{\substack{\beta\in H_2(X,\mathbb{Z})\setminus Cen(f)\\ \int_\beta c_1(X)=0}} v^{\beta}\sum_{g\in\mathbb{Z}}\sum_{r\in div(\beta)} n^X_{g,\frac{\beta}{r}}\cdot\frac{(-1)^{g-1}}{r}\Big[(-q)^r-2+(-q)^{-r}\Big]^{g-1}$$

$$+\sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_\beta c_1(X)>0}} v^{\beta}\sum_{g\in\mathbb{Z}}\sum_{e_0,\ldots,e_L\in\mathbb{Z}_{\geqslant 0}} n^X_{g,\beta}(\prod_{i=0}^{L}T_i^{e_i})\prod_{i=0}^{L}\frac{t_i^{e_i}}{e_1!}$$

$$\times(-1)^{g-1}\Big[(-q)-2+(-q)^{-1}\Big]^{g-1}(1+q)^{\int_\beta c_1(X)}\Bigg\}.$$

Using the identification $\mathscr{F}: H_2(X,\mathbb{Z})\xrightarrow{\cong} H_2(X',\mathbb{Z})$, we also have

(18)

$$\frac{\sum\limits_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_{\mathscr{F}\beta} c_1(X')=0}} v^{\beta} Z'_{DT}(X';q|)_{\mathscr{F}\beta}+\sum\limits_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_\beta c_1(X)>0}} v^{\beta}\sum\limits_{e_0,\ldots,e_L\in\mathbb{Z}_{\geqslant 0}} Z'_{DT}(X';q|\prod\limits_{i=0}^{L}\tilde{\tau}_0(\mathscr{F}T_i)^{e_i})_{\mathscr{F}\beta}\prod\limits_{i=0}^{L}\frac{t_i^{e_i}}{e_1!}}{\sum\limits_{\beta\in Cen(f)} v^{\beta} Z'_{DT}(X';q|)_{\mathscr{F}\beta}}$$

$$=\exp\Bigg\{\sum_{\substack{\beta\in H_2(X,\mathbb{Z})\setminus Cen(f)\\ \int_\beta c_1(X)=0}} v^{\beta}\sum_{g\in\mathbb{Z}}\sum_{r\in div(\beta)} n^{X'}_{g,\frac{\mathscr{F}\beta}{r}}\cdot\frac{(-1)^{g-1}}{r}\Big[(-q)^r-2+(-q)^{-r}\Big]^{g-1}$$

$$+\sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \int_{\mathscr{F}\beta} c_1(X')>0}} v^{\beta}\sum_{g\in\mathbb{Z}}\sum_{e_0,\ldots,e_L\in\mathbb{Z}_{\geqslant 0}} n^{X'}_{g,\mathscr{F}\beta}(\prod_{i=0}^{L}(\mathscr{F}T_i)^{e_i})\prod_{i=0}^{L}\frac{t_i^{e_i}}{e_1!}$$

$$\times(-1)^{g-1}\Big[(-q)-2+(-q)^{-1}\Big]^{g-1}(1+q)^{\int_{\mathscr{F}\beta} c_1(X')}\Bigg\}.$$

Note that

$$\int_\beta c_1(X) = \int_{\phi^!_{w-1}\cdots\phi^!_0\beta} c_1(X_w) = \int_{(\phi'_{w-1})^!\cdots(\phi'_0)^!\mathscr{F}\beta} c_1(X_w) = \int_{\mathscr{F}\beta} c_1(X').$$

So (3) follows from (1), (17) and (18).

For (4), using the identification $-\mathscr{F} : H_2(X,\mathbb{Z}) \xrightarrow{\cong} H_2(X',\mathbb{Z})$, we get from (16)

(19)

$$\sum_{\beta\in Cen(f)} v^{-\beta} Z'_{DT}(X';q|)_{\mathscr{F}\beta}$$
$$= \exp\left\{ \sum_{\beta\in Cen(f)} v^{-\beta} \sum_{g\in\mathbb{Z}} \sum_{r\in div(\mathscr{F}\beta)} n^X_{g,\frac{\mathscr{F}\beta}{r}} \cdot \frac{(-1)^{g-1}}{r} \Big[(-q)^r - 2 + (-q)^{-r}\Big]^{g-1} \right\},$$

So (4) follows from (2), (16) and (19).

5. GW/DT correspondence

In this section, we give a proof of Corollary 1.4. We first review basic materials in Gromov-Witten theory, and describe the change of Gromov-Witten theory under flops. Then we follow [21] to recall the conjectural formulae for the GW/DT correspondence, and use these formulae to prove Corollary 1.4.

Let $\overline{M}_{g,m}(X,\beta)$ be the moduli space of m-pointed stable maps from connected, genus g curves to X, representing the class $\beta \in H_2(X,\mathbb{Z})$. Let $ev_i : \overline{M}_{g,m}(X,\beta) \to X$ be the evaluation map at the i-th marked point, and set

$$\psi_i := c_1(L_i) \in H^2(\overline{M}_{g,m}(X,\beta),\mathbb{Q}),$$

where L_i is the cotangent line bundle associated to the i-th marked point. For $\gamma_1,\ldots,\gamma_m \in H^*(X,\mathbb{Q})$ and $d_1,\ldots,d_m \in \mathbb{Z}_{\geqslant 0} (m \geqslant 0)$, define the (connected) correlator by

$$\left\langle \prod_{i=1}^m \tau_{d_i}(\gamma_i) \right\rangle^X_{g,\beta} := \int_{[\overline{M}_{g,m}(X,\beta)]^{vir}} \prod_{i=1}^m \psi_i^{d_i} ev_i^*(\gamma_i).$$

The conjectural GW/DT correspondence compares partition functions of disconnected Gromov-Witten invariants with reduced Donaldson-Thomas partition function. Let $\{T_i\}_{0\leqslant i\leqslant N}$ be a basis of $H^*(X,\mathbb{Q})$, and the disconnected partition functions in Gromov-Witten theory are given by the following identity:

$$1+\sum_{\beta\in H_2(X,\mathbb{Z})\setminus\{0\}} v^\beta \sum_{e_{d,i}\in\mathbb{Z}_{\geqslant 0}} Z'_{GW}(X;u|\prod_{\substack{d\geqslant 0\\ 0\leqslant i\leqslant N}}\tau_d(T_i)^{e_{d,i}})_\beta \prod_{\substack{d\geqslant 0\\ 0\leqslant i\leqslant N}}\frac{t_{d,i}^{e_{d,i}}}{e_{d,i}!}$$

$$=\exp\left\{\sum_{\beta\in H_2(X,\mathbb{Z})\setminus\{0\}} v^\beta \sum_{g\in\mathbb{Z}_{\geqslant 0}} u^{2g-2}\sum_{e_{d,i}\in\mathbb{Z}_{\geqslant 0}}\langle\prod_{\substack{d\geqslant 0\\ 0\leqslant i\leqslant N}}\tau_d(T_i)^{e_{d,i}}\rangle^X_{g,\beta}\prod_{\substack{d\geqslant 0\\ 0\leqslant i\leqslant N}}\frac{t_{d,i}^{e_{d,i}}}{e_{d,i}!}\right\}.$$

For the change of Gromov-Witten theory under flops, we have the following theorem.

Theorem 5.1. *(Theorem A in [18]) Let f be a general flop. Let $\gamma_1,\ldots,\gamma_m\in H^*(X,\mathbb{Q})$ and $d_1,\ldots,d_m\in\mathbb{Z}_{\geqslant 0}(m\geqslant 0)$, such that γ_i has support away from the center of f. Then*

$$\frac{1+\sum\limits_{\beta\in H_2(X,\mathbb{Z})\setminus\{0\}} v^\beta Z'_{GW}(X;u|\prod\limits_{i=1}^m\tilde{\tau}_{d_i}(\gamma_i))_\beta}{1+\sum\limits_{\beta\in Cen(f)\setminus\{0\}} v^\beta Z'_{GW}(X;u|)_\beta} \tag{20}$$

$$=\frac{1+\sum\limits_{\beta\in H_2(X,\mathbb{Z})\setminus\{0\}} v^\beta Z'_{GW}(X';u|\prod\limits_{i=1}^m\tilde{\tau}_{d_i}(\mathscr{F}\gamma_i))_{\mathscr{F}\beta}}{1+\sum\limits_{\beta\in Cen(f)\setminus\{0\}} v^\beta Z'_{GW}(X';u|)_{\mathscr{F}\beta}},$$

$$Z'_{GW}(X;u|)_\beta = Z'_{GW}(X';u|)_{-\mathscr{F}\beta},\qquad \forall\beta\in Cen(f)\setminus\{0\}. \tag{21}$$

Remark 5.2. *Theorem A in [18] only deals with the case $d_1=\cdots=d_m=0$, but the generalization is straightforward.*

Now we give precise formulae for the conjectural GW/DT correspondence. For primary insertions, we have the following conjecture.

Conjecture 5.3. *(Conjecture 2 in [21]) Suppose that*

$$\gamma_1,\ldots,\gamma_m\in H^*(X,\mathbb{Q})(m\geqslant 0).$$

Then after the change of variables $q = -e^{\sqrt{-1}u}$, we have

$$(-\sqrt{-1}u)^{\int_\beta c_1(X)} Z'_{GW}\left(X; u \Big| \prod_{i=1}^{m} \tau_0(\gamma_i)\right)_\beta = (-q)^{-\frac{1}{2}\int_\beta c_1(X)} Z'_{DT}\left(X; q \Big| \prod_{i=1}^{m} \tilde{\tau}_0(\gamma_i)\right)_\beta .$$

The authors of [21] conjectured that the descendent Gromov-Witten theory of X is equivalent to the descendent Donaldson-Thomas theory of X in a subtle way. In the general case, they did not find a complete formula for the conjectural correspondence. However, we have the following precise conjecture for the descendants of the point class.

Conjecture 5.4. *(Conjecture $4'$ in [21]) Let P be the class of a point in X. Suppose that $\gamma_1, \dots, \gamma_m \in H^{>0}(X, \mathbb{Q})(m \geqslant 0)$, and $d_1, \dots, d_n \in \mathbb{Z}_{\geqslant 0}(n \geqslant 0)$. Then after the change of variables $q = -e^{\sqrt{-1}u}$, we have*

$$(-\sqrt{-1}u)^{\int_\beta c_1(X) - \sum_{i=1}^{n} d_i} Z'_{GW}\left(X; u \Big| \prod_{i=1}^{m} \tau_0(\gamma_i) \prod_{i=1}^{n} \tau_{d_i}(P)\right)_\beta = (-q)^{-\frac{1}{2}\int_\beta c_1(X)} Z'_{DT}\left(X; q \Big| \prod_{i=1}^{m} \tilde{\tau}_0(\gamma_i) \prod_{i=1}^{n} \tilde{\tau}_{d_i}(P)\right)_\beta .$$

To prove Corollary 1.4, note that by Lemma 3.1 in [6], we only need to consider insertions whose pullback classes have degree> 2. Without loss of generality, let $\{T_i\}_{0 \leqslant i \leqslant L}$ be a basis of $H^*(X, \mathbb{Q})$, where T_0 is the class of a point. Then the assumption in Corollary 1.4 gives

$$1 + \sum_{\beta \in H_2(X,\mathbb{Z}) \setminus \{0\}} v^\beta (-\sqrt{-1}u)^{\int_\beta c_1(X)} \times \sum_{\substack{e_{0,1},\dots,e_{0,L} \in \mathbb{Z}_{\geqslant 0} \\ e_{d,0} \in \mathbb{Z}_{\geqslant 0}}} Z'_{GW}\left(X; u \Big| \prod_{i=1}^{L} \tau_0(T_i)^{e_{0,i}} \prod_{d=0}^{\infty} \tau_d(T_0)^{e_{d,0}}\right)_\beta \times \prod_{i=1}^{L} \frac{t_{0,i}^{e_{0,i}}}{e_{0,i}!} \prod_{d=0}^{\infty} \frac{((-\sqrt{-1}u)^{-1} t_{d,0})^{e_{d,0}}}{e_{d,0}!}$$

$$= \sum_{\beta \in H_2(X,\mathbb{Z})} v^\beta (-q)^{-\frac{1}{2}\int_\beta c_1(X)}$$
$$\times \sum_{\substack{e_{0,1},\ldots,e_{0,L} \in \mathbb{Z}_{\geqslant 0} \\ e_{d,0} \in \mathbb{Z}_{\geqslant 0}}} Z'_{DT}\left(X; q \middle| \prod_{i=1}^{L} \tilde{\tau}_0(T_i)^{e_{0,i}} \prod_{d=0}^{\infty} \tilde{\tau}_d(T_0)^{e_{d,0}}\right)_\beta \prod_{i=1}^{L} \frac{t_{0,i}^{e_{0,i}}}{e_{0,i}!} \prod_{d=0}^{\infty} \frac{t_{d,0}^{e_{d,0}}}{e_{d,0}!}.$$

Note that the map $v^\beta \mapsto v^\beta(-\sqrt{-1}u)^{\int_\beta c_1(X)}$ gives an isomorphism in the Novikov ring of X, and then (20) implies that

$$\frac{1 + \sum_{\beta \in H_2(X,\mathbb{Z})\setminus\{0\}} v^\beta (-\sqrt{-1}u)^{\int_\beta c_1(X)} \sum_{\substack{e_{0,1},\ldots,e_{0,L} \in \mathbb{Z}_{\geqslant 0} \\ e_{d,0} \in \mathbb{Z}_{\geqslant 0}}} Z'_{GW}(X; u| \prod_{i=1}^{L} \tau_0(T_i)^{e_{0,i}} \prod_{d=0}^{\infty} \tau_d(T_0)^{e_{d,0}})_\beta \prod_{i=1}^{L} \frac{t_{0,i}^{e_{0,i}}}{e_{0,i}!} \prod_{d=0}^{\infty} \frac{((-\sqrt{-1}u)^{-1} t_{d,0})^{e_{d,0}}}{e_{d,0}!}}{1 + \sum_{\beta \in Cen(f)\setminus\{0\}} v^\beta Z'_{GW}(X; u|)_\beta}$$
$$= \frac{1 + \sum_{\beta \in H_2(X,\mathbb{Z})\setminus\{0\}} v^\beta (-\sqrt{-1}u)^{\int_\beta c_1(X)} \sum_{\substack{e_{0,1},\ldots,e_{0,L} \in \mathbb{Z}_{\geqslant 0} \\ e_{d,0} \in \mathbb{Z}_{\geqslant 0}}} Z'_{GW}(X'; u| \prod_{i=1}^{L} \tau_0(\mathscr{F}T_i)^{e_{0,i}} \prod_{d=0}^{\infty} \tau_d(\mathscr{F}T_0)^{e_{d,0}})_{\mathscr{F}\beta} \prod_{i=1}^{L} \frac{t_{0,i}^{e_{0,i}}}{e_{0,i}!} \prod_{d=0}^{\infty} \frac{((-\sqrt{-1}u)^{-1} t_{d,0})^{e_{d,0}}}{e_{d,0}!}}{1 + \sum_{\beta \in Cen(f)\setminus\{0\}} v^\beta Z'_{GW}(X'; u|)_{\mathscr{F}\beta}}.$$

Here we have used the change of variables $t_{d,0} \mapsto (-\sqrt{-1}u)^{-1} t_{d,0}$. Similarly, note that the change of variables $v^\beta \mapsto v^\beta(-q)^{-\frac{1}{2}\int_\beta c_1(X)}$ gives an isomorphism of the Novikov ring of X, and then (1) gives

$$\frac{\sum_{\beta \in H_2(X,\mathbb{Z})} v^\beta (-q)^{-\frac{1}{2}\int_\beta c_1(X)} \sum_{\substack{e_{0,1},\ldots,e_{0,L} \in \mathbb{Z}_{\geqslant 0} \\ e_{d,0} \in \mathbb{Z}_{\geqslant 0}}} Z'_{DT}(X; q| \prod_{i=1}^{L} \tilde{\tau}_0(T_i)^{e_{0,i}} \prod_{d=0}^{\infty} \tilde{\tau}_d(T_0)^{e_{d,0}})_\beta \prod_{i=1}^{L} \frac{t_{0,i}^{e_{0,i}}}{e_{0,i}!} \prod_{d=0}^{\infty} \frac{t_{d,0}^{e_{d,0}}}{e_{d,0}!}}{\sum_{\beta \in Cen(f)} v^\beta Z'_{DT}(X; q|)_\beta}$$
$$= \frac{\sum_{\beta \in H_2(X,\mathbb{Z})} v^\beta (-q)^{-\frac{1}{2}\int_\beta c_1(X)} \sum_{\substack{e_{0,1},\ldots,e_{0,L} \in \mathbb{Z}_{\geqslant 0} \\ e_{d,0} \in \mathbb{Z}_{\geqslant 0}}} Z'_{DT}(X'; q| \prod_{i=1}^{L} \tilde{\tau}_0(\mathscr{F}T_i)^{e_{0,i}} \prod_{d=0}^{\infty} \tilde{\tau}_d(\mathscr{F}T_0)^{e_{d,0}})_{\mathscr{F}\beta} \prod_{i=1}^{L} \frac{t_{0,i}^{e_{0,i}}}{e_{0,i}!} \prod_{d=0}^{\infty} \frac{t_{d,0}^{e_{d,0}}}{e_{d,0}!}}{\sum_{\beta \in Cen(f)} v^\beta Z'_{DT}(X'; q|)_{\mathscr{F}\beta}}.$$

Now from (2), (21) and the assumption in Corollary 1.4, we obtain

$$\begin{aligned} Z'_{GW}(X'; u|)_{\mathscr{F}\beta} &= Z'_{GW}(X; u|)_{-\beta} \\ &= Z'_{DT}(X; q|)_{-\beta} = Z'_{DT}(X'; q|)_{\mathscr{F}\beta}, \quad \forall \beta \in Cen(f) \setminus \{0\}, \end{aligned}$$

and then the desired result follows from the above three long equalities.

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